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# HYPERSURFACES WITH MEAN CURVATURE GIVEN BY AN AMBIENT SOBOLEV FUNCTION

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## Abstract

We consider $n$-hypersurfaces $\Sigma_j$ with interior $E_j$ whose mean curvature are given by the trace of an ambient Sobolev function $u_j \in W^{1,p}(\mathbb{R}^{n+1})$

$$\vec{\mathbf{H}}_{\Sigma_j} = u_j \nu_{E_j} \quad \text{on } \Sigma_j, \tag{0.1}$$

where $\nu_{E_j}$ denotes the inner normal of $\Sigma_j$. We investigate (0.1) when $\Sigma_j \to \Sigma$ weakly as varifolds and prove that $\Sigma$ is an integral $n$-varifold with bounded first variation which still satisfies (0.1) for $u_j \to u, E_j \to E$. $p$ has to satisfy

$$p > \frac{1}{2}(n+1)$$

and $p \geq \frac{4}{3}$ if $n = 1$. The difficulty is that in the limit several layers can meet at $\Sigma$ which creates cancellations of the mean curvature.

## 1. Introduction

**1.1. The problem.** We consider the reduced boundaries $\partial^* E_j$ of sets $E_j \subseteq \Omega$ of finite perimeter in an open set $\Omega \subseteq \mathbb{R}^{n+1}, n \geq 1$. We define the corresponding unit-density $n$-varifolds

$$V_j := \mathbf{v}(\partial^* E_j, 1), \quad \mu_{V_j} = \mathcal{H}^n \lfloor \partial^* E_j = |\nabla \chi_{E_j}| \tag{1.1}$$

in $\Omega$, with support

$$\Sigma_j := \operatorname{spt} V_j = \operatorname{spt} \mu_{V_j} = \overline{\partial^* E_j} \cap \Omega. \tag{1.2}$$

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The mean curvature of $V_j$ is assumed to be given by the trace on $\partial^* E_j$ of a function $u_j$, defined in the ambient space

$$\vec{\mathbf{H}}_{V_j} = u_j \nu_{E_j} \quad \text{on } \partial^* E_j, \tag{1.3}$$

where $\nu_{E_j} = \dfrac{\nabla \chi_{E_j}}{|\nabla \chi_{E_j}|}$ denotes the inner normal of $\partial^* E_j$. This is assumed to hold in a weak sense of the form

$$\delta V_j(\eta) = \int\limits_\Omega \operatorname{div}_V \eta \, d\mu_V = \int\limits_\Omega \operatorname{div}(u_j \eta) \chi_{E_j} \quad \text{for } \eta \in C_0^1(\Omega). \tag{1.4}$$

We assume that

$$u_j \in W^{1,p}(\Omega) \tag{1.5}$$

with

$$\frac{1}{2}(n+1) < p < (n+1) \quad \text{and} \quad p \geq \frac{4}{3} \text{ if } n = 1. \tag{1.6}$$

We specify the exponent of the Sobolev trace mapping on the boundary

$$1 - \frac{n+1}{p} =: -\frac{n}{s}.$$

(1.6) determines its range as

$$n < s < \infty, s \geq 2.$$

In particular, we have imposed $p \geq \frac{4}{3}$ if $n = 1$ to get $s \geq 2$.

We put

$$\alpha := 1 - \frac{n}{s} \in \, ]0, 1[.$$

If $\Sigma_j$ are smooth then (1.3), when satisfied in the classical way, implies (1.4) via integration by parts. Conversely, we will see below in Theorem 1.3 that (1.4) implies $\vec{\mathbf{H}}_{V_j} \in L^s_{\text{loc}}(\mu_{V_j})$ and (1.3).

We impose the bounds

$$\| u_j \|_{W^{1,p}(\Omega)}, \int\limits_\Omega |\nabla \chi_{E_j}| = \mu_{V_j}(\Omega) \leq \Lambda, \tag{1.7}$$

for some $\Lambda < \infty$ and want to investigate the limit behaviour of (1.3) when

$$u_j \to u \quad \text{weakly in } W^{1,p}(\Omega),$$
$$\chi_{E_j} \to \chi_E \quad \text{strongly in } L^1(\Omega),$$
$$V_j \to V \quad \text{weakly as varifolds.}$$

We put

$$\Sigma := \operatorname{spt} V = \operatorname{spt} \mu_V.$$

Our main theorem states that (1.3) still holds in the limit.

For notions in geometric measure theory and in BV-context, we refer to [13], [14], [16] and [33].

**Theorem 1.1.** *Under the above assumptions, $V$ is an integral $n$-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with locally bounded first variation and*

$$\vec{\mathbf{H}}_V \in L^s_{\text{loc}}(\mu_V). \tag{1.8}$$

*$E$ is a set of finite perimeter in $\Omega$ and its reduced boundary is contained in the support $\Sigma$ of $V$ that is*

$$\partial^* E \subseteq \Sigma = \operatorname{spt} V. \tag{1.9}$$

*The mean curvature vector satisfies*

$$\vec{\mathbf{H}}_V = u\, \nu_E \quad \mu_V\text{-almost everywhere on } \Sigma, \tag{1.10}$$

*where $\nu_E = \dfrac{\nabla \chi_E}{|\nabla \chi_E|}$ denotes the generalized normal of $\partial^* E$, which is put equal to $0$ outside of $\partial^* E$.*

Equation (1.9) is obtained from the lower semicontinuity of the perimeter observing for $B^{n+1}_\varrho(x_0) \subset\subset \Omega$ with $\mu_V(\partial B^{n+1}_\varrho(x_0)) = 0$ that

$$\int_{B^{n+1}_\varrho(x_0)} |\nabla \chi_E| \le \liminf_{j\to\infty} \int_{B^{n+1}_\varrho(x_0)} |\nabla \chi_{E_j}| = \lim_{j\to\infty} \mu_{V_j}(B^{n+1}_\varrho(x_0)) = \mu_V(B^{n+1}_\varrho(x_0)).$$

We remark that the closure of the reduced boundary of $E$ does not have to coincide with the support $\Sigma$ of $V$. This is due to cancellations when parts of $\Sigma_j$ meet which have opposite normals. Then the respective parts of $\Sigma$ do not separate its interior from its exterior, but can instead be considered as hidden boundary of $E$. The result of these cancellations depend on the number how many layers of $\Sigma_j$ meet to one layer on $\Sigma$.

In Theorem 1.1, two different limit procedures are involved which are not immediately compatible: we pass to the limit for $\Sigma_j$ in the sense of varifolds, but for $E_j$ we pass to the limit in the sense of currents.

