

Hypersingular Integrals and Their Applications

Stefan G. Samko

Rostov State University, Russia
and
University of Algarve, Portugal



London and New York

Contents

Preface	xv
Notation	1
Part 1. Hypersingular integrals	3
Chapter 1. Some basics from the theory of special functions and operator theory	5
1. Some special functions	5
1.1. Euler gamma- and beta-functions	5
1.2. Orthogonal Gegenbauer and Chebyshev polynomials $C_m^\lambda(t)$, $\lambda > 0$, and $T_m(t)$	7
1.3. Bessel functions $J_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$	8
1.4. The Gauss hypergeometric function	9
2. Basics of the theory of spherical harmonics	10
2.1. The basis of spherical harmonics	10
2.2. The addition theorem and the Funk–Hekke formula	12
2.3. The connection between the smoothness of a function and the convergence of its Fourier–Laplace series	14
3. On fractional integration and differentiation of functions of one variable	16
4. Fourier transforms and the Wiener ring; Fourier multipliers	18
4.1. The Wiener ring	19
4.2. Fourier multipliers	24
5. Miscellaneous	24
5.1. On estimation of the Taylor remainder for Hölderian functions	24
5.2. Analyticity of integrals depending on a parameter	25
5.3. Hadamard’s approach to divergent integrals and their regularization	26
5.4. Estimates of some multidimensional integrals	30
6. Bibliographical notes to Chapter 1	34
Chapter 2. The Riesz potential operator and Lizorkin type invariant spaces Φ_V	37
1. The Riesz potential operator	37
1.1. Definitions and the Fourier transform of the Riesz potential in the case $0 < \Re\alpha < \frac{n-1}{2}$	37
1.2. Mapping properties of the operator I^α in the spaces $L_p(\mathbb{R}^n)$ and $L_p(\mathbb{R}^n; \rho)$	38
2. The Lizorkin space Φ	39

3.	The Riesz potential in the general case $\Re\alpha > 0$ and invariance of the space Φ	43
3.1.	The function $1/ x ^\alpha$ as a distribution in $\mathcal{S}'(R^n)$ or $\Phi'(R^n)$	43
3.2.	The Fourier transform of the distribution $1/ x ^\alpha$	44
3.3.	The Riesz kernel and the Riesz potential in the general case	45
4.	Lizorkin-type spaces Φ_V	46
4.1.	Definitions, examples and simple properties	46
4.2.	Denseness of Ψ_V and Φ_V in $L_p(R^n)$	50
5.	Connections of the Riesz potential with the Poisson and Gauss-Weierstrass semigroups of operators	53
6.	Bibliographical notes to Chapter 2	54
Chapter 3. Hypersingular integrals with constant characteristics		55
1.	The Riesz fractional differentiation \mathbb{D}^α	56
1.1.	Fourier transform of the hypersingular integral $\mathbb{D}^\alpha f$	57
1.2.	Zeros of the function $d_{n,\ell}(\alpha)$	58
1.3.	The phenomenon of annihilation of hypersingular integrals	61
1.4.	Convergence of the hypersingular integral $\mathbb{D}^\alpha f$ in the case of smooth functions; moments of hypersingular integrals	62
1.5.	Diminution of order $\ell > \Re\alpha$ to $\ell > 2 \left[\frac{\Re\alpha}{2} \right]$ in the case of a non-centred difference	64
2.	The hypersingular operator \mathbb{D}^α as inverse to the Riesz potential operator J^α	65
2.1.	Auxiliary functions $\Delta_{\ell,\alpha}(x, h)$, $k_{\ell,\alpha}(x)$ and $\mathcal{K}_{\ell,\alpha}(x)$	65
2.2.	The Inversion theorem	70
2.3.	Almost everywhere convergence of the Riesz derivative	71
2.4.	Reducing the order α : the case of complex α	73
3.	More on the independence of $\mathbb{D}^\alpha f$ on the order ℓ of finite differences	73
4.	A Zygmund type estimate for the continuity modulus of hypersingular integrals	75
5.	Connection of hypersingular integrals with fractional powers of operators	76
6.	Hypersingular integrals with generalized differences	78
6.1.	Generalized differences	78
6.2.	Modified hypersingular integrals and their Fourier transforms	80
6.3.	Properties of the function $\tilde{A}_\ell(\alpha)$	81
6.4.	Modified hypersingular integral as inverse to the Riesz potential operator	83
7.	Bibliographical notes to Chapter 3	84
Chapter 4. Potentials and hypersingular integrals with homogeneous characteristics		87
1.	Preliminaries on some means of functions on the unit sphere in R^n	87
1.1.	Some means of functions on the unit sphere	87
1.2.	On the choice of a rotation which is smooth with respect to the parameter	89
1.3.	On smoothness of the means $M_\omega(x', t)$	90
1.4.	Cavalieri type principle and the means of spherical harmonics	93
2.	Potential type operators with homogeneous characteristics	94
2.1.	The symbol in the case $0 < \Re\alpha < 1$	95

