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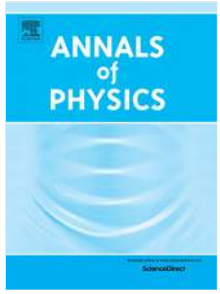

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Hyperspherical δ-δ' potentials

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Abstract

The spherically symmetric potential $a\,\delta(r-r_0)+b\,\delta'(r-r_0)$ is generalised for the d-dimensional space as a characterisation of a unique selfadjoint extension of the free Hamiltonian. For this extension of the Dirac delta, the spectrum of negative, zero and positive energy states is studied in $d \geq 2$, providing numerical results for the expectation value of the radius as a function of the free parameters of the potential. Remarkably, only if $d = 2$ the δ-δ' potential for arbitrary $a > 0$ admits a bound state with zero angular momentum.

Keywords:
Contact interactions; selfadjoint extensions; singular potentials; spherical potentials

1. Introduction

The presence of boundaries has played a central role in many areas of physics for many years. In this respect, concerning the quantum world, one of the most significant phenomenon is due to the interaction of quantum vacuum fluctuations of the electromagnetic field with two conducting ideal plane parallel plates: the Casimir effect [1], meassured by Sparnaay in 1959 [2]. Frontiers are also essential in the theory of quantum black holes, where one of the most remarkable results is the brick wall model developed by G. 't Hooft [3, 4], in which boundary conditions are used to implement the interaction of quantum massless particles with the black hole horizon observed from far away. In addition, the propagation of plasmons over the graphene sheet and the surprising scattering properties through abrupt defects [5] can be understood by using boundary conditions to represent the defects. In all these situations, the physical properties of the frontiers and their interaction with quantum objects of the bulk are mimicked by different boundary conditions. Many of these effects concerning condensed matter quantum field theory can be reproduced in the laboratory.

Moreover, point potentials or potentials supported on a point have attracted much attention over the years (see [6] for a review). These kind of potentials, also called contact interactions, enables us to build integrable toy-model approximations for very

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localised interactions. The most known example of such kind of interaction is the Dirac-δ potential, that has been extensively studied in the literature (see, e.g. [7–9]). Since this potential admits only one bound state when it has negative coupling [8, 9], it can represent Hydrogen-like nuclei in interaction with a classical background. Dirac-δ potentials can also be used to represent extended plates in effective scalar quantum field theories to compute the quantum vacuum interaction for semi-transparent plates in flat spacetime or curved backgrounds [10–13].

From what has been mentioned above, it is intuitively clear that quantum boundaries and contact interactions are almost the same. The rigorous mathematical framework to study them is the use of selfadjoint extensions to represent extended objects and point supported potentials (see [14–16] for a more physical point of view), such as plates in the Casimir effect setup [17, 18], or contact interactions more general than the Dirac-δ in quantum mechanics and effective quantum field theories [19–25]. The theory of selfadjoint extensions for symmetric operators has been well known to mathematicians for many years. However, it only became a valuable tool for modern quantum physics after the seminal works of Asorey *et al.* [15, 17, 18], in which the problem was re-formulated in terms of physically meaningful quantities for relevant operators in quantum mechanics and quantum field theory [26–31].

One of the most immediate extensions of the Dirac-δ potential is the δ'-potential $V_{\delta'} = b\delta'(x)$. Over the last years, there has been some controversy about the definition of this potential in one dimension (see the discussion in [32, 33]), and yet it is not clear how $V_{\delta'}$ should be characterised. The aim of this paper is not to discuss this definition but to use the one introduced in [32], including a Dirac-δ to regularise the potential, and study its generalisation as a hyperspherical potential in dimension $d > 1$. We will fully solve the non-relativistic quantum mechanical problem associated with the spherically symmetric potential

$$\widehat{V}_{\delta\text{-}\delta'}(r) = a\,\delta(r - r_0) + b\,\delta'(r - r_0), \quad a, b \in \mathbb{R}, \; r_0 > 0. \tag{1}$$

Due to the radial symmetry of the problem, we will end up having a family of one-dimensional Hamiltonians (the radial Hamiltonian), for which a generalisation of the definition given in [19, 32] is needed.

This paper is organised as follows. Section 2 defines the spherically symmetric δ-δ' potential in arbitrary dimension based on the work for one dimensional systems performed in [19]. Having determined the properties which characterise the potential, we carry out a thorough study of the bound states structure in Section 3 and of the zero-modes and scattering states in Section 4. In the latter, we also compute some numerical results concerning the mean value of the position (radius) operator. Through these two sections we specially focus on the peculiarities of the two dimensional case. Finally, in Section 5 we present our concluding remarks.

2. The δ-δ' interaction in the d-dimensional Schrödinger equation

We consider a non-relativistic quantum particle of mass m moving in $\mathbb{R}^d$ ($d = 2, 3, \ldots$) under the influence of the spherically symmetric potential $\widehat{V}_{\delta\text{-}\delta'}(r)$ given in

(1). The quantum Hamiltonian operator that governs the dynamics of the system is

$$\mathbf{H} = \frac{-\hbar^2}{2\,m}\widehat{\Delta}_d + \widehat{V}_{\delta\text{-}\delta'}(r), \tag{2}$$

where $\widehat{\Delta}_d$ is the d-dimensional Laplace operator. To start with, let us analyse the dimensions of the free parameters a and b that appear in our system. Using the properties of the Dirac-δ under dilatations and knowing that the δ' has to have the same units as the formal expression $d\delta(x)/dx$ it is straightforward to see that the dimensions of the parameters a and b are

$$[a] = L^3T^{-2}M, \quad [b] = L^4T^{-2}M. \tag{3}$$

Hence, we can introduce the following dimensionless quantities:

$$\mathbf{h} \equiv \frac{2}{mc^2}\mathbf{H}, \quad w_0 \equiv \frac{2a}{\hbar c}, \quad w_1 \equiv \frac{bm}{\hbar^2}, \quad x \equiv \frac{mc}{\hbar}r. \tag{4}$$

With the previous definitions, the dimensionless quantum Hamiltonian reads

$$\mathbf{h} = -\Delta_d + w_0\,\delta(x - x_0) + 2w_1\,\delta'(x - x_0). \tag{5}$$

Introducing hyperspherical coordinates, $(x, \Omega_d \equiv \{\theta_1, \ldots, \theta_{d-2}, \phi\})$, the d-dimensional Laplace operator Δ_d is written as

$$\Delta_d = \frac{1}{x^{d-1}}\frac{\partial}{\partial x}\left(x^{d-1}\frac{\partial}{\partial x}\right) + \frac{\Delta_{S^{d-1}}}{x^2}, \tag{6}$$

where $\Delta_{S^{d-1}} = -\mathbf{L}_d^2$ is the Laplace-Beltrami operator in the hypersphere S^{d-1}, and minus the square of the generalised dimensionless angular momentum operator [34]. In hyperspherical coordinates, the eigenvalue equation for $\mathbf{h}$ in (5) is separable, and therefore we can write the solutions as

$$\psi_{\lambda\ell}(x, \Omega_d) = R_{\lambda\ell}(x)Y_\ell(\Omega_d), \tag{7}$$

where $R_{\lambda\ell}(x)$ is the radial wave function and $Y_\ell(\Omega_d)$ are the hyperspherical harmonics which are the eigenfunctions of $\Delta_{S^{d-1}}$ with eigenvalue (see [35] and references therein)

$$\chi(d, \ell) \equiv -\ell(\ell + d - 2). \tag{8}$$

The degeneracy of $\chi(\ell, d)$ is given by [36]

$$\deg(d, \ell) = \begin{cases} \dfrac{(d+\ell-3)!}{(d-2)!\ell!}(d + 2(\ell - 1)) & \text{if} \quad d \neq 2 \ \text{ and } \ \ell \neq 0, \\ 1 & \text{if} \quad d = 2 \ \text{ and } \ \ell = 0. \end{cases} \tag{9}$$

In three dimensions we come up with $\chi(3, \ell) = -\ell(\ell + 1)$ and $\deg(3, \ell) = 2\ell + 1$ as expected. Taking into account the eigenvalue equation for (5) and equations (7, 8) the radial wave function fulfils

$$\left[-\frac{d^2}{dx^2} - \frac{d-1}{x}\frac{d}{dx} + \frac{\ell(\ell + d - 2)}{x^2} + V_{\delta\text{-}\delta'}(x)\right]R_{\lambda\ell}(x) = \lambda R_{\lambda\ell}, \tag{10}$$

being

$$V_{\delta\text{-}\delta'}(x) = w_0\delta(x - x_0) + 2w_1\delta'(x - x_0). \tag{11}$$