The reduced boundary of $E_j$ can be considered as an $n$-current

$$\partial^* E_j = \partial [\![ E_j ]\!] \tag{1.11}$$

as well. We pass to the limit in (1.4), on the left side with convergence of varifolds, on the right with the convergence of currents which is given by $\chi_{E_j} \to \chi_E$ strongly in $L^1(\Omega)$, and get

$$\delta V(\eta) = \int\limits_\Omega \operatorname{div}(u\eta)\chi_E \quad \text{for } \eta \in C_0^1(\Omega). \tag{1.12}$$

At this stage, we assume that $V$ is integral, has bounded first variation with $\vec{\mathbf{H}}_V \in L^s_{\text{loc}}(\mu_V)$ and that $u \in L^s_{\text{loc}}(|\nabla \chi_E|)$. This will be justified later. Under these assumptions, we obtain

$$\vec{\mathbf{H}}_V \ \mu_V = u \ \nabla \chi_E. \tag{1.13}$$

Since $s > n$, we get from [33] Corollary 17.8 that $\theta^n(\mu_V)$ is upper semicontinuous. Therefore $\mu_V = \theta^n(\mu_V)\mathcal{H}^n \lfloor \Sigma$ for the $n$-dimensional density $\theta^n(\mu_V) \in \mathbb{N}$ of $\mu_V$ and $\nabla \chi_E = \nu_E \ \mathcal{H}^n \lfloor \partial^* E$. Hence, we conclude that

$$\begin{gathered} \theta^n(\mu_V) \ \vec{\mathbf{H}}_V = u \ \nu_E \quad \mu_V\text{-almost everywhere on } \partial^* E \\ \text{and} \\ \vec{\mathbf{H}}_V = 0 \quad \mu_V\text{-almost everywhere on } \Sigma - \partial^* E. \end{gathered} \tag{1.14}$$

Clearly, the density $\theta^n(\mu_V)$ corresponds to the number of layers which meet at $\Sigma$.

We distinguish three cases:

(i) $\theta^n(\mu_V) = 1$.
Here $\Sigma$ separates $E$ from its complement and $\Sigma = \partial^* E$. In this case, (1.14) already provides a proof of (1.10), of course, so far under the assumption that $\vec{\mathbf{H}}_V \in L^s_{\text{loc}}(\mu_V)$.

(ii) $\theta^n(\mu_V) \neq 1$ is odd.
For example if $u_j, u \geq 0$ the picture could look like


Since $\Sigma_j = \overline{\partial^* E_j}$ has interior $E_j$, the layers which meet have opposite mean curvature vector. Therefore one expects also cancellations for the mean curvature. For three layers, (1.14) gives

$$\vec{\mathbf{H}}_V = \frac{1}{3}\, u\, \nu_E.$$

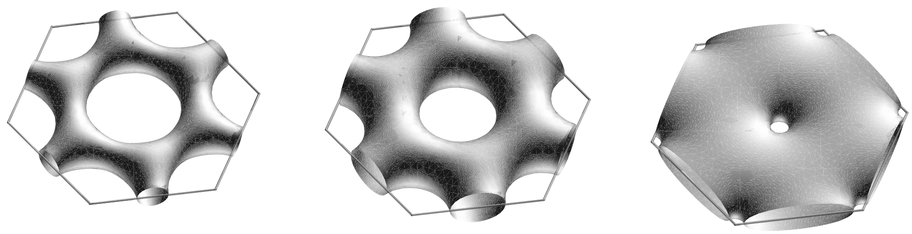
On the other hand, the opposite layers create an obstacle for each other if the mean curvature is not equal to zero. Actually, we will prove (1.10) by showing that

$$\vec{\mathbf{H}}_V,\ u = 0,$$

when $\theta^n(\mu_V) \neq 1$ is odd.

(iii) $\theta^n(\mu_V)$ is even.


Here $\Sigma$ does not separate $E$ from its complement and $\partial^* E = \emptyset$. It can be considered as a hidden boundary. (1.14) shows $\vec{\mathbf{H}}_V = 0$, in coincidence with (1.10). No claim is made whether $u = 0$ on $\Sigma$. Actually, this cannot be proved in dimensions $n \geq 2$ as the following example communicated by Karsten Große-Brauckmann; see [17] and [18] Theorem 2, shows. This is a one parameter family of surfaces with constant mean curvature equal to $H \equiv 1$. These



surfaces can periodically be extended to complete surfaces when reflected at the planes vertical at the hexagon edges. Then they converge to a double-density plane.

We summarize (i)–(iii) in the following supplement to Theorem 1.1.

**Theorem 1.2.** *In the situation of Theorem 1.1, we get for $\mu_V$-almost all $x \in \Sigma$ with integral density $\theta_0 := \theta^n(\mu_V, x) \in \mathbb{N}$ that:*

(i) *If $\theta_0 = 1$ then $\vec{\mathbf{H}}_V(x) = u(x)\ \nu_E(x)$.*

(ii) *If $\theta_0 \neq 1$ is odd, then $\vec{\mathbf{H}}_V(x),\ u(x) = 0$.*

*Moreover, in both cases* (i) *and* (ii), $\theta^{n+1}(E, x) = \frac{1}{2}$.

(iii) *If $\theta_0$ is even, then $\vec{\mathbf{H}}_V(x) = 0$,*
*and either*

$$\theta^{n+1}(E, x) = 1 \quad \textit{and} \quad u(x) \leq 0,$$

*or*

$$\theta^{n+1}(E, x) = 0 \quad \textit{and} \quad u(x) \geq 0.$$

The main task for proving Theorem 1.1 is to establish (1.10). In order to do this, we will approximate $V$ locally by a $C^{1,1}$-graph $M$ near points $x_0 \in \Sigma$ which have a tangent plane $T_{x_0}V = \theta_0 P$, for some $n$-hyperplane $P \in G(n+1, n)$, and which have full density

$$\lim_{\varrho \downarrow 0} \frac{\mu_V([\theta^n(\mu_V) = \theta_0] \cap B_\varrho^{n+1}(x_0))}{\mu_V(B_\varrho^{n+1}(x_0))} = 1$$

with respect to $\mu_V$ in the set of points which have the same n-dimensional density $\theta^n(\mu_V)$ as $x_0$. The $C^{1,1}$-approximation has to be of second order in the sense that not only $TV = TM$, but also

$$\vec{\mathbf{H}}_V = \vec{\mathbf{H}}_M \tag{1.15}$$

on a set with positive density near $x_0$. Such a second order approximation seems only possible if the varifold itself behaves in a second order way on a reasonably large set, more precisely, if the tilt-excess of $V$, defined for any $T \in G(n+1,n)$ by

$$\operatorname{tiltex}_V(x_0,\varrho,T) := \varrho^{-n} \int\limits_{B_\varrho^{n+1}(x_0)} \| \, T_x V - T \, \|^2 \; d\mu_V(x),$$

has a quadratic decay

$$\operatorname{tiltex}_V(x_0,\varrho,T_{x_0}V) = O_{x_0}(\varrho^2). \tag{1.16}$$

Tilt-excess decay estimates for varifolds, or more precisely height-excess decay estimates, where

$$\operatorname{heightex}_V(x_0,\varrho,T) := \varrho^{-n-2} \int\limits_{B_\varrho^{n+1}(x_0)} |\operatorname{dist}(x-x_0,T)|^2 \; d\mu_V(x),$$

were established in Allard's Regularity Theorem for unit-density, see [1] Theorem 8.16 where it is proved without specifying the respectives planes that

$$\operatorname{heightex}_V(x_0,\tau\varrho) \le \tau^{2\alpha} \operatorname{heightex}_V(x_0,\varrho)$$

and were extended to the higher density case in [3] Theorem 5.6 to

$$\operatorname{heightex}_V(x_0,\tau\varrho) \le C\tau^2 \operatorname{heightex}_V(x_0,\varrho).$$

Moreover, it is stated in [3] on p. 157 when combined with Lemma 5.5 that

$$\operatorname{tiltex}_V(x_0,\varrho,T_{x_0}V) = o_{x_0}(\varrho^{2-\varepsilon})$$

for $\mu_V$-almost all $x_0$ if $\vec{\mathbf{H}}_V \in L^2_{\mathrm{loc}}(\mu_V)$, which is slightly weaker than (1.16). For a complete statement of all assumptions for the above two estimates, see [1] and [3].

