J. DIFFERENTIAL GEOMETRY
1 (1967) 369-370

HYPERSURFACES OF COMPLEX PROJECTIVE SPACE WITH CONSTANT SCALAR CURVATURE

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1. Introduction

In his dissertation, B. Smyth [3] classified the complex hypersurfaces of the simply connected complex space forms which are complete and Einsteinian.[1] In particular, he proved the following theorem:

Let M be a complete complex hypersurface of the complex projective space $P_{n+1}(C)$ of dimension $n+1$ for $n \geq 2$. If M is an Einstein space with respect to the metric induced from the Fubini-Study metric of $P_{n+1}(C)$, then M is either a complex hyperplane $P_n(C)$ or a complex quadric in $P_{n+1}(C)$.

The purpose of this note is to point out that the theorem of Smyth combined with the theorem of Riemann-Roch-Hirzebruch yields the following:

Let M be a complete complex hypersurface of $P_{n+1}(C)$. If M has constant scalar curvature with respect to the induced metric, then M is either a complex hyperplane $P_n(C)$ or a complex quadric in $P_{n+1}(C)$.

2. Kähler manifolds with constant scalar curvature

Let M be a Kähler manifold with metric $ds^2 = 2 \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ and the fundamental 2-form $\Phi = \frac{2}{i} \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$. The first Chern class $c_1(M)$ of M is represented by the closed 2-form

$$\gamma_1 = \frac{1}{2\pi i} \sum_{\alpha,\beta} R_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta ,$$

where $R_{\alpha\bar{\beta}}$ denotes the Ricci tensor. We denote by $[\Phi]$ and $[\gamma_1]$ the cohomology classes represented by Φ and γ_1, respectively, so that $c_1(M) = [\gamma_1]$.

If M is an Einstein space, then its scalar curvature $2 \sum_{\alpha,\beta} g^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}$ is constant and $[\gamma_1] = k[\Phi]$ for some constant k. Conversely, we have

Received September 18, 1967. Supported by NSF Grant GP-5798.

[1] In [1] Chern showed that even the corresponding local result is true. Takahashi [4] obtained a partial generalization of the result of Smyth by showing that if a hypersurface in a space of constant holomorphic sectional curvature has parallel Ricci tensor, then it is Einsteinian and symmetric.

Lemma. *If M is a compact Kähler manifold such that its scalar curvature is constant and $[\gamma_1] = k[\Phi]$, then M is an Einstein space.*

In fact, from the harmonic integral theory we see easily (by local calculation) that γ_1 is harmonic if and only if the scalar curvature is constant. If γ_1 is harmonic and $[\gamma_1] = k[\Phi]$, then $\gamma_1 = k\Phi$. This proves the lemma.

3. The first Chern class of a hypersurface

Let h be the generator of $H^2(P_{n+1}(C); Z)$ corresponding to the divisor class of a hyperplane $P_n(C)$ so that the first Chern class $c_1(P_{n+1}(C))$ of $P_{n+1}(C)$ is given by

$$c_1(P_{n+1}(C)) = (n + 2)h \, .$$

Let M be a complete complex hypersurface in $P_{n+1}(C)$. By the well known theorem of Chow M is algebraic. Let d be the degree of M. Then (cf. Hirzebruch [2, p. 159]) the first Chern class $c_1(M)$ of M is given by

$$c_1(M) = (n - d + 2)\bar{h} \, ,$$

where $\bar{h}$ is the image of h under the natural homomorphism $H^*(P_{n+1}(C); Z) \to H^*(M; Z)$ induced by the imbedding $M \to P_{n+1}(C)$.

Let Ψ be the fundamental 2-form of $P_{n+1}(C)$. Since $\dim H^*(P_{n+1}(C); R) = 1$, it follows that $[\Psi] = ah$, where a is a constant. Since the fundamental 2-form Φ of M is the restriction of Ψ to M, we have $[\Phi] = a\bar{h}$. This, together with $[\gamma_1] = c_1(M) = (n - d + 2)h$, implies that $[\gamma_1] = k[\Phi]$ for some constant k. Our assertion in §1 now follows from the lemma.

Remark. The same reasoning as above, together with the result on p. 159 of Hirzebruch [2], gives the following: Let M_n be a complete intersection of r non-singular hypersurfaces in $P_{n+1}(C)$. If M has constant scalar curvature, then M is an Einstein space.

Bibliography

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[4] T. Takahashi, *Hypersurface with parallel Ricci tensor in a space of constant holomorphic sectional curvature,* J. Math. Soc. Japan **19** (1967) 199–204.

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