To solve the eigenvalue equation (10), we first need to define the potential $V_{\delta\text{-}\delta'}$. In order to characterise the potential $V_{\delta\text{-}\delta'}(x)$ as a selfadjoint extension following [19, 32], we introduce the reduced radial function

$$u_{\lambda\ell}(x) \equiv x^{\frac{d-1}{2}} R_{\lambda\ell}(x), \tag{12}$$

to remove the first derivative from the one dimensional radial operators in (10). Taking into account (10) and (12), we obtain the eigenvalue problem that this function satisfies

$$(\mathcal{H}_0 + V_{\delta\text{-}\delta'}(x))\, u_{\lambda\ell}(x) = \lambda_\ell u_{\lambda\ell}(x), \tag{13}$$

where

$$\mathcal{H}_0 \equiv -\frac{d^2}{dx^2} + \frac{(d + 2\ell - 3)(d + 2\ell - 1)}{4x^2}. \tag{14}$$

Thus, as in [19], we define the potential $V_{\delta\text{-}\delta'}$ through a set of matching conditions on the eigenfunction of $\mathcal{H}_0$ at $x = x_0^{\pm}$. The discussion for the one dimensional case imposes that the wave function ψ must belong to the Sobolev space $W_2^2(\mathbb{R}\setminus\{0\})$ in order to ensure that $\psi''(x)$ is a square integrable function, i.e., the mean value of the kinetic energy is finite. To generalise this condition to higher dimensional Hamiltonians with spherical symmetry, we need to impose that the domain of wave functions where the operator $\mathcal{H}_0$ is selfadjoint when it is defined on $\mathbb{R}_{>0}$ is

$$W(\mathcal{H}_0, \mathbb{R}_{>0}) \equiv \{f(x) \in L^2(\mathbb{R}_{>0})\,|\, \langle \mathcal{H}_0 \rangle_{f(x)} < \infty\}, \tag{15}$$

where the expectation value of $\mathcal{H}_0$ is defined as usual

$$\langle \mathcal{H}_0 \rangle_{f(x)} \equiv \int_0^\infty f^*(x)\,(\mathcal{H}_0 f(x))\; dx.$$

When we remove the point $x = x_0$ the operator $\mathcal{H}_0$ is no longer selfadjoint on the space of functions $W(\mathcal{H}_0, \mathbb{R}_{x_0}) \equiv \{f(x) \in L^2(\mathbb{R}_{x_0})\,|\, \langle \mathcal{H}_0 \rangle_{f(x)} < \infty\}$ since

$$\int_0^\infty dx\, \phi^*\,(\mathcal{H}_0\varphi) - \int_0^\infty dx\, \varphi\,(\mathcal{H}_0\phi)^* \neq 0, \quad \varphi, \phi \in W(\mathcal{H}_0, \mathbb{R}_{x_0}),$$

due to the boundary terms appearing when integrating by parts twice. Nevertheless, $\mathcal{H}_0$ is symmetric on the subspace given by the closure of the $L^2(\mathbb{R}_{x_0})$ functions with compact support in $\mathbb{R}_{x_0}$. This situation generalises the initial conditions given in [19], and matches the geometric view in [14, 15]. Hence, the domain of the selfadjoint extension $\mathcal{H}_0 + V_{\delta\text{-}\delta'}$ of the operator $\mathcal{H}_0$ defined on $\mathbb{R}_{x_0}$ is given by

$$\mathcal{D}(\mathcal{H}_0 + V_{\delta\text{-}\delta'}) = \left\{ f \in W(\mathcal{H}_0, \mathbb{R}_{x_0})\,|\, \begin{pmatrix} f(x_0^+) \\ f'(x_0^+) \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix} \begin{pmatrix} f(x_0^-) \\ f'(x_0^-) \end{pmatrix} \right\}, \tag{16}$$

where we have introduced the values

$$\alpha \equiv \frac{1 + w_1}{1 - w_1}, \quad \beta \equiv \frac{w_0}{1 - w_1^2}. \tag{17}$$

Now, using (12) in (16) we obtain the following matching conditions for the radial wave function $R_{\lambda\ell}$:

$$\begin{pmatrix} R_{\lambda\ell}(x_0^+) \\ R'_{\lambda\ell}(x_0^+) \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \tilde\beta & \alpha^{-1} \end{pmatrix} \begin{pmatrix} R_{\lambda\ell}(x_0^-) \\ R'_{\lambda\ell}(x_0^-) \end{pmatrix}, \tag{18}$$

where the effective couplings $\tilde\beta$ and $\tilde w_0$ are

$$\tilde\beta \equiv \beta - \frac{\left(\alpha^2-1\right)(d-1)}{2\,\alpha\,x_0} = \frac{\tilde w_0}{1-w_1^2} \quad \Rightarrow \quad \tilde w_0 = \frac{2(1-d)\,w_1}{x_0} + w_0. \tag{19}$$

Observe that when we turn off the δ' contribution, $w_1 = 0$ or $\alpha = 1$, the finite discontinuity in the derivative that characterises the δ-potential arises

$$R_{\lambda\ell}(x_0^+) = R_{\lambda\ell}(x_0^-) \quad \text{and} \quad R'_{\lambda\ell}(x_0^+) - R'_{\lambda\ell}(x_0^-) = w_0 R_{\lambda\ell}(x_0).$$

On the other hand, when $w_1 = \pm1$ the matching condition matrix is ill defined because it does not relate the boundary data on x_0^- with those on x_0^+. This case is treated in detail in [10], where it is demonstrated that $w_1 = \pm1$ leads to Robin and Dirichlet boundary conditions in each side of the singularity $x = x_0$. Specifically,

$$\begin{aligned} R_{\lambda\ell}(x_0^+) - \frac{4}{\tilde w_0^+} R'_{\lambda\ell}(x_0^+) &= 0, \qquad R_{\lambda\ell}(x_0^-) = 0 \quad \text{if} \quad w_1 = 1, \\ R_{\lambda\ell}(x_0^-) + \frac{4}{\tilde w_0^-} R'_{\lambda\ell}(x_0^-) &= 0, \qquad R_{\lambda\ell}(x_0^+) = 0 \quad \text{if} \quad w_1 = -1, \end{aligned} \tag{20}$$

where $\tilde w_0^\pm = w_0 \pm 2(1-d)/x_0$. Recently the potential (11) was studied for two and three dimensions in [37] where the matching conditions used for $R_{\lambda\ell}$ are those in (16) instead of (18) which is valid under the approximation $\tilde w_0 \simeq w_0$, only satisfied if

$$x_0\,|w_0| \gg |w_1| \tag{21}$$

Throughout the text we will point out the equations that are valid even when the previous inequality does not hold.

A remark on selfadjoint extensions and point supported potentials

The operator $\mathcal{H}_0$ defined as a one dimensional Hamiltonian over the physical space $\mathbb{R}_{x_0}$ is not selfadjoint as we have seen. In order to define a true Hamiltonian as a selfadjoint operator one has to select a selfadjoint extension of $\mathcal{H}_0$ in the way explained above for the particular case of the potential $V_{\delta\text{-}\delta'}(x)$. More generally, the set of all selfadjoint extensions is in one-to-one correspondence with the set of unitary matrices $U(2)$. As was demonstrated in [15], for a given unitary matrix $G \in U(2)$ there is a unique selfadjoint extension $\mathcal{H}_0^G$ of $\mathcal{H}_0$. In this sense, the selfadjoint extension $\mathcal{H}_0^G$ can be thought in a more physically meaningful way as a potential $V_G(x-x_0)$ supported on a point x_0 for the quantum Hamiltonian $\mathcal{H}_0$ and write $\mathcal{H}_0 + V_G(x-x_0) \equiv \mathcal{H}_0^G$. Physically one would just think on $V_G(x-x_0)$ as a potential term in the same way as the Dirac-δ potential [8]. In this view, once the operator $\mathcal{H}_0$ is fixed, the selfadjoint extensions can be seen as potentials supported on a point, and the other way around because of the one-to-one correspondence demonstrated in [15] (and recently reviewed in [14]).