In order to establish (1.16), we will instead apply tools from fully non-linear elliptic equations. Assuming $x_0 = 0$ and $P = \mathbb{R}^n \times \{0\}$, we consider the upper height function

$$\varphi_+(y) := \sup\{t \mid (y,t) \in \Sigma \cap B_{\varrho_0}^{n+1}(0)\} \tag{1.17}$$

for $y \in B^n_{\varrho_0}(0) \subseteq \mathbb{R}^n$ and some $\varrho_0 > 0$ small, and we will show that $\varphi_+$ is a $W^{2,s}$-viscosity subsolution of the inhomogeneous minimal surface equation

$$-\nabla \left( \frac{\nabla \varphi_+}{\sqrt{1 + |\nabla \varphi_+|^2}} \right) \leq \pm u(., \varphi_+).$$

For the notion of viscosity solutions, we refer to [10], [6] and [7].

Then we will apply Caffarelli's and Trudinger's result in [5] respectively in [35] (see also [6] Lemma 7.8 and [7]), saying that subsolutions of uniformly elliptic equations with right-hand side in $L^n$, are almost everywhere touched from above by paraboloids or likewise have second order superdifferentials almost everywhere. This yields quadratic decay of the distance $\operatorname{dist}(x - x_0, T_{x_0} V)$, in particular it establishes quadratic decay of the height-excess and then (1.16) by standard results. Actually, we will apply this result to a sup-convolution of order one of $\varphi_+$ in order to get a uniformly elliptic equation.

Finally, we remark that integrality of $V$ and (1.8) are obtained from Allard's Integral Compactness Theorem; see [1] Theorem 6.4 or [33] Remark 42.8, when we establish bounds on $\vec{\mathbf{H}}_{V_j}$ in $L^s_{\text{loc}}(\mu_{V_j})$. These bounds are provided by combining (1.4), (1.7) and the following theorem which is new to our knowledge.

**Theorem 1.3.** *Let $W$ be an $n$-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$, $v \in W^{1,p}(\Omega)$, $\frac{1}{2}(n+1) < p < (n+1)$ and $\chi_F \in BV(\Omega)$ for some $F \subseteq \Omega$ satisfying*

$$\delta W(\eta) = \int_\Omega \operatorname{div}(v\eta)\chi_F \quad \text{for } \eta \in C^1_0(\Omega), \tag{1.18}$$

$$|\nabla \chi_F| \leq \mu_W, \tag{1.19}$$

$$\| \, v \, \|_{W^{1,p}(\Omega)}, \; \mu_W(\Omega) \leq \Lambda, \tag{1.20}$$

*for some $\Lambda < \infty$.*

*Then $W$ has locally bounded first variation and satisfies*

$$\| \, \vec{\mathbf{H}}_W \, \|_{L^s(\mu_W \lfloor B^{n+1}_d(x_0))}, \; \| \, v \, \|_{L^s(\mu_W \lfloor B^{n+1}_d(x_0))} \leq C_{n,p}(d) \; \Lambda^{1+\frac{1}{s}} \tag{1.21}$$

*for any $B^{n+1}_{2d}(x_0) \subseteq \Omega$ with $d > 0$.*

Applying this theorem to $W = V_j, v = u_j$ and $F = E_j$ satisfying (1.4), we see that $\vec{\mathbf{H}}_{V_j} \in L^s_{\text{loc}}(\mu_{V_j})$, and (1.3) holds via integration by parts.

**1.2. The BV-context and applications.** We have seen from (1.14) that in case of unit-density the conclusion of Theorem 1.1 is easily obtained, and actually it is well known in this case in the BV-context. A condition which ensures the varifold $V$ to have unit-density is preservation of the perimeter of the set $E_j$ that is

$$\int_\Omega |\nabla \chi_E| = \lim_{j\to\infty} \int_\Omega |\nabla \chi_{E_j}|. \tag{1.22}$$

Indeed, from weak varifold convergence, which preserves mass always, and from the lower semicontinuity of the perimeter, we get for $B_\varrho^{n+1}(x_0) \subset\subset \Omega$, satisfying $\mu_V(\partial B_\varrho^{n+1}(x_0)), \int_{\partial B_\varrho^{n+1}(x_0)} |\nabla \chi_E| = 0$, that

$$\begin{aligned} \mu_V(B_\varrho^{n+1}(x_0)) &= \lim_{j\to\infty} \mu_{V_j}(B_\varrho^{n+1}(x_0)) \\ &= \lim_{j\to\infty} \int_{B_\varrho^{n+1}(x_0)} |\nabla \chi_{E_j}| = \int_{B_\varrho^{n+1}(x_0)} |\nabla \chi_E|, \end{aligned}$$

hence

$$\mathcal{H}^n \lfloor \partial^* E = |\nabla \chi_E| = \mu_V$$

and $V$ has unit-density. Plugging this into the first variation formula for $V$ (1.12), we get

$$\int_\Omega (\operatorname{div} \eta - \nu_E D\eta \nu_E)|\nabla \chi_E| = \int_\Omega \operatorname{div}(u\eta)\chi_E \quad \text{for } \eta \in C_0^1(\Omega, \mathbb{R}^{n+1}). \tag{1.23}$$

This coincides with the weak form (1.4) of (1.3) and can be considered as a weak form of (1.10) in the BV-context; see [22]. (1.23) does not make use of varifold notions. Also the convergence procedure outlined above can be carried out without referring to varifolds, but instead using a theorem of Reshetnyak in [29], of course still under the assumption (1.22). On the other hand, for a set of finite perimeter $E$ satisfying (1.23), Theorem 1.3 implies that its reduced boundary, considered as a unit-density varifold $V$, has bounded first variation and $\vec{\mathbf{H}}_V \in L^s_{\text{loc}}(\mu_V)$, hence rigorously satisfies (1.10).

The above convergence procedure was applied in BV-context by Luckhaus in [22] to get solutions for the Stefan problem with Gibbs-Thomson law, which reads

$$\begin{aligned} &\partial_t(u+\varphi) - \Delta u = f, \\ &H_{\partial[\varphi=1]} = u, \end{aligned} \tag{1.24}$$

and where $\varphi \in BV$ takes only the values $\pm 1$. The second equation imposes that the melting temperature $u$ on the free boundary $\partial[\varphi = 1]$ equals the mean curvature of the free boundary. This is called Gibbs-Thomson law. In [22], a time-discrete approximation was set up, and due to minimizations at each time step, it was established that the perimeter is preserved when passing to the limit.

