KYUNGPOOK Math. J. 49(2009), 533-543

Hypersurfaces of an almost r-paracontact Riemannian Manifold Endowed with a Quarter Symmetric Non-metric Connection

Mobin Ahmad and Abdul Haseeb
Department of Applied Mathematics, Integral University, Kursi Road, Lucknow-226026, India
e-mail : mobinahmad@rediffmail.com

Cihan Özgür*
Department of Mathematics, Balıkesir University, 10145, Çağış, Balıkesir, Turkey
e-mail : cozgur@balikesir.edu.tr

Abstract. We define a quarter symmetric non-metric connection in an almost r-paracontact Riemannian manifold and we consider invariant, non-invariant and anti-invariant hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter symmetric non-metric connection.

1. Introduction

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

In [8], S. Golab introduced the idea of a quarter symmetric linear connection in a differentiable manifold. A linear connection is said to be a quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = u(Y)\varphi X - u(X)\varphi Y, \tag{1.1}$$

where u is a 1-form and φ is a (1,1)-tensor field. In [8], [11] some properties of some kinds of quarter symmetric non-metric connections were studied.

---

* Corresponding author.

Received July 1, 2008; received in revised form May 13, 2009; accepted May 21, 2009.
2000 Mathematics Subject Classification: 53D12, 53C05.
Key words and phrases: hypersurfaces, almost r-paracontact Riemannian manifold, quarter symmetric non-metric connection.

A. Bucki and A. Miernowski defined an almost r-paracontact structures and studied some properties of invariant hypersurfaces of an almost r-paracontact structures in [5] and [6] respectively. A. Bucki also studied almost r-paracontact structures of P-Sasakian type in [3]. I. Mihai and K. Matsumoto studied submanifolds of an almost r-paracontact Riemannian manifold of P-Sasakian type in [10]. Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter-symmetric metric connection were studied by first and third author and J. B. Jun in [2]. Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a semi-symmetric metric connection were studied by J. B. Jun and the first author in [9]. Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a semi-symmetric non-metric connection were studied by first and second author in [1].

Motivated by the studies of the above authors, in this paper, we study quarter symmetric non-metric connection in an almost r-paracontact Riemannian manifold. We consider invariant, non-invariant and anti-invariant hypersurfaces of almost r-paracontact Riemannian manifold endowed with a quarter symmetric non-metric connection.

The paper is organized as follows: In Section 2, we give a brief introduction about an almost r-paracontact Riemannian manifold. In Section 3, we show that the induced connection on a hypersurface of an almost r-paracontact Riemannian manifold with a quarter symmetric non-metric connection with respect to the unit normal is also a quarter symmetric non-metric connection. We find the characteristic properties of invariant, non-invariant and anti-invariant hypersurfaces of almost r-paracontact Riemannian manifold endowed with a quarter symmetric non-metric connection.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold with a positive definite metric g. If there exist a tensor field φ of type (1,1), r vector fields $\xi_1, \xi_2, \cdots, \xi_r$ $(n > r)$, and r 1-forms $\eta^1, \eta^2, \cdots, \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad \alpha, \beta \in (r) = \{1, 2, 3, \cdots, r\}, \tag{2.1}$$

$$\varphi^2(X) = X - \eta^\alpha(X)\xi_\alpha, \tag{2.2}$$

$$\eta^\alpha(X) = g(X, \xi_\alpha), \quad \alpha \in (r), \tag{2.3}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{2.4}$$

where X and Y are vector fields on M, then the structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is said to be *an almost r-paracontact Riemannian structure* and M is *an almost r-paracontact Riemannian manifold* [5]. From (2.1)-(2.4), we have

$$\varphi(\xi_\alpha) = 0, \quad \alpha \in (r), \tag{2.5}$$

$$\eta^\alpha \circ \varphi = 0, \quad \alpha \in (r), \tag{2.6}$$

$$\Psi(X,Y) \overset{def}{=} g(\varphi X, Y) = g(X, \varphi Y). \tag{2.7}$$

For Riemannian connection $\overset{*}{\nabla}$ on M, the tensor N is given by

$$\begin{aligned} N(X,Y) &= \left(\overset{*}{\nabla}_{\varphi Y}\varphi\right)X - \left(\overset{*}{\nabla}_X\varphi\right)\varphi Y - \left(\overset{*}{\nabla}_{\varphi X}\varphi\right)Y \\ &\quad + \left(\overset{*}{\nabla}_Y\varphi\right)\varphi X + \eta^\alpha(X)\overset{*}{\nabla}_Y\xi_\alpha - \eta^\alpha(Y)\overset{*}{\nabla}_X\xi_\alpha. \end{aligned} \tag{2.8}$$