3. Bound states with the free Hamiltonian and the singular interaction

In this section we will analyse in detail the discrete spectrum of negative energy states (bound states) for the δ-δ' potential. In particular, we will give an analytic formula for the number of them as a function of the parameters $\{w_0, w_1, x_0\}$. As the eigenvalue equation for the bound states is (10) with $\lambda < 0$, we define $\lambda \equiv -\kappa^2$ with $\kappa > 0$, and replace the subindex λ by κ in the wave functions all over this section. The general form of the solutions of equation (10) is

$$R_{\kappa\ell}(x) = \begin{cases} A_1\, \mathcal{I}_\ell(\kappa x) + B_1\, \mathcal{K}_\ell(\kappa x) & \text{if} \quad x \in (0, x_0) \\ A_2\, \mathcal{I}_\ell(\kappa x) + B_2\, \mathcal{K}_\ell(\kappa x) & \text{if} \quad x \in (x_0, \infty), \end{cases} \tag{22}$$

being $\mathcal{I}_\ell(z)$ and $\mathcal{K}_\ell(z)$, up to a constant factor, the modified hyperspherical Bessel functions of the first and second kind respectively

$$\mathcal{I}_\ell(z) \equiv \frac{1}{z^\nu}\, I_{\ell+\nu}(x), \quad \mathcal{K}_\ell(z) \equiv \frac{1}{z^\nu}\, K_{\ell+\nu}(z) \quad \text{with} \quad \nu \equiv \frac{d-2}{2}, \tag{23}$$

Similarly from Eq.(12) the general form of the reduced radial function is

$$u_{\kappa\ell}(x) = \sqrt{x} \begin{cases} A_1\, I_{\ell+\nu}(\kappa x) + B_1\, K_{\ell+\nu}(\kappa x) & \text{if} \quad x \in (0, x_0), \\ A_2\, I_{\ell+\nu}(\kappa x) + B_2\, K_{\ell+\nu}(\kappa x) & \text{if} \quad x \in (x_0, \infty). \end{cases} \tag{24}$$

The integrability condition on the reduced radial function

$$\int_0^\infty |u_{\kappa\ell}(x)|^2\, dx < \infty$$

imposes $A_2 = 0$. Moreover, the solution multiplied by B_1 is not square integrable except for zero angular momentum in two and three dmensions [38]. The regularity condition of the wave function at the origin $u_{\kappa\ell}(x = 0) = 0$, sets $B_1 = 0$ for $d = 3$. It would seem that the two solutions in the inner region are admissible when $d = 2$, but $B_1 \neq 0$ would lead to a normalizable bound state with arbitrary negative energy [39]. In addition, for any wave function ψ, the following identity involving the mean value of the kinetic energy operator:

$$\frac{1}{2\,m} \langle \psi | P^2 | \psi \rangle = \frac{1}{2\,m} (\langle \psi | \boldsymbol{P}) \cdot (\boldsymbol{P} | \psi \rangle), \tag{25}$$

holds if we impose certain conditions on the wave function at the boundary $x = 0$, which are not satisfied by $\mathcal{K}_0$. Hence, we conclude that B_1 should be zero for all the cases. With the previous analysis and (18) we obtain the matching condition

$$B_2 \begin{pmatrix} \mathcal{K}_\ell(\kappa x_0) \\ \kappa\, \mathcal{K}'_\ell(\kappa x_0) \end{pmatrix} = A_1 \begin{pmatrix} \alpha & 0 \\ \tilde{\beta} & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{I}_\ell(\kappa x_0) \\ \kappa\, \mathcal{I}'_\ell(\kappa x_0) \end{pmatrix}, \tag{26}$$

from which the secular equation is obtained

$$\alpha\, \frac{d}{dx} \log \mathcal{K}_\ell(\kappa x)\Big|_{x=x_0} = \tilde{\beta} + \alpha^{-1}\, \frac{d}{dx} \log \mathcal{I}_\ell(\kappa x)\Big|_{x=x_0}. \tag{27}$$

The solutions for $\kappa > 0$ of the previous equation give the energies of the bound states accounting for $\lambda = -\kappa^2$. The equation (27) can be written as

$$F(y_0) \equiv -y_0 \left(\frac{I_{\nu+\ell-1}(y_0)}{\alpha\, I_{\nu+\ell}(y_0)} + \frac{\alpha\, K_{\nu+\ell-1}(y_0)}{K_{\nu+\ell}(y_0)} \right) - \left(\alpha - \alpha^{-1}\right)\ell = 2\nu\left(\alpha - \alpha^{-1}\right) + \tilde{\beta} x_0, \quad (28)$$

where $y_0 \equiv \kappa x_0$ and the right hand side is independent of the energy and the angular momentum. For $d = 2, 3$ the results of [37] are obtained as a limiting case ($x_0|w_0| \gg |w_1|$). In particular, the secular equation for the δ-potential ($\alpha = 1$ and $\tilde{\beta} = w_0$) is

$$-y_0 \left(\frac{I_{\nu+\ell-1}(y_0)}{I_{\nu+\ell}(y_0)} + \frac{K_{\nu+\ell-1}(y_0)}{K_{\nu+\ell}(y_0)} \right) = w_0 x_0.$$

3.1. On the number of bound states

Although equation (28) can not be solved analytically in κ, it can be used to characterise some fundamental aspects of the set of positive solutions of (28). The main feature is the number of bound states that exist for d and ℓ

$$N_\ell^d = n_\ell^d \deg(d, \ell),$$

where n_ℓ^d is the number of negative energy eigenvalues and $\deg(d, \ell)$ is the degeneracy associated with ℓ in d dimensions (9). In this way, we first delimit the possible values of n_ℓ^d.

Proposition 1. In the d-dimensional quantum system described by the Hamiltonian (5) the number n_ℓ^d is at most one, i.e. $n_\ell^d \in \{0, 1\}$.
Proof. From (28) and applying the properties of the Bessel functions, the derivative of $F(\kappa x_0)$ with respect to κ is

$$x_0 F'(y_0) = -y_0 \left[\alpha \left(\frac{K_{\nu+\ell-1}(y_0) K_{\nu+\ell+1}(y_0)}{K_{\nu+\ell}(y_0)^2} - 1 \right) + \alpha^{-1} \left(1 - \frac{I_{\nu+\ell-1}(y_0) I_{\nu+\ell+1}(y_0)}{I_{\nu+\ell}(y_0)^2} \right) \right],$$

and, as it is proven in [40] and the references cited therein,

$$\begin{aligned} K_{n-1}(y_0)K_{n+1}(y_0) &> K_n(y_0)^2, \quad \text{if} \quad y_0 > 0,\ n \geq -1/2, \\ I_{n-1}(y_0)I_{n+1}(y_0) &< I_n(y_0)^2, \quad \text{if} \quad y_0 > 0,\ n \in \mathbb{R}. \end{aligned}$$

In the present case $n = \nu + \ell \geq 0$, therefore we can conclude that

$$\operatorname{sgn}\left(F'(y_0)\right) = -\operatorname{sgn}(\alpha). \quad (29)$$

Hence, except for $\alpha = 0$ (ill defined matching conditions) $F(y_0)$ is a strictly monotone function and the proposition is proved. ■

This result is in agreement with the Bargmann's inequalities for a general potential in three dimensional systems

$$n_\ell^{d=3} < \frac{1}{2\,\ell + 1} \int_0^\infty x|V(x)|\, dx,$$

which guarantees a finite number of bound states when the integral is convergent [41]. Moreover, this inequality was generalised for arbitrary dimensional systems with spherical symmetry [36]

$$n_\ell^d < \frac{1}{2\,\ell + d - 2} \int_0^\infty x|V(x)|\,dx \quad \text{if} \quad \int_0^\infty x|V(x)|\,dx < \infty \quad \text{and} \quad d + 2\ell - 2 \geq 1. \tag{30}$$

In the case $d = 2$ and $\ell = 0$, a stronger condition is imposed on the potential being the upper bound of the inequality different [36]. In fact, when the potential is a linear combination of Dirac-δ potentials sufficiently distant from each other n_ℓ^d tends to the r.h.s. of the inequality (30) (see Ref. [42]). The following result also matches with the properties of such potentials [8].

Proposition 2. The d-dimensional quantum system described by the Hamiltonian (5) admits bound states with angular momentum ℓ if, and only if,

$$\ell_{max} \neq L_{max}, \quad \text{and} \quad \ell \in \{0, 1, \ldots, \ell_{max}\} \quad (\ell_{max} > -1), \tag{31}$$

where

$$\ell_{max} \equiv \lfloor L_{max} \rfloor, \quad L_{max} \equiv -\frac{w_1 - x_0\, w_0/2}{w_1^2 + 1} + \frac{2 - d}{2}, \tag{32}$$

being $\lfloor\cdot\rfloor$ the integer part. In addition, if $\lambda_\ell = -\kappa_\ell^2$ is the energy of the bound state with angular momentum ℓ the following inequality holds

$$\lambda_\ell < \lambda_{\ell+1} < 0, \quad \ell \in \{0, 1, \ldots, \ell_{max} - 1\}.$$

Proof. We analyse the behaviour of $F(y_0)$ for $y_0 \sim 0$. The solutions of (22) satisfy

$$\lim_{\kappa\to 0^+} \frac{d}{d\kappa} \log \mathcal{I}_\ell(\kappa x_0) = \frac{\ell}{x_0}, \quad \lim_{\kappa\to 0^+} \frac{d}{d\kappa} \log \mathcal{K}_\ell(\kappa\, x_0) = -\frac{d + \ell - 2}{x_0},$$

therefore the secular equation (27) for $\kappa \to 0^+$ becomes

$$F_0(\ell) \equiv \lim_{\kappa\to 0^+} F(y_0) = \alpha^{-1}(d + \ell - 2) + \alpha\ell = \left(\alpha - \alpha^{-1}\right)(d - 2) + \tilde{\beta} x_0. \tag{33}$$

The function $F_0(\ell)$ is a strictly monotone function of ℓ, increasing if $\alpha > 0$ and decreasing if $\alpha < 0$. In addition, from (29) we can conclude that there are no bound states for $\ell > \ell_{max}$. Using the definitions of (17), the solution of (33) is L_{max} given by (32). Finally, with the previous analysis it is clear that $\kappa_\ell > \kappa_{\ell+1}$. ■

From (32), it can be seen that as the dimension of the system increases, the maximum angular momentum reached by the system decreases. This happens because the centrifugal potential in (14) becomes more repulsive as d grows. In Fig.1 we plot two configurations in two dimensions which illustrate the results of Propositions 1 and 2.

