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Hyperplane Projections of the Unit Ball of ℓ_p^n *

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Abstract. Let $B_p^n = \{x \in \mathbb{R}^n;\ \sum_{i=1}^n |x_i|^p \le 1\}$, $1 \le p \le +\infty$. We study the extreme values of the volume of the orthogonal projection of B_p^n onto hyperplanes $H \subset \mathbb{R}^n$. For a fixed H, we prove that the ratio $\mathrm{vol}(P_H B_p^n)/\mathrm{vol}(B_p^{n-1})$ is non-decreasing in $p \in [1, +\infty]$.

1. Introduction

Computing the volume of sections or projections of convex sets is not easy, even in specific cases. However, in the last decades several authors managed to produce workable formulas, often related to probability and to Fourier analysis, and to determine extremal volumes of sections of certain bodies. The example of the cube $B_\infty^n = [-1, 1]^n$ was settled first: Hadwiger's result in [11] implies that the hyperplane sections through the origin have no less volume than the canonical sections. Vaaler [22] was able to show this for sections of arbitrary dimension. The largest hyperplane section of the cube was found by Ball [2], an important result which led to a negative answer to the Busemann–Petty problem in dimensions larger than 10 (see [3] for results in larger codimension). Denoting for $1 \le k \le n$, $H_k = \{x \in \mathbb{R}^n;\ \sum_{i=1}^k x_i = 0\}$, the best control on hyperplane sections of the cube reads as

$$\mathrm{vol}(B_\infty^n \cap H_1) \le \mathrm{vol}(B_\infty^n \cap H) \le \mathrm{vol}(B_\infty^n \cap H_2),$$

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for every vector-hyperplane $H \subset \mathbb{R}^n$. Here "vol" stands for the Lebesgue measure on the corresponding subspace.

Vaaler's result was considerably extended by Meyer and Pajor [14], who studied the unit balls of ℓ_p^n for $p \geq 1$, $B_p^n = \{x \in \mathbb{R}^n;\ \sum_{i=1}^n |x_i|^p \leq 1\}$. They showed that for any k-dimensional vector subspace E, the ratio

$$\frac{\mathrm{vol}(B_p^n \cap E)}{\mathrm{vol}(B_p^k)}$$

is a non-decreasing function of $p \geq 1$. Since this quantity is always equal to 1 for $p = 2$, this settles the question of minimal sections for $p \geq 2$ and of maximal sections for $p \leq 2$. These results were later extended to $p \in (0, 1)$ by Caetano [7] and the first named author [5]. Meyer and Pajor also found the extremal hyperplane sections for $p = 1$. They proved that for any hyperplane $H \subset \mathbb{R}^n$,

$$\mathrm{vol}(B_1^n \cap H_n) \leq \mathrm{vol}(B_1^n \cap H) \leq \mathrm{vol}(B_1^n \cap H_1),$$

and conjectured the same lower bound for $p \in (1, 2)$, which was proved by Koldobsky [12] even for $p \in (0, 2)$. We mention related works by Webb [23] about the sections of the regular simplex and by Oleszkiewicz and Pełczyński [16], concerning a complex version of Ball's upper bound.

The study of extremal volume projections is much less advanced, even though sections and projections are related via duality. The problem is that volume does not behave well under duality. Hence results for sections do not transfer to projections. However, in the few known cases, the results for hyperplane projections are in perfect duality with the ones for sections. The case of the Euclidean ball B_2^n is trivial, the one of the cube is very simple: if $H = \{a\}^\perp$ where $a \in \mathbb{R}^n$ satisfies $\sum_{i=1}^n a_i^2 = 1$, then denoting by P_H the orthogonal projection onto H, one has $\mathrm{vol}(P_H B_\infty^n) = \mathrm{vol}(B_\infty^{n-1})(\sum_{i=1}^n |a_i|)$ and therefore

$$\mathrm{vol}(B_\infty^{n-1}) = \mathrm{vol}(P_{H_1} B_\infty^n) \leq \mathrm{vol}(P_H B_\infty^n) \leq \mathrm{vol}(P_{H_n} B_\infty^n) = \sqrt{n}\,\mathrm{vol}(B_\infty^{n-1}).$$