An almost r-paracontact Riemannian manifold M with structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be of *para-contact type* if

$$2\Psi(X,Y) = \left(\overset{*}{\nabla}_X\eta^\alpha\right)Y + \left(\overset{*}{\nabla}_Y\eta^\alpha\right)X, \quad \text{for all } \alpha \in (r).$$

If all η^α are closed, then the last equation reduces to

$$\Psi(X,Y) = \left(\overset{*}{\nabla}_X\eta^\alpha\right)Y, \quad \text{for all } \alpha \in (r) \tag{2.9}$$

and M satisfying this condition is called an *almost r-paracontact Riemannian manifold of s-paracontact type* [3]. An almost r-paracontact Riemannian manifold M with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be *P-Sasakian* if it satisfies (2.6) and

$$\begin{aligned} \left(\overset{*}{\nabla}_Z\Psi\right)(X,Y) &= -\sum_\alpha \eta^\alpha(X)\left[g(Y,Z) - \sum_\beta \eta^\beta(Y)\eta^\beta(Z)\right] \\ &\quad -\sum_\alpha \eta^\alpha(Y)\left[g(X,Z) - \sum_\beta \eta^\beta(Y)\eta^\beta(Z)\right] \end{aligned} \tag{2.10}$$

for all vector fields X, Y and Z on M [3]. The conditions (2.9) and (2.10) are equivalent to

$$\varphi X = \overset{*}{\nabla}_X\xi_\alpha, \quad \text{for all } \alpha \in (r) \tag{2.11}$$

and

$$\begin{aligned} \left(\overset{*}{\nabla}_Y\varphi\right)X &= -\sum_\alpha \eta^\alpha(X)\left[Y - \sum_\beta \eta^\alpha(Y)\xi_\alpha\right] \\ &\quad -\left[g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)\right]\sum_\beta \xi_\beta, \end{aligned} \tag{2.12}$$

respectively.

We define a quarter symmetric non-metric connection ∇ on M by

$$\nabla_X Y = \overset{*}{\nabla}_X Y + \eta^\alpha(Y)\varphi X, \tag{2.13}$$

for any $\alpha \in (r)$. Using (2.13) we get

$$\begin{aligned}(\nabla_Y \varphi)X &= -\sum_\alpha \eta^\alpha(X)\left[Y - \eta^\alpha(Y)\xi_\alpha\right] \\ &\quad - \left[g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)\right]\sum_\beta \xi_\beta.\end{aligned} \tag{2.14}$$

and

$$\nabla_X \xi_\alpha = 2\varphi X. \tag{2.15}$$

3. Hypersurfaces of almost r-paracontact Riemannian manifold with a quarter-symmetric non-metric connection

Let $\widetilde{M}^{n+1}$ be an almost r-paracontact Riemannian manifold with a positive definite metric g and M^n be a hypersurface immersed in $\widetilde{M}^{n+1}$ by immersion $f : M^n \to \widetilde{M}^{n+1}$. If B denote the differential of f then any vector field $\overline{X} \in \chi(M^n)$ implies $B\overline{X} \in \chi(\widetilde{M}^{n+1})$. We denote the object belonging to M^n by the mark of hyphen placed over them, e.g, $\overline{\varphi}, \overline{X}, \overline{\eta}, \overline{\xi}$ etc.

Let N be the unit normal field to M^n. Then the induced metric $\overline{g}$ on M^n is defined by

$$\overline{g}(\overline{X}, \overline{Y}) = g(\overline{X}, \overline{Y}). \tag{3.1}$$

Then we have [7]

$$g(\overline{X}, N) = 0 \quad \text{and} \quad g(N, N) = 1. \tag{3.2}$$

Equation of Gauss with respect to Riemannian connection $\overset{*}{\nabla}$ is given by

$$\nabla_{\overline{X}}\overline{Y} = \overset{*}{\nabla}_{\overline{X}}\overline{Y} + h(\overline{X}, \overline{Y})N. \tag{3.3}$$

