

HYPERSURFACES IN A SPHERE WITH CONSTANT MEAN CURVATURE

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ABSTRACT. Let M^n be a closed hypersurface of constant mean curvature immersed in the unit sphere S^{n+1} . Denote by S the square of the length of its second fundamental form. If $S < 2\sqrt{n-1}$, M is a small hypersphere in S^{n+1} . We also characterize all M^n with $S = 2\sqrt{n-1}$.

1. INTRODUCTION

Let M^n be a closed submanifold with parallel mean curvature vector field immersed in the unit sphere S^{n+p} . Denote by H the length of the mean curvature vector field and by S the square of the length of the second fundamental form of M^n . It is important to characterize those M immersed as n -spheres in S^{n+p} by H and S .

When M is minimal, J. Simons [9] obtained a pinching constant $n/(2-1/p)$ of S and Chern-do Carmo-Kobayashi [3] showed that it is sharp and characterized all M with $S = n/(2-1/p)$. M. Okumura [6, 7] first discussed the general case and gave a pinching constant of S , but it is not sharp. Recently the sharp ones were obtained by H. Alencar-M. do Carmo [1] for $p = 1$, W. Santos [8] for $p > 1$ and H. W. Xu [11] for $p \geq 1$ respectively. But all of them were expressed by the mean curvature H . S. T. Yau [12] obtained a pinching constant for $p > 1$ which depended only on n and p . H. W. Xu [10] improved Yau's result, but far from sharpness.

In the present paper, we shall give a pinching constant for $p = 1$ which depends only on n and show the sharpness of it. More precisely, we want to prove the following theorems:

Theorem A. *Let M^n be a hypersurface of constant mean curvature immersed in S^{n+1} with constant length of the second fundamental form. Then:*

- (1) *If $S < 2\sqrt{n-1}$, M^n is locally a piece of small hypersphere $S^n(r)$ of radius $r = \sqrt{n/(n+S)}$.*
- (2) *If $S = 2\sqrt{n-1}$, M is locally a piece of either $S^n(r_0)$ or $S^1(r) \times S^{n-1}(s)$ where $r_0^2 = n/(n+2\sqrt{n-1})$, $r^2 = 1/(\sqrt{n-1}+1)$ and $s^2 = \sqrt{n-1}/(\sqrt{n-1}+1)$.*

Theorem A'. *Let M^n be a closed hypersurface of constant mean curvature immersed in S^{n+1} . Then:*

- (1) *If $S < 2\sqrt{n-1}$, M^n is a small hypersphere $S^n(r)$ of radius $r = \sqrt{n/(n+S)}$.*

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- (2) If $S = 2\sqrt{n-1}$, M is either a small hypersphere $S^n(r_0)$ or a $H(r)$ -torus $S^1(r) \times S^{n-1}(s)$, where r_0, r and s are taken as before.

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2. PROOF OF THE THEOREMS

Let M be a closed hypersurface immersed in the unit sphere S^{n+1} . Take a local orthonormal coframe field $\{\omega_i\}_{i=1}^n$ on M . Then the second fundamental form can be expressed as $L = (h_{ij})_{n \times n}$. The mean curvature H and the square of the length of the second fundamental form S are defined by $H = \frac{1}{n} \sum_{(i)} h_{ii}$, $S = \sum_{(i,j)} (h_{ij})^2$.

From now on, we shall always use i, j, k, \dots for indices running from 1 to n .

Denote the covariant differentials of $\{h_{ij}\}$ by $\{h_{ijk}\}$ and $\{h_{ijkl}\}$. Then the Laplacian of h_{ij} is defined by $\Delta h_{ij} = \sum_{(k)} h_{ijkk}$. It follows that

$$(1) \quad \sum_{(i,j)} h_{ij} \Delta h_{ij} = nS + nHf - n^2H^2 - S^2,$$

where $f = \text{Tr } L^3$ (cf. e.g. [2] or [7]).