We refer to [8] for more details on this and projections onto lower dimensions, and to [9] for projections of the regular simplex. The case of the unit ball of ℓ_1^n is more interesting. It is well known to be related to Khinchine inequalities, see [4]. Since all the facets of B_1^n have the same volume, and their outer normals are corresponding to the vertices of the cube (its dual body), the usual formula for volumes of projections of polytopes yields

$$\frac{\mathrm{vol}(P_H B_1^n)}{\mathrm{vol}(B_1^{n-1})} = \frac{\mathbb{E}|\sum_{i=1}^n \varepsilon_i a_i|}{\sqrt{\mathbb{E}|\sum_{i=1}^n \varepsilon_i a_i|^2}},$$

where ε_i are independent symmetric Bernoulli variables. Finding extremal projections reduces to computing the sharp constants in the Khinchine inequality for these variables, which was done by Szarek [21] (see also [13] for a short proof). The result for projections is

$$\mathrm{vol}(P_{H_2} B_1^n) \leq \mathrm{vol}(P_H B_1^n) \leq \mathrm{vol}(P_{H_1} B_1^n).$$

The aim of the present paper is to study the extremal projections of B_p^n for $p \in (1, +\infty)$. We bring the knowledge on this problem to the same level as it is for sections. Although we do not use duality in our proof, the reader will see that there is a continuous analogy between our methods and the ones used for sections, but not an obvious one.

2. A Khinchine Formula for Volumes of Projections

In this section we derive a simple formula for the volume of a hyperplane projection of the unit ball of ℓ_p^n. In what follows we work on the standard Euclidean space $(\mathbb{R}^n, \|\cdot\|_2, \langle\cdot,\cdot\rangle)$.

Let K be a convex symmetric body in $\mathbb{R}^n$. We denote by $\text{area}(K)$ the surface area of K. Let σ_K be the normalized surface area measure on ∂K. One can define another natural probability measure on ∂K, the so-called "cone measure," which we denote by μ_K. For any $A \subset \partial K$, $\mu_K(A)$ is defined as follows:

$$\mu_K(A) = \frac{\text{vol}([0,1]A)}{\text{vol}(K)} = \frac{\text{vol}(ta;\ a \in A,\ 0 \le t \le 1)}{\text{vol}(K)},$$

i.e., $\mu_K(A)$ is the volume of the cone with base A and cusp 0, normalized by the volume of K. It was proved in [15] that for almost every $x \in \partial K$,

$$\frac{d\sigma_K}{d\mu_K}(x) = \frac{n \cdot \text{vol}(K)}{\text{area}(K)} \|\nabla(\|\cdot\|_K)(x)\|_2,$$

where $\|\cdot\|_K$ is the Minkowski functional of K. Fix some a in the unit sphere S^{n-1}. By a well-known formula of Cauchy,

$$2\,\text{vol}(P_{a^\perp}K) = \text{area}(K) \int_{\partial K} |\langle \mathbf{n}(x), a\rangle|\, d\sigma_K(x),$$

where $\mathbf{n}(x)$ is the outer unit normal to ∂K at x, see, e.g., [20]. Hence,

$$\begin{aligned} 2\,\text{vol}(P_{a^\perp}K) &= \text{area}(K) \int_{\partial K} \frac{\left|\langle \nabla(\|\cdot\|_K)(x), a\rangle\right|}{\|\nabla(\|\cdot\|_K)(x)\|_2} \cdot \frac{d\sigma_K}{d\mu_K}(x)\, d\mu_K(x) \\ &= n \cdot \text{vol}(K) \int_{\partial K} \left|\langle \nabla(\|\cdot\|_K)(x), a\rangle\right| d\mu_K(x). \end{aligned}$$

Specializing to $K = B_p^n$ we get that for some constant $C(p,n)$,

$$\text{vol}(P_{a^\perp}B_p^n) = C(p,n) \int_{\partial B_p^n} \left| \sum_{i=1}^n |x_i|^{p-1} \,\text{sign}(x_i) a_i \right| d\mu_{B_p^n}(x).$$

This formula is useful since $\mu_{B_p^n}$ has a concrete probabilistic description. Let g be a random variable with density $1/(2\Gamma(1+1/p))e^{-|t|^p}$ $(t \in \mathbb{R})$. If $g_1, \ldots, g_n$ are i.i.d. copies of g, set

$$S = \left(\sum_{i=1}^n |g_i|^p \right)^{1/p},$$

and consider the random vector

$$Z = \left(\frac{g_1}{S}, \dots, \frac{g_n}{S}\right) \in \mathbb{R}^n.$$

The following result appeared in [19], and later independently also in [18]:

Theorem 1. *The random vector Z is independent of S. Moreover, for every measurable $A \subset \partial B_p^n$ we have*

$$\mu_{B_p^n}(A) = P(Z \in A).$$

Plugging this in the above formula for $\mathrm{vol}(P_{a^\perp} B_p^n)$ we get

$$\begin{aligned}
\mathrm{vol}(P_{a^\perp} B_p^n) &= C(p,n)\mathbb{E}\left[\frac{\left|\sum_{i=1}^n |g_i|^{p-1}\,\mathrm{sign}(g_i/S)a_i\right|}{S^{p-1}}\right] \\
&= C(p,n)\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p-1}} \cdot \mathbb{E}\left[\frac{\left|\sum_{i=1}^n |g_i|^{p-1}\,\mathrm{sign}(g_i/S)a_i\right|}{S^{p-1}}\right] \\
&= \frac{C(p,n)}{\mathbb{E}S^{p-1}}\mathbb{E}\left|\sum_{i=1}^n |g_i|^{p-1}\,\mathrm{sign}(g_i)a_i\right|,
\end{aligned}$$

where we have used in the last equality the independence of S and Z.

Let X be the random variable $|g|^{p-1}\,\mathrm{sign}(g)$. For $p = 1$, X is a Rademacher. It is easy to check that for $p > 1$ the density of X is

$$\frac{p}{2(p-1)\Gamma(1/p)}|t|^{(2-p)/(p-1)}e^{-|t|^{p/(p-1)}}.$$

Summing up, we have proved the following extension of the formula for the volume of projections of B_1^n which appeared in the introduction:

Proposition 2. *Let $X_1, \dots, X_n$ be i.i.d. random variables with density proportional to $|t|^{(2-p)/(p-1)}e^{-|t|^{p/(p-1)}}$, $p > 1$. Then for every $a \in S^{n-1}$,*

$$\frac{\mathrm{vol}(P_{a^\perp} B_p^n)}{\mathrm{vol}(B_p^{n-1})} = \frac{\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|}{\mathbb{E}|X_1|}.$$

Remark. In the case $0 < p < 1$, B_p^n is no longer convex, and the Cauchy formula fails. Although some estimates can be made, more work needs to be done in the study of hyperplane projections of B_p^n when $0 < p < 1$.

3. An Analogue of the Meyer–Pajor Theorem

The aim of this section is to establish the following:

Theorem 3. *Let $1 \leq p \leq q \leq +\infty$ and let H be a hyperplane in $\mathbb{R}^n$. Then*

$$\frac{\mathrm{vol}(P_H B_p^n)}{\mathrm{vol}(B_p^{n-1})} \leq \frac{\mathrm{vol}(P_H B_q^n)}{\mathrm{vol}(B_q^{n-1})}.$$

We prove this fact by an induction argument, which is nicely explained in terms of the Choquet ordering of measures. The Choquet ordering originated from the proof of the classical Choquet representation theorem, where the main interest focused on the study of maximal measures, see, e.g., [17] and [10]. It turns out that this notion has some purely probabilistic applications. We start with some definitions and useful facts. Since we are interested in symmetric measures, we formulate the definition in this case only.

Definition 4. Let μ and ν be symmetric Radon measures on $\mathbb{R}^n$. We say that μ is smaller than ν with respect to the symmetric Choquet order and write $\mu \prec \nu$ if, for every even non-negative convex function $c\colon \mathbb{R}^n \to [0, +\infty]$, one has

$$\int_{\mathbb{R}^n} c \, d\mu \leq \int_{\mathbb{R}^n} c \, d\nu.$$

Switching to probabilistic notation, for any two symmetric random vectors $U, V \in \mathbb{R}^n$, we say that $U \prec V$ if for every even non-negative convex function $c\colon \mathbb{R}^n \to \mathbb{R}$ we have $\mathbb{E}\, c(U) \leq \mathbb{E}\, c(V)$.

