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HYPERSURFACES OF ALMOST r-PARACONTACT RIEMANNIAN MANIFOLD ENDOWED WITH A QUARTER SYMMETRIC METRIC CONNECTION

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Abstract. We define a quarter symmetric metric connection in an almost r-paracontact Riemannian manifold and we consider invariant, non-invariant and anti-invariant hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter symmetric metric connection.

1. Introduction

In [1], T. Adati studied Hypersurfaces of almost paracontact Riemannian manifolds. In [3], A. Bucki, considered hypersurfaces of almost r-paracontact Riemannian manifold. Some properties of invariant hypersurfaces of an almost r-paracontact Riemannian manifold were investigated in [4] by A. Bucki and A. Miernowski. In [2], M. Ahmad, C. Ozgur, and A. Haseeb studied hypersurfaces of almost r-paracontact Riemannian manifold with quarter symmetric non-metric connection. Moreover in [7], I. Mihai and K. Matsumoto studied submanifolds of an almost r-paracontact Riemannian manifold of P-Sasakian type.

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T of ∇ is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

The connection ∇ is *symmetric* if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is *metric* if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection. In [6], S. Golab introduced the idea of a quarter symmetric linear connection if its torsion tensor T is of the form

$$T(X,Y) = u(Y)\phi X - u(X)\phi Y,$$

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where u is a 1-form and ϕ is a tensor field of the type (1,1). In [8], R. S. Mishra and S. N. Pandey considered a quarter symmetric metric F-connection and studied some of its properties. In [8], [9] and [10], some kinds of quarter symmetric metric connection were studied.

In this paper, we study quarter symmetric metric connection in an almost r-paracontact Riemannian manifold. We consider invariant, non-invariant and anti-invariant hypersurfaces of almost r-paracontact Riemannian manifold endowed with a quarter symmetric metric connection.

The paper is organized as follows: In Section 2, we give a brief introduction about an almost r-paracontact Riemannian manifold. In Section 3, we show that the induced connection on a hypersurface of an almost r-paracontact Riemannian manifold with quarter symmetric metric connection with respect to the normal is also a quarter symmetric metric connection. We find the characteristic properties of invariant, non-invariant and anti-invariant hypersurfaces of almost r-paracontact Riemannian manifold endowed with a quarter symmetric metric connection.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold with a positive definite metric g. If there exist a tensor field ϕ of type (1,1), r vector fields $\xi_1, \xi_2, \ldots, \xi_r$ $(n > r)$, r 1-forms $\eta^1, \eta^2, \ldots, \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta,\ \alpha, \beta \in (r) = \{1, 2, 3, \ldots, r\}, \tag{2.1}$$

$$\phi^2(X) = X - \eta^\alpha(X)\xi_\alpha, \tag{2.2}$$

$$\eta^\alpha(X) = g(X, \xi_\alpha), \alpha \in (r), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{2.4}$$

where X and Y are vector fields on M, then the structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be an *almost r-paracontact Riemannian structure* and M is an *almost r-paracontact Riemannian manifold* [3]. From (2.1) through (2.4), we have for $\alpha \in (r)$

$$\phi(\xi_\alpha) = 0, \qquad \eta^\alpha \circ \phi = 0, \tag{2.5}$$

$$\Phi(X, Y) = g(\phi X, Y) = g(X, \phi Y).$$

An almost r-paracontact Riemannian manifold M with structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be of *S-paracontact type* if [4]

$$\Phi(X, Y) = (\nabla^*{}_Y \eta^\alpha)(X),\ \alpha \in (r) \tag{2.6}$$

for the Riemannian connection ∇^* on M. An almost r-paracontact Riemannian manifold M with a structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be of *P-Sasakian* *type* if it satisfies (2.6) and (2.7)

$$(\nabla^*{}_Z\Phi)(X,Y) = -\sum_\alpha \eta^\alpha(X)[g(Y,Z) - \sum_\beta \eta^\beta(Y)\eta^\beta(Z)] - \sum_\alpha \eta^\alpha(Y)[g(X,Z) - \sum_\beta \eta^\beta(X)\eta^\beta(Z)] \tag{2.7}$$

for all vector fields X, Y and Z on M [7]. The conditions (2.6) and (2.7) are equivalent respectively to

$$\phi X = \nabla^*_X \xi_\alpha, \ \alpha \in (r), \tag{2.8}$$

$$(\nabla^*_Y \phi)(X) = -\sum_\alpha \eta^\alpha(X)[Y - \eta^\alpha(Y)\xi_\alpha] - [g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)] \sum_\beta \xi_\beta. \tag{2.9}$$