Time-discrete approximations with a minimization at each time step were also set up for mean curvature flow in BV-setting and for the two-phase Mullins-Sekerka problem in [24], and for the multiphase Mullins-Sekerka problem in [4]. But in these problems, preservation of perimeter could not be shown, and existence was only obtained conditionally under the assumption of (1.22).

It was one of the original motivations for this article to give tools for the justification of the above convergence procedures without preservation of perimeter. A second motivation was a parabolic approximation of the Stefan problem with Gibbs-Thomson law which we describe now.

The phase-field equations

$$\begin{aligned} &\partial_t(u_\varepsilon+\varphi_\varepsilon) - \Delta u_\varepsilon = f, \\ &\alpha(\varepsilon)\varepsilon\partial_t\varphi_\varepsilon - 2\varepsilon\Delta\varphi_\varepsilon + \tfrac{1}{\varepsilon}W'(\varphi_\varepsilon) = u_\varepsilon, \end{aligned} \tag{1.25}$$

where $W(t) := (t^2-1)^2$ is a double-well potential, are proposed in [8] to approximate the Stefan problem with Gibbs-Thomson law, when $\alpha(\varepsilon) \to 0$ for $\varepsilon \to 0$ or $\alpha(\varepsilon) = 0$ for the quasi-stationary phase-field equations. Here, a parabolic system is used to approximate a sharp interface problem.

Existence of solutions for the quasi-stationary phase-field equations and convergence of these solutions to the Stefan problem with Gibbs-Thomson law when $\varepsilon \to 0$ was proved by Plotnikov and Starovoitov in [27] and by the author in [31]. Since energy is preserved for the quasi-stationary phase-field equations when $\varepsilon \to 0$, the following theorem of Luckhaus and Modica in [23] was applicable.

**Theorem** ([23]). *Let $u_\varepsilon, \varphi_\varepsilon$ satisfy*

$$-2\varepsilon\Delta\varphi_\varepsilon + \tfrac{1}{\varepsilon}W'(\varphi_\varepsilon) = u_\varepsilon \quad \text{in } \Omega,$$

$$u_\varepsilon \to u \quad \text{weakly in } W^{1,2}(\Omega),$$

$$\varphi_\varepsilon \to \varphi \quad \text{strongly in } L^1(\Omega),$$

*and suppose the energy is preserved, that is*

$$F_\varepsilon(\varphi_\varepsilon) := \int_\Omega \varepsilon|\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}W(\varphi_\varepsilon) \to \frac{4}{3}\int_\Omega |\nabla\varphi|, \tag{1.26}$$

*then the Gibbs-Thomson law is valid in the limit*

$$\frac{4}{3}\int_\Omega (\operatorname{div}\eta - \nu_\varphi D\eta\nu_\varphi)|\nabla\varphi| = \int_\Omega \operatorname{div}(u\eta)\varphi \quad \text{for } \eta \in C_0^1(\Omega, \mathbb{R}^{n+1}),$$

*where $\nu_\varphi = \frac{\nabla\varphi}{|\nabla\varphi|}$ is a Radon-Nikodym derivative.*

Clearly, (1.26) corresponds to (1.22).

This theorem does not remain true when the assumption of energy preservation is dropped, as it was shown by the author in [30]. The reason is that the BV-statement (1.23) breaks down because hidden boundaries are lost in the BV-setting. Since the varifold $V$ keeps also the information of hidden boundaries, it seemed quite natural to investigate this situation in measure-theoretic context. Theorem 1.1 encourages us to pose the question whether a convergence theorem for the above described elliptic approximation of the Gibbs-Thomson law is valid for the varifold $V$ when energy is not preserved. Such a theorem would yield applications to the phase-field equations, where, in contrast to the quasi-stationary case, the energy need not be preserved when $\varepsilon \to 0$.

Finally, we want to mention references where measure-theoretic methods were already successfully applied to prove approximations of sharp interface problems by elliptic or parabolic regularizations in cases without preservation of mass. In all four of the following articles, no assumptions were made on existence or regularity for the solutions of the limit problem.

Ilmanen has proved in [20] that solutions of the Allen-Cahn equation (this is the second equation in (1.25))

$$\varepsilon\partial_t\varphi_\varepsilon - 2\varepsilon\Delta\varphi_\varepsilon + \frac{1}{\varepsilon}W'(\varphi_\varepsilon) = 0,$$

converge to a Brakke-solution of mean curvature flow.

In [26] and [19], Hutchinson, Padilla and Tonegawa have proved the theorem of Luckhaus and Modica for measure theoretic limits without the assumption of energy preservation for constant right-hand sides $u_\varepsilon = \lambda_\varepsilon$.

Convergence of solutions of a variant of the phase-field equations (1.25) with $\alpha(\varepsilon) = 1$ to weak solutions of the Mullins-Sekerka problem with kinetic undercooling, also called the Stefan problem with kinetic undercooling, was proved by Soner in [34].

The fourth-order counterpart of the Allen-Cahn equation is the Cahn-Hilliard equation

$$\partial_t \varphi_\varepsilon = \Delta \psi_\varepsilon,$$

$$\psi_\varepsilon = -2\varepsilon \Delta \varphi_\varepsilon + \tfrac{1}{\varepsilon} W'(\varphi_\varepsilon).$$

Convergence of solutions of the Cahn-Hilliard equation to weak solutions of the Mullins-Sekerka problem was proved by Chen in [9].

**1.3. Notation.** We fix some notations.

$G(n+1,n)$ denotes the set of unoriented $n$-planes in $\mathbb{R}^{n+1}$. For any set $Q \subseteq \mathbb{R}^{n+1}$, we put $G(Q) := Q \times G(n+1,n)$. In particular, we fix $P := \mathbb{R}^n \times \{0\}$ and $\pi : \mathbb{R}^{n+1} \to P$ the orthogonal projection onto $P$. Usually, we will not distinguish between the plane, its orthogonal projection and the corresponding matrix.

Open balls in dimension $n$ and $n+1$ will be denoted by $B^n_\varrho(x) \subseteq \mathbb{R}^n$ and by $B^{n+1}_\varrho(x) \subseteq \mathbb{R}^{n+1}$. The cylinder over $P$ is $Z_\varrho := B^n_\varrho(0) \times ]-\varrho, \varrho[$.

$\mathcal{L}^n$ and $\mathcal{L}^{n+1}$ are the Lebesgue-measures in dimension $n$ and $n+1$, respectively. $\mathcal{H}^n$ and $\mathcal{H}^0$ are the $n$ and 0-dimensional Hausdorff-measure in any metric space. The volume of the $n-$ and $n+1$-dimensional unit-ball is abbreviated by $\omega_n := \mathcal{L}^n(B^n_1(0))$ and $\omega_{n+1} := \mathcal{L}^{n+1}(B^{n+1}_1(0))$, respectively.

For two real-valued functions, we put $[f > g] := \{x \mid f(x) > g(x)\}$ and similarly for analogous expressions.

$S(n)$ denotes the set of all symmetric $n \times n$-matrices.