This ordering behaves well under products:

Lemma 5. *Let μ, ν be symmetric Radon measures on $\mathbb{R}^n$ such that $\mu \prec \nu$. Then for any $k \geq 2$, the product measures compare*:

$$\mu^k \prec \nu^k.$$

Proof. By induction, it is enough to show the following: if $\mu \prec \nu$ are symmetric measures on $\mathbb{R}^n$ and λ is a symmetric measure on R^ℓ, then $\mu \otimes \lambda \prec \nu \otimes \lambda$. To see this, consider an even non-negative convex function c on $\mathbb{R}^{n+\ell}$ and notice that the function

$$s(x) := \int_{R^\ell} c(x, y) \, d\lambda(y)$$

is also convex and even, because both c and λ are symmetric with respect to the origin. Therefore

$$\int_{\mathbb{R}^{n+\ell}} c \; d\mu \; d\lambda = \int_{\mathbb{R}^n} s \, d\mu \leq \int_{\mathbb{R}^n} s \, d\nu = \int_{\mathbb{R}^{n+\ell}} c \; d\nu \; d\lambda. \qquad \square$$

To apply this lemma, we need to characterize the symmetric Choquet ordering for measures on $\mathbb{R}$.

Lemma 6. *Let U and V be symmetric, real-valued random variables with $\mathbb{E}|U| = \mathbb{E}|V| < \infty$. Then $U \prec V$ if and only if for every $t \geq 0$,*

$$\mathbb{E}\big[(|U| - t)\cdot 1_{\{|U|\geq t\}}\big] \leq \mathbb{E}\big[(|V| - t)\cdot 1_{\{|V|\geq t\}}\big].$$

Proof. Let $c\colon \mathbb{R} \to \mathbb{R}$ be an even, non-negative, convex, twice differentiable function. Taylor's formula gives

$$c(a) = c(0) + c'(0)a + \int_0^a c''(t)(a-t)\,dt.$$

Hence, by Fubini's theorem,

$$\mathbb{E}\,c(U) = \mathbb{E}\,c(|U|) = c(0) + c'(0)\mathbb{E}|U| + \int_0^\infty c''(t)\mathbb{E}\big[(|U| - t)\cdot 1_{\{|U|\geq t\}}\big]dt,$$

and similarly for V. Hence, by approximating a general non-negative even convex function on $\mathbb{R}$ by a twice differentiable one, we get that $U \prec V$ if and only if for every measurable $\theta\colon [0,\infty) \to [0,\infty)$,

$$\int_0^\infty \theta(t)\mathbb{E}\big[(|U| - t)\cdot 1_{\{|U|\geq t\}}\big]dt \leq \int_0^\infty \theta(t)\mathbb{E}\big[(|V| - t)\cdot 1_{\{|V|\geq t\}}\big]dt,$$

and this implies the required result. □

For absolutely continuous measures, the above condition could be expressed more explicitly. For any integrable $f\colon \mathbb{R} \to \mathbb{R}$ denote $f^{(-1)}(x) = -\int_x^\infty f(t)\,dt$, and if $f^{(-1)}$ is also integrable, set $f^{(-2)} = [f^{(-1)}]^{(-1)}$.

Corollary 7. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be non-negative even integrable functions which are continuous on $\mathbb{R}\backslash\{0\}$ and such that*

$$\int_0^\infty f(t)\,dt = \int_0^\infty g(t)\,dt < \infty \quad \text{and} \quad \int_0^\infty t f(t)\,dt = \int_0^\infty t g(t)\,dt < \infty.$$

Then $f\,dx \prec g\,dx$ if and only if $f^{(-2)} \leq g^{(-2)}$ on $\mathbb{R}^+$.

Proof. One just has to notice that

$$f^{(-2)}(t) = \int_t^\infty \int_s^\infty f(u)\,du\,ds = \int\int_{\{u>s>t\}} f(u)\,ds\,du = \int_t^\infty f(u)(u-t)\,du. \quad \square$$

We apply this by using the following convenient necessary condition:

Lemma 8. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be non-negative even integrable functions such that*

$$\int_0^\infty f(t)\,dt = \int_0^\infty g(t)\,dt < \infty \quad \text{and} \quad \int_0^\infty t f(t)\,dt = \int_0^\infty t g(t)\,dt < \infty.$$

Assume that there are $0 < x < y < \infty$ such that $\{t \geq 0;\ g(t) < f(t)\} = (x, y)$. Then $f\,dx \prec g\,dx$.