A quarter symmetric metric connection ∇ on M is defined as

$$\nabla_{\bar{X}}\bar{Y} = \nabla^*_{\bar{X}}\bar{Y} + \eta^\alpha(\bar{Y})\phi\bar{X} - g(\phi\bar{X},\bar{Y})\xi_\alpha, \ \alpha \in (r). \tag{2.10}$$

Using (2.10) in (2.8) and (2.9), we get

$$\nabla_X \xi_\alpha = 2\phi X, \tag{2.11}$$

$$(\nabla_Y \phi)(X) = -\sum_\alpha \eta^\alpha(X)[Y - \eta^\alpha(Y)\xi_\alpha] - [g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)] \sum_\beta \xi_\beta - g(X,Y)\xi_\alpha + \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)\xi_\alpha. \tag{2.12}$$

3. Hypersurfaces of almost r-paracontact Riemannian manifold endowed with a quarter symmetric metric connection

Let M^{n+1} be an almost r-paracontact Riemannian manifold with a positive definite metric g and M^n be the hypersurface immersed in M^{n+1} by the immersion $\tau : M^n \to M^{n+1}$. If B denotes the differential of τ, then any vector field $\bar{X} \in M^n$ implies $B\bar{X} \in M^{n+1}$. We denote the objects belonging to M^n by the mark of hyphen placed over them, for example $\bar{\phi}, \bar{X}, \bar{\eta}, \bar{\xi}$. Let N be the unit normal vector field to M^n. Then the induced metric $\bar{g}$ on M^n is defined by

$$\bar{g}(\bar{X},\bar{Y}) = g(\bar{X},\bar{Y}). \tag{3.1}$$

Then we have [5]

$$g(\bar{X}, N) = 0, \quad g(N,N) = 1. \tag{3.2}$$

If $\bar{\nabla}^*$ is the induced connection on hypersurface from ∇^* with respect to the unit normal vector N, then the Gauss formula is given by

$$\nabla^*_{\bar{X}}\bar{Y} = \bar{\nabla}^*_{\bar{X}}\bar{Y} + h(\bar{X},\bar{Y})N, \tag{3.3}$$

where h is the second fundamental tensor satisfying

$$h(\bar{Y},\bar{X}) = h(\bar{X},\bar{Y}) = \bar{g}(H\bar{X},\bar{Y}).$$

If $\bar{\nabla}$ is the induced connection on hypersurface from ∇ with respect to the unit normal vector N, then we have

$$\nabla_{\bar{X}}\bar{Y} = \bar{\nabla}_{\bar{X}}\bar{Y} + m(\bar{X},\bar{Y})N, \tag{3.4}$$

where m is a tensor field of type (0, 2) of hypersurface M^n. From (2.10), we obtain

$$\nabla_{\bar{X}}\bar{Y} = \nabla^*_{\bar{X}}\bar{Y} + \eta^\alpha(\bar{Y})(\bar{\phi}\bar{X} + b(\bar{X})N) - \bar{g}(\bar{\phi}\bar{X},\bar{Y})\xi_\alpha, \tag{3.5}$$

where $\phi\bar{X} = \bar{\phi}\bar{X} + b(\bar{X})N$. From equations (3.3), (3.4) and (3.5), we get

$$\begin{aligned}\bar{\nabla}_{\bar{X}}\bar{Y} + m(\bar{X},\bar{Y})N = {}& \bar{\nabla}^*_{\bar{X}}\bar{Y} + h(\bar{X},\bar{Y})N + \eta^\alpha(\bar{Y})\bar{\phi}\bar{X} \\ & + \bar{\eta}^\alpha(\bar{Y})b(\bar{X})N - \bar{g}(\bar{\phi}\bar{X},\bar{Y})(\bar{\xi}_\alpha + a_\alpha N),\end{aligned}$$