If $\overline{\overset{*}{\nabla}}$ is the induced connection on hypersurface from $\overset{*}{\nabla}$ with respect to unit normal N, then Gauss equation is given by

$$\overset{*}{\nabla}_{\overline{X}}\overline{Y} = \overline{\overset{*}{\nabla}}_{\overline{X}}\overline{Y} + h(\overline{X}, \overline{Y})N, \tag{3.4}$$

where, h is second fundamental tensor satisfying

$$h(\overline{X},\overline{Y}) = h(\overline{Y},\overline{X}) = g(H(\overline{X}),Y) \tag{3.5}$$

and H is the shape operator of M^n in $\widetilde{M}^{n+1}$. If $\overline{\nabla}$ is the induced connection on hypersurface from the quarter symmetric non-metric connection ∇ with respect to unit normal N, then we have

$$\nabla_{\overline{X}}\overline{Y} = \overline{\nabla}_{\overline{X}}\overline{Y} + m(\overline{X},\overline{Y})N, \tag{3.6}$$

where m is a tensor field of type (0, 2) on hypersurface M^n. From (2.13), using $\varphi\overline{X} = \overline{\varphi}\overline{X} + b(\overline{X})N$ we obtain

$$\nabla_{\overline{X}}\overline{Y} = \overset{*}{\nabla}_{\overline{X}}\overline{Y} + \overline{\eta}^{\alpha}(\overline{Y})\left(\overline{\varphi}\overline{X} + b(\overline{X})N\right). \tag{3.7}$$

From equations (3.4), (3.6) and (3.7), we get

$$\overline{\nabla}_{\overline{X}}\overline{Y} + m(\overline{X},\overline{Y})N = \overline{\overset{*}{\nabla}}_{\overline{X}}\overline{Y} + h(\overline{X},\overline{Y})N + \overline{\eta}^{\alpha}(\overline{Y})\overline{\varphi}\overline{X} + \overline{\eta}^{\alpha}(\overline{Y})b(\overline{X})N.$$

By taking tangential and normal parts from both the sides, we obtain

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\overset{*}{\nabla}}_{\overline{X}}\overline{Y} + \overline{\eta}^{\alpha}(\overline{Y})\overline{\varphi}\overline{X} \tag{3.8}$$

and

$$m(\overline{X},\overline{Y}) = h(\overline{X},\overline{Y}) + \overline{\eta}^{\alpha}(\overline{Y})b(\overline{X}). \tag{3.9}$$

Thus we get the following theorem:

Theorem 3.1. *The connection induced on a hypersurface of an almost r-paracontact Riemannian manifold with a quarter-symmetric non-metric connection with respect to the unit normal is also a quarter-symmetric non-metric connection.*

From (3.6) and (3.9), we have

$$\nabla_{\overline{X}}\overline{Y} = \overline{\nabla}_{\overline{X}}\overline{Y} + h(\overline{X},\overline{Y})N + \overline{\eta}^{\alpha}(\overline{Y})b(\overline{X}), \tag{3.10}$$

which is Gauss equation for a quarter symmetric non-metric connection. The equation of Weingarten with respect to the Riemannian connection $\overset{*}{\nabla}$ is given by

$$\overset{*}{\nabla}_{\overline{X}}N = -H\overline{X} \tag{3.11}$$

for every $\overline{X}$ in M^n. From equation (2.13), we have

$$\nabla_{\overline{X}}N = \overset{*}{\nabla}_{\overline{X}}N + a_{\alpha}\overline{\varphi}\overline{X} + a_{\alpha}b(\overline{X})N, \tag{3.12}$$

where

$$\eta^{\alpha}(N) = a_{\alpha} = m(\xi_{\alpha}). \tag{3.13}$$

From (3.11) and (3.12), we have

$$\nabla_{\overline{X}} N = -M\overline{X}, \tag{3.14}$$

where $M\overline{X} = H\overline{X} - a_{\alpha}\overline{\varphi}\overline{X} - a_{\alpha}b(\overline{X})N$, which is Weingarten equation with respect to quarter symmetric non-metric connection.