Proof. For every $t \geq 0$ define $\varphi(t) = g^{(-2)}(t) - f^{(-2)}(t)$. By the construction, $\lim_{t\to\infty} \varphi(t) = 0$. Moreover, as explained before, $f^{(-2)}(0) = \int_0^\infty t f(t)\, dt = g^{(-2)}(0)$, so that $\varphi(0) = 0$. Now, $\varphi'' = g - f$ so that by our assumption φ' is increasing on $[0, x]$, decreasing on (x, y), and increasing again on $[y, \infty)$. However, by the definition, $\lim_{t\to\infty} \varphi'(t) = 0$ and $\varphi'(0) = \int_0^\infty (f - g) = 0$. Thus there is some $u > 0$ such that $\varphi' \geq 0$ on $[0, u]$ and $\varphi' \leq 0$ on $[u, \infty)$. Finally, since φ equals zero at 0 and ∞, first increases, and then decreases, we conclude that $\varphi \geq 0$, and the previous corollary gives the result. □

The latter criterion is easily checked for exponential type densities:

Lemma 9. *Let α_1, α_2 be real numbers and let $\beta_1, \beta_2, c_1, c_2, d_1, d_2$ be positive real numbers. For $i = 1, 2$, consider the function*

$$f_i(t) = c_i t^{\alpha_i} e^{-(t/d_i)^{\beta_i}}, \qquad t \in (0, +\infty).$$

If $\alpha_1 < \alpha_2$ and $0 < \beta_1 < \beta_2$, then either the function $f_1 - f_2$ is non-negative or there exist $0 < x < y < +\infty$ such that $f_1 - f_2$ is negative exactly on the set (x, y).

Proof. Taking logarithms, $f_1(t) \geq f_2(t)$ amounts to $\varphi(t) \geq 0$, where for $t > 0$,

$$\varphi(t) = \log\left(\frac{c_1}{c_2}\right) + (\alpha_1 - \alpha_2)\log t + \left(\frac{t}{d_2}\right)^{\beta_2} - \left(\frac{t}{d_1}\right)^{\beta_1}.$$

This function clearly tends to $+\infty$ at 0 and $+\infty$. We study the sign of its derivative, or rather of the more convenient function

$$\psi(t) = t\varphi'(t) = \alpha_1 - \alpha_2 + \beta_2\left(\frac{t}{d_2}\right)^{\beta_2} - \beta_1\left(\frac{t}{d_1}\right)^{\beta_1}.$$

This function has a negative value at 0 and tends to $+\infty$ at $+\infty$. Next,

$$\psi'(t) = \beta_2^2 \frac{t^{\beta_2 - 1}}{d_2^{\beta_2}} - \beta_1^2 \frac{t^{\beta_1 - 1}}{d_1^{\beta_1}}$$

is non-negative if and only if $t^{\beta_2 - \beta_1} \geq \beta_1^2 d_2^{\beta_2} \beta_2^{-2} d_1^{-\beta_1}$. Since $\beta_2 > \beta_1$, this happens exactly on an interval of the form $[z, +\infty)$ for some $z > 0$. Hence, ψ starts from a negative value at zero, decreases, and then increases to $+\infty$. It is therefore first negative and then positive. So the original function φ is $+\infty$ at 0, first decreases, and then increases to $+\infty$. The conclusion follows. □

Proof of Theorem 3. We can restrict to $1 < p < q < \infty$. For $r > 1$, let $X^{(r)}$ be the random variable with density:

$$\frac{r}{2(r-1)\Gamma(1/r)} |t|^{(2-r)/(r-1)} e^{-|t|^{r/(r-1)}},$$

and let $Y^{(r)} = X^{(r)}/(\mathbb{E}|X^{(r)}|)$. Its density is given by

$$\frac{r'}{\Gamma(1/r)^{r'}}|t|^{(2-r)/(r-1)} \exp\left[-\left(\frac{|t|}{\Gamma(1/r)}\right)^{r'}\right],$$

where $1/r + 1/r' = 1$. Using the previous lemmas, we get $Y^{(p)} \prec Y^{(q)}$. Tensorizing this inequality we get that $(Y_1^{(p)}, \ldots, Y_n^{(p)}) \prec (Y_1^{(q)}, \ldots, Y_n^{(q)})$ where $Y_1^{(p)}, \ldots, Y_n^{(p)}$ are i.i.d. copies of $Y^{(p)}$ and similarly for q. Therefore for every $a \in S^{n-1}$, one has

$$\frac{\mathbb{E}\left|\sum_{i=1}^n a_i X_i^{(p)}\right|}{\mathbb{E}|X^{(p)}|} = \mathbb{E}\left|\sum_{i=1}^n a_i Y_i^{(p)}\right| \le \mathbb{E}\left|\sum_{i=1}^n a_i Y_i^{(q)}\right| = \frac{\mathbb{E}\left|\sum_{i=1}^n a_i X_i^{(q)}\right|}{\mathbb{E}|X^{(q)}|}.$$