The density of a set $Q \subseteq \mathbb{R}^{n+1}$ in $x \in \mathbb{R}^{n+1}$ with respect to a Radon measure $\mu$ on $\mathbb{R}^{n+1}$ is defined by

$$\theta(\mu, Q, x) := \lim_{\varrho \to 0} \frac{\mu(B^{n+1}_\varrho(x) \cap Q)}{\mu(B^{n+1}_\varrho(x))}$$

if this limit exists.

In contrast, we define the $k$-dimensional density for $k \in \{n, n+1\}$ by

$$\theta^k(\mu, Q, x) := \lim_{\varrho \to 0} \frac{\mu(B_\varrho^{n+1}(x) \cap Q)}{\omega_k \varrho^k}$$

if this limit exists.

We will denote any module of continuity by $\omega(\varrho)$ that means $\omega(\varrho) \to 0$ for $\varrho \to 0$.

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## 2. Monotonicity formula and bounded first variation

The aim of this section is to prove Theorem 1.3. To this end, we first establish a monotonicity formula from the first variation formula (1.18). For the monotonicity formula for varifolds with bounded first variation, we refer to [1] §5.1 and [33] §17.

We have the general assumptions for this section that

$$\begin{aligned} &W \text{ is a } n-\text{varifold in } \Omega \subseteq \mathbb{R}^{n+1}, \\ &v \in W^{1,p}(\Omega), \quad \tfrac{1}{2}(n+1) < p < (n+1), \\ &\chi_F \in BV(\Omega), \text{ for some } F \subseteq \Omega. \end{aligned} \tag{2.1}$$

We specify the Sobolev mappings for any ball $B_d^{n+1} \subseteq \mathbb{R}^{n+1}$ and deduce the ranges of the Sobolev-exponents from (2.1):

$$W^{1,p}(B_d^{n+1}) \hookrightarrow L^q(B_d^{n+1}) \quad \text{for} \quad 1 - \frac{n+1}{p} = -\frac{n+1}{q} \quad \text{and} \quad n+1 < q < \infty, \tag{2.2}$$

and for $0 < \varrho < d$

$$W^{1,p}(B_d^{n+1}) \to L^s(\partial B_\varrho^{n+1}) \quad \text{for} \quad 1 - \frac{n+1}{p} = -\frac{n}{s} \quad \text{and} \quad n < s < \infty. \tag{2.3}$$

We determine bounds for the norm of the mappings in (2.2) and (2.3). Extending $v \in W^{1,p}(B_d^{n+1})$ to $\tilde{v} \in W_0^{1,p}(B_{2d}^{n+1}) \subseteq W^{1,p}(\mathbb{R}^{n+1})$, we get after rescaling in $d$ that

$$\begin{aligned} \parallel \nabla \tilde{v} \parallel_{L^p(\mathbb{R}^{n+1})} &\leq C(\parallel \nabla v \parallel_{L^p(B_d^{n+1})} + d^{-1} \parallel v \parallel_{L^p(B_d^{n+1})}) \\ &\leq C_{n,p} \min(1,d)^{-1} \parallel v \parallel_{W^{1,p}(B_d^{n+1})} . \end{aligned}$$

Since $q$ and $s$ are the Sobolev-exponents, the norms of the Sobolev mappings depend only on $n$ and $p$, but not on $\varrho$. Therefore,

$$\parallel v \parallel_{L^q(B_d^{n+1})}, \parallel v \parallel_{L^s(\partial B_\varrho^{n+1})} \leq C_{n,p} \min(1,d)^{-1} \parallel v \parallel_{W^{1,p}(B_d^{n+1})} . \tag{2.4}$$

Finally, we put

$$\alpha := 1 - \frac{n}{s} \in ]0,1[$$

and assume the following bounds

$$\parallel v \parallel_{W^{1,p}(\Omega)}, \mu_W(\Omega) \leq \Lambda \tag{2.5}$$

for some $\Lambda < \infty$.

**Lemma 2.1.** *Under the above assumptions, let the first variation of the $n$-varifold $W$ be given by*

$$\delta W(\eta) = \int_\Omega \operatorname{div}(v\eta)\chi_F \quad \text{for } \eta \in C_0^1(\Omega). \tag{2.6}$$

*Then for $x_0 \in \Omega, 0 < \varrho < d := \operatorname{dist}(x_0, \partial\Omega)$ the function*

$$(\varrho \mapsto \varrho^{-n}\mu_W(B_\varrho^{n+1}(x_0)) + C_{n,p}\min(1,d)^{-1}\Lambda\varrho^\alpha) \tag{2.7}$$

*is nondecreasing, where $C_{n,p}$ depends only on $n$ and $p$.*

*Proof.* The proof follows [33] §17.

We put $r := |x - x_0|$, choose $0 < \varrho < d$ and test (2.6) with

$$\eta_\varrho(x) := \Phi(r/\varrho)(x - x_0),$$

where we choose $\Phi \in C^1([0,\infty[)$ satisfying

$$\Phi' \leq 0, \qquad \Phi(t) \equiv 1 \quad \text{for } t \leq \frac{1}{2}, \quad \Phi(t) \equiv 0 \quad \text{for } t \geq 1.$$

Putting

$$I(\varrho) := \int\limits_{\Omega} \Phi(r/\varrho) d\mu_W, \quad J(\varrho) := \int\limits_{G(\Omega)} \Phi(r/\varrho) |S^{\perp}(\nabla r)|^2 dW(x,S),$$

where $S^{\perp}$ denotes the orthogonal projection of $\mathbb{R}^{n+1}$ onto the orthogonal complement of $S$, we arrive, as in [33] §17, at

$$\begin{aligned} \frac{d}{d\varrho}(\varrho^{-n} I(\varrho)) =& \varrho^{-n} J'(\varrho) - \varrho^{-n-1} \delta W(\eta_\varrho) \\ \geq& - \varrho^{-n-1} \int\limits_{\Omega} \operatorname{div}(v\eta_\varrho)\chi_F, \end{aligned} \tag{2.8}$$

since $J'(\varrho) = -\int\limits_{G(\Omega)} \frac{r}{\varrho^2} \Phi'(r/\varrho) |S^{\perp}(\nabla r)|^2 dW(x,S) \geq 0$, as $\Phi' \leq 0$, and where we have used (2.6).

We estimate the integrand of the right-hand side for $x \in B^{n+1}_{\varrho}(x_0)$ by

$$\begin{aligned} |\operatorname{div}(v\eta_\varrho)\chi_F|(x) \leq& |\nabla v(x)||x - x_0|\Phi(r/\varrho) \\ &+ |v(x)|(n+1)\Phi(r/\varrho) + |v(x)| \frac{(x-x_0)^2}{\varrho r} |\Phi'(r/\varrho)| \\ \leq& \varrho |\nabla v(x)| + (n+1)|v(x)| + |v(x)||\Phi'(r/\varrho)|, \end{aligned}$$

and get

$$\begin{aligned}
&\left| \varrho^{-n-1} \int\limits_\Omega \operatorname{div}(v\eta_\varrho)\chi_F \right| \\
&\quad \le \varrho^{-n} \int\limits_{B_\varrho^{n+1}(x_0)} |\nabla v| + \varrho^{-n-1}(n+1) \int\limits_{B_\varrho^{n+1}(x_0)} |v| + \varrho^{-n-1} \\
&\qquad \cdot \int\limits_0^\varrho |\Phi'(r/\varrho)| \int\limits_{\partial B_r^{n+1}(x_0)} |v| d\mathcal{H}^n dr \qquad (2.9)\\
&\quad \le C_{n,p}\Big( \varrho^{(1-(n+1)/p)} \parallel \nabla v \parallel_{L^p(B_\varrho)} + \varrho^{-(n+1)/q} \parallel v \parallel_{L^q(B_\varrho)} \\
&\qquad + \varrho^{-n/s} \sup_{0\le r\le \varrho} \parallel v \parallel_{L^s(\partial B_r)} \Big) \\
&\quad \le C_{n,p} \min(1,d)^{-1} \varrho^{\alpha-1} \parallel v \parallel_{W^{1,p}(B_d^{n+1}(x_0))} \\
&\quad \le C_{n,p} \min(1,d)^{-1} \Lambda \varrho^{\alpha-1},
\end{aligned}$$

since $\int_0^\varrho |\Phi'(r/\varrho)| dr = \varrho$, and using (2.4).