Now, suppose that $\sum = (\varphi, \xi_{\alpha}, \eta^{\alpha}, g)_{\alpha\in(r)}$ is an almost r-paracontact Riemannian structure on $\widetilde{M}^{n+1}$, then every vector field X on $\widetilde{M}^{n+1}$ is decomposed as

$$X = \overline{X} + l(X)N, \tag{3.15}$$

where l is a 1-form on $\widetilde{M}^{n+1}$ and for any vector field $\overline{X}$ on M^n and normal N, we have

$$\varphi\overline{X} = \overline{\varphi}\overline{X} + b(\overline{X})N, \tag{3.16}$$

$$\varphi N = \overline{N} + KN, \tag{3.17}$$

where $\overline{\varphi}$ is a tensor field of type (1,1) on hypersurface M^n, b is a 1-form on M^n and K is a scalar function on M^n. For each $\alpha \in (r)$, we have

$$\xi_{\alpha} = \overline{\xi}_{\alpha} + a_{\alpha}N, \tag{3.18}$$

where $a_{\alpha} = m(\xi_{\alpha}) = \eta^{\alpha}(N)$, $\alpha \in (r)$. Now, we define $\overline{\eta}^{\alpha}$ by

$$\overline{\eta}^{\alpha}(\overline{X}) = \eta^{\alpha}(\overline{X}), \quad \alpha \in (r). \tag{3.19}$$

Making use of (3.16), (3.17), (3.18) and (3.13), from (2.1)-(2.5), we obtain

$$\overline{\varphi}^2\overline{X} + b(\overline{X})N = \overline{X} - \overline{\eta}^{\alpha}(\overline{X})\overline{\xi}_{\alpha}, \tag{3.20}$$

$$b(\overline{\varphi}\overline{X}) + Kb(\overline{X}) = -a_{\alpha}\overline{\eta}^{\alpha}(\overline{X}) \tag{3.21}$$

$$\overline{\varphi}\overline{N} + K\overline{N} = -\sum_{\alpha} a_{\alpha}\overline{\xi}_{\alpha} \tag{3.22}$$

$$b(\overline{N}) + K^2 = 1 - \sum_{\alpha}(a_{\alpha})^2 \tag{3.23}$$

$$\overline{\varphi}(\overline{\xi}_{\alpha}) + a_{\alpha}\overline{N} = 0, \tag{3.24}$$

$$Ka_{\alpha} + b(\overline{\xi}_{\alpha}) = 0, \tag{3.25}$$

$$(\overline{\eta}^{\alpha} \circ \overline{\varphi})(\overline{X}) + b(\overline{X})a_{\alpha} = 0, \tag{3.26}$$

$$\overline{\eta}^{\alpha}(\overline{\xi}_{\beta}) + a_{\alpha}a_{\beta} = \delta^{\alpha}_{\beta}, \tag{3.27}$$

$$\overline{\eta}^{\alpha}(\overline{X}) = \overline{g}(\overline{X}, \overline{\xi}_{\alpha}), \tag{3.28}$$

$$\overline{g}(\overline{\varphi}\overline{X},\overline{\varphi}\overline{Y})-b(\overline{X})b(\overline{Y})=\overline{g}(\overline{X},\overline{Y})-\sum_{\alpha}\overline{\eta}^{\alpha}(\overline{X})\overline{\eta}^{\alpha}(\overline{Y}), \tag{3.29}$$

and

$$\Psi(\overline{X},\overline{Y})=\overline{g}(\overline{\varphi}\overline{X},\overline{Y})=\overline{g}(\overline{X},\overline{\varphi}\overline{Y})=\overline{\Psi}(\overline{X},\overline{Y}). \tag{3.30}$$

where $\alpha,\beta\in(r)$ Using (3.1), (3.2), (3.17), (3.18) and (2.7), we have

$$g(\overline{\varphi}\overline{X},N)=g(\varphi\overline{X},N)-b(\overline{X})=g(\overline{X},\varphi N)-b(\overline{X})=0.$$