where $\xi_\alpha = \bar{\xi}_\alpha + a_\alpha N$ and $\bar{\eta}^\alpha(\bar{X}) = \eta^\alpha(\bar{X})$ for each $\alpha \in (r)$. By taking the tangential and normal parts from the both sides, we get respectively

$$\begin{aligned}\bar{\nabla}_{\bar{X}}\bar{Y} &= \bar{\nabla}^*_{\bar{X}}\bar{Y} + \bar{\eta}^\alpha(\bar{Y})\bar{\phi}\bar{X} - \bar{g}(\bar{\phi}\bar{X},\bar{Y})\bar{\xi}_\alpha, \\ m(\bar{X},\bar{Y}) &= h(\bar{X},\bar{Y}) + \bar{\eta}^\alpha(\bar{Y})b(\bar{X}) - a_\alpha\bar{g}(\bar{\phi}\bar{X},\bar{Y}).\end{aligned} \tag{3.6}$$

Thus we get the following theorem.

Theorem 3.1. *The connection induced on a hypersurface of an almost r-paracontact Riemannian manifold endowed with a quarter symmetric metric connection with respect to the unit normal vector is also a quarter symmetric metric connection.*

From (3.4) and (3.6), we have

$$\nabla_{\bar{X}}\bar{Y} = \bar{\nabla}_{\bar{X}}\bar{Y} + \{h(\bar{X},\bar{Y}) - a_\alpha\bar{g}(\bar{\phi}\bar{X},\bar{Y}) + \bar{\eta}^\alpha(\bar{Y})b(\bar{X})\}N, \tag{3.7}$$

which is the Gauss formula for a quarter symmetric metric connection. The Weingarten formula with respect to the Riemannian connection ∇^* is given by

$$\nabla^*_{\bar{X}}N = -H\bar{X} \tag{3.8}$$

for every $\bar{X}$ in M^n, where H is a tensor field of type (1,1) of M^n given by

$$\bar{g}(H\bar{X},\bar{Y}) = h(\bar{X},\bar{Y}) = h(\bar{Y},\bar{X}). \tag{3.9}$$

From equation (2.10), we have

$$\nabla_{\bar{X}}N = \nabla^*_{\bar{X}}N + a_\alpha\bar{\phi}\bar{X} - b(\bar{X})\bar{\xi}_\alpha, \tag{3.10}$$

where we have put

$$\eta^\alpha(N) = a_\alpha = m(\xi_\alpha). \tag{3.11}$$

From (3.8) and (3.10), we have

$$\nabla_{\bar{X}} N = -H\bar{X} + a_\alpha \bar{\phi}\bar{X} - b(\bar{X})\bar{\xi}_\alpha, \tag{3.12}$$

which is the Weingarten formula with respect to the quarter symmetric metric connection.

Now, suppose that $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is an almost r-paracontact Riemannian structure on M^{n+1}. Then every vector field X on M^{n+1} is decomposed as

$$X = \bar{X} + \lambda(X)N,$$

where λ is an 1-form on M^{n+1} and $\bar{X}$ is any vector field and N is normal vector on M^n. Also we have

$$\phi\bar{X} = \bar{\phi}\bar{X} + b(\bar{X})N, \tag{3.13}$$

$$\phi N = \bar{N} + KN, \tag{3.14}$$

where $\bar{\phi}$ is a tensor field of type (1,1), b is an 1-form and K is a scalar function on M^n. For each $\alpha \in (r)$, we have

$$\xi_\alpha = \bar{\xi}_\alpha + a_\alpha N, \tag{3.15}$$

where $a_\alpha = m(\xi_\alpha) = \eta^\alpha(N)$. Now, we define $\bar{\eta}^\alpha$ as

$$\bar{\eta}^\alpha(\bar{X}) = \eta^\alpha(\bar{X}),\ \alpha \in (r). \tag{3.16}$$

Making use of (3.13), (3.14), (3.15) and (3.11), we obtain from (2.1) through (2.5) for $\alpha \in (r)$

$$b(\bar{N}) + K^2 = 1 - \sum_\alpha (a_\alpha)^2, \tag{3.17}$$

$$Ka_\alpha + b(\bar{\xi}_\alpha) = 0, \tag{3.18}$$

$$\Phi(\bar{X}, \bar{Y}) = \bar{g}(\bar{\phi}\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \bar{\phi}\bar{Y}) = \bar{\Phi}(\bar{X}, \bar{Y}). \tag{3.19}$$