M. Okumura [7] established the following lemma (see also [1] or [11]).

Lemma. Let $\{a_i\}_{i=1}^n$ be a set of real numbers satisfying $\sum_{(i)} a_i = 0$, $\sum_{(i)} a_i^2 = t^2$, where $t \geq 0$. Then we have

$$(2) \quad -\frac{n-2}{\sqrt{n(n-1)}}t^3 \leq \sum_{(i)} a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}t^3,$$

and equalities hold if and only if at least $(n-1)$ of the a_i 's are equal to one another.

Suppose that $\lambda_1, \lambda_2, \dots, \lambda_n$ are the principal curvatures of M . Then we have

$$(3) \quad nH = \sum_{(i)} \lambda_i, \quad S = \sum_{(i)} \lambda_i^2, \quad f = \sum_{(i)} \lambda_i^3.$$

Set $\tilde{S} = S - nH^2$, $\tilde{f} = f - 3HS + 2nH^3$ and $\tilde{\lambda}_i = \lambda_i - H$ ($1 \leq i \leq n$). Then (3) changes into

$$(4) \quad 0 = \sum_{(i)} \tilde{\lambda}_i, \quad \tilde{S} = \sum_{(i)} \tilde{\lambda}_i^2, \quad \tilde{f} = \sum_{(i)} \tilde{\lambda}_i^3.$$

By applying Okumura's Lemma to \tilde{f} in (4), we have

$$\tilde{f} \geq -\frac{n-2}{\sqrt{n(n-1)}}\tilde{S}\sqrt{\tilde{S}} \iff f \geq 3HS - 2nH^3 - \frac{n-2}{\sqrt{n(n-1)}}\tilde{S}\sqrt{\tilde{S}}.$$

Substituting this into (1), we have

$$(5) \quad \sum_{(i,j)} h_{ij} \Delta h_{ij} \geq \tilde{S} \left\{ n - (\tilde{S} - nH^2) - (n-2)H\sqrt{\frac{n}{n-1}}\tilde{S} \right\}.$$

Consider the quadratic form $Q(u, t) = u^2 - \frac{n-2}{\sqrt{n-1}}ut - t^2$. By the orthogonal transformation

$$\begin{cases} \tilde{u} = \frac{1}{\sqrt{2n}}\{(1 + \sqrt{n-1})u + (1 - \sqrt{n-1})t\}, \\ \tilde{t} = \frac{1}{\sqrt{2n}}\{(\sqrt{n-1} - 1)u + (\sqrt{n-1} + 1)t\}, \end{cases}$$

$Q(u, t)$ turns into $Q(u, t) = \frac{n}{2\sqrt{n-1}}(\tilde{u}^2 - \tilde{t}^2)$, where $\tilde{u}^2 + \tilde{t}^2 = u^2 + t^2 = S$.

Take $t = \sqrt{S}$ and $u = \sqrt{n}H$ in $Q(u, t)$, and substitute it into (5). We can see

$$(6) \quad \sum_{(i,j)} h_{ij} \Delta h_{ij} \geq \tilde{S} \left(n - \frac{n}{2\sqrt{n-1}} S + \frac{n}{\sqrt{n-1}} \tilde{u}^2 \right) \geq \tilde{S} \left(n - \frac{n}{2\sqrt{n-1}} S \right).$$

Therefore we have

$$(7) \quad \frac{1}{2} \Delta S = \sum_{(i,j,k)} h_{ijk}^2 + \sum_{(i,j)} h_{ij} \Delta h_{ij} \geq \tilde{S} \left(n - \frac{n}{2\sqrt{n-1}} S \right).$$

Theorem A. *Let M^n be a hypersurface of constant mean curvature immersed in S^{n+1} with constant length of the second fundamental form. Then:*

- (1) *If $S < 2\sqrt{n-1}$, M is locally a piece of a small hypersphere $S^n(r)$ in S^{n+1} , where $r = \sqrt{n/(n+S)}$.*
- (2) *If $S = 2\sqrt{n-1}$, M is locally a piece of either $S^n(r_0)$ or $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = n/(n+2\sqrt{n-1})$, $r^2 = 1/(\sqrt{n-1}+1)$ and $s^2 = \sqrt{n-1}/(\sqrt{n-1}+1)$.*

