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SHORTER NOTES

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HYPERINVARIANT SUBSPACES FOR APPROXIMATELY IDEMPOTENT OPERATORS[1]

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ABSTRACT. Operators which are "approximately idempotent" have hyperinvariant subspaces.

Let $\mathfrak{X}$ be a Banach space. $\mathfrak{B}(\mathfrak{X})$ denotes the set of all bounded linear operators on $\mathfrak{X}$. A closed subspace $\mathfrak{M}$ of $\mathfrak{X}$ is "hyperinvariant" for $E \in B(X)$ if $\mathfrak{M}$ is invariant for every operator commuting with E.

Theorem 1. *Suppose E belongs to $\mathfrak{B}(\mathfrak{X})$, $|E| > \frac{1}{2}$, $|I - E| > \frac{1}{2}$, and $|E^2 - E| \le \frac{1}{4}$. Then the norm-closed algebra generated by E and I contains an operator X with nontrivial kernel. In addition, $|X - (I - E)| \le \frac{1}{2}$. (The kernel of X is then a hyperinvariant subspace for E.)*

We say that an operator E has "pinched spectrum" if its spectrum may be translated and dilated to fit into $\{z \mid |z^2 - z| \le \frac{1}{4}\}$ without being disjoint from $\{z \mid \operatorname{Re} z > \frac{1}{2}\}$ or $\{z \mid \operatorname{Re} z < \frac{1}{2}\}$. If the spectral radius of $E^2 - E(r(E^2 - E))$ is equal to its norm and the spectrum of E is pinched, then E satisfies the hypotheses of Theorem 1. It follows that if $r(E^2 - E) = |T^{-1}(E^2 - E)T|$ for some $T \in \mathfrak{B}(\mathfrak{X})$ and E has pinched spectrum, then E has a hyperinvariant subspace.

Theorem 1 is a consequence (with $B = I - E$, $C = -E$) of the apparently more general

Theorem 2. *Suppose B and C belong to $\mathfrak{B}(\mathfrak{X})$, $BC = CB$, and $A = B - C$ is invertible. If*

a. $|A^{-1}|\,|A^{-1}CB| \le \frac{1}{4}$,

b. $|C|\,|A^{-1}| > \frac{1}{2}$, *and*

c. $|B|\,|A^{-1}| > \frac{1}{2}$,

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[1] These results are part of the author's Ph.D. dissertation written at the University of Virginia under the guidance of Professor Marvin Rosenblum.

then the norm-closed algebra $\mathfrak{A}$ *generated by* I, B, C, *and* A^{-1} *contains an operator* X *with nontrivial kernel and* $|X - B| \leq \frac{1}{2}|A^{-1}|$.

The proof of Theorem 2 is a straightforward application of Kantorovich's theorem on the convergence of Newton's method [1] to the equation $X^2 - BX - CX = 0$ in $\mathfrak{A}$. Starting the iteration at B, we conclude that there exists a solution $X \neq 0$ or $B + C$ with $|X - B| \leq \frac{1}{2}|A^{-1}|$. But $(X - (B + C))X = 0$ then implies that the kernel of X is not $\{0\}$.

REFERENCE

1. R. A. Tapia, *The Kantorovich theorem for Newton's method*, Amer. Math. Monthly **78** (1971), 389–392.

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