Making use of (3.1), (3.2), (3.13), (3.14) and (2.5), we have

$$g(\bar{\phi}\bar{X}, N) = g(\phi\bar{X}, N) - b(\bar{X}) = g(\bar{X}, \phi N) - b(\bar{X}) = 0.$$

Hence we get

$$g(\bar{X}, \bar{N}) = b(\bar{X}). \tag{3.20}$$

Differentiating covariantly (3.13) and (3.14) along M^n and making use of (3.7) and (3.12), we get respectively

$$\begin{aligned}(\nabla_{\bar{Y}}\phi)\bar{X} = {} & (\bar{\nabla}_{\bar{Y}}\bar{\phi})\bar{X} - (h(\bar{X},\bar{Y}) - a_\alpha\bar{g}(\bar{\phi}\bar{Y},\bar{X}) + \bar{\eta}^\alpha(\bar{X})b(\bar{Y}))\bar{N} \\ & + [(\bar{\nabla}_{\bar{Y}}b)(\bar{X}) + h(\bar{\phi}\bar{X},\bar{Y}) - (h(\bar{X},\bar{Y}) \\ & - a_\alpha\bar{g}(\bar{\phi}\bar{Y},\bar{X}) + \bar{\eta}^\alpha(\bar{X})b(\bar{Y}))K - a_\alpha\bar{g}(\bar{X},\bar{Y}) \\ & + a_\alpha\sum_\alpha\bar{\eta}^\alpha(\bar{X})\bar{\eta}^\alpha(\bar{Y}) - 2a_\alpha b(\bar{X})b(\bar{Y})]N \\ & - b(\bar{X})(H\bar{Y}) - b(\bar{X})b(\bar{Y})\bar{\xi}_\alpha - a_\alpha b(\bar{X})\bar{\phi}\bar{Y},\end{aligned} \tag{3.21}$$

$$\begin{aligned}(\nabla_{\bar{Y}}\phi)N = {} & \bar{\nabla}_{\bar{Y}}\bar{N} + [\bar{Y}(K) + 2(a_\alpha)^2\bar{\eta}^\alpha(\bar{Y}) + h(\bar{X},\bar{N}) + b(H\bar{Y})]N \\ & + \bar{\phi}(H\bar{Y}) - K(H\bar{Y}) + a_\alpha(\bar{Y} - \bar{\eta}^\alpha(\bar{Y})\bar{\xi}_\alpha) \\ & + K(\bar{\phi}\bar{Y}) - Kb(\bar{Y})\bar{\xi}_\alpha.\end{aligned} \tag{3.22}$$

From (3.11) and (3.15), we have

$$\begin{aligned}\nabla_{\bar{Y}}\xi_\alpha = {} & \bar{\nabla}_{\bar{Y}}\bar{\xi}_\alpha - a_\alpha(H\bar{Y}) + (a_\alpha)^2\bar{\phi}\bar{Y} - b(\bar{Y})a_\alpha\bar{\xi}_\alpha \\ & + [\bar{Y}(a_\alpha) + h(\bar{Y},\bar{\xi}_\alpha) + b(\bar{Y}) - (a_\alpha)^2 b(\bar{Y}) - a_\alpha\bar{g}(\bar{\phi}\bar{Y},\bar{\xi}_\alpha)]N,\end{aligned} \tag{3.23}$$

$$\begin{aligned}(\nabla_{\bar{Y}}\eta^\alpha)(\bar{X}) = {} & (\bar{\nabla}_{\bar{Y}}\bar{\eta}^\alpha)(\bar{X}) - a_\alpha h(\bar{Y},\bar{X}) \\ & - a_\alpha\bar{\eta}^\alpha(\bar{X})b(\bar{Y}) + (a_\alpha)^2\bar{g}(\bar{\phi}\bar{Y},\bar{X}).\end{aligned} \tag{3.24}$$