Hence we get

$$g(\overline{X},\overline{N})=b(\overline{X}), \tag{3.31}$$

(see [4]). Differentiating (3.16) and (3.17) along M^n and making use of (3.10), (3.29) and (3.1) we get

$$\begin{aligned}(\nabla_{\overline{Y}}\varphi)\overline{X} &= (\overline{\nabla}_{\overline{Y}}\overline{\varphi})\overline{X}-h(\overline{X},\overline{Y})\overline{N}+\overline{\eta}^{\alpha}(\overline{\varphi}\overline{X})b(\overline{Y})-b(\overline{X})\left[H(\overline{Y})-a_{\alpha}\overline{\varphi}\overline{Y}\right]\\ &\quad+\left[h(\overline{\varphi}\overline{X},\overline{Y})+(\overline{\nabla}_{\overline{Y}}b)\overline{X}-Kh(\overline{X},\overline{Y})+a_{\alpha}b(\overline{X})b(\overline{Y})\right]N,\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}(\nabla_{\overline{Y}}\varphi)N &= \overline{\nabla}_{\overline{Y}}\overline{N}+\overline{\varphi}H(\overline{Y})-a_{\alpha}\overline{Y}+a_{\alpha}\overline{\eta}^{\alpha}(\overline{Y})\overline{\xi}_{\alpha}-b(\overline{Y})\left(a_{\alpha}\overline{N}-\overline{\eta}^{\alpha}(\overline{N})\right)\\ &\quad+K\left(a_{\alpha}\overline{\varphi}\overline{Y}-H(\overline{Y})\right)+\left[h(\overline{Y},\overline{N})+\overline{Y}(K)+bH(\overline{Y})\ .\right.\\ &\quad\left.-a_{\alpha}\left(b(\overline{Y})-b(\overline{\varphi}\overline{Y})\right)\right]N.\end{aligned} \tag{3.33}$$

From (3.18) and (3.13), we have

$$\begin{aligned}\nabla_{\overline{Y}}\xi_{\alpha} &= \overline{\nabla}_{\overline{Y}}\overline{\xi}_{\alpha}-a_{\alpha}H(\overline{Y})+(a_{\alpha})^{2}\overline{\varphi}\overline{Y}+\overline{\eta}^{\alpha}(\overline{\xi}_{\alpha})b(\overline{Y})\\ &\quad+\left[(a_{\alpha})^{2}b(\overline{Y})+\overline{Y}(a_{\alpha})+h(\overline{Y},\overline{\xi}_{\alpha})\right]N\end{aligned} \tag{3.34}$$

and

$$\left(\nabla_{\overline{Y}}\eta^{\alpha}\right)\overline{X}=\left(\overline{\nabla}_{\overline{Y}}\overline{\eta}^{\alpha}\right)\overline{X}-h(\overline{Y},\overline{X})a_{\alpha}. \tag{3.35}$$

From identity

$$\left(\nabla_{Z}\Psi\right)(X,Y)=g(\left(\nabla_{Z}\varphi\right)X,Y),$$

using (3.30), (3.31) and (3.32) we have

$$\begin{aligned}\left(\nabla_{\overline{Z}}\Psi\right)\left(\overline{X},\overline{Y}\right) &= \left(\overline{\nabla}_{\overline{Z}}\overline{\Psi}\right)\left(\overline{X},\overline{Y}\right)-h(\overline{X},\overline{Z})b(\overline{Y})\\ &\quad-b(\overline{X})h(\overline{Z},\overline{Y})+a_{\alpha}b(\overline{X})\overline{\Psi}(Y,Z).\end{aligned} \tag{3.36}$$

Theorem 3.2. *If M^n is an invariant hypersurface immersed in an almost r-paracontact Riemannian manifold $\widetilde{M}^{n+1}$ endowed with a quarter symmetric non-metric connection with structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$, then either*
(i) *All ξ_α are tangent to M^n and M^n admits an almost r-paracontact Riemannian structure $\sum_1 = \left(\overline{\varphi}, \overline{\xi}_\alpha, \overline{\eta}^\alpha, \overline{g}\right)_{\alpha\in(r)}$, $(n-r>2)$ or*
(ii) *One of ξ_α (say ξ_r) is normal to M^n and remaining ξ_α are tangent to M^n and M^n admits an almost $(r-1)$-paracontact Riemannian structure $\sum_2 = \left(\overline{\varphi}, \overline{\xi}_i, \overline{\eta}^i, \overline{g}\right)_{i\in(r)}$, $(n-r>1)$.*