The results obtained in [37] for $d = 2$ and $d = 3$ are recovered when $|w_0|x_0 \gg |w_1|$ (for $d = 3$ there is minus sign and the integer part missing). To end this section, let us briefly study the behaviour of the number of negative energy eigenvalues as a function

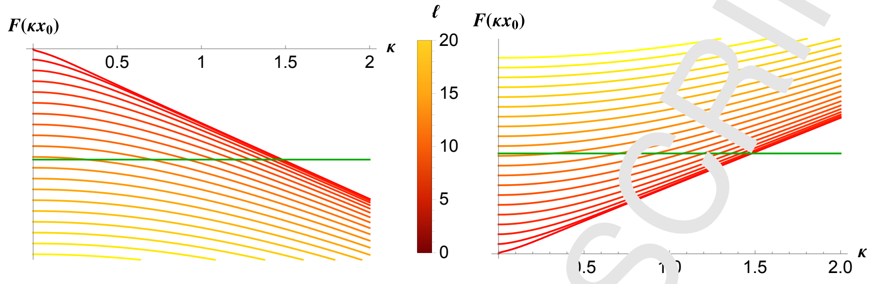

Figure 1: Each curve represents $F(\kappa\, x_0)$ in (28) for different values of the angular momentum. The green horizontal line is the r.h.s. of (28). LEFT: $d = 2$, $\alpha = 0.8$, $\tilde{\beta} = -3$ and $x_0 = 7$. RIGHT: $d = 2$, $\alpha = -0.8$, $\tilde{\beta} = 3$ and $x_0 = 7$.

of the dimension d and the angular momentum ℓ. As was shown above, the number of bound states depends on $\deg(d, \ell)$ (9). The increments with respect to d and ℓ are

$$\deg(d+1,\ell) - \deg(d,\ell) = \frac{(d+\ell-3)!(d+2\ell-3)}{(\ell-1)!(d-1)!},$$
$$\deg(d,\ell+1) - \deg(d,\ell) = \frac{(d+\ell-3)!(d+2\ell-1)}{(\ell+1)!(d-3)!},$$

therefore, both quantities are positive if $d \geq 3$ and $\ell \geq 1$. This ensures the growth of the number of bound states with the dimension and the angular momentum, except for $\ell = 0$ where the degeneracy is always 1 (ground state) and for $d = 2$ where $\deg(2,\ell) = 2$ for $\ell \geq 1$.

3.2. Special feature of two dimensions

It is known that the existence of bound states with $V_\delta = w_0\delta(x - x_0)$ necessarily imposes $w_0 < 0$ for any dimension d. This fact can be easily proved with the results obtained above. The maximum angular momentum for this potential is

$$\ell_{max} \equiv \left| \frac{-x_0\, w_0}{2} + \frac{2-d}{2} \right| \leq L_{max} = \frac{-x_0\, w_0}{2} + \frac{2-d}{2} < \frac{2-d}{2} \leq 0 \quad \text{if} \quad w_0 > 0, \quad (34)$$

which means that there are no bound states if $w_0 > 0$. The next proposition shows that this condition on the coupling w_0 does not remain valid for all the cases when we add the δ'-potential, allowing the existence of a bound state with arbitrary positive w_0 for $d = 2$ with $\ell = 0$. This result is quite surprising taking into account the usual interpretation of the Dirac-δ potential as a infinitely thin potential barrier if $w_0 > 0$. The key point to understand it is that only for $d = 2$ and $\ell = 0$ the centrifugal potential in (14) is attractive (centripetal), since $d + 2\ell - 3 = -1 < 0$.

Proposition 3. The quantum Hamiltonian (5) admits a bound state for any $w_0 > 0$ only if $d = 2$ and $\ell = 0$.

Proof. From Proposition 2 we conclude that

$$L_{max} = \frac{1}{2}\left(2 - d - \frac{x_0\, w_0}{w_1^2+1} + \frac{2w_1}{w_1^2+1}\right) \leq 1/2 \qquad \text{if} \qquad w_0 \leq 0$$

since $2w_1/(1+w_1^2) \in [-1,1]$ $w_1 \in \mathbb{R}$. Therefore, bound states with $\ell \geq 1$ are not physically admissible. For higher dimensions this state can not be achieved since

$$L_{max} \leq 0 \qquad \text{if} \qquad d \geq 3$$

The equality is reached only if $w_0 = 0$, $w_1 = 1$ and $d = 3$ being $\ell_{max} = L_{max} = 0$. In this case the selfadjoint extension of $\mathcal{H}_0$ which defines the potential $V_{\delta\text{-}\delta'}$ can not be characterised in terms of the matching conditions (18). In conclusion, with $V_{\delta\text{-}\delta'}$ described by (18) this bound state appears only if $d = 2$ and $\ell = 0$.

■

It is of note that the condition $2\,w_1 > x_0\,w_0$ ensures the existence of this bound state for arbitrary $w_0 > 0$. In addition, we must mention that the appearance of such bound state is significant because of two reasons. In one dimension, and with the definition of the δ' given by (16), this potential can not introduce bound states by itself [32]. Furthermore, when $w_0 > 0$ the Dirac-δ potential $w_0\delta(x - x_0)$ can be interpreted as an infinitely thin barrier, which contributes to the disappearance of bound states from the system. The result from Proposition 3 is illustrated in Fig.2. At the end of the next section we will compute some numerical results that point out more differences with respect to the one dimensional $V_{\delta\text{-}\delta'}$ potential.

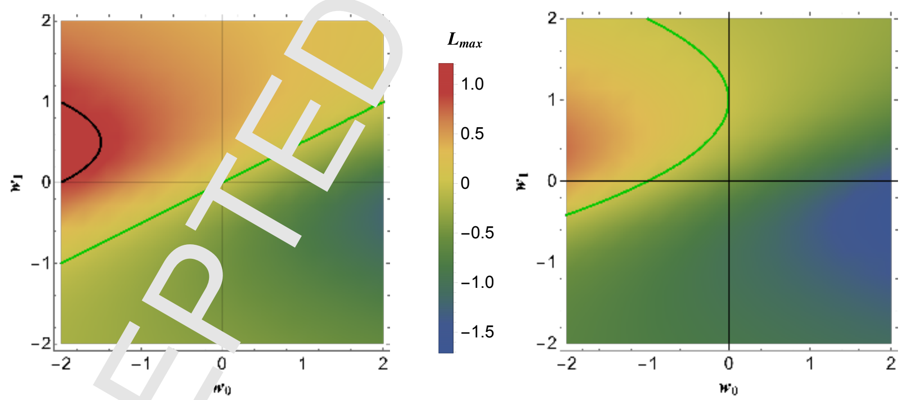

Figure 2: Plots of L_{max} (32), with $x_0 = 1$, as a function of w_0 and w_1. LEFT: $d = 2$ showing $L_{max} = 0$ (green line) and $L_{max} = 1$ (black curve). RIGHT: $d = 3$ showing $L_{max} = 0$ (green curve). There is a bound state with $\ell = 0$ in two dimensions for $w_0 > 0$, but this is not the case in three or higher dimensions.