From the identity $(\nabla_Z\Phi)(X,Y) = g((\nabla_Z\phi)(X),Y)$, making use of (3.19), (3.20) and (3.21), we have

$$\begin{aligned}(\nabla_{\bar{Z}}\Phi)(\bar{X},\bar{Y}) = {} & (\bar{\nabla}_{\bar{Z}}\bar{\Phi})(\bar{X},\bar{Y}) - b(\bar{X})h(\bar{Z},\bar{Y}) - b(\bar{Y})h(\bar{Z},\bar{X}) \\ & + a_\alpha b(\bar{X})\bar{\Phi}(\bar{Y},\bar{Z}) + a_\alpha b(\bar{Y})\bar{\Phi}(\bar{X},\bar{Z}) \\ & - b(\bar{X})b(\bar{Z})\bar{\eta}^\alpha(\bar{Y}) - b(\bar{Y})b(\bar{Z})\bar{\eta}^\alpha(\bar{X}).\end{aligned} \tag{3.25}$$

From the above identities, we have the followings.

Theorem 3.2. *If M^n is an invariant hypersurface immersed in an almost r-paracontact Riemannian manifold M^{n+1} endowed with a quarter symmetric metric connection with structure $\sum = (\phi,\xi_\alpha,\eta^\alpha,g)_{\alpha\in(r)}$, then either*

(i) *All ξ_α are tangent to M^n and M^n admits an almost r-paracontact Riemannian structure $\sum_1 = (\bar{\phi},\bar{\xi}_\alpha,\bar{\eta}^\alpha,\bar{g})_{\alpha\in(r)}(n-r>2)$ or*

(ii) *One of ξ_α (say, ξ_r) is normal to M^n and remaining ξ_α are tangent to M^n and M^n admits an almost $(r-1)$-paracontact Riemannian structure $\sum_2 = (\bar{\phi},\bar{\xi}_i,\bar{\eta}^i,\bar{g})_{i\in(r)}(n-r>1)$.*

Proof. From (3.18), $Ka_\alpha = 0, \alpha\in(r)$. Hence we have the two possibilities when $K=0$ or $K\neq 0$.

(i) If $K\neq 0$, then $a_\alpha = 0$ and $\xi_\alpha = \bar{\xi}_\alpha$ (all ξ_α are tangent to M^n) and the structure $(\bar{\phi},\bar{\xi}_\alpha,\bar{\eta}^\alpha,\bar{g})_{\alpha\in(r)}$ is an almost r-paracontact Riemannian structure on M^n.

(ii) If $K = 0$, then $\phi(N) = 0$. Let $N = \xi_r$, then $\bar{\xi}_r = 0, a_r = 1, \bar{\eta}^r = 0$. From (3.17) $\sum_\alpha (a_\alpha)^2 = 1$ and since $a_r = 1, \sum_i (a_i)^2 = 0, i \in (r-1)$. Thus $a_i = 0$ for all $i \in (r-1)$. Thus, $\xi_i = \bar{\xi}_i, \xi_r = N$ (all ξ_α but one tangent to M^n). Hence structure $(\bar{\phi}, \bar{\xi}_i, \bar{\eta}^i, \bar{g})_{i\in(r-1)}$ is an almost $(r-1)$-paracontact structure on M^n. □

Corollary 3.1. *If M^n is a hypersurface immersed in an almost r-paracontact Riemannian manifold M^{n+1} with a structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ endowed with a quarter symmetric metric connection, then the following statements are equivalent*:

(a) *M^n is invariant.*
(b) *The Normal field N is an eigenvector of ϕ.*
(c) *All ξ_α are tangent to M^n if and only if M^n admits an almost r-paracontact Riemannian structure $\sum_1$, or one of ξ_α is normal and $(r-1)$ remaining ξ_i are tangent to M^n if and only if M^n admits an almost $(r-1)$-paracontact Riemannian structure $\sum_2$.*

Theorem 3.3. *If M^n is an invariant hypersurface immersed in an almost r-paracontact Riemannian manifold of P-Sasakian type endowed with a quarter symmetric metric connection, then the induced almost r-paracontact Riemannian structure $\sum_1$ or $(r-1)$-paracontact Riemannian structure $\sum_2$ are also of P-Sasakian type.*

Proof. Making use of (3.1), (3.16), (3.19), (3.24) and (3.25), we observe that the conditions (2.11) and (2.12) are satisfied for both $\sum_1$ and $\sum_2$. □

Lemma 3.1. $\bar{\nabla}_{\bar{X}}(\operatorname{trace}\bar{\phi}) = \operatorname{trace}(\bar{\nabla}_{\bar{X}}\bar{\phi})$.