Proof. Since S is constant, the left-hand side of (7) is zero. When $S \leq 2\sqrt{n-1}$, we have

$$(8) \quad \tilde{S} \left(n - \frac{n}{2\sqrt{n-1}} S \right) = 0, \quad h_{ijk} = 0, \quad 1 \leq i, j, k \leq n.$$

If $S < 2\sqrt{n-1}$, we have $\tilde{S} = 0$, which means that M is totally umbilical and hence is locally a piece of hypersphere $S^n(r)$ where $r = \sqrt{n/(n+S)}$.

Suppose $S = 2\sqrt{n-1}$. Then all of the inequalities in (5)–(7) become equal ones. Okumura's Lemma implies that at least $(n-1)$ of λ_i 's are equal to one another. When $\lambda_1 = \lambda_2 = \dots = \lambda_n$, M is totally umbilical and hence is locally a piece of hypersphere $S^n(r)$ where $r^2 = n/(n+2\sqrt{n-1})$. When M is not totally umbilical, there are exactly $(n-1)$ of λ_i 's that are equal to one another. The same arguments as those developed by Chern-do Carmo-Kobayashi (see [3], p. 68) show that M is locally a piece of $S^1(r) \times S^{n-1}(s)$ in S^{n+1} . To determine the radii r and s , we refer to the examples of K. Nomizu and B. Smyth [5], from which we have

$$H = -\frac{1}{n} \left(\frac{s}{r} \right) + \frac{n-1}{n} \left(\frac{r}{s} \right), \quad S = \left(\frac{s}{r} \right)^2 + (n-1) \left(\frac{r}{s} \right)^2.$$

It is easy to see that

$$\left(\frac{s}{r} \right)^2 + (n-1) \left(\frac{r}{s} \right)^2 \geq 2\sqrt{n-1}$$

and equality holds if and only if $\left(\frac{s}{r} \right)^2 = \sqrt{n-1}$. Therefore we have $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$. \square

When M is closed, the integral of the left-hand side of (7) on M is equal to zero, and so is that of the right-hand side. After the same deduction as in the proof of Theorem A, we can obtain the following:

Theorem A'. *Suppose M is a closed hypersurface of constant mean curvature immersed in S^{n+1} . Then:*

- (1) *If $S < 2\sqrt{n-1}$, M is a small hypersphere $S^n(r)$, where $r = \sqrt{n/(n+S)}$.*
- (2) *If $S = 2\sqrt{n-1}$, M is either a small hypersphere $S^n(r_0)$ or $S^1(r) \times S^{n-1}(s)$, where r_0, r and s are taken as in Theorem A.*

We can show an application of Theorem A'. H. W. Xu [10] proved the following:

Proposition (Xu). *Let M^n be an n -dimensional compact submanifold with parallel mean curvature vector field in S^{n+p} and $p > 1$. If*

$$S \leq \min \left\{ \frac{2n}{1 + \sqrt{n}}, \frac{n}{2 - (p-1)^{-1}} \right\},$$

and the Gauss mapping of M is relatively affine, then M^n is a standard hypersphere in a totally geodesic S^{n+1} of S^{n+p} .

By Theorem A', we can remove the assumption that the Gauss mapping is relatively affine. Namely we can obtain the following

Corollary. *Let M^n be an n -dimensional compact submanifold with parallel mean curvature vector field in S^{n+p} and $p > 1$. If*

$$S \leq \min \left\{ \frac{2n}{1 + \sqrt{n}}, \frac{n}{2 - (p-1)^{-1}} \right\},$$

then M^n is a standard hypersphere in a totally geodesic S^{n+1} of S^{n+p} .

Proof. It is easy to check that $(\sqrt{n} + 1)/n > 1/\sqrt{n-1}$. Therefore we have

$$\sqrt{n-1} > \frac{n}{\sqrt{n}+1} \iff 2\sqrt{n-1} > \frac{2n}{\sqrt{n}+1} \geq S.$$

□

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