2.2. The Hadamard constructions of divergent integrals over the sphere in the case of singularity at an equator	95
2.3. On the symbol $\mathcal{K}_\omega^\alpha(x)$ in the case $\Re\alpha \geq 1$	99
2.4. Smoothness properties of the symbol $\mathcal{K}_\omega^\alpha(x)$	100
2.5. On the inversion of the operator "characteristic \rightarrow symbol"	102
3. Hypersingular integrals with homogeneous characteristics	103
3.1. Classification of hypersingular integrals	104
3.2. Symbol of a hypersingular integral	106
3.3. A hypersingular integral as a convolution with the function $\frac{\Omega(x')}{ x ^{n+\alpha}}$	108
3.4. Symbol of a hypersingular operator as the Fourier transform of a distribution	110
3.5. Some estimates of hypersingular integrals of smooth functions vanishing at infinity	112
3.6. Smoothness properties of the symbol $\mathcal{D}_\Omega^\alpha(x)$	113
4. Representation of homogeneous differential operators in partial derivatives by means of hypersingular operators	114
4.1. Hypersingular integrals with harmonic characteristics	114
4.2. On a certain integral equation of the first kind on the unit sphere: the case of a non-integer α	116
4.3. The case of integer α and the main representation theorem	118
5. Bibliographical notes to Chapter 4	120
Chapter 5. Hypersingular integrals with non-homogeneous characteristics	121
1. On the connection $\mathbb{D}_{\Omega,\epsilon}^\alpha f = A_\epsilon \mathbb{D}_\epsilon^\alpha f$	121
2. The limiting operator $A = \lim_{\epsilon \rightarrow 0} A_\epsilon$ and justification of the convergence	127
2.1. Integrability of the kernel $a(x)$ and its Fourier transform	127
2.2. Justification of the convergence $A_\epsilon \rightarrow A$ as $\epsilon \rightarrow 0$	129
3. The main theorems on the convergence of $\mathbb{D}_{\Omega,\epsilon}^\alpha f$	132
4. Representation of a hypersingular integral as a convolution with a function of the type $\frac{\Psi(x)}{ x ^{n+\alpha}}$ and the inverse problem	133
4.1. On a <i>f.p.</i> -convolution	134
4.2. Representation of a hypersingular integral as a <i>f.p.</i> -convolution	134
4.3. On a functional equation	136
4.4. On the inverse problem	139
5. On the continuity modulus of a hypersingular integral	143
6. Bibliographical notes to Chapter 5	143
Chapter 6. Hypersingular integrals on the unit sphere	145
1. Spherical convolution operators. Operators commuting with rotations	145
1.1. Spherical convolution operators on harmonics	145
1.2. Operators commuting with rotations	147
1.3. Generalized functions (distributions) on the sphere	149
1.4. Boundedness of convolution operators in $L_p(S^{n-1})$	149
2. Spherical potentials	150
2.1. The spherical Riesz potential operator	151
2.2. Connection with the spatial Riesz potential	153
2.3. Other potentials	153
3. Spherical hypersingular integrals and inversion of the spherical Riesz potential	159