Proof. The proof is similar to the proof of Theorem 3.3 in [4]. □

Corollary 3.3. *If M^n is a hypersurface immersed in an almost r-paracontact Riemannian manifold $\widetilde{M}^{n+1}$ with a structure $\sum = \left(\overline{\varphi}, \overline{\xi}_\alpha, \overline{\eta}^\alpha, \overline{g}\right)_{\alpha\in(r)}$ endowed with a quarter symmetric non-metric connection, then the following statements are equivalent.*
(i) *M^n is invariant,*
(ii) *The normal field N is an eigenvector of φ,*
(iii) *All ξ_α are tangent to M^n if and only if M^n admits an almost r-paracontact Riemannian structure $\sum_1$, or one of ξ_α is normal and $(r-1)$ remaining ξ_i are tangent to M^n if and only if M^n admits an almost $(r-1)$ paracontact Riemannian structure $\sum_2$.*

Theorem 3.4. *If $M^n$is an invariant hypersurface immersed in an almost r-paracontact Riemannian manifold $\widetilde{M}^{n+1}$ of P-Sasakian type endowed with a quarter symmetric non-metric connection then the induced almost r-paracontact Riemannian structure $\sum_1$ or $(r-1)$ paracontact Riemannian structure $\sum_2$ are also of P-Sasakian type.*

Proof. The computations are similar to the proof of Theorem 3.1 in [4]. □

Lemma 3.5([4]). $\overline{\nabla}_{\overline{X}}(trace\overline{\varphi}) = trace\left(\overline{\nabla}_{\overline{X}}\overline{\varphi}\right)$.

Theorem 3.6. *Let M^n be a non-invariant hypersurface of an almost r-paracontact Riemannian manifold $\widetilde{M}^{n+1}$ with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ satisfying $\nabla\varphi = 0$ along M^n then M^n is totally geodesic if and only if*

$$\left(\overline{\nabla}_{\overline{Y}}\overline{\varphi}\right)\overline{X} + b(\overline{X})a_\alpha\overline{\varphi}\overline{Y} + \overline{\eta}^\alpha\left(\overline{\varphi}\overline{X}\right)b(\overline{Y}) = 0.$$

Proof. From (3.32) we have

$$\left(\overline{\nabla}_{\overline{Y}}\overline{\varphi}\right)\overline{X} - h(\overline{Y},\overline{X})\overline{N} - b(\overline{X})\left(H(\overline{Y}) - a_\alpha\varphi\overline{Y}\right) = 0 \tag{3.37}$$

and

$$h(\overline{\varphi}\overline{X},\overline{Y}) + (\overline{\nabla}_{\overline{Y}}b)\overline{X} - Kh(\overline{Y},\overline{X}) + a_\alpha b(\overline{X})b(\overline{Y}) = 0. \tag{3.38}$$

If M^n is totally geodesic, then $h=0$ and $H=0$. So from (3.37), we get

$$\left(\overline{\nabla}_{\overline{Y}}\overline{\varphi}\right)\overline{X} + b(\overline{X})a_\alpha\overline{\varphi}\overline{Y} + \overline{\eta}^\alpha\left(\overline{\varphi}\overline{X}\right)b(\overline{Y}) = 0.$$

Conversely, if $\left(\overline{\nabla}_{\overline{Y}}\overline{\varphi}\right)\overline{X}+b(\overline{X})a_\alpha\overline{\varphi}\overline{Y}+\overline{\eta}^\alpha\left(\overline{\varphi}\overline{X}\right)b(\overline{Y})=0$, then

$$h(\overline{Y},\overline{X})\overline{N}+b(\overline{X})H(\overline{Y})=0. \tag{3.39}$$

Making use of (3.31) and (3.5) we have

$$b(\overline{X})h(\overline{Y},\overline{Z})+b(\overline{Z})h(\overline{X},\overline{Z})=0. \tag{3.40}$$

Using (3.39), we get from (3.5)

$$b(\overline{X})h(\overline{Y},\overline{Z})=b(\overline{Z})h(\overline{X},\overline{Z}).$$

So from (3.40) and (3.41) we get $b(\overline{Z})h(\overline{X},\overline{Y})=0$. This gives us $h=0$ since $b\neq 0$. Using $h=0$ in (3.40), we get $H=0$. Thus, $h=0$ and $H=0$. Hence M^n is totally geodesic. This completes the proof of the theorem. □

We have also the following:

Theorem 3.7. *Let M^n be a non-invariant hypersurface of an almost r-paracontact Riemannian manifold $\widetilde{M}^{n+1}$ with a quarter symmetric non-metric connection and satisfying $\nabla\varphi=0$ along M^n. If $trace\overline{\varphi}=constant$, then*

$$h(\overline{X},\overline{N})=\frac{1}{2}a_\alpha\sum_a b(e_a)\overline{\Psi}(e_a,\overline{X}). \tag{3.41}$$

