J. DIFFERENTIAL GEOMETRY
5 (1971) 479-486

HYPERSURFACES OF ODD-DIMENSIONAL SPHERES

DAVID E. BLAIR, GERALD D. LUDDEN & KENTARO YANO

A structure similar to an almost complex structure was shown in [2] to exist on a hypersurface of an almost contact manifold or a submanifold of codimension 2 of an almost complex space. This structure on a manifold M has been studied in [1], [5], [6] from two points of view, namely, that the structure exists on M because M is a submanifold of some ambient space, and also that the structure exists intrinsically on M.

The odd-dimensional sphere S^{2n+1} has an almost contact structure which is naturally induced from the Kaehler structure of Euclidean space E^{2n+2}. The purpose of this paper is to study complete hypersurfaces immersed in S^{2n+1}. In § 3 it is shown that if the Weingarten map of the immersion and f commute then the hypersurface is a sphere whose radius is determined. Here, f is a tensor field of type (1,1) on the hypersurface, which is part of the induced structure. That the hypersurface satisfying this condition is a sphere follows from the results in [6], however a new proof is given here for completeness. In § 4 it is shown that if the Weingarten map K of the immersion and f satisfy $fK + Kf = 0$, and the hypersurface is of constant scalar curvature, then it is a great sphere or $S^n \times S^n$.

1. Hypersurfaces of a sphere

Let S^{2n+1} be the natural sphere of dimension $2n + 1$ in Euclidean $(2n + 2)$-space E^{2n+2}. Let (ϕ, ξ, η, g) be the normal, almost contact metric structure (see [4]) induced on S^{2n+1} by the Kaehler structure on E^{2n+2}. That is to say, ϕ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form and g is a Riemannian metric on S^{2n+1} satisfying

$$\begin{gathered} \phi^2 = -I + \eta \otimes \xi , \\ \phi\xi = 0 , \qquad \eta \circ \phi = 0 , \\ \eta(\xi) = 1 , \\ g(\phi\tilde{X}, \phi\tilde{Y}) + \eta(\tilde{X})\eta(\tilde{Y}) = g(\tilde{X}, \tilde{Y}) , \\ [\phi, \phi] + d\eta \otimes \xi = 0 , \end{gathered} \tag{1}$$

where $[\phi, \phi]$ is the Nijenhuis torsion tensor of ϕ, and $\tilde{X}$ and $\tilde{Y}$ are arbitrary vector fields on S^{2n+1}.

Communicated December 17, 1970.

Suppose $\pi\colon M^{2n} \to S^{2n+1}$ is an embedding of the orientable manifold M^{2n} in S^{2n+1}. The tensor G defined on M^{2n} by

$$G(X, Y) = g(\pi_* X, \pi_* Y) \tag{2}$$

is a Riemannian metric on M^{2n}, where π_* denotes the differential of the embedding π. If C is a field of unit normals defined on M^{2n}, and $\tilde{\nabla}$ is the Riemannian connection of g, then the Gauss and Weingarten equations can be written as

$$\begin{aligned} \tilde{\nabla}_{\pi_* X}\pi_* Y &= \pi_*(\nabla_X Y) + k(X, Y)C \ , \\ \tilde{\nabla}_{\pi_* X} C &= \pi_*(KX) \ . \end{aligned} \tag{3}$$

Then ∇ is the Riemannian connection of G, k is a symmetric tensor of type (0,2) on M^{2n}, and $G(KX, Y) = k(X, Y)$. Furthermore, if we set

$$\begin{aligned} \phi\pi_* X &= \pi_* fX + v(X)C \ , \qquad \xi = \pi_* U + \lambda C \ , \\ \phi C &= -BV \ , \qquad u(X) = \eta(\pi_* X) \ , \end{aligned} \tag{4}$$

then f is a tensor field of type (1,1), U and V are vector fields, u and v are 1-forms, and λ is a function satisfying

$$\begin{aligned} f^2 &= -I + u \otimes U + v \otimes V \ , \\ u \circ f &= \lambda v \ , \qquad v \circ f = -\lambda u \ , \\ fU &= -\lambda V \ , \qquad fV = \lambda U \ , \\ u(U) &= v(V) = 1 - \lambda^2 \ , \qquad u(V) = v(U) = 0 \ , \\ G(fX, fY) &= G(X, Y) - u(X)u(Y) - v(X)v(Y) \ . \end{aligned} \tag{5}$$