Plugging (2.9) in (2.8) and letting $\Phi \nearrow \chi_{[0,1[}$, we get in the distributional sense that

$$\frac{d}{d\varrho}(\varrho^{-n}\mu_W(B_\varrho^{n+1}(x_0))) \ge -C_{n,p}\min(1,d)^{-1}\Lambda\varrho^{\alpha-1}.$$

Integrating yields (2.7). q.e.d.

For the proof of Theorem 1.3, we will use the following result which is part of a theorem of Meyers and Ziemer; see [25] Theorem 4.7 and [37] Theorem 5.12.4.

**Theorem** ([25]). *Let $\mu$ be a positive Radon measure on $\mathbb{R}^{n+1}$ satisfying*

$$M(\mu) := \sup_{x\in\mathbb{R}^{n+1},\varrho>0} \varrho^{-n}\mu(B_\varrho^{n+1}(x)) < \infty. \qquad (2.10)$$

*Then*

$$\left| \int\limits_{\mathbb{R}^{n+1}} \varphi d\mu \right| \le C_n M(\mu) \parallel \nabla\varphi \parallel_{L^1(\mathbb{R}^{n+1})} \quad \textit{for all } \varphi \in C_0^1(\mathbb{R}^{n+1}), \qquad (2.11)$$

*where $C_n$ depends only on $n$.*

*Proof of Theorem* 1.3. From Lemma 2.1, we see for $B^{n+1}_{2d}(x_0) \subseteq \Omega$ with $d > 0$ and $\mu = \mu_W \lfloor B^{n+1}_d(x_0)$ that

$$M(\mu_W \lfloor B^{n+1}_d(x_0)) \leq 2^n(d^{-n} + C_{n,p}\min(1,d)^{-1}d^\alpha)\Lambda,$$

hence from the theorem above

$$\begin{aligned}\| v \|_{L^s(\mu_W \lfloor B^{n+1}_d(x_0))} &\leq C_{n,p}(d) M(\mu_W \lfloor B^{n+1}_d(x_0))^{\frac{1}{s}} \| v \|_{W^{1,p}(B^{n+1}_d(x_0))} \\ &\leq C_{n,p}(d)\Lambda^{1+\frac{1}{s}}.\end{aligned}$$

Moreover, since $|\nabla \chi_F| \leq \mu_W$, we get $v \in L^s_{\text{loc}}(|\nabla \chi_F|)$ and for $\eta \in C^1_0(B^{n+1}_d(x_0))$

$$\begin{aligned}|\delta W(\eta)| &= \left| \int_\Omega v \eta \nabla \chi_F \right| \leq \| v \|_{L^s(|\nabla \chi_F| \lfloor B^{n+1}_d(x_0))} \| \eta \|_{L^{s^*}(|\nabla \chi_F| \lfloor B^{n+1}_d(x_0))} \\ &\leq C_{n,p}(d)\Lambda^{1+\frac{1}{s}} \| \eta \|_{L^{s^*}(\mu_W \lfloor B^{n+1}_d(x_0))}\end{aligned}$$

with $\frac{1}{s} + \frac{1}{s^*} = 1$, hence

$$\| \vec{\mathbf{H}}_W \|_{L^s(\mu_W \lfloor B^{n+1}_d(x_0))} \leq C_{n,p}(d)\Lambda^{1+\frac{1}{s}},$$

and Theorem 1.3 follows. q.e.d.

We draw several conclusions.

(i) From Theorem 1.3, (1.4) and (1.7), we get the uniform estimate

$$\| \vec{\mathbf{H}}_{V_j} \|_{L^s(\mu_W \lfloor B^{n+1}_d(x_0))} \leq C_{n,p}(d)\Lambda^{1+\frac{1}{s}}. \tag{2.12}$$

As pointed out before the statement of Theorem 1.3, this implies that $V$ is integral with locally bounded first variation and that (1.8) holds. Further it gives the justification for the argument leading to (1.13) and (1.14) which are proved now.

(ii) Since $\vec{\mathbf{H}}_V \in L^s_{\text{loc}}(\mu_V)$, we get from [33] Corollary 17.8 that $\theta^n(\mu_V)$ is upper semicontinuous, and since $V$ is integral, we get $\mu_V = \theta^n(\mu_V)\mathcal{H}^n \lfloor \Sigma$ and $\theta^n(\mu_V) \geq 1$ on $\Sigma$. Therefore, $\mathcal{H}^n(\Sigma \cap \Omega') < \infty$ for any $\Omega' \subset\subset \Omega$, hence

$$\mathcal{L}^{n+1}(\Sigma) = 0.$$

On the other hand, we have already seen after the statement of Theorem 1.1 that $|\nabla\chi_E| \leq \mu_V$ and $\partial^* E \subseteq \Sigma = \operatorname{spt} V$. Therefore, we may assume after changing $E$ on a set of measure zero; see [16] Proposition 3.1, that

$$E \text{ is open }, \quad \partial E = \overline{\partial^* E} \subseteq \Sigma = \operatorname{spt} V. \tag{2.13}$$

Likewise, we may assume that

$$E_j \text{ is open }, \quad \partial E_j = \overline{\partial^* E_j} = \Sigma_j = \operatorname{spt} V_j. \tag{2.14}$$

Indeed, since $\mu_{V_j} = |\nabla\chi_{E_j}| = \mathcal{H}^n \lfloor \partial^* E_j$ and $\theta^n(\mu_{V_j}) \geq 1$ on $\partial E_j$, as $\theta^n(\mu_{V_j})$ is upper semicontinuous by [33] Corollary 17.8 and $\vec{\mathbf{H}}_{V_j} \in L^s_{\text{loc}}(\mu_W)$ by Theorem 1.3, we get additionally that $\mathcal{H}^n(\partial E_j - \partial^* E_j) = 0$ and

$$|\nabla\chi_{E_j}| = \mathcal{H}^n \lfloor \partial E_j = \mu_{V_j}. \tag{2.15}$$

From (2.12) and Allard's Regularity Theorem; see [1] Theorem 8.16 or [33] Theorem 23.1, we see that $\partial E_j$ is smooth on the reduced boundary, since $\theta^n(\mu_{V_j}) = 1$ on $\partial^* E_j$; see [16] Theorem 3.8 or [33] Theorem 14.3. Hence $\partial E_j$ is smooth $\mu_{V_j}$-almost everywhere.