4. Scattering states, zero-modes, and some numerical reuslts

4.1. Scattering States

To complete the general spectral study of the potential (11) it is necessary to characterise its positive energy states, i.e., the scattering states. These states are always present in the system weather there exist negative energy states or not. In addition, when the parameters w_0 and w_1 are such that the potential $V_{\delta\text{-}\delta'}$ does not admit bound states, the Schrödinger Hamiltonian (5) can be re-interpreted as the one particle states operator of an effective quantum field theory (see e.g. [10, 12] and references therein), where the scattering states are the one particle states of the scalar quantum vacuum fluctuations produced by the field. With this interpretation the explicit knowledge of the scattering states, specially the phase shifts, enables to obtain one loop calculations of the quantum vacuum energy acting on the internal and external wall of the singularity at $x = x_0$ [43]. In this case, defining[1] $k = \sqrt{\lambda} > 0$, the general solution of (10) is

$$R_{k\ell}(x) = \begin{cases} A_1\,\mathcal{J}_\ell(kx) + B_1\,\mathcal{Y}_\ell(kx) & \text{if} \quad x \in (0, x_0), \\ A_2\,\mathcal{J}_\ell(kx) + B_2\,\mathcal{Y}_\ell(kx) & \text{if} \quad x \in (x_0, \infty), \end{cases} \tag{35}$$

being $\mathcal{J}_\ell(z)$ and $\mathcal{Y}_\ell(z)$, up to a constant factor, the hyperspherical Bessel functions of the first and second kind respectively

$$\mathcal{J}_\ell(z) \equiv \frac{1}{z^\nu} J_{\ell+\nu}(z), \qquad \mathcal{Y}_\ell(z) \equiv \frac{1}{z^\nu} Y_{\ell+\nu}(z).$$

Proposition 4. The phase shift $\delta_\ell(k)$ for the ℓ-wave in a d-dimensional system described by a central potential with finite support V is given by

$$\tan\delta_\ell(k, V) = -B_{ext}/A_{ext}, \tag{36}$$

where A_{ext} and B_{ext} are defined from the asymptotic behaviour of the radial function as

$$R_{k\ell}(x) \underset{x\to\infty}{\sim} x^{\frac{1}{2}-\frac{d}{2}} \left(A_{ext}\cos\mu_\ell + B_{ext}\sin\mu_\ell\right), \qquad \mu_\ell \equiv kx - \frac{\pi}{2}(\ell + \nu + \frac{1}{2}). \tag{37}$$

Proof. Far away from the origin the central potential is identically zero, consequently the scattering solution will be a linear combination of $\mathcal{J}_\ell(kx)$ and $\mathcal{Y}_\ell(kx)$ which satisfy

$$\mathcal{J}_\ell(kx) \underset{x\to\infty}{\sim} \sqrt{\frac{2}{\pi}}(kx)^{\frac{1}{2}-\frac{d}{2}}\cos\mu_\ell, \qquad \mathcal{Y}_\ell(kx) \underset{x\to\infty}{\sim} \sqrt{\frac{2}{\pi}}(kx)^{\frac{1}{2}-\frac{d}{2}}\sin\mu_\ell.$$

On the other hand, from partial wave analysis, the asymptotic behaviour of $R_{k\ell}(x)$ is proportional to $\cos(\mu_\ell + \delta_\ell)$ [44], where δ_ℓ is the phase shift for the ℓ-wave. Gathering both equations,

$$A_{ext}\cos\mu_\ell + B_{ext}\sin\mu_\ell = C_{ext}\,\cos\left(\mu_\ell + \delta_\ell\right),$$

from which the result (36) is obtained. ∎

The previous result can be easily generalised to central potentials satisfying $x^2\,V(x) \to 0$ as $x \to \infty$ (see [44]). For the potential $V_{\delta\text{-}\delta'}$, the square integrability condition on the

[1] By using this definition we recover the usual relation between k (scattering states) and κ (bound sates): $k \to i\kappa$ as we go from $\lambda > 0$ to $\lambda < 0$.

radial wave function in any finite region sets $B_1 = 0$, except for $d = 2$ and $\ell = 0$ where the argument developed in Section 3, imposes $B_1 = 0$ [39]. In this way, using (18) and (35) the exterior coefficients $\{A_2, B_2\}$ can be expressed as

$$\begin{aligned}\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} &= A_1 \begin{pmatrix} \mathcal{J}_\ell(kx_0) & \mathcal{Y}_\ell(kx_0) \\ k\,\mathcal{J}'_\ell(kx_0) & k\,\mathcal{Y}'_\ell(kx_0) \end{pmatrix}^{-1} \begin{pmatrix} \alpha & 0 \\ \tilde{\beta} & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{J}_\ell(kx_0) \\ k\mathcal{J}'_\ell(kx_0) \end{pmatrix} \\ &= \frac{1}{2k}\pi(kx_0)^{d-1} A_1 \begin{pmatrix} k\mathcal{Y}'_\ell(kx_0) & -\mathcal{Y}_\ell(kx_0) \\ -k\,\mathcal{J}'_\ell(kx_0) & \mathcal{J}_\ell(kx_0) \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ \tilde{\beta} & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{J}_\ell(kx_0) \\ k\mathcal{J}'_\ell(kx_0) \end{pmatrix}.\end{aligned}$$

From this result and (36) we get

$$\tan\delta_\ell(k, V_{\delta\text{-}\delta'}) = -\frac{\mathcal{J}_\ell(kx_0)\Big(\big(1-\alpha^2\big)k\mathcal{J}'_\ell(kx_0) + \alpha\tilde{\beta}\mathcal{J}_\ell(kx_0)\Big)}{-k\mathcal{J}'_\ell(kx_0)\mathcal{Y}_\ell(kx_0) + \mathcal{J}_\ell(kx_0)\Big(\alpha^2 k\mathcal{Y}'_\ell(kx_0) - \alpha\tilde{\beta}\mathcal{Y}_\ell(kx_0)\Big)}. \tag{38}$$

In the spherical wave basis, the scattering matrix is diagonal and its eigenvalues can be written as

$$\exp\left(2i\delta_\ell(k, V_{\delta\text{-}\delta'})\right) = (1 + 2i\tan\delta_\ell - \tan^2\delta_\ell)/(1 + \tan^2\delta_\ell). \tag{39}$$

Note that for the potential $V_{\delta\text{-}\delta'}$, the secular equation (28) can be re-obtained as the positive imaginary poles of (39) using (38) (for details see [44]).

To complete this section, let us show explicit formulas of the phase shift for some particular cases of the potential $V_{\delta\text{-}\delta'}$ previously studied in the literature:

- The δ-potential ($w_1 = 0, \beta = w_0$) phase shift is

$$\tan\delta_\ell(k, V_\delta) = \frac{\pi\, w_0\, x_0\, J_{\ell+\nu}\,(k\,x_0)^{\,2}}{\pi\, w_0\, x_0\, J_{\ell+\nu}\,(k\,x_0)\, Y_{\ell+\nu}\,(k\,x_0) - 2},$$

which matches for $d = 2, 3$ with the results obtained in [45] and [43] respectively.

- The hard hypersphere defined as

$$V_{hh}(x) = \begin{cases} \infty, & x \le x_0, \\ 0, & x > x_0, \end{cases}$$

imposes Dirichlet boundary conditions for the wave function on the exterior region, $R(x_0^+) = 0$. The same result can be obtained from the δ-δ' potential setting $w_1 \to -1$ (20). Thus, the phase shift is

$$\tan\delta_\ell(k, V_{hh}) = \lim_{w_1\to -1} \tan\delta_\ell(k, V_{\delta\text{-}\delta'}) = \frac{J_{\ell+\nu}\,(kx_0)}{Y_{\ell+\nu}\,(kx_0)}.$$

For two and three dimensional systems it coincides with [45] (hard circle) and [46] (hard sphere) respectively.

- When we turn off the Dirac-δ term ($w_0 = 0 \Rightarrow \beta = 0$) we have that there is an effective δ potential coupling characterised by

$$\tilde{\beta} = -\frac{(\alpha - \alpha^{-1})(d-1)}{2x_0},$$

therefore from (38) we obtain that the phase shift for the pure δ' is

$$\tan \delta_\ell(k, V_{\delta'}) = -\frac{\left(1-\alpha^2\right)\mathcal{J}_\ell(z_0)((d-1)\mathcal{J}_\ell(z_0) + 2z_0\mathcal{J}'_\ell(z_0))}{(\alpha^2-1)(d-1)\mathcal{J}_\ell(z_0)\mathcal{Y}_\ell(z_0) + 2z_0\left(\alpha^2\mathcal{Y}'_\ell(z_0)\mathcal{J}_\ell(z_0) - \mathcal{J}'_\ell(z_0)\mathcal{Y}_\ell(z_0)\right)},$$

where $z_0 \equiv kx_0$. As can be seen, $\delta_\ell(k, V_{\delta'})$ depends on the energy through z_0 unlike it happens with the scattering amplitudes for the pure δ' potential in one dimension, where there is no dependence on the energy [[illegible], 32, 33]. Nevertheless, what is maintained is the conformal invariance of the system, i.e., the phase shift is invariant under

$$x_0 \to \Lambda x_0, \quad k \to \frac{k}{\Lambda}, \quad w_1 \to w_1. \tag{40}$$