It was shown in [2] that the following relations hold

$$\begin{aligned} (\nabla_X f)Y &= G(X, Y)U - u(Y)X - k(X, Y)V + v(Y)KX \ , \\ \nabla_X U &= -fX - \lambda KX \ , \\ \nabla_X V &= -\lambda X + fKX \ , \\ \nabla_X \lambda &= v(X) + k(U, X) \ . \end{aligned} \tag{6}$$

2. Case I: $Kf - fK = 0$

We will prove the following theorem.

Theorem 1. *If M^{2n} is an orientable submanifold of S^{2n+1} satisfying $Kf = fK$, and $\lambda \neq$ constant, K being the Weingarten map of the embedding, and f and λ being defined in* (4), *then M^{2n} is a sphere of radius $1/\sqrt{1 + \alpha^2}$, where α is some constant determined by the embedding.*

Proof. We have that $0 = G((Kf - fK)U, U)$, so that

$$\begin{aligned} 0 &= G(KfU, U) - G(fKU, U) \\ &= -\lambda G(V, KU) + G(KU, fU) \\ &= -\lambda G(V, KU) - \lambda G(KU, V) . \end{aligned}$$

Therefore we see $\lambda = 0$ or $k(U, V) = 0$. By continuity, since λ is non-constant,

$$k(U, V) = 0 . \tag{7}$$

In a similar fashion we obtain

$$k(U, U) = k(V, V) . \tag{8}$$

Now $fKU + \lambda KV = 0$, so that

$$\begin{aligned} 0 &= -KU + u(KU)U + v(KU)V + \lambda KfV \\ &= -KU + u(KU)U + \lambda^2 KU , \end{aligned}$$

and hence

$$(1 - \lambda^2)KU = k(U, U)U .$$

Similarly, we obtain

$$(1 - \lambda^2)KV = k(U, U)V .$$

At points where $\lambda \neq \pm 1$, we have $KU = \alpha U$ for $\alpha = k(U, U)/(1 - \lambda^2)$, which implies

$$(\nabla_X K)U + K(-fX - \lambda KX) = \nabla_X \alpha \cdot U + \alpha(-fX - \lambda KX) .$$

The Codazzi equation for an embedding gives that $(\nabla_X K)(Y) = (\nabla_Y K)(X)$ so that we have

$$2G(KfZ, X) = (\nabla_X \alpha)u(Z) - (\nabla_Z \alpha)u(X) + 2\alpha G(fZ, X) .$$

If we set Z equal to U, then

$$-2\alpha\lambda u(X) = (\nabla_X \alpha)(1 - \lambda^2) - (\nabla_U \alpha)u(X) - 2\lambda u(X) ,$$

so that $\nabla_X \alpha$ and $u(X)$ are proportional. Therefore $Kf = \alpha f$, and hence

$$-KX + u(X)KU + v(X)KV = \alpha(-X + u(X)U + v(X)V) .$$

Thus $KX = \alpha X$ for all X, and by the Codazzi equation α is constant. From (6) we have that $\nabla_X \lambda = v(X) + \alpha u(X)$, and therefore that

$$\begin{aligned}\nabla_Y\nabla_X\lambda - d\lambda(\nabla_Y X) &= \nabla_Y(v(X) + \alpha u(X)) - (v(\nabla_Y X) + \alpha u(\nabla_Y X)) \\ &= -\lambda G(X, Y) + \alpha G(X, fY) - \alpha G(X, fY) - \alpha\lambda G(KY, X) \\ &= -\lambda(1 + \alpha^2)G(X, Y) .\end{aligned}$$

By the following lemma of Obata [3], M^{2n} is sphere of radius $(1 + \alpha^2)^{-1/2}$.