(iii) From the monotonicity formula in Lemma 2.1 (2.7) or likewise from (2.12), we get that

$$\operatorname{spt} V_j \to \operatorname{spt} V \quad \text{locally in Hausdorff-distance} \tag{2.16}$$

in the sense that

$$x_j \in \operatorname{spt} V_j, x_j \to x_0 \in \Omega \Rightarrow x_0 \in \operatorname{spt} V,$$

$$\forall (x_0 \in \operatorname{spt} V) : \exists (x_j \in \operatorname{spt} V_j) : x_j \to x_0.$$

Since the monotonicity formula rescales when blowing-up, we get for $x_0 \in \operatorname{spt} V$ with tangent plane $T_{x_0} V$ that

$$\begin{aligned} &\operatorname{spt} V_{x_0,\varrho} \to T_{x_0} V \\ &\qquad \text{locally in Hausdorff-distance for } \varrho \to 0, \end{aligned} \tag{2.17}$$

where $V_{x_0,\varrho} := \zeta_{x_0,\varrho,\#} V$ and $\zeta_{x_0,\varrho}(x) := \varrho^{-1}(x - x_0)$.

## 3. Lipschitz-Approximation

In this section, we apply the Lipschitz-Approximation Theorem due to Brakke; see [3] Theorem 5.4, to get some preliminary information on the varifold $V$.

We fix some notions.

**Definition 3.1.** Let $W$ be a $n$-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$. We define the tilt, tilt-excess and the Lipschitz-approximation constant for $x_0 \in \Omega, \varrho > 0, B_\varrho^{n+1}(x_0) \subseteq \Omega$ and $T \in G(n+1,n)$ by

$$\begin{aligned}
\operatorname{tilt}_W(x_0,\varrho,T) :=& \varrho^{-n-2} \int\limits_{B_\varrho^{n+1}(x_0)} \operatorname{dist}(x-x_0,T)^2 d\mu_W(x),\\
\operatorname{tiltex}_W(x_0,\varrho,T) :=& \varrho^{-n} \int\limits_{G(B_\varrho^{n+1}(x_0))} \parallel S-T \parallel^2 dW(x,S),\\
\operatorname{lipapp}_W(x_0,\varrho,T) :=& \operatorname{tilt}_W(x_0,\varrho,T) + \operatorname{tiltex}_W(x_0,\varrho,T)\\
&+ \varrho^{2-n} \int\limits_{B_\varrho^{n+1}(x_0)} |\vec{\mathbf{H}}_W|^2 d\mu_W.
\end{aligned}$$

We put $\operatorname{lipapp}_W(x_0,\varrho,T) = \infty$ if $\vec{\mathbf{H}}_W \notin L^2(\mu_W \lfloor B_\varrho^{n+1}(x_0))$.

We see for $x_0 \in \Omega, 0 < \varrho < \varrho_0$ with $B_{2\varrho_0}^{n+1}(x_0) \subseteq \Omega$ that

$$\begin{aligned}
&\varrho^{2-n} \int\limits_{B_\varrho^{n+1}(x_0)} |\vec{\mathbf{H}}_{V_j}|^2 d\mu_{V_j}\\
&\qquad \le \varrho^{2-n} \mu_{V_j}\left(B_\varrho^{n+1}(x_0)\right)^{1-\frac{2}{s}} \left( \int\limits_{B_{\varrho_0}^{n+1}(x_0)} |\vec{\mathbf{H}}_{V_j}|^s d\mu_{V_j} \right)^{\frac{2}{s}}\\
&\qquad \le C_{n,p}(\Lambda,\varrho_0)\varrho^{2\alpha}
\end{aligned}$$

since $s \ge 2$ and where we have used (2.7) and (2.12). Therefore

$$\varrho^{2-n} \int\limits_{B_\varrho^{n+1}(x_0)} |\vec{\mathbf{H}}_V|^2 d\mu_V, \varrho^{2-n} \int\limits_{B_\varrho^{n+1}(x_0)} |\vec{\mathbf{H}}_{V_j}|^2 d\mu_{V_j} \le C_{n,p}(\Lambda,\varrho_0)\varrho^{2\alpha} \tag{3.1}$$

is getting small for $\varrho$ small.

Now we state the Lipschitz-Approximation Theorem; see [3] Theorem 5.4. Actually, the statement in [3] is more general.

**Theorem** ([3]). *Let $W$ be an integral $n$-varifold in $B_7^{n+1}(0)$ and $\theta_0 \in \mathbb{N}$ such that,*

$$(3.2)\qquad \begin{aligned} (\theta_0 - \frac{1}{2})\omega_n \leq& \mu_W(B_1^{n+1}(0)), \\ \mu_W(B_3^{n+1}(0)) \leq& (\theta_0 + \frac{1}{2})3^n\omega_n, \\ \mathrm{lipapp}_W(0,7,P) \leq& \varepsilon \quad \text{for } P = \mathbb{R}^n \times \{0\}. \end{aligned}$$

*Then there exist $\theta_0$-Lipschitz maps*

$$f_1 \leq \ldots \leq f_{\theta_0} : B_1^n(0) \to \mathbb{R}$$

*such that*

$$\mathrm{Lip}\, f_i \leq C(\theta_0), \quad \| f_i \|_{L^\infty} \leq \omega(\varepsilon),$$

*with $\omega(\varepsilon) \to 0$ for $\varepsilon \to 0$.*

*Next putting $F_i(y) := (y, f_i(y))$ and*

$$(3.3)\qquad Y := \{y \in B_1^n(0) | \theta^n(\mu_W, (y,t)) = \#\{i | f_i(y) = t\} \text{ for all } t \in ]-1,1[\},$$

$$(3.4)\qquad X := \mathrm{spt}\, W \cap Z_1 \cap p^{-1}(Y) = \cup_{i=1}^{\theta_0} F_i(Y),$$

*for the orthogonal projection $\pi : \mathbb{R}^{n+1} \to P$, we get*

$$(3.5)\qquad \mu_W(Z_1 - X) + \mathcal{L}^n(B_1^n(0) - Y) \leq C\varepsilon,$$

*where $C$ depends only on $n$ and $\theta_0$.*

We first apply the Lipschitz-Approximation Theorem in the following lemma.

**Lemma 3.2.** *We consider $x_0 \in \Sigma = \mathrm{spt}\, V$ such that the tangent plane $T_{x_0}V$ exists with density $\theta_0 := \theta^n(\mu_V, x_0)$. Then $\theta_0 \in \mathbb{N}$ and*

$$\theta_0 \text{ is odd} \Longleftrightarrow \theta^{n+1}(E, x_0) = \tfrac{1}{2},$$

$$\theta_0 \text{ is even} \Longleftrightarrow \theta^{n+1}(E, x_0) \in \{0, 1\};$$

*in particular,*

$$\theta_0 \text{ is odd} \Longleftrightarrow x_0 \in \partial_* E,$$

*where* $\partial_* E := \{x | \theta^{n+1,*}(E, x_0), \theta^{n+1,*}(\mathbb{R}^{n+1} - E, x_0) > 0\}$ *is the measure-theoretic boundary of* $E$*; see* [13] §5.8.