Proof. Let $\{e_1, e_2, e_3, \ldots, e_n\}$ be an orthogonal basis of TM^n, then $\operatorname{trace}\bar{\phi} = \sum_a \bar{g}(\bar{\phi}(e_a, e_a))$ for $a \in (n-1)$. Let $\bar{\nabla}_{\bar{X}} e_a = A_a^b e_b$ and $\phi(e_a) = B_a^b e_b$, then from $0 = \bar{g}(\bar{\nabla}_{\bar{X}} e_a, e_b) + \bar{g}(e_a, \bar{\nabla}_{\bar{X}} e_b)$ and from $\bar{g}(\bar{\phi}(e_a), e_b) = \bar{g}(e_a, \bar{\phi}(e_b))$, we obtain $A_a^b - A_b^a = 0$ and $B_b^a = B_a^b$. Hence $\sum_\alpha \bar{g}(\bar{\phi}(e_a), \bar{\nabla}_{\bar{X}} e_a) = \sum_{a,b} A_b^a B_b^a = 0$ and we have

$$\bar{\nabla}_{\bar{X}}(\operatorname{trace}\bar{\phi}) = \sum_a \bar{g}((\bar{\nabla}_{\bar{X}}\bar{\phi})(e_a), e_a) + 2\sum_a \bar{g}(\bar{\phi}(e_a), \bar{\nabla}_{\bar{X}} e_a) = \operatorname{trace}(\bar{\nabla}_{\bar{X}}\bar{\phi}).$$

□

Theorem 3.4. *Let M^n be a non-invariant hypersurface of an almost r-paracontact Riemannian manifold M^{n+1} endowed with a quarter symmetric metric connection with a structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ satisfying $\nabla\phi = 0$ along M^n, then M^n is totally geodesic if and only if*

$$(\bar{\nabla}_{\bar{Y}}\bar{\phi})\bar{X} + a_\alpha \bar{g}(\bar{\phi}\bar{Y}, \bar{X})\bar{N} + a_\alpha b(\bar{X})\bar{\phi}\bar{Y} - b(\bar{X})b(\bar{Y})\bar{\xi}_\alpha + \bar{\eta}^\alpha(\bar{X})b(\bar{Y})\bar{N} = 0.$$

Proof. From (3.21) we have

$$(3.26)\qquad \begin{aligned}(\bar{\nabla}_{\bar{Y}}\phi)\bar{X} = {} & (\bar{\nabla}_{\bar{Y}}\bar{\phi})\bar{X} - (h(\bar{X},\bar{Y}) - a_\alpha \bar{g}(\bar{\phi}\bar{Y},\bar{X}) + \bar{\eta}^\alpha(\bar{X})b(\bar{Y}))\bar{N} \\ & + [(\bar{\nabla}_{\bar{Y}}b)(\bar{X}) + h(\bar{\phi}\bar{X},\bar{Y}) - (h(\bar{X},\bar{Y}) \\ & - a_\alpha \bar{g}(\bar{\phi}\bar{Y},\bar{X}) + \bar{\eta}^\alpha(\bar{X})b(\bar{Y}))K \\ & - a_\alpha \bar{g}(\bar{X},\bar{Y}) + a_\alpha \sum_\alpha \bar{\eta}^\alpha(\bar{X})\bar{\eta}^\alpha(\bar{Y}) - 2a_\alpha b(\bar{X})b(\bar{Y})]N \\ & - b(\bar{X})(H\bar{Y}) - b(\bar{X})b(\bar{Y})\bar{\xi}_\alpha + b(\bar{X})a_\alpha \bar{\phi}\bar{Y}.\end{aligned}$$

If M^n is totally geodesic, then $h = 0$ and $H = 0$. Thus from (3.26), we get

$$(\bar{\nabla}_{\bar{Y}}\phi)\bar{X} + a_\alpha \bar{g}(\bar{\phi}\bar{Y},\bar{X})\bar{N} + a_\alpha b(\bar{X})\bar{\phi}\bar{Y} - b(\bar{X})b(\bar{Y})\bar{\xi}_\alpha + \bar{\eta}^\alpha(\bar{X})b(\bar{Y})\bar{N} = 0.$$