Lemma. *A complete connected Riemannian manifold M admits a non-trivial solution λ of $\nabla_Y\nabla_X\lambda - d\lambda(\nabla_Y X) = -k\lambda G(X, Y)$ for some real number $k > 0$ if and only if M is globally isometric to a Euclidean sphere of radius $k^{-1/2}$.*

Corollary. *Let M^{2n} be an orientable submanifold of S^{2n+1} with $\lambda \neq$ constant. Then $Kf - fK = 0$ if and only if M^{2n} is a totally umbillical submanifold of S^{2n+1}.*

Remark. In [5], there was introduced the idea of *normality* of an (f, G, u, v, λ)-structure, which is of a manifold M^{2n} with tensors satisfying (5). This condition is

$$[f, f] + du \otimes U + dv \otimes V = 0 .$$

We have the following proposition.

Proposition. *Let M^{2n} be a hypersurface of S^{2n+1} with $\lambda \neq$ constant. The (f, G, u, v, λ)-structure on M^{2n} is normal if and only if $fK - Kf = 0$.*

Proof. Let

$$S(X, Y) = [f, f](X, Y) + du(X, Y)U + dv(X, Y)V .$$

Using (5) it can be shown that

$$S(X, Y) = v(Y)(Kf - fK)X - v(X)(Kf - fK)Y ,$$

and hence the "if" part of the proposition is proved. On the other hand, assume $S(X, Y) = 0$ for all X and Y and let $PX = (Kf - fK)X$. Then

$$v(V)PX = v(X)PV .$$

Also, it can be shown that

$$G(PX, Y) = G(X, PY)$$

so that

$$v(X)G(PV, Y) = v(Y)G(PV, X) ,$$

that is to say,

$$G(PV, Y) = \alpha v(Y)$$

for some α. Thus we have that

$$v(V)G(PX, Y) = v(X)G(PV, Y) = \alpha v(X)v(Y) ,$$

but since the trace of P is 0, we have $\alpha = 0$ and thus $P = 0$.

3. Case II: $Kf + fK = 0$

In this section we prove the following theorem.

Theorem 2. *If M^{2n} is a complete orientable submanifold of S^{2n+1} with constant scalar curvature satisfying $Kf + fK = 0$ and $\lambda \neq$ constant, where K is the Weingarten map of the embedding, and f and λ are defined in* (4), *then M^{2n} is a natural sphere S^{2n} or $M^{2n} = S^n \times S^n$.*

Proof. We have that

$$0 = (Kf + fK)U = -\lambda KV + fKU ,$$
$$0 = (Kf + fK)V = \lambda KU + fKV ,$$

so that

$$\begin{aligned} 0 &= -\lambda k(V, V) + G(fKU, V) \\ &= -\lambda k(V, V) - G(KU, fV) \\ &= -\lambda k(V, V) - \lambda k(U, U) , \end{aligned}$$

and hence

$$k(U, U) + k(V, V) = 0 \tag{9}$$

by continuity. Also

$$\begin{aligned} 0 &= -\lambda fKV + f^2KU \\ &= \lambda^2KU + (-KU + u(KU)U + v(KU)V) , \end{aligned}$$

that is,

$$(1 - \lambda^2)KU = k(U, U)U + k(U, V)V , \tag{10}$$

and similarly

$$(1 - \lambda^2)KV = k(U, V)U + k(V, V)V . \tag{11}$$

At points where $\lambda \neq \pm 1$, write equations (10) and (11) as

$$KU = \alpha U + \beta V , \tag{10'}$$
$$KV = \beta U - \alpha V . \tag{11'}$$

If we apply ∇_X to equation (10′), use equation (6) for $\nabla_X U$ and $\nabla_X V$, and use the fact that $(\nabla_X K)Y = (\nabla_Y K)X$ because of the Codazzi equation, then we find that