*Proof.* We assume $x_0 = 0, T_{x_0} V = \theta_0 P$ for $P = \mathbb{R}^n \times \{0\}$. We get

$$V_\varrho := \zeta_{\varrho,\#} V \to \theta_0 P \quad \text{as } \varrho \downarrow 0$$

for $\zeta_\varrho(x) := \varrho^{-1} x$. From (3.1), we see that $V_\varrho$ has locally bounded first variation, hence $\theta_0 \in \mathbb{N}$ by Allard's Integral Compactness Theorem.

By local Hausdorff-convergence; see (2.17), we get for $\varepsilon > 0$ that $\Sigma \cap B_\varrho^{n+1}(0) \subseteq \{|x_{n+1}| \leq \varepsilon \varrho\}$ if $\varrho$ is small enough, hence

$$\Sigma \cap B_\varrho^{n+1}(0) \subseteq \{|x_{n+1}| \leq \varepsilon |x|\}.$$

Since $\partial E \subseteq \Sigma$ by (2.13), we get four cases

$$\begin{aligned} &\{x_{n+1} > \varepsilon |x|\} \cap B_\varrho^{n+1}(0) \subseteq E \text{ or } E^c \quad \text{and} \\ &\{x_{n+1} < -\varepsilon |x|\} \cap B_\varrho^{n+1}(0) \subseteq E \text{ or } E^c. \end{aligned} \tag{3.6}$$

This yields $\theta^{n+1}(E, 0) \in \{0, \frac{1}{2}, 1\}$ and

$$\begin{aligned} &\chi_{\varrho^{-1}E} \to 0 \text{ or } 1 \quad \text{in } L^1(B_1^{n+1}(0)), \\ &\text{or} \\ &\chi_{\varrho^{-1}E} \to \{x | x\nu \leq 0\} \quad \text{in } L^1(B_1^{n+1}(0)), \end{aligned} \tag{3.7}$$

where $\nu$ is a normal at $T_{x_0} V$.

Next we apply the Lipschitz-Approximation Theorem to $V_{j_\varrho}$ in $B_7^{n+1}(0)$. First, we observe that

$$\operatorname{lipapp}_{V_\varrho}(0, 7, P) = \operatorname{lipapp}_V(0, 7\varrho, P) \leq \omega(\varrho),$$

with $\omega(\varrho) \to 0$ is $\varrho \to 0$. Indeed, $\operatorname{tiltex}_{V_\varrho}(0, 7, P) \to 0$, since $V_\varrho \to \theta_0 P$, and the last term in $\operatorname{lipapp}_V$ is estimated in (3.1).

As clearly $\limsup_{j \to \infty} \operatorname{tiltex}_{V_j}(0, \varrho, P) \leq \operatorname{tiltex}_V(0, 2\varrho, P)$ for fixed $\varrho$ and again appealing to (3.1), we get for $j \geq j_\varrho$ that

$$\operatorname{lipapp}_{V_{j_\varrho}}(0, 7, P) \leq \omega(\varrho)$$

where $V_{j_\varrho} := \zeta_{\varrho,\#} V_j$.

Now we apply the Lipschitz-Approximation Theorem to $V_{j_\varrho}$ and get for the sets $Y, X$ defined in (3.3) and (3.4) that

$$\mu_V(Z_1 - X) + \mathcal{L}^n(B_1^n(0) - Y) \leq \omega(\varrho);$$

see (3.5).

Next, we put

$$\tilde{Y} := \pi(\{x \in \operatorname{spt} V_{j_\varrho} = \varrho^{-1}\partial E_j | \operatorname{spt} V_{j_\varrho} \text{ is not smooth in } x, \text{ or } \nu_{V_{j_\varrho}}(x) e_{n+1} = 0\}$$

and observe by Co-Area formula; see [33] §12, and the almost everywhere regularity of $\partial E_j$ by Allard's Regularity Theorem and (2.15) that

$$\mathcal{L}^n(\tilde{Y}) = 0.$$

For $\varrho$ small, we select

$$y_0 \in Y - \tilde{Y} \neq \emptyset.$$

We see from (3.4) that

$$\operatorname{spt} V_{j_\varrho} \cap Z_1 \cap p^{-1}(y_0) = \{y_0\} \times \{t_1 < \ldots < t_{\theta_0}\}$$

with $|t_i| \leq \omega(\varrho)$ and $\nu_{\varrho^{-1}\partial E_j}(y_0, t_i) e_{n+1} \neq 0$. Therefore tracing $(\varrho y_0, \varrho t)$ from $t = -\frac{1}{2}$ to $t = \frac{1}{2}$, we jump from $E_j$ to ${E_j}^c$ or from ${E_j}^c$ to $E_j$ exactly at $t_1 < \ldots < t_{\theta_0}$. If $\theta_0$ is even, the same set $E_j$ or ${E_j}^c$ is at the top and bottom, whereas if $\theta_0$ is odd, the opposite set $E_j$ or ${E_j}^c$ is at the top and bottom. Now for $j \geq j_\varrho$, we get from (2.16) and (3.6) that

$$\begin{aligned} &\{x_{n+1} > 2\varepsilon\varrho\} \cap B_\varrho^{n+1}(0) \subseteq E_j \text{ or } {E_j}^c, \text{ and} \\ &\{x_{n+1} < -2\varepsilon\varrho\} \cap B_\varrho^{n+1}(0) \subseteq E_j \text{ or } {E_j}^c, \end{aligned}$$

and $E_j$ and $E$ are in the same of the four cases. The conclusion follows now from (3.7). q.e.d.

The upper height function, as defined in (1.17), is in general not upper semicontinuous. Therefore, we define it now locally near points which have a tangent plane.

We call $x_0 \in \Sigma = \operatorname{spt} V$ generic if

$$\begin{aligned} &T_{x_0} V \quad \text{exists}, \quad \theta_0 := \theta^n(\mu_V, x_0) \in \mathbb{N} \quad \text{and} \\ &\theta(\mu_V, [\theta^n(\mu_V) = \theta_0], x_0) = 1. \end{aligned} \tag{3.8}$$

We observe from [14] Theorem 2.9.11 that $\mu_V$-almost all points are generic. For simplicity, we assume $x_0 = 0, T_{x_0}V = P = \mathbb{R}^n \times \{0\}$ and

$$B^{n+1}_{2\varrho_0}(0) \subset\subset \Omega \tag{3.9}$$

for some $0 < \varrho_0 < 1$.

From the local Hausdorff-convergence in (2.17) for $0 < \varrho < \varrho_0$, we can choose $\varrho_0$ small enough such that

$$\Sigma \cap B^{n+1}_{2\varrho_0}(0) \subseteq \{|x_{n+1}| < \varrho_0/2\}. \tag{3.10}$$

This vertical cutoff makes the height functions semicontinuous.

**Definition 3.3.** Let $0 \in \Sigma$ be generic with $T_0V = P$ and $0 < \varrho_0 < 1$ such that (3.9