Proof. From (3.37) we have

$$\overline{g}\left(\left(\overline{\nabla}_{\overline{Y}}\overline{\varphi}\right)\overline{X},\overline{X}\right)=2h(\overline{X},\overline{Y})b(\overline{X})-a_\alpha b(\overline{X})g(\overline{X},\overline{Y})$$

and using $\overline{N}=\sum_a b(e_a)e_a$

$$\overline{\nabla}_{\overline{X}}(trace\overline{\varphi})=2h(\overline{X},\overline{N})-a_\alpha\sum_a b(e_a)\overline{\Psi}(e_a,\overline{X}).$$

Using Lemma 3.5, we get (3.41), where $\overline{N}=\sum_a b(e_a)e_a$. Thus our theorem is proved. □

Let M^n be an almost r-paracontact Riemannian manifold of S-paracontact type, then from (2.11), (3.16) and (3.34), we get

$$\overline{\varphi}\overline{X}=\frac{1}{2}\left[\overline{\nabla}_{\overline{X}}\overline{\xi}_\alpha-a_\alpha H(\overline{X})+(a_\alpha)^2\varphi\overline{X}+\eta^\alpha(\overline{\xi}_\alpha)b(\overline{X})\right],\quad \alpha\in(r) \tag{3.42}$$

$$b(\overline{X})=\frac{1}{2}\left[\overline{X}(a_\alpha)+h(\overline{X},\overline{\xi}_\alpha)+(a_\alpha)^2 b(\overline{X})\right],\quad \alpha\in(r). \tag{3.43}$$

Making use of (3.43), if M^n is totally geodesic then $a_\alpha=0$ and $h=0$. Hence $b=0$, that is, M^n is invariant.

So we have the following Proposition:

Proposition 3.8. *If M^n is totally geodesic hypersurface of an almost r-paracontact Riemannian manifold $\widetilde{M}^{n+1}$ with a quarter symmetric non-metric connection of S-paracontact type with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ and all ξ_α are tangent to M^n, then M^n is invariant.*

Theorem 3.9. *If M^n is an anti-invariant hypersurface of an almost r-paracontact Riemannian manifold $\widetilde{M}^{n+1}$ with a quarter symmetric non-metric connection of S-paracontact type with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ then $\overline{\nabla}_{\overline{X}}\overline{\xi}_\alpha + b(\overline{X}) = 0$.*

Proof. If M^n is anti-invariant then $\overline{\varphi} = 0$, $a_\alpha = 0$ and from (3.42), we have

$$\overline{\nabla}_{\overline{X}}\overline{\xi}_\alpha + b(\overline{X}) = 0.$$

This completes the proof of the theorem. □

Now, let M^n be an almost r-paracontact Riemannian manifold of P-Sasakian type. Then from (2.14) and (3.32), we have

$$\begin{aligned}&(\overline{\nabla}_{\overline{Y}}\overline{\varphi})\overline{X} - h(\overline{X},\overline{Y})\overline{N} - b(\overline{X})H(\overline{Y}) + a_\alpha b(\overline{X})\overline{\varphi}\overline{Y} + \overline{\eta}^\alpha(\overline{\varphi}\overline{X})b(\overline{Y})\\ &\quad + \left[h(\overline{\varphi}\overline{X},\overline{Y}) + (\overline{\nabla}_{\overline{Y}}b)\overline{X} - Kh(\overline{X},\overline{Y}) + a_\alpha b(\overline{X})b(\overline{Y})\right]N\\ &\quad = -\sum_\alpha \overline{\eta}^\alpha(\overline{X})\left[\overline{Y} - \overline{\eta}^\alpha(\overline{Y})\overline{\xi}_\alpha\right] - \left[\overline{g}(\overline{X},\overline{Y}) - \sum_\alpha \overline{\eta}^\alpha(\overline{X})\overline{\eta}^\alpha(\overline{Y})\right]\sum_\beta \overline{\xi}_\beta.\end{aligned}$$