$$(\nabla_X\alpha)u(Y) - (\nabla_Y\alpha)u(X) + (\nabla_X\beta)v(Y) - (\nabla_Y\beta)v(X) - 2\alpha F(X, Y) = 0 , \tag{12}$$

where $F(X, Y) = G(fX, Y)$. Setting $X = U$ and $Y = V$ and using the fact that $\lambda \neq$ constant we see that

$$-\nabla_V\alpha + \nabla_U\beta + 2\alpha\lambda = 0 . \tag{13}$$

From equations (12) and (13) we obtain

$$(1 - \lambda^2)\nabla_Y\alpha = (\nabla_U\alpha)u(Y) + (\nabla_V\alpha)v(Y) , \tag{14}$$

$$(1 - \lambda^2)\nabla_Y\beta = (\nabla_U\beta)u(Y) + (\nabla_V\beta)v(Y) , \tag{15}$$

$$2\alpha(1 - \lambda^2)F(X, Y) = (u(Y)v(X) - v(Y)u(X))(\nabla_V\alpha - \nabla_U\beta) . \tag{16}$$

However, since the rank of f is $\geq 2n - 2$, equation (16) implies that $\alpha = 0$ and $\nabla_U\beta = 0$ if $n \neq 1$. Thus equation (12) becomes

$$(\nabla_X\beta)v(Y) = (\nabla_Y\beta)v(X) , \tag{12'}$$

or

$$(1 - \lambda^2)\nabla_X\beta = (\nabla_V\beta)v(X) . \tag{12''}$$

Applying ∇_X to equation (11′), and using the fact that $\alpha = 0$ and the Codazzi equation, we find that

$$(\nabla_X\beta)u(X) - (\nabla_Y\beta)u(X) - 2\beta F(X, Y) - 2F(KX, KY) = 0 . \tag{17}$$

Setting $Y = U$ and using (12″) we have that $2\beta^2\lambda = 2\beta\lambda - \nabla_V\beta$ so that $\beta =$ constant implies that $\beta = 0$ or $\beta = 1$.

Replace Y by fY in equation (17) and use equation (12″) to obtain

$$2(1 - \lambda^2)F(KX, KfY) = (\nabla_V\beta)(v(X)u(fY) - v(fY)u(X)) - 2\beta(1 - \lambda^2)F(X, fY) ,$$

that is,

$$-2(1 - \lambda^2)[G(KX, KY) - u(KX)u(KY) - v(KX)v(KY)] = \nabla_V\beta[\lambda v(X)v(Y) + \lambda u(X)u(Y)] - 2\beta(1 - \lambda^2)[G(X, Y) - u(X)u(Y) - v(X)v(Y)] ,$$

from which follows

$$(18) \qquad (1-\lambda^2)K^2 = (\beta^2-\beta)(u\otimes U + v\otimes V) + \beta(1-\lambda^2)I\ .$$

From (18) and a previous remark we see that if $\beta =$ constant then $K^2 = 0$ or $K^2 = I$. If $K^2 = 0$, then $K = 0$ since K is symmetric. In this case, M^{2n} is a totally geodisic submanifold of S^{2n+1} and hence $M^{2n} = S^{2n}$. In the case where $K^2 = I$, K gives an almost product structure on M^{2n}.

We have

$$\begin{aligned} k(fX, fY) &= G(KfX, fY) = G(Kf^2X, Y) \\ &= -G(KX, Y) + u(X)G(KU, Y) + v(X)G(KV, Y) \\ &= -k(X, Y) + \beta(u(X)v(Y) + v(X)u(Y))\ . \end{aligned}$$

Now since $k(U, U) + k(V, V) = 0$ and $G(U, V) = 0$, the last equation can be used to show that the trace of K is 0, that is, M^{2n} is a minimal hypersurface (note that this last conclusion holds whether or not $\beta =$ constant). In the case where $K^2 = I$, $\operatorname{tr} K = 0$ implies that the global distributions on M^{2n} given by $\frac{1}{2}(K + I)$ and $\frac{1}{2}(I - K)$ are both of dimension n.