4.2. *On the existence of zero-modes*

In this section we will deduce the conditions which ensure the existence of states with zero energy for the δ-δ' potential[1]. The presence of an energy gap between the discrete spectrum of negative energy levels and the continuum spectrum of positive energy levels is of great importance in some areas of fundamental physics (see, e.g. [47]), specially when we promote non-relativistic quantum Hamiltonians to effective quantum field theories under the influence of a given classical background. To start with, we solve (13) for $k = 0$

$$\left[\frac{d^2}{dx^2} - \frac{(2-\eta)(4-\eta)}{4x^2}\right] u_{0\,\ell}(x) = 0 \quad \text{with} \quad \eta \equiv 5 - (d + 2\ell). \tag{41}$$

The general solution of the zero-mode differential equation is given by

$$v_\eta(x) \equiv u_{0\,\ell}(x) = \begin{cases} c_1\, x^{\frac{\eta-2}{2}} + c_2\, x^{\frac{4-\eta}{2}} & \text{if} \quad \eta \neq 3, \\ c_1\sqrt{x} + c_2\sqrt{x}\log x & \text{if} \quad \eta = 3. \end{cases} \tag{42}$$

It must be emphasized that $\eta = 3$ corresponds to $d = 2$ and $\ell = 0$. In order to determine the integration constants of the general solution (42) we must impose two requirements. The first condition is square integrability

$$\int_0^\infty |v_\eta(x)|^2\, dx = \int_0^{x_0} |v_\eta(x)|^2\, dx + \int_{x_0}^\infty |v_\eta(x)|^2\, dx < \infty, \tag{43}$$

where both integrals should be finite. The second one is the matching condition that defines the δ-δ' singular potential (16). Depending on η, i.e., the angular momentum ℓ and the dimension of the physical space d, we can distinguish two cases.

Case 1: $\eta \in \{1, 2, 3\}$. After imposing (43) we end up with the reduced radial wave functions

$$v_\eta(x) = \begin{cases} \sqrt{x}\,(c_1 + c_2\ \log x) & \text{if} \quad \eta = 3, \\ c_1 + c_2\, x & \text{if} \quad \eta = 2, \\ c_1 x^{3/2} & \text{if} \quad \eta = 1, \end{cases} \quad \text{for} \quad x < x_0 \quad \text{and} \quad v_\eta(x) = 0 \ \text{ for } \ x > x_0.$$

In this case the matching conditions of (16) are satisfied if, and only if, $c_1 = c_2 = 0$. Therefore there are no zero energy states.

Case 2: $\eta \leq 0$. In this situation, the square integrable solution and matching conditions result in

$$v_\eta(x) = \begin{cases} c_2\, x^{\frac{4-\eta}{2}} & x < x_0, \\ c_1\, x^{\frac{\eta-2}{2}} & x > x_0, \end{cases} ; \quad c_1 \begin{pmatrix} x_0^{\frac{\eta-2}{2}} \\ \frac{\eta-2}{2} x_0^{\frac{\eta-2}{2}-1} \end{pmatrix} = c_2 \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix} \begin{pmatrix} x_0^{\frac{4-\eta}{2}} \\ \frac{4-\eta}{2} x_0^{-\frac{\eta-2}{2}} \end{pmatrix}. \tag{44}$$

A non trivial solution exists if, and only if, the system satisfies

$$\beta = \frac{-2\alpha^2 + \alpha^2\eta + \eta - 4}{2\alpha x_0} \quad \Rightarrow \quad c_2 = x_0^{\eta-3}\alpha^{-1} c_1. \tag{45}$$

In addition, the regularity condition at $x = 0$ is also satisfied: $v_\eta(x = 0) = 0$. Hence, for a given dimension d and an angular momentum ℓ such that $5 \leq d + 2\ell$, there is a zero mode given by (44) with $c_2 = x_0^{\eta-3}\alpha^{-1} c_1$ if and only if the couplings α and β satisfy the relation (45). Indeed, if the previous equation is inserted in (32) we obtain

$$L_{max} = \ell_{max} = \ell,$$

which is in agreement with our previous analysis of the energy levels, i.e., if $L_{max} = \ell_{max}$ the left hand side of the secular equation (28), $F(\kappa_\ell\, x_0)$, reaches the right hand side at $\kappa_\ell = 0$. The reverse is also true.

4.3. The mean value of the position operator

In this section we will show some numerical results concerning the expectation value of x for the bound states that satisfy $\eta < 0$ (as a function of the parameters w_0 and w_1). Once the dimension d, the radius x_0 and the angular momentum ℓ are fixed, the plane w_0-w_1 is divided into two zones: one in which the bounds states do not exist and another one in which they do. The limit between these two zones corresponds to the zero-mode states[2]. The existence of zero-modes is of critical importance to compute numerically the expectation value of the dimensionless radius x when the parameters w_0 and w_1 are close to the common boundary of the regions mentioned.

For a given bound state of energy $\lambda_\ell = -\kappa_\ell^2$, the general expression for the expectation value $\langle x \rangle_{\kappa\ell} \equiv \langle \Psi_{\kappa\ell} | \, x \, | \Psi_{\kappa\ell} \rangle$ is given in terms of the reduced radial wave function

[2] This is ensured by the condition $\eta \leq 0$. If $\eta > 0$ the limit between the two zones does not correspond to a physically meaningful state as it was previously demonstrated.

as

$$\langle x\rangle_{\kappa\ell} = \frac{1}{\kappa_\ell}\frac{\displaystyle\int_0^{\kappa_\ell x_0} z^2 I^2_{\ell+\nu}(z)dz + \left(\frac{\alpha\, I_{\ell+\nu}(\kappa_\ell x_0)}{K_{\ell+\nu}(\kappa_\ell x_0)}\right)^2 \int_{\kappa_\ell x_0}^{\infty} z^2 K^2_{\ell+\nu}(z)dz}{\displaystyle\int_0^{\kappa_\ell x_0} z I^2_{\ell+\nu}(z)dz + \left(\frac{\alpha\, I_{\ell+\nu}(\kappa_\ell x_0)}{K_{\ell+\nu}(\kappa_\ell x_0)}\right)^2 \int_{\kappa_\ell x_0}^{\infty} z K^2_{\ell+\nu}(z)dz}. \tag{46}$$

The last expression does not depend explicitly on β (17), but it does through κ_ℓ. If we take the limit $\kappa_\ell \to 0^+$ we obtain

$$\frac{\langle x\rangle_{0\ell}}{x_0} \equiv \lim_{\kappa_\ell\to 0^+}\frac{\langle x\rangle_{\kappa\ell}}{x_0} = \begin{cases} \dfrac{\eta-1}{\eta}\left(1-\left|\dfrac{2(\eta-3)}{(\eta-6)\left(\alpha^2(\eta-5)+\eta-1\right)}\right|\right) & \eta\leq -1, \\ \infty & \eta\in\{0,1,2,3\}. \end{cases}$$

As expected, this result coincides with the calculation of the mean value for the zero-modes, carried out with the wave functions in (44). As can be seen, when there exist zero-modes with $\eta < 0$, the expectation value $\langle x\rangle_{0\ell}$ is finite, but when the system does not admit them, or $\eta = 0$ the limit $\langle x\rangle_{0\ell}$ is divergent. Somehow, the zero-modes with $\eta = 0$ are semi-bound states in the sense that the expectation value is divergent. This behaviour gives rise to three different situations:

- When there are zero-modes with $\eta < 0$, the mean value $\langle x\rangle_{\kappa\ell}$ for the bound states has a finite upper bound

$$\lim_{w_1\to 1}\frac{\langle x\rangle_{0\ell}}{x_0} = (\eta-1)/\eta. \tag{47}$$

- If there is a semi-bound zero mode, i.e., $\eta = 0$, the upper bound imposed by $\langle x\rangle_{0\ell}$ is infinite: $\langle x\rangle_{\kappa\ell}$ diverges as $\lambda_\ell \to 0^-$ in the w_0-w_1 plane.
- When there are no zero-modes, $\langle x\rangle_{\kappa\ell}$ does not have an upper bound and therefore, as λ_ℓ goes to zero the expectation value goes to infinity. This fact can be interpreted as the state disappearing from the system: when $\lambda_\ell \to 0^-$ the corresponding wave function becomes identically zero.