The theorem then follows from Proposition 2. □

4. Extremal Projections when $p \ge 2$

This section is devoted to the proof of the following result:

Theorem 10. *Let $p \ge 2$ and H be a hyperplane of $\mathbb{R}^n$, then*

$$\mathrm{vol}(P_{H_1} B_p^n) \le \mathrm{vol}(P_H B_p^n) \le \mathrm{vol}(P_{H_n} B_p^n).$$

The lower bound is a consequence of Theorem 3 applied with 2 and p. It also follows from Meyer and Pajor's result [14]. Indeed, for every hyperplane H containing the origin $B_p^n \cap H \subset P_H B_p^n$ so $\mathrm{vol}(P_H B_p^n) \ge \mathrm{vol}(B_p^n \cap H) \ge \mathrm{vol}(B_p^{n-1})$. Our argument for the upper bound will provide yet another proof of the lower bound.

We begin with a lemma, which originates from Koldobsky's article [12]. First recall that an infinitely differentiable function $f\colon (0, +\infty) \to \mathbb{R}^+$ is said to be *completely monotonic* if for every $n = 0, 1, 2, \ldots, (-1)^n f^{(n)} \ge 0$. Direct differentiation shows that $f(t) = t^{-\alpha}$, $\alpha > 0$, is completely monotonic. A straightforward induction shows also that $f(t) = e^{-t^\beta}$ is completely monotonic provided $0 < \beta \le 1$. Similarly, the product of completely monotonic functions is still completely monotonic. A classical theorem of Bernstein, see, for example, [24], asserts that f is completely monotonic if and only if there is a non-negative Borel measure μ on $\mathbb{R}^+$ such that $\mu([0, \infty)) = f(0^+)$ and for each $x > 0$,

$$f(x) = \int_0^\infty e^{-tx}\, d\mu(t).$$

For complete proofs of the above facts, refer to [24].

Lemma 11. *Let $g\colon \mathbb{R} \to \mathbb{R}^+$ be an even integrable function such that $g(\sqrt{t})$ is completely monotonic. Then the function*

$$t \mapsto \log \hat{g}(\sqrt{t}), \qquad t > 0,$$

is convex. Here $\hat{g}(\xi) = \int_{\mathbb{R}} e^{is\xi} g(s)\, ds$ is the Fourier transform of g.

Proof. By Bernstein's theorem, there is a non-negative measure μ on $\mathbb{R}^+$ such that for every $t \geq 0$,

$$g(\sqrt{t}) = \int_0^\infty e^{-tx}\, d\mu(x).$$

So for every $t \in \mathbb{R}$,

$$g(t) = \int_0^\infty e^{-t^2 x}\, d\mu(x).$$

Taking Fourier transforms in t and using Fubini's theorem, we get

$$\hat{g}(u) = \int_0^\infty e^{-u^2/(4x)} \sqrt{\frac{\pi}{x}}\, d\mu(x).$$

Hence, for $u \geq 0$,

$$\hat{g}(\sqrt{u}) = \int_0^\infty \left(e^{-1/(4x)}\right)^u \sqrt{\frac{\pi}{x}}\, d\mu(x),$$

which is log-convex by Hölder's inequality. □

Remark. It follows from the above proof that if $g(\sqrt{u})$ is completely monotonic, then also $\hat{g}(\sqrt{u})$ is completely monotonic.

Proof of Theorem 10. Let a be a unit vector orthogonal to H. Using the representation

$$|s| = c \int_0^\infty \frac{1 - \cos(us)}{u^2}\, du,$$

and the notation of Proposition 2, we get that

$$\mathbb{E}\left|\sum_{k=1}^n a_k X_k\right| = c\,\mathbb{E}\int_0^\infty \frac{1 - \mathrm{Re}\left(e^{iu\sum_{k=1}^n a_k X_k}\right)}{u^2}\, du$$
$$= c \int_0^\infty \frac{1 - \prod_{k=1}^n \mathbb{E}\left(e^{iua_k X}\right)}{u^2}\, du,$$