Now to find the scalar curvature of M^{2n} by the Gauss equation, let $\tilde{R}$ and R denote the curvature tensors of g and G respectively. Then the Gauss equation is

$$(19) \qquad \begin{aligned} &\tilde{R}(\pi_*X, \pi_*Y, \pi_*Z, \pi_*W) \\ &\quad = R(X, Y, Z, W) - (k(Y, Z)k(X, W) - k(Y, W)k(X, Z))\ . \end{aligned}$$

Using (18) and the fact that S^{2n+1} is of constant curvature equal to 1, for $1 - \lambda^2 \neq 0$ we have

$$\begin{aligned} \bar{R}(X, Y) &= (2n-1)g(X, Y) \\ &\quad - \left[\beta g(X, Y) + \frac{\beta^2-\beta}{1-\lambda^2}(u(X)u(Y) + v(X)v(Y))\right]\ , \end{aligned}$$

where $\bar{R}$ is the Ricci tensor of G. From this it follows that the scalar curvature of M^{2n} is equal to $2n(2n-1) - \beta(2n-2) - 2\beta^2$, and therefore that $\beta =$ constant.

If we apply ∇_X to equation (19) and use the second Bianchi identity and $\operatorname{tr} K = 0$, then we obtain that

$$(\nabla_X K)Y + (\nabla_Y K)X = 0\ ,$$

and thus $\nabla_X K = 0$ by the Codazzi equation.

Therefore, if $\beta = 1$, the almost product structure K is decomposable. Hence by completeness, M^{2n} is a product $M^n \times \bar{M}^n$. Now we have, by equation (17),

$$
\begin{aligned}
(20)\quad & G(f(K \pm I)X, (K \pm I)Y) \\
& \qquad = F(KX, KY) + F(X, Y) \pm (F(KX, Y) + F(X, KY)) = 0 ,
\end{aligned}
$$

and, by equation (6),

$$
\begin{aligned}
\nabla_Y \nabla_X \lambda - d\lambda(\nabla_Y(X)) &= \nabla_Y(v(X) + k(U, X)) - (v(\nabla_Y X) + k(U, \nabla_Y X)) \\
&= (\nabla_Y v)X + k(\nabla_Y U, X) = -2\lambda G(X, Y) - 2G(fY, KX) .
\end{aligned}
$$

From equation (20) we see that if X and Y are both in the distribution $I + K$ or $I - K$, then $g(fY, KX) = 0$ so that

$$\nabla_Y \nabla_X \lambda - d\lambda(\nabla_Y X) = -2\lambda G(X, Y) .$$

Thus, M^n and $\bar{M}^n$ are both spheres of radius $1/\sqrt{2}$ by the lemma of Obata.

References

[1] D. E. Blair & G. D. Ludden, *On Intrinsic structures similar to those on S^{2n}*, Kōdai Math. Sem. Rep.

[2] D. E. Blair, G. D. Ludden & K. Yano, *Induced structures on submanifolds,* Kōdai Math. Sem. Rep. **22** (1970) 188–198.

[3] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere,* J. Math. Soc. Japan **14** (1962) 333–340.

[4] Y. Tashiro, *On contact structure of hypersurfaces in complex manifolds.* I, Tôhoku Math. J. **2** (1963) 62–78.

[5] K. Yano & M. Okumura, *On (f, g, u, v, λ)-structures,* Kōdai Math. Sem. Rep. **22** (1970) 401–423.

[6] ——, *On normal (f, g, u, v, λ)-structures on submanifolds of codimension 2 in a Euclidean space,* to appear.

MICHIGAN STATE UNIVERSITY