In Fig.3 we have plotted the mean value of two configurations as a function of the couplings w_0 and w_1 for values of d and ℓ such that $\eta < 0$ (there is a zero-mode). We have distinguished the region in which the expectation value of x lies outside the δ-δ' horizon and the one with $\langle x\rangle < x_0$. The former, bearing in mind the original ideas by G. 't Hooft [3, 4], would correspond to the states of quantum particles falling into a black hole that would be observed by a distant observer. Indeed, the amount of bound states in two and three dimensions is proportional to x_0 and x_0^2 respectively and as it is mentioned[3] in [37] these bound states would give an area law for the corresponding entropy in quantum field theory when they are interpreted as micro-states of the black hole horizon.

[3] Although the formulas for ℓ_{max} presented in [37] are only valid when (21) is satisfied, the behavior of the total amount of bound states as a function of x_0 does not change (as long as x_0 is large enough). Consequently, the argument for the area law remains valid.

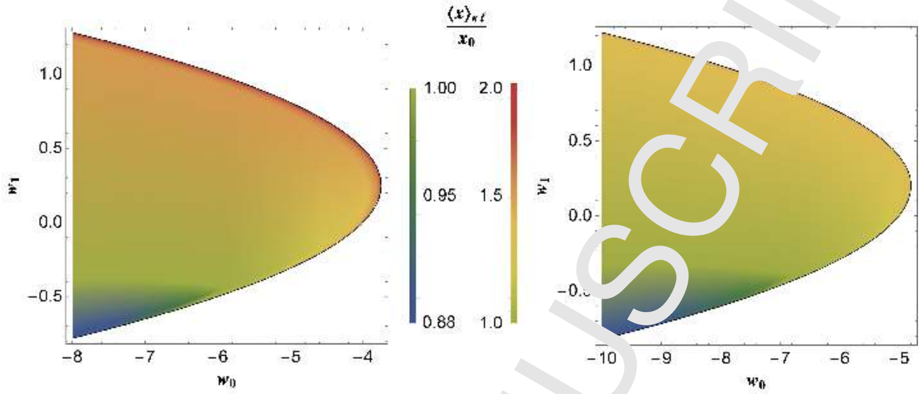

Figure 3: Mean value of the dimensionless radius operator $\langle x\rangle/x_0$ given in (46). LEFT: $x_0 = 1$, $\ell = 2$ and $d = 2$ being $\eta = -1$. RIGHT: $x_0 = 1$, $\ell = 2$ and $d = 3$ being $\eta = -2$. The limit $\kappa \to 0^+$ in (46) fits with (4.3). The black curve satisfies $L_{max} = \ell$ in each case.

On the energy shifts produced by the δ'

It is worth mentioning two central differences between the present analysis in arbitrary dimension with the hyperspherical δ-δ' potential ($d \geq 2$) and the one dimensional point analog [32]. In the latter, the δ' by itself ($w_0 = 0$) only gives rise to a pure continuum spectrum of positive energy levels (scattering states). In addition, for the one dimensional case when $w_0 < 0$ the appearance of the δ'-term in the potential increases the energies of the bound states because it breaks parity symmetry, which does not happen for $d \geq 2$. These two properties are not maintained in general for $d \geq 2$. For example, in two dimensions there is a bound state with energy $\lambda_{\ell=0} = -1.205$ if $w_1 = 0.9$ ($x_0 = 0.15$ and, of course, $w_0 = 0$). Secondly, the previous case and all the study of Section 3.2 prove that there are bound states with lower energy when the δ' is added to the δ potential. In view of the above, it could be thought that it only takes place when $w_0 \geq 0$ (since the δ potential presents no bound states). However, if we consider a three dimensional system with $x_0 = 1$ and $w_0 = -1.85$, a single bound state with energy $\lambda_{\ell=0} = -0.514$ appears when $w_1 = 0.437$ and with $\lambda_{\ell=0} = -0.482$ if we turn off the δ'. What we can conclude from the numerical results is that the δ-δ' potential can give rise to a lower energy fundamental state than if it has only the δ potential for $d \geq 2$.

5. Concluding remarks

Our study provides novel results with δ-δ' hyperspherical potentials. Firstly, on the basis of this paper in arbitrary dimension, a careful study of the applications that we have already reported (and others) can be performed. The special attention paid on bound states is justified: as was shown in [37] the bound states can be thought of, in a quantum field theoretical view, as photon states falling into a black hole for an observer far away from the event horizon. In this sense, the δ-δ' potential generalises the brick wall model by G. 't Hooft [3, 4]. In addition, the knowledge of the bound state spectrum of the system plays an essential role in the study of fluctuations around classical solutions and in the Casimir effect when the Schödinger operator $-\Delta_d + V_{\delta\text{-}\delta'}$ is reinterpreted as the one particle Hamiltonian of an effective quantum field theory.

Our first achievement is the generalisation of the results given in [19] for the one dimensional δ'-potential. We have introduced a rigorous and consistent definition of the potential $V_{\delta\text{-}\delta'} = w_0\delta(x - x_0) + 2w_1\delta'(x - x_0)$ in arbitrary dimension, characterizing a selfadjoint extension of the Hamiltonian $\mathcal{H}_0$ (14) defined on $\mathbb{R}^d$. In doing so, we have corrected the matching conditons in [37] for the two and three dimensional $V_{\delta\text{-}\delta'}$ potential. We have shown that the Dirac-δ coupling requires a re-definition which also depends on the radius x_0 and the δ' coupling w_1.

We have also characterised the spectrum of bound states in arbitrary dimension, computing analytically the amount of bound states for any values of the free parameters w_0, w_1 and x_0 that appear in the Hamiltonian. One of the most interesting and counterintuitive results we have found is the existence of a negative energy level for $d = 2$ and $\ell = 0$ when the Dirac-δ coupling w_0 is positive. In such a situation, the Dirac-δ potential $w_0\delta(x - x_0)$, with $w_0 > 0$, is an infinitely thin potential barrier, therefore bound states in the regime $w_0 > 0$ are not expected (as it happens for the one dimensional analog [10]).

As a limiting case of the spectrum of bound states for the Hamiltonian (5), we have obtained the spectrum of zero-modes of the system in terms of the parameter $\eta = 5-(d+2\ell)$. We have shown that the conditions on w_0, w_1 and x_0 for the existence of zero-modes are $\ell_{max} = L_{max}$ and $\eta \leq 0$. In addition, we have computed numerically the expectation value $\langle x\rangle_{\kappa\ell}/x_0$ for the bound states with energy $\lambda = -\kappa^2$ and angular momentum ℓ as a function of w_0 and w_1. This calculation has enabled us to realise that the zero-modes with $\eta < 0$ behave as bound states in the sense that $\langle x\rangle_{0\ell} < \infty$, and the zero-modes corresponding to $\eta = 0$ behave as semi-bound states due to $\langle x\rangle_{0\ell} = \infty$. These results determine the topological properties of the space of states of the system since the existence of zero modes characterises the space of couplings.

To complete our study of the Hamiltonian (5) we have obtained an analytical expression for all the phase shifts which describe all the scattering states of the system. This calculation is of central importance when we promote (5) to an effective quantum field theory (see [1[illegible]]) under the influence of a classical background. In this scenario, the knowledge of the phase shifts allows us to compute the zero point energy [43]. In addition, as it is shown in [43] the phase shifts contain in their asymptotic behaviour all the heat kernel coefficients of the asymptotic expansion of the heat trace.

Further work for the future could usefully be to add a non-singular hyperspherical background potential $V_0(x)$ to the $V_{\delta\text{-}\delta'}(x)$. For example, the spectrum of $|x|$ plus the δ-δ' potential at the origin is studied for one dimensional systems in [30]. For these cases, first of all, we would have to define the selfadjoint extension which characterizes $V_{\delta\text{-}\delta'}$ considering $\mathcal{H}_{V_0} \equiv \mathcal{H}_0 + V_0(x)$ instead of $\mathcal{H}_0$. If $V_0(x)$ satisfies the hypothesis of the Kato-Rellich theorem, the selfdadjointness of $\mathcal{H}_{V_0}$ is guaranteed by the selfdadjointness of $\mathcal{H}_0$ [48]. In this way, for this kind of potentials it seems reasonable that the analysis carried out in Section 3 can be generalised by just exchanging the modified hyperspherical Bessel functions ($V_0 = 0$) by the corresponding general solutions of the background potential $V_0(x)$. Of course, most potentials can not be solved analytically [49], but it is worth exploring the (solvable) Coulomb potential $V_0(x) = -\gamma/x$ with

$\gamma \in \mathbb{R}_{>0}$ in arbitrary dimension. In addition to its known applications in a multitude of disciplines, this potential has recently been shown to play a central role in condensed matter physics to mimic impurities in real graphene sheets and other two-dimensional systems [50–52]. For the Coulomb potential, the general solution can be written in terms of Whittaker functions which are closely related to the modified Bessel functions studied in the free case [38, 53]. Some important differences with respect to the latter are expected, e.g., an infinite number of negative energy levels ($\ell_{max} \to \infty$) with, possibly, an accumulation point not necessarily at zero energy.