3.1. Spherical hypersingular integrals of order $0 < \Re\alpha < 2$	159
3.2. The multiplier of the spherical hypersingular operator	160
3.3. Justification of the inversion in $L_p(S^{n-1})$; identity approximation on the sphere	160
3.4. Inversion of the spherical Riesz potential in the case $\Re\alpha > 2$	165
4. Zygmund type estimates for spherical potential operators and spherical hypersingular integrals	167
4.1. The continuity modulus of functions on the sphere	167
4.2. Technical lemma	167
4.3. The case of potentials	168
4.4. The case of hypersingular integrals	170
5. Bibliographical notes to Chapter 6	170
Part 2. Applications of hypersingular integrals	173
Chapter 7. Characterization of some function spaces in terms of hypersingular integrals	175
1. The spaces $I^\alpha(L_p)$ of Riesz potentials	175
1.1. Definitions and local behaviour of Riesz potentials	175
1.2. Characterization of the space $I^\alpha(L_p)$ in terms of convergence of hypersingular integrals	179
2. The spaces $B^\alpha(L_p)$ of Bessel potentials	184
2.1. Denseness of $C_0^\infty(R^n)$ in the space $L_p^\alpha(R^n)$	185
2.2. Proof of the main result $B^\alpha(L_p) = L_p^\alpha(R^n)$	186
3. The spaces $L_{p,r}^\alpha(R^n)$	187
3.1. Connection with the space of Riesz potentials	188
3.2. Imbeddings in the spaces $L_{p,r}^\alpha(R^n)$	189
3.3. Weak derivatives of functions $f \in L_{p,r}^\alpha(R^n)$	190
3.4. Riesz derivatives $\mathbb{D}^\alpha f$ of integer order and powers of the Laplace operator, $f \in I^\alpha(L_p)$	192
3.5. The spaces $W_{p,r}^k$ and the characterization of the space $I^\alpha(L_p)$ in terms of higher derivatives	192
3.6. Spaces $L_{p,r}^\alpha(\rho_p, \rho_r)$ with Muckenhoupt weight functions	193
4. Simultaneous approximation of functions and their Riesz derivatives in different L_p -norms	196
5. Convergence in L_p of hypersingular integrals with non-stabilizing characteristics	204
6. Hypersingular integrals and differences of fractional order	207
6.1. Fractional differentiation in a given direction	208
6.2. The case of ordinary fractional differences	209
6.3. The case of mixed fractional differences	213
6.4. The case of operator fractional difference	214
6.5. Uniform estimates	216
7. Spaces of Riesz potentials with densities in Orlicz spaces	220
7.1. Definitions	221
7.2. On boundedness of the Riesz potential operator in Orlicz spaces	221
7.3. The space $I^\alpha(E_M)$ of Riesz potentials	222
7.4. The spaces $E_{M_1, M_2}^\alpha(R^n)$	222
8. Characterization of the spaces $L_p^\alpha(S^{n-1})$	223

8.1. Characterization of the range $\mathfrak{R}^\alpha(L_p)$	223
8.2. Proof of the coincidence $\mathfrak{R}^\alpha(L_p) = L_p^\alpha(S^{n-1})$	225
9. Bibliographical notes to Chapter 7	226
Chapter 8. Solution of multidimensional integral equations of the first kind with a potential type kernel	231
1. Inversion of potential type operators with homogeneous characteristics in the elliptic case	231
1.1. General remarks on the inversion	231
1.2. Structure of the Fourier transform of the reciprocal of the symbol of a potential	232
1.3. Representation of $F^{-1}\left(\frac{1}{\kappa_\alpha}\right)$ by an f.p.-integral over the sphere	233
1.4. The associated characteristics	234
1.5. Inversion of potentials of non-integer order α by hypersingular opera- tors	236
1.6. Inversion of potentials of even order $\alpha = 2, 4, 6, \dots$ by hypersingular operators	237
1.7. Inversion of potentials of odd order $\alpha = 1, 3, 5, \dots$ by hypersingular operators	238
1.8. On the inversion of potentials of integer order in the general case	239
2. Extension to the case of densities in $L_p(R^n)$	241
2.1. The annihilation of the kernel of a potential by the associated hyper- singular operator	242
2.2. Fundamental solutions of hypersingular operators	244
2.3. Integral representation of truncated hypersingular integrals	245
2.4. The Fourier transform of the kernel of the representation	246
2.5. The kernel of the representation as an identity approximation kernel in the case of associated characteristics	247
2.6. Convergence in L_p of the truncated hypersingular integrals and inversion of potentials within the framework of L_p -spaces	248
3. Inversion of potentials with homogeneous harmonic characteristics	249
3.1. The general case	250
3.2. The case of a linear characteristic	251
3.3. Realization of the inversion for densities in $L_p(R^n)$ in the case of a linear characteristic	253
4. Inversion of potentials with homogeneous characteristics in the case of linear degeneracy of the symbol	257
4.1. Characteristics of potentials with a linear degeneracy of the symbol	257
4.2. Inversion in the case of "nice" functions	257
4.3. The inversion in the case of functions $\varphi \in L_p$	258
5. Inversion of potentials with radial difference characteristics	260
5.1. What is to be done?	260
5.2. The class $C^{m;\gamma}(\hat{R}_+^1)$	261
5.3. On a Hankel transformation	263
5.4. The analytical continuation of the Fourier transform of the function $ x ^\alpha(1+ x ^2)^{-\frac{\alpha}{2}}$	264
5.5. Construction of the inverse operator	265