From above equation we have

$$\begin{aligned}&\left(\overline{\nabla}_{\overline{Y}}\overline{\varphi}\right)\overline{X} - h(\overline{X},\overline{Y})\overline{N} - b(\overline{X})H(\overline{Y}) + a_\alpha b(\overline{X})\overline{\varphi}\overline{Y} + \overline{\eta}^\alpha(\overline{\varphi}\overline{X})b(\overline{Y})\\ &\quad = -\sum_\alpha \overline{\eta}^\alpha(\overline{X})\left[\overline{Y} - \overline{\eta}^\alpha(\overline{Y})\overline{\xi}_\alpha\right] - \left[\overline{g}(\overline{X},\overline{Y}) - \sum_\alpha \overline{\eta}^\alpha(\overline{X})\overline{\eta}^\alpha(\overline{Y})\right]\sum_\beta \overline{\xi}_\beta.\end{aligned} \tag{3.44}$$

Theorem 3.10. *Let $\widetilde{M}^{n+1}$ be an almost r-paracontact Riemannian manifold of P-Sasakian type with a quarter symmetric non-metric connection with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ and let M^n be a hypersurface immersed in $\widetilde{M}^{n+1}$ such that none of ξ_α is tangent to M^n. Then M^n is totally geodesic if and only if.*

$$\begin{aligned}\left(\overline{\nabla}_{\overline{Y}}\overline{\varphi}\right)\overline{X} + a_\alpha b(\overline{X})\overline{\varphi}\overline{Y} &= -\sum_\alpha \overline{\eta}^\alpha(\overline{X})\left[\overline{Y} - \overline{\eta}^\alpha(\overline{Y})\overline{\xi}_\alpha\right]\\ &\quad -\overline{\eta}^\alpha(\overline{\varphi}\overline{X})b(\overline{Y}) - \left[\overline{g}(\overline{X},\overline{Y}) - \sum_\alpha \overline{\eta}^\alpha(\overline{X})\overline{\eta}^\alpha(\overline{Y})\right]\sum_\beta \overline{\xi}_\beta.\end{aligned} \tag{3.45}$$

Proof. If (3.45) is satisfied then from (3.45), we get $b(\overline{Z})h(\overline{X},\overline{Y}) = 0$. Since $b \neq 0$ hence $h(\overline{X},\overline{Y}) = 0$. Conversely, let M^n be totally geodesic, that is, $h(\overline{X}, \overline{Y}) = 0$, $H = 0$, then (3.45) is satisfied. From (3.43), $b(\overline{X}) = \frac{1}{2}\left[\overline{X}(a_\alpha) + h(\overline{Y}, \overline{\xi}_\alpha) + (a_\alpha)^2 b(\overline{X})\right]$. If $a_\alpha = h = 0$ then $b = 0$, which is a contradiction. Hence all ξ_α are not tangent to M^n. So we get the result as required. □

References

[1] Ahmad, M. and Özgür, C., *Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a semi- symmetric non-metric connection*, Results in Mathematics, (2009) (Accepted).

[2] Ahmad, M., Jun, J. B and Haseeb, A., *Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter symmetric metric connection*, Bull. Korean Math. Soc., **46(0)**(2009), 1-10.

[3] Bucki, A., *Almost r-paracontact structures of P-Sasakian type*, Tensor, N.S., **42**(1985), 42-54.

[4] Bucki, A., *Hypersurfaces of almost r-paracontact Riemannian manifold*, Tensor, N.S., **48**(1989), 245-251.

[5] Bucki, A. and Miernowski, A., *Almost r-paracontact structures*, Ann. Univ. Mariae Curie-Sklowska, Sect. A, **39**(1985), 13-26.

[6] Bucki, A. and Miernowski, A. *Invariant hypersurfaces of an almost r-paracontact manifold,* Demonstratio Math., **19**(1986), 113-121.

[7] Chen, B.Y., Geometry of Submanifolds. Marcel Dekker, New York, 1973.

[8] Golab, S., *On semi-symmetric and quarter-symmetric linear connections*, Tensor (N.S.), **29(3)**(1975), 249-254.

[9] Jun, J. B. and Ahmad, M., *Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a semi- symmetric metric connection*, Bull. Korean Math. Soc., (2009) (Accepted).

[10] Mihai, I. and Matsumoto, K., *Submanifolds of an almost r-paracontact Riemannian manifold of P-Sasakian type.* Tensor (N.S.), **48(2)**(1989), 136-142.

[11] Tripathi, M. M., *A new connection in a Riemannian manifold*, Int. Elec. J. Geom., **1(10)**(2008), 15-24.