Conversely, if

$$(\bar{\nabla}_{\bar{Y}}\phi)\bar{X} + a_\alpha \bar{g}(\bar{\phi}\bar{Y},\bar{X})\bar{N} + a_\alpha b(\bar{X})\bar{\phi}\bar{Y} - b(\bar{X})b(\bar{Y})\bar{\xi}_\alpha + \bar{\eta}^\alpha(\bar{X})b(\bar{Y})\bar{N} = 0,$$

then it holds

$$(3.27)\qquad h(\bar{Y},\bar{X})\bar{N} + b(\bar{X})H(\bar{Y}) = 0.$$

Making use of (3.9) and (3.20), we have

$$(3.28)\qquad h(\bar{X},\bar{Y})b(\bar{Z}) + h(\bar{X},\bar{Z})b(\bar{Y}) = 0.$$

Using (3.27), we get from (3.9)

$$(3.29)\qquad h(\bar{X},\bar{Z})b(\bar{Y}) = h(\bar{X},\bar{Y})b(\bar{Z}).$$

From (3.28) and (3.29), we get $b(\bar{Z})h(\bar{X},\bar{Y}) = 0$ which gives $h = 0$ as $b \neq 0$. Using $h = 0$ in (3.27), we get $H = 0$. Thus $h = 0$ and $H = 0$. Hence M^n is totally geodesic. □

Theorem 3.5. *Let M^n be a non-invariant hypersurface of an almost r-para-contact Riemannian manifold M^{n+1} endowed with a quarter symmetric metric connection satisfying $\nabla\phi = 0$ along M^n and if* trace $\bar{\phi}$ *= constant, then*

$$h(\bar{X},\bar{N}) = \sum_a [a_\alpha b(e_a)\Phi(e_a,\bar{X}) - b(\bar{X})b(e_a)\bar{\eta}^\alpha(e_a)],$$

where $\bar{N} = \sum_a b(e_a)e_a$.

Proof. From (3.26) we have

$$\bar{g}((\bar{\nabla}_{\bar{Y}}\bar{\phi})\bar{X},\bar{X}) = 2b(\bar{X})h(\bar{X},\bar{Y}) - 2a_\alpha b(\bar{X})\bar{\Phi}(\bar{X},\bar{Y}) + 2b(\bar{X})b(\bar{Y})\bar{\eta}^\alpha(\bar{X})$$

and

$$\bar{\nabla}_{\bar{X}}(\text{trace}\,\bar{\phi}) = 2h(\bar{X},\bar{N}) - 2a_\alpha \sum_a b(e_a)z\bar{\Phi}(e_a,\bar{X}) + 2\sum_a b(\bar{X})b(e_a)\bar{\eta}^\alpha(e_a).$$

Using Lemma 3.1, we get

$$h(\bar{X},\bar{N}) = \sum_a [a_\alpha b(e_a)\bar{\Phi}(e_a,\bar{X}) - b(\bar{X})b(e_a)\bar{\eta}^\alpha(e_a)],$$

where $\bar{N} = \sum_a b(e_a)e_a$. □

Let M^n be an almost r-paracontact Riemannian manifold of S-paracontact type with a quarter symmetric metric connection, then from (2.11), (3.13) and (3.23), we get

$$\bar{\phi}\bar{X} = \frac{1}{2}[\bar{\nabla}_{\bar{X}}\bar{\xi}_\alpha - a_\alpha(H\bar{X}) + (a_\alpha)^2\bar{\phi}(\bar{X}) - a_\alpha b(\bar{X})\bar{\xi}_\alpha],\ \alpha \in (r), \tag{3.30}$$

$$b(\bar{X}) = \frac{1}{2}[\bar{X}(a_\alpha) + h(\bar{X}, \bar{\xi}_\alpha) + (1-(a_\alpha)^2)b(\bar{X}) - a_\alpha\bar{g}(\bar{\phi}\bar{X}, \bar{\xi}_\alpha)],\ \alpha \in (r). \tag{3.31}$$

Making use of (3.31), we have that if M^n is totally geodesic, then $a_\alpha = 0$ and $h = 0$. Hence $b = 0$, that is, M^n is invariant. Thus we have the following.