where we also used the symmetry of X. Let f be the density of X. For $t \geq 0$,

$$f(\sqrt{t}) = c_p t^{(2-p)/(2p-2)} e^{-t^{p/(2p-2)}}.$$

Since, for $p > 2$, $(2-p)/(2p-2) < 0$ and $p/(2p-2) \in (0, 1]$, we get by the preceding remarks that $f(\sqrt{t})$ is a product of two completely monotonic functions, and is therefore completely monotonic. Lemma 11 implies that $\log \hat{f}(\sqrt{t})$ is convex. So for every $u \geq 0$, the function $(\alpha_j)_{j=1}^n \mapsto \sum_{j=1}^n \log \hat{f}(\sqrt{\alpha_j u})$ is convex on the simplex $\{\alpha \geq 0;\ \sum_{j=1}^n \alpha_j = 1\}$. Thus, it attains its minimum at the barycenter of the simplex and its maximum at the vertices. Since $\sum_{j=1}^n a_j^2 = 1$, we get that

$$\hat{f}\left(\frac{u}{\sqrt{n}}\right)^n \leq \prod_{j=1}^n \hat{f}(a_j u) \leq \hat{f}(u)\,\hat{f}(0)^{n-1} = \hat{f}(u).$$

Combining this estimate with the relation

$$\mathrm{vol}(P_{a^\perp} B_p^n) = c'_{p,n} \int_0^\infty \frac{1 - \prod_{j=1}^n \hat{f}(a_j u)}{u^2}\, du,$$

we obtain that

$$\mathrm{vol}(P_{(1,0,\ldots,0)^\perp} B_p^n) \le \mathrm{vol}(P_{a^\perp} B_p^n) \le \mathrm{vol}(P_{(1/\sqrt{n},\ldots,1/\sqrt{n})^\perp} B_p^n). \qquad \square$$

5. The Case $1 \le p \le 2$

Theorem 12. *Let $1 \le p \le 2$ and let H be a hyperplane of $\mathbb{R}^n$, then*

$$\max\left(\frac{1}{\sqrt{2}}, \left(\frac{n-1}{n}\right)^{(n-1)(1/p-1/2)}\right) \mathrm{vol}(B_p^{n-1}) \le \mathrm{vol}(P_H B_p^n) \le \mathrm{vol}(P_{H_1} B_p^n).$$

The lower bound is sharp only for $p = 1$ or 2.

Proof. The upper bound follows from Theorem 3 with $q = 2$. The $1/\sqrt{2}$ lower bound can be viewed as a consequence of it too, and of the optimal lower bound on projections of B_1^n:

$$\frac{\mathrm{vol}(P_H B_p^n)}{\mathrm{vol}(B_p^{n-1})} \ge \frac{\mathrm{vol}(P_H B_1^n)}{\mathrm{vol}(B_1^{n-1})} \ge \frac{\mathrm{vol}(P_{H_2} B_1^n)}{\mathrm{vol}(B_1^{n-1})} = \frac{1}{\sqrt{2}}.$$

There is however a shorter argument: introduce $\varepsilon_1, \ldots, \varepsilon_n$ symmetric i.i.d. Bernoulli variables, independent from the vector $(X_1, \ldots, X_n)$. Since the X_i's are symmetric, the vectors $(\varepsilon_1|X_1|, \ldots, \varepsilon_n|X_n|)$ and $(X_1, \ldots, X_n)$ have the same law, therefore, using only the Khinchine inequality for Bernoulli laws,

$$\frac{\mathrm{vol}(P_{a^\perp} B_p^n)}{\mathrm{vol}(B_p^{n-1})} = \frac{\mathbb{E}_\varepsilon \mathbb{E}_X \left| \sum_{i=1}^n a_i \varepsilon_i |X_i| \right|}{\mathbb{E}|X_1|} \ge \frac{\mathbb{E}_\varepsilon \left| \sum_{i=1}^n a_i \varepsilon_i \mathbb{E}|X_i| \right|}{\mathbb{E}|X_1|}$$

$$\ge \frac{1}{\sqrt{2}} \left(\mathbb{E} \left(\sum_{i=1}^n a_i \varepsilon_i \right)^2 \right)^{1/2} = \frac{1}{\sqrt{2}},$$

for any $a \in S^{n-1}$. The other part of the lower bound is proved in [6] using the reverse Brascamp–Lieb inequality. $\square$

The calculation of the minimal volume hyperplane projection of B_p^n, $1 < p < 2$, seems more difficult. This reflects the situation for maximal sections of B_p^n in the case $2 < p < \infty$. Ball's calculation of the maximal section of the cube has not been extended so far to any $p < \infty$. Moreover, recent results of Oleszkiewicz (private communication) show that for $2 < p < 24$ and large n, $\mathrm{vol}(B_p^n \cap H_n) > \mathrm{vol}(B_p^n \cap H_2)$, so that the direction of the maximal hyperplane changes with p.