5.6. Hypersingular integrals as inverses: the cases $0 < \alpha < 1$ and $1 < \alpha < 2$	267
5.7. Justification of the inversion in the Lizorkin space $\Phi(R^n)$	267
5.8. Justification of the inversion in the space L_p : the cases $0 < \alpha < 1$ and $1 < \alpha < 2$	268
6. Bibliographical notes to Chapter 8	269
Chapter 9. Hypersingular operators as positive fractional powers of some operators of mathematical physics	271
1. Inversion of the Bessel potentials	271
1.1. The idea of the construction	271
1.2. Properties of the function $\omega_{-\alpha}(r)$	272
1.3. The operator $(I - \Delta)^{\alpha/2}$ inverse to the Bessel potential operator	274
1.4. Justification of the inversion in the case of "nice" functions	276
1.5. The case $0 < \Re\alpha < 2$	276
1.6. Justification of the inversion in L_p : the cases $0 < \Re\alpha < 1$ and $1 < \Re\alpha < 2$	278
1.7. Another form for $(I - \Delta)^{\alpha/2}$	279
1.8. Another expression for $(I - \Delta)^{\alpha/2}$	281
2. Parabolic (heat) hypersingular integrals	282
2.1. Parabolic fractional potentials	282
2.2. Positive fractional powers $(\frac{\partial}{\partial t} - \Delta_x)^{\alpha/2}$ and $(I + \frac{\partial}{\partial t} - \Delta_x)^{\alpha/2}$, $\alpha > 0$	284
2.3. Justification of the inversion in the case of "nice" functions	286
2.4. The case of functions in L_p	288
2.5. The spaces of parabolic potentials	288
3. Fractional powers of the hyperbolic (wave) operator	290
3.1. The Riesz hyperbolic potential in the case $n = 2$	291
3.2. The hyperbolic hypersingular integral: the case $n = 2$	292
3.3. Hyperbolic hypersingular integrals: the general case $n \geq 2$	294
4. Fractional powers of the Schrödinger operator	296
4.1. The fractional Schrödinger potential operator	296
4.2. Analytical continuation of the Schrödinger fractional potential	297
4.3. Positive fractional powers of the Schrödinger operator	299
5. Fractional powers of some other differential operators	300
6. Bibliographical notes to Chapter 9	302
Chapter 10. Regularization of multidimensional integral equations of the first kind with a potential type kernel	303
1. Regularization of the equations in R^n	303
1.1. The case $0 < \alpha < 1$	304
1.2. The case $\alpha \geq 1$	309
2. Application of spherical hypersingular operators to regularization of integral equations of the first kind on the unit sphere	310
3. Bibliographical notes to Chapter 10	314
Chapter 11. Some modifications of hypersingular integrals and their applications	315
1. The case of the Riesz potential operator	316

1.1. General requirements for the kernel $q_\alpha(y)$	316
1.2. Convergence in (11.4) on "nice" functions	318
1.3. Construction (11.4) as the inverse operator to I^α on $I^\alpha(L_p)$	318
1.4. The approximative inverse operator under the choice $k(x) = P(x, 1)$	319
1.5. The approximative inverse operator under the direct choice of $q_\alpha(x)$	321
2. Inversion of potential type operators with homogeneous characteristics by approximative inverses	325
2.1. General requirements for the kernel $q_\alpha(x)$ in (11.4)	326
2.2. Approximative inverses under the choice $\mathcal{K}(\xi) = e^{- \xi }$	326
2.3. Inversion in the non-elliptic case	327
3. Inversion of Bessel potentials by approximative inverses	329
3.1. Approximative inverses under the choice $\mathcal{K}_\epsilon(\xi) = e^{-\epsilon \xi }$	329
3.2. The approximative inverses under the choice $\mathcal{K}_\epsilon(\xi) = \frac{1}{(1+\epsilon^2 \xi ^2)^{\frac{\alpha}{2}}}$	330
4. Inversion of the acoustic potentials by approximative inverses	331
4.1. Some preliminaries	331
4.2. The choice of approximative inverses	333
4.3. The representation for $(A^\alpha)_\epsilon^{-1}A^\alpha, \epsilon > 0$	333
4.4. The inversion theorem	335
5. Inversion of potentials with oscillating symbols by approximative inverses	336
5.1. The kernel corresponding to the oscillating symbol $\mathcal{B}_{\alpha,\gamma}(\xi)$	336
5.2. Passage to the symbol	337
5.3. The inversion	338
6. Other applications of the approximative inverses method	339
7. Bibliographical notes to Chapter 11	340
References	343
Author index	355
Subject index	357
Index of Symbols	359