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References

References

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. 51 (1948) 793.

[2] M. Sparnaay, Physica 24 (6) (1958) 751–764. doi:10.1016/S0031-8914(58)80090-7.

[3] G. 't Hooft, Nucl. Phys. B 256 (1985) 727–745. doi:10.1016/0550-3213(85)90418-3.

[4] G. 't Hooft, Int. J. Mod. Phys. A 11 (26) (1996) 4623–4688. doi:10.1142/S0217751X96002145.

[5] J. L. Garcia-Pomar, A. Y. Nikitin, L. Martin-Moreno, ACS Nano 7 (6) (2013) 4988–4994. doi:10.1021/nn400342v.

[6] M. Belloni, R. Robinett, Phys. Rept. 540 (2) (2014) 25–122. doi:10.1016/j.physrep.2014.02.005.

[7] R. de L. Kronig, W. G. Penney, Proc. R. Soc. London, Ser. A 130 (814) (1931) 499–513. doi:10.1098/rspa.1931.0019.

[8] A. Galindo, P. Pascual, Quantum Mechanics I, Texts and Monographs in Physics, Springer-Verlag, 1990. doi:10.1007/978-3-642-83854-5.

[9] R. Jackiw, Delta-function potentials in two-and three-dimensional quantum mechanics, Diverse Topics in Theoretical and Mathematical Physics I.3 (1995) 35–54. doi:10.1142/9789814261494_0001.

[10] J. M. Muñoz-Castañeda, J. M. Guilarte,Phys. Rev. D 91 (2) (2015) 025028. doi:10.1103/PhysRevD.91.025028.

[11] J. M. Guilarte, J. M. Munoz-Castaneda, Int. J. Theor. Phys. 50 (2011) 2227–2241. doi:10.1007/s10773-011-0723-0.

[12] J. M. Munoz-Castaneda, J. M. Guilarte, A. M. Mosquera, Phys. Rev. D 87 (2013) 105020. doi:10.1103/PhysRevD.87.105020.

[13] M. Bordag, D. Hennig, D. Robaschik, J. Phys. A: Math. Gen. 25 (16) (1992) 4483.

[14] M. Asorey, A. Ibort, G. Marmo, Int. J. Geom. Meth. Mod. Phys. 12 (6) (2015) 1561007. doi:10.1142/S0219887815610071.

[15] M. Asorey, A. Ibort, G. Marmo, Int. J. Mod. Phys. A20 (2005) 1001–1026. doi:10.1142/S0217751X05019798.

[16] J. Alvarez, M. Gadella, L. Lara, F. Maldonado-Villamizar, Phys. Lett. A 377 (38) (2013) 2510–2519. doi:10.1016/j.physleta.2013.07.045.

[17] M. Asorey, D. Garcia-Alvarez, J. Munoz-Castaneda, J. Phys. A: Math. Gen. 39 (2006) 6127–6136. doi:10.1088/0305-4470/39/21/S03.

[18] M. Asorey, J. Munoz-Castaneda, Nucl. Phys. B 874 (2013) 852–876. doi:10.1016/j.nuclphysb.2013.06.014.

[19] P. Kurasov, J. Math. Anal. Appl. 201 (1) (1996) 297–323. doi:10.1006/jmaa.1996.0256.

[20] J. Dittrich, P. Exner, P. Seba, J. Math. Phys. 33 (6) (1992) 2207–2214. doi:10.1063/1.529641.

[21] D. Gitman, I. Tyutin, B. Voronov, Self-adjoint Extensions in Quantum Mechanics: General Theory and Applications to Schrödinger and Dirac Equations with Singular Potentials, Progress in Mathematical Physics, Birkhäuser Boston, 2012. doi:10.1007/978-0-8176-4662-2.

[22] S. Albeverio, P. Kurasov, Singular Perturbations of Differential Operators, Vol. 271 of London Mathematical Society: Lecture Notes Series, Cambridge University Press, 2000. doi:10.1017/CBO9780511758904.

[23] A. Ibort, F. Lledó, J. M. Pérez-Pardo, Ann. Henri Poincaré 16 (10) (2015) 2367-2397. doi:10.1007/s00023-014-0379-4.

[24] G. Fucci, K. Kirsten, P. Morales, Springer Proc. Phys. 137 (2011) 313–322. doi:10.1007/978-3-642-19760-4_29.

[25] G. Fucci, K. Kirsten, J. Phys. A: Math. Theor 44 (2011) 295403. doi:10.1088/1751-8113/44/29/295403.

[26] T. R. Govindarajan, R. Tibrewala, Phys. Rev. D 92 (2015) 045040. doi:10.1103/PhysRevD.92.045040.

[27] T. R. Govindarajan, V. P. Nair, Phys. Rev. D 89 (2014) 025020. doi:10.1103/PhysRevD.89.025020.

[28] G. Fucci, Int. J. Mod. Phys. A 32 (31) (2017) 1750182. doi:10.1142/S0217751X17501822.

[29] G. Fucci, Nucl. Phys. B 891 (2015) 676–699. doi:10.1016/j.nuclphysb.2014.12.023.

[30] S. Fassari, M. Gadella, L. Glasser, L. Nieto, Ann. Phys. 389 (2018) 48–62. doi:10.1016/j.aop.2017.12.006.

[31] P. Facchi, G. Garnero, G. Marmo, J. Samuel, S. Sinha, Ann. Phys. 394 (2018) 139–154. doi:10.1016/j.aop.2018.04.030.

[32] M. Gadella, J. Negro, L. Nieto, Phys. Lett. A 373 (15) (2009) 1310–1313. doi:10.1016/j.physleta.2009.02.025.

[33] M. Gadella, J. Mateos-Guilarte, J. M. Muñoz-Castañeda, L. M. Nieto, J. Phys. A: Math. Theor. 49 (1) (2016) 015204. doi:10.1088/1751-8113/49/1/015204.

[34] J. S. Avery, J. Comput. Appl. Math. 233 (6) (2010) 1366–1379. doi:10.1016/j.cam.2009.02.057.

[35] K. Kirsten, Spectral functions in mathematics and physics, Chapman & Hall/CRC, Boca Raton, FL, 2001.

[36] N. Setô, Publ. RIMS Kyoto Univ. 9 (2) (1974) 429–461. doi:10.2977/prims/1195192566.

[37] T. Govindarajan, J. M. Munõz-Castañeda, Mod. Phys. Lett. A 31 (39) (2016) 1650210. doi:10.1142/S0217732316502102.

[38] F. W. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, NIST Handbook of Mathematical Functions, 1st Edition, Cambridge University Press, New York, NY, USA, 2010.

[39] M. Berry, A. O. de Almeida, J. Phys. A: Math. Nuc. Gen. 6 (10) (1973) 1451.

[40] A. Laforgia, P. Natalini, J. Inequal. Appl. (1) (2010) 253035. doi:10.1155/2010/253035.

[41] V. Bargmann, Proc. Natl. Acad. Sci. U.S.A. 38 (11) (1952) 961–966.

[42] J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. 47 (1) (1961) 122–129.

[43] M. Beauregard, M. Bordag, K. Kirsten, J. Phys. A: Math. Theor. 48 (9) (2015) 095401. doi:10.1088/1751-8113/48/9/095401.

[44] J. Taylor, Scattering Theory: The Quantum Theory of Nonrelativistic Collisions, Dover Books on Engineering, Dover Publications, 2012.

[45] I. R. Lapidus, Am. J. Phys. 54 (5) (1986) 459–461.

[46] K. Huang, C. N. Yang, Phys. Rev. 105 (3) (1957) 767.

[47] P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, E. Witten (Eds.), Quantum fields and strings: A course for mathematicians. Vol. 1, 2, 1999.

[48] C. de Oliveira, Intermediate Spectral Theory and Quantum Dynamics, Progress in Mathematical Physics, Birkhäuser Basel, 2009. doi:10.1007/978-3-7643-8795-2.

[49] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden, Solvable models in quantum mechanics, 2nd Edition, AMS Chelsea Publishing, Providence, RI, 2005.

[50] V. M. Pereira, J. Nilsson, A. H. Castro Neto, Phys. Rev. Lett. 99 (2007) 166802. doi:10.1103/PhysRevLett.99.166802.

[51] N. N. Konobeeva, M. B. Belonenko, Opt. Spectrosc. 120 (6) (2016) 940–943. doi:10.1134/S0030400X16060126.

[52] J. Wang, H. A. Fertig, G. Murthy, L. Brey, Phys. Rev. B 83 (2011) 035404. doi:10.1103/PhysRevB.83.035404.

[53] S. Richard, J. Derezinski, arXiv:1712.04068 [math-ph].