A similar phenomenon occurs for hyperplane projections in the case $1 \le p \le 2$. As remarked in the Introduction, the minimal hyperplane projection for $p = 1$ is orthogonal to the direction $(1, 1, 0, \dots, 0)$. Now, by the central limit theorem,

$$\lim_{n\to\infty} \frac{\mathrm{vol}(P_{H_n} B_p^n)}{\mathrm{vol}(B_p^{n-1})} = \lim_{n\to\infty} \frac{\mathbb{E}\left|(\sum_{i=1}^n X_i)/\sqrt{n}\right|}{\mathbb{E}|X|}$$

$$= \frac{(\mathbb{E}X^2)^{1/2}}{\mathbb{E}|X|} \int_{-\infty}^{\infty} |u| e^{-u^2/2} \frac{du}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi} \cdot \Gamma\left(\frac{1}{p}\right) \Gamma\left(2 - \frac{1}{p}\right)}.$$

On the other hand,

$$\frac{\mathrm{vol}(P_{H_2} B_p^n)}{\mathrm{vol}(B_p^{n-1})} = \frac{\mathbb{E}\left|(X_1 + X_2)/\sqrt{2}\right|}{\mathbb{E}|X|} = \frac{\mathrm{vol}(P_{(1,1)^\perp} B_p^2)}{\mathrm{vol}(B_p^1)} = 2^{1/2 - 1/p}.$$

So that

$$\lim_{n\to\infty} \frac{\mathrm{vol}(P_{H_n} B_p^n)}{\mathrm{vol}(P_{H_2} B_p^n)} = \frac{2^{1/p}}{\sqrt{\pi}} \cdot \sqrt{\Gamma\left(\frac{1}{p}\right) \Gamma\left(2 - \frac{1}{p}\right)}.$$

Set $y = 1 - 1/p \in [0, 1)$. The previous limit is

$$\psi(y) = \frac{2^{1-y}}{\sqrt{\pi}} \cdot \sqrt{\Gamma(1-y)\,\Gamma(1+y)}.$$

The function ψ is clearly strictly log-convex, $\psi(0) > 1 = \psi(\frac{1}{2})$ and $\lim_{y\to 1} \psi(y) = \infty$. Moreover, by the complement formula (see [1]),

$$\psi(\tfrac{1}{4}) = \frac{2^{3/4}}{\sqrt{\pi}} \sqrt{\Gamma(\tfrac{3}{4})\Gamma(\tfrac{1}{4})/4} = \frac{2^{-1/4}}{\sqrt{\pi}} \sqrt{\frac{\pi}{\sin(\pi/4)}} = 1 = \psi(\tfrac{1}{2}).$$

It follows that $\psi(y) < 1$ for $\frac{1}{4} < y < \frac{1}{2}$ and $\psi(y) > 1$ for $0 < y < \frac{1}{4}$. Hence for $1 < p < \frac{4}{3}$ and n large enough, $\mathrm{vol}(P_{H_2} B_p^n) < \mathrm{vol}(P_{H_n} B_p^n)$, whereas the reverse inequality holds for $\frac{4}{3} < p < 2$ and large n.

It is plausible that for $p < 2$ close to 2, the minimal volume projection is the one onto H_n. Indeed, for any hyperplane H and $p \ge 2$, we have shown that $\mathrm{vol}(P_H B_p^n) \le \mathrm{vol}(P_{H_n} B_p^n)$. Since these quantities are differentiable in p and coincide for $p = 2$, it follows that

$$\frac{d}{dp}\mathrm{vol}(P_H B_p^n)|_{p=2} \le \frac{d}{dp}\mathrm{vol}(P_{H_n} B_p^n)|_{p=2},$$

which also gives information for $p < 2$ very close to 2. If the previous inequality were strict for some direction H, then $\mathrm{vol}(P_H B_p^n) \ge \mathrm{vol}(P_{H_n} B_p^n)$ would hold for $p \in (2-\varepsilon, 2]$. It would be nice to prove the strict inequality for hyperplanes which are not orthogonal to the main diagonal. Note that the same reasoning applies for sections and that, for once, any result on projections would yield the corresponding result for sections and vice versa. Indeed, for any H and p, $\mathrm{vol}(B_p^n \cap H) \le \mathrm{vol}(P_H B_p^n)$ with equality for $p = 2$, so their derivatives at $p = 2$ coincide.

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