Proposition 3.1. *If M^n is totally geodesic hypersurface of an almost r-paracontact Riemannian manifold M^{n+1} endowed with a quarter symmetric metric connection of S-paracontact type with a structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ and all ξ_α are tangent to M^n, then M^n is invariant.*

Theorem 3.6. *If M^n is an anti-invariant hypersurface of an almost r-paracontact Riemannian manifold M^{n+1} endowed with a quarter symmetric metric connection of S-paracontact type with a structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$, then all $\bar{\xi}_\alpha$ are parallel to M^n.*

Proof. If M^n is anti-invariant, then $\bar{\phi} = 0$ and $a_\alpha = 0$ and from (3.30) we have $\bar{\nabla}_{\bar{X}}\bar{\xi}_\alpha = 0$. ☐

Now, let M^n be an almost r-paracontact Riemannian manifold of P-Sasakian type endowed with a quarter symmetric metric connection. Then from (2.12) and (3.21), we have

$$\begin{aligned}
&(\bar{\nabla}_{\bar{Y}}\bar{\phi})\bar{X} - [h(\bar{X}, \bar{Y}) - a_\alpha\bar{g}(\bar{\phi}\bar{Y}, \bar{X}) + \bar{\eta}^\alpha(\bar{X})b(\bar{Y})]\bar{N} \\
&\quad - b(\bar{X})(H\bar{Y}) + a_\alpha b(\bar{X})\bar{\phi}\bar{Y} - b(\bar{X})b(\bar{Y})\bar{\xi}_\alpha \\
&= -\sum_\alpha \bar{\eta}^\alpha(\bar{X})(\bar{Y} - \bar{\eta}^\alpha(\bar{X})\bar{\xi}_\alpha) \\
&\quad - [\bar{g}(\bar{X}, \bar{Y}) - \sum_\alpha \bar{\eta}^\alpha(\bar{X})\bar{\eta}^\alpha(\bar{Y})]\sum_\beta \bar{\xi}_\beta - \bar{g}(\bar{X}, \bar{Y})\bar{\xi}_\alpha \\
&\quad + \sum_\alpha \bar{\eta}^\alpha(\bar{X})\bar{\eta}^\alpha(\bar{Y})\bar{\xi}_\alpha.
\end{aligned} \tag{3.32}$$

Theorem 3.7. *Let M^{n+1} be an almost r-paracontact Riemannian manifold of P-Sasakian type endowed with a quarter symmetric metric connection with a structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$, and let M^n be a hypersurface immersed in M^{n+1} such that none of ξ_α are tangent to M^n. Then M^n is totally geodesic if* *and only if*

$$(3.33)\quad (\bar{\nabla}_{\bar{Y}}\bar{\phi})\bar{X} = -a_\alpha b(\bar{X})\bar{\phi}\bar{Y} - a_\alpha \bar{g}(\bar{\phi}\bar{Y},\bar{X})\bar{N} - \sum_\alpha \bar{\eta}^\alpha(\bar{X})[\bar{Y} - \bar{\eta}^\alpha(\bar{Y})\bar{\xi}_\alpha] + b(\bar{X})b(\bar{Y})\bar{\xi}_\alpha - [\bar{g}(\bar{X},\bar{Y}) - \sum_\alpha \bar{\eta}^\alpha(\bar{X})\bar{\eta}^\alpha(\bar{Y})]\sum_\beta \bar{\xi}_\beta - \bar{g}(\bar{X},\bar{Y})\bar{\xi}_\alpha + \sum_\alpha \bar{\eta}^\alpha(\bar{X})\bar{\eta}^\alpha(\bar{Y})\bar{\xi}_\alpha.$$

Proof. If (3.33) is satisfied, then from (3.32), we get $h(\bar{X},\bar{Y})\bar{N}+b(\bar{X})H(\bar{Y})=0$. Since $b \neq 0$, so that $h(\bar{X},\bar{Y})=0$. Hence M^n is totally geodesic. Conversely, let M^n is totally geodesic, that is $H=0$, then from (3.32) we get (3.33) and from (3.31) we have $b=0$, which is contradiction. Hence ξ_α are not tangent to M^n. □

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