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Hypersurface of an even-dimensional sphere satisfying a certain commutative condition

By David E. BLAIR, Gerald D. LUDDEN
and Masafumi OKUMURA

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Introduction.

Let $\bar{M}$ be a Riemannian manifold which admits a linear transformation $\bar{f}$ of its tangent bundle $T(\bar{M})$. Then, the tangent bundle $T(M)$ of a hypersurface M of $\bar{M}$ naturally admits a linear transformation f induced from that of the tangent bundle of the ambient space.

On the other hand, on any hypersurface of a Riemannian manifold there is the linear transformation H of its tangent bundle which is defined by the second fundamental tensor.

It seems to be an interesting problem to consider relations between some linear algebraic conditions of these two transformations and properties of the hypersurface.

In this direction one of the authors started to study the case where the ambient space is an even-dimensional Euclidean space [4] and thereafter Yamaguchi [7], Yano [3] and the present authors [3, 5] studied the case where the ambient manifold is an odd-dimensional sphere and these transformations are commutative or anti-commutative.

However, until recently there was no known linear transformation of the tangent bundle of an even-dimensional sphere except in dimensions 2 and 6. Recently two of the present authors and Yano [1, 2] found a linear transformation of the tangent bundle of certain even-dimensional manifolds including an even-dimensional sphere.

In this paper, using this linear transformation, we study a hypersurface of an even-dimensional sphere for which this transformation commutes with the transformation H.

§ 1. Hypersurface of an even-dimensional sphere.

Let S^{2n} be an even-dimensional sphere of radius 1. Then on S^{2n} there exist a (1, 1)-tensor field $\bar{f}$, two vector fields $\bar{U}$, $\bar{V}$, two 1-forms u, v and a function λ satisfying the following conditions [**1**, **2**]:

$$\bar{f}^2\bar{X}=-\bar{X}+\bar{u}(\bar{X})\bar{U}+\bar{v}(\bar{X})\bar{V}, \tag{1.1}$$

$$\bar{u}(\bar{f}\bar{X})=\lambda\bar{v}(\bar{X}), \qquad \bar{f}\bar{U}=-\lambda\bar{V}, \tag{1.2}$$

$$\bar{v}(\bar{f}\bar{X})=-\lambda\bar{u}(\bar{X}), \qquad \bar{f}\bar{V}=\lambda\bar{U}, \tag{1.3}$$

$$\bar{u}(\bar{U})=\bar{v}(\bar{V})=1-\lambda^2, \tag{1.4}$$

$$\bar{v}(\bar{U})=\bar{u}(\bar{V})=0, \tag{1.5}$$

where $\bar{X}$ is a vector field on S^{2n}.

The Riemannian metric $\bar{g}$ of S^{2n} satisfies

$$\bar{g}(\bar{U},\bar{X})=\bar{u}(\bar{X}), \qquad \bar{g}(\bar{V},\bar{X})=\bar{v}(\bar{X}), \tag{1.6}$$

$$\bar{g}(\bar{f}\bar{X},\bar{f}\bar{Y})=\bar{g}(\bar{X},\bar{Y})-\bar{u}(\bar{X})\bar{u}(\bar{Y})-\bar{v}(\bar{X})\bar{v}(\bar{Y}), \tag{1.7}$$

for any vector fields $\bar{X}$ and $\bar{Y}$. In this case

$$\bar{\omega}(\bar{X},\bar{Y})=\bar{g}(\bar{f}\bar{X},\bar{Y})$$

is a 2-form, that is, $\bar{f}$ is a skew-symmetric linear transformation of the tangent bundle of S^{2n}.

In general, the set $(\bar{f},\bar{U},\bar{V},\bar{u},\bar{v},\lambda,\bar{g})$ satisfying (1.1)~(1.7) is said to be an $(\bar{f},\bar{g},\bar{u},\bar{v},\lambda)$-*structure*. Moreover, it is known on S^{2n} that the $(\bar{f},\bar{g},\bar{u},\bar{v},\lambda)$-structure satisfies the following relations [**1**, **8**]:

$$(\bar{\nabla}_{\bar{X}}\bar{f})(\bar{Y})=-\bar{g}(\bar{X},\bar{Y})\bar{V}+\bar{v}(\bar{Y})\bar{X}, \tag{1.8}$$

$$\bar{\nabla}_{\bar{X}}\bar{U}=-\lambda\bar{X}, \tag{1.9}$$

$$\bar{\nabla}_{\bar{X}}\bar{V}=\bar{f}\bar{X}, \tag{1.10}$$

$$\bar{X}\lambda=\bar{u}(\bar{X}), \tag{1.11}$$

where $\bar{\nabla}_{\bar{X}}$ denotes the operator of the covariant differentiation with respect to $\bar{g}$.

Let M be a hypersurface of S^{2n} and B the differential of the imbedding i of M into S^{2n}.

Applying $\bar{f}$ to BX and to the unit normal vector N to M, we obtain two vector fields $\bar{f}BX$ and $\bar{f}N$ which can be represented as a sum of their tangential and normal parts. Thus we write

$$\bar{f}BX=BfX+w(X)N, \tag{1.12}$$

$$\bar{f}N=-BW. \tag{1.13}$$

Then f defines a linear transformation of the tangent bundle of M, w and W define respectively a 1-form and a vector field on M. Moreover, we see easily that

$$g(W, X)=w(X),$$

where g is the induced Riemannian metric on M.

The vector fields $\bar{U}$ and $\bar{V}$ can be decomposed as

$$\bar{U}=BU+\alpha N, \tag{1.14}$$

$$\bar{V}=BV+\beta N, \tag{1.15}$$

where U, V are vector fields and α, β are functions on M. Now we define two 1-forms on M by

$$u(X)=\bar{u}(BX), \tag{1.16}$$

$$v(X)=\bar{v}(BX). \tag{1.17}$$

Then we can easily see that

$$g(U, X)=u(X), \qquad g(V, X)=v(x).$$

Furthermore we have

$$\bar{u}(N)=\bar{g}(\bar{U}, N)=\bar{g}(BU+\alpha N, N)=\alpha, \tag{1.18}$$

$$\bar{v}(N)=\bar{g}(\bar{V}, N)=\bar{g}(BV+\beta N, N)=\beta. \tag{1.19}$$

We denote by ∇_X the operator of covariant differentiation with respect to the induced Riemannian connection of M. Then the Gauss and Weingarten equations are given by

$$\bar{\nabla}_{BX}BY=B\nabla_X Y+h(X, Y)N, \tag{1.20}$$

$$\bar{\nabla}_{BX}N=-BHX, \tag{1.21}$$

where h is the second fundamental tensor of the hypersurface and satisfies

$$h(X, Y)=g(HX, Y)=g(X, HY)=h(Y, X).$$

The equations of Gauss, Mainardi-Codazzi are respectively given by

$$R(X, Y)Z=g(Y, Z)X-g(X, Z)Y+h(Y, Z)X-h(X, Z)Y, \tag{1.22}$$

$$(\nabla_X H)Y-(\nabla_Y H)X=0, \tag{1.23}$$

since the ambient manifold S^{2n} is a manifold of constant curvature 1. From (1.23) we have easily

$$\nabla_Y \operatorname{tr} H=\sum_i (\nabla_{E_i}h)(E_i, Y)=\sum_i g((\nabla_{E_i}H)E_i, Y), \tag{1.24}$$

where E_i $(i=1, \cdots, 2n-1)$ is an orthonormal frame of $T(M)$.

§ 2. Some formulas.

We apply $\bar{f}$ to both sides of (1.12). Then by virtue of (1.1), (1.12) and (1.13) we get

$$-BX+\bar{u}(BX)\bar{U}+\bar{v}(BX)\bar{V}=Bf^2X+w(fX)N-w(X)BW.$$

Substituting (1.14), (1.15), (1.16) and (1.17) into the last equation, we have

$$-BX+u(X)(BU+\alpha N)+v(X)(BV+\beta N)=Bf^2X+w(fX)N-w(X)BW,$$

from which we obtain

$$f^2X=-X+u(X)U+v(X)V+w(X)W \tag{2.1}$$

and

$$w(fX)=\alpha u(X)+\beta v(X), \tag{2.2}$$

that is,

$$fW=-\alpha U-\beta V, \tag{2.2$'$}$$

because of (1.6).

Applying $\bar{f}$ to both sides of (1.13), we find

$$-N+\bar{u}(N)\bar{U}+\bar{v}(N)\bar{V}=-BfW-w(W)N,$$

from which we have that

$$-N+\alpha(BU+\alpha N)+\beta(BV+\beta N)=-BfW-w(W)N.$$

Thus we get

$$w(W)=g(W, W)=1-\alpha^2-\beta^2. \tag{2.3}$$

Now write (1.2) and (1.3) in the following way

$$\bar{f}\bar{U}=\bar{f}(BU+\alpha N)=-\lambda(BV+\beta N),$$

$$\bar{f}\bar{V}=\bar{f}(BV+\beta N)=\lambda(BU+\alpha N),$$

which imply that

$$BfU+w(U)N-\alpha BW=-\lambda BV-\lambda\beta N$$

and

$$BfV+w(V)N-\beta BW=\lambda BU+\lambda\alpha N.$$

That is, we have

$$fU=-\lambda V+\alpha W, \tag{2.4}$$

$$fV=\lambda U+\beta W, \tag{2.5}$$

$$w(U)=u(W)=g(U, W)=-\lambda\beta, \tag{2.6}$$

(2.7) $$w(V)=v(W)=g(W, V)=\lambda\alpha .$$

Substituting (1.14), (1.15) into (1.4), (1.5), we find

(2.8) $$u(U)=g(U, U)=1-\alpha^2-\lambda^2 ,$$

(2.9) $$v(V)=g(V, V)=1-\beta^2-\lambda^2 ,$$

(2.10) $$u(V)=v(U)=g(U, V)=-\alpha\beta .$$

Now we calculate the length of the vectors fW, fU and fV.

$$\begin{aligned} g(fW, fW) &= g(-\alpha U-\beta V, -\alpha U-\beta V) \\ &= \alpha^2 g(U, U)+2\alpha\beta g(U, V)+\beta^2 g(V, V) . \end{aligned}$$

Substituting (2.8), (2.9), (2.10) into the last equation, we get

(2.11) $$g(fW, fW)=(\alpha^2+\beta^2)(1-\alpha^2-\beta^2-\lambda^2) .$$

Similarly we have

(2.12) $$\begin{cases} g(fU, fU)=(\alpha^2+\lambda^2)(1-\alpha^2-\beta^2-\lambda^2) , \\ g(fV, fV)=(\beta^2+\lambda^2)(1-\alpha^2-\beta^2-\lambda^2) . \end{cases}$$

We also have

(2.13) $$\begin{cases} g(fU, V)=-\lambda(1-\alpha^2-\beta^2-\lambda^2) , & g(fU, W)=\alpha(1-\alpha^2-\beta^2-\lambda^2) , \\ g(fV, U)=\lambda(1-\alpha^2-\beta^2-\lambda^2) , & g(fV, W)=\beta(1-\alpha^2-\beta^2-\lambda^2) , \\ g(fW, U)=-\alpha(1-\alpha^2-\beta^2-\lambda^2) , & g(fW, V)=-\beta(1-\alpha^2-\beta^2-\lambda^2) . \end{cases}$$

Next, differentiating (1.12) covariantly and making use of (1.8), (1.20) and (1.21), we find that

$$\begin{aligned} -\bar{g}(BY, BX)\bar{V}+\bar{v}(BX)BY+\bar{f}(B\nabla_Y X+h(X, Y)N) \\ = B\nabla_Y(fX)+h(fX, Y)N+Y(w(X))N-w(X)BHY , \end{aligned}$$

from which we have

(2.14) $$(\nabla_Y f)X=-g(X, Y)V+v(X)Y-h(X, Y)W+w(X)HY$$

and

(2.15) $$(\nabla_Y w)X=-\beta g(X, Y)-h(fX, Y) ,$$

by virtue of (1.14) and (1.15).

Differentiating (1.14) covariantly and making use of (1.9), we have

$$-\lambda BX=B\nabla_X U+h(X, U)N+(X\alpha)N-\alpha BHX .$$

That is,

(2.16) $$\nabla_X U=-\lambda X+\alpha HX ,$$

(2.17) $$X\alpha=-h(X, U)=-u(HX).$$

Similarly from (1.15) we have

(2.18) $$\nabla_X V=fX+\beta HX,$$

(2.19) $$X\beta=w(X)-v(HX)=w(X)-h(V, X),$$

and from (1.11)

(2.20) $$X\lambda=u(X).$$

§ 3. Hypersurfaces satisfying $Hf=fH$.

In the following discussion we assume that the hypersurface M satisfies the condition

(3.1) $$Hf=fH.$$

This means that

(3.2) $$g(HX, fY)+g(HY, fX)=0.$$

Thus putting $X=U$, $Y=U$, we have

(3.3) $$g(HU, fU)=0.$$

Similarly we have

(3.4) $$g(HV, fV)=0,$$

(3.5) $$g(HW, fW)=0.$$

Using (3.2), we can also prove

(3.6) $$g(HV, fW)+g(HW, fV)=0,$$

(3.7) $$g(HW, fU)+g(HU, fW)=0,$$

(3.8) $$g(HU, fV)+g(HV, fU)=0.$$

Next differentiating

$$HfX=fHX$$

covariantly and making use of (2.14), we get

$$(\nabla_Y H)fX-g(X, Y)HV+v(X)HY-g(HX, Y)HW+w(X)H^2Y$$
$$=-g(HX, Y)V+v(HX)Y-g(H^2X, Y)W+w(HX)HY+f(\nabla_Y H)X,$$

or

$$g((\nabla_Y H)fX, Z)-g(X, Y)g(HV, Z)+v(X)g(HY, Z)$$
$$-g(HX, Y)g(HW, Z)+w(x)g(H^2Y, Z)$$

$$=-g(HX, Y)g(V, Z)+v(HX)g(Y, Z)$$
$$-g(H^2X, Y)g(W, Z)+w(HX)g(HY, Z)+g(f(\nabla_Y H)X, Z).$$

Replacing Y and Z by an orthonormal frame E_i and making use of the symmetric property of $\nabla_Y H$, we find

$$\sum_i \{g(fX, (\nabla_{E_i}H)E_i)-g(X, E_i)g(HV, E_i)+v(X)g(HE_i, E_i)$$
$$-g(HX, E_i)g(HW, E_i)+w(X)g(H^2E_i, E_i)\}$$
$$=\sum_i \{-g(HX, E_i)g(V, E_i)+v(HX)g(E_i, E_i)$$
$$-g(H^2X, E_i)g(W, E_i)+w(HX)g(HE_i, E_i)+g(f(\nabla_{E_i}H)X, E_i)\},$$

from which

$$\operatorname{tr} \nabla_{fX}H-g(HV, X)+v(X) \operatorname{tr} H-g(HX, HW)+w(X) \operatorname{tr} H^2$$
$$=-g(HX, V)+(2n-1)v(HX)-g(H^2X, W)+w(HX) \operatorname{tr} H+\operatorname{tr} f\nabla_X H$$

because of (1.24). Hence we get

$$\operatorname{tr} \nabla_{fX}H+v(X) \operatorname{tr} H+w(x) \operatorname{tr} H^2=(2n-1)v(HX)+w(HX) \operatorname{tr} H, \tag{3.9}$$

since f is skew-symmetric and $\nabla_X H$ is symmetric.

§ 4. Determination of the hypersurfaces of constant mean curvature which satisfy $Hf=fH$.

Now we assume that the hypersurface M is of constant mean curvature and satisfies $Hf=fH$. Then by (3.9), we have

$$v(X) \operatorname{tr} H+w(X) \operatorname{tr} H^2=(2n-1)v(HX)+w(HX) \operatorname{tr} H. \tag{4.1}$$

Replacing X in (4.1) by fW and making use of (2.13) and (3.5), we find

$$\beta(1-\alpha^2-\beta^2-\lambda^2) \operatorname{tr} H=(2n-1)g(HV, fW). \tag{4.2}$$

Replacing X in (4.1) by fV and making use of (2.13) and (3.4), we find

$$\beta(1-\alpha^2-\beta^2-\lambda^2) \operatorname{tr} H^2=(\operatorname{tr} H)g(HW, fV) \tag{4.3}$$

from which

$$\beta(1-\alpha^2-\beta^2-\lambda^2) \operatorname{tr} H^2=(\operatorname{tr} H)g(HV, fW), \tag{4.4}$$

by (3.6).

Combining (4.3) and (4.4), we have

$$\begin{aligned} &\beta(1-\alpha^2-\beta^2-\lambda^2)(\operatorname{tr} H^2-(1/(2n-1))(\operatorname{tr} H)^2) \\ &=\beta(1-\alpha^2-\beta^2-\lambda^2) \operatorname{tr} (H-(1/(2n-1))(\operatorname{tr} H)I)^2 \\ &=\beta(1-\alpha^2-\beta^2-\lambda^2) \operatorname{tr} \{(H-(1/(2n-1))(\operatorname{tr} H)I)^t(H-(1/(2n-1))(\operatorname{tr} H)I)\}=0. \end{aligned} \tag{4.5}$$

Thus we know that, at a point x, if $\beta(1-\alpha^2-\beta^2-\lambda^2)\neq 0$, this point must be an umbilical point. Now, let M_1 be a set of all umbilical points of M and $M_2=M-M_1$. Furthermore we assume that the vector field $\bar{V}$ is not tangent almost everywhere to M. Then M_2 is an open submanifold of M_0 and any point of M_2 must satisfy $1-\alpha^2-\beta^2-\lambda^2=0$. Hence, by means of (2.11) and (2.12), we get

$$fW=fU=fV=0. \tag{4.6}$$

Differentiating covariantly $fW=0$, we have

$$\nabla_X(fW)=\nabla_X(\alpha U+\beta V)=(X\alpha)U+\alpha\nabla_X U+(X\beta)V+\beta\nabla_X V=0,$$

which, together with (2.16) and (2.18), implies that

$$(X\alpha)U+\alpha(-\lambda X+\alpha HX)+(X\beta)V+\beta(fX+\beta HX)=0.$$

Transforming the last equation by f and taking account of (4.6), we have

$$\alpha(-\lambda fX+\alpha fHX)+\beta(f^2X+\beta fHX)=0.$$

That is,

$$-\alpha\lambda g(fX, Y)+\alpha^2 g(fHX, Y)+\beta g(f^2X, Y)+\beta^2 g(fHX, Y)=0.$$

If we replace X and Y in the last equation by an orthonormal frame E_i and sum over i, then we get

$$-\alpha\lambda \operatorname{tr} f+\alpha^2 \operatorname{tr}(fH)+\beta \operatorname{tr} f^2+\beta^2 \operatorname{tr}(fH)=\beta \operatorname{tr} f^2=0,$$

since f is a skew-symmetric and H is a symmetric. Thus, in M_2 we have

$$X=u(X)U+v(X)V+w(X)W,$$

from which

$$\sum_i g(E_i, E_i)=\sum_i \{g(U, g(U, E_i)E_i)+g(V, g(V, E_i)E_i)+g(W, g(W, E_i)E_i)\}$$
$$=g(U, U)+g(V, V)+g(W, W).$$

That is

$$2n-1=3-2(\alpha^2+\beta^2+\lambda^2)=1,$$

because of (2.8), (2.9) and (2.10).

So, if the dimension of the hypersurface is greater than 1, this shows that $M=M_1$. Thus we obtain

THEOREM. *Let M be a hypersurface with constant mean curvature of S^{2n} $(n>1)$. If M satisfies the condition $Hf=fH$ and the vector field $\bar{V}$ is not tangent almost everywhere to M, then each point of M is umbilical. Moreover, if M is complete, then M is a great or a small sphere in S^{2n}.*

Department of Mathematics
Michigan State University
East Lansing, Michigan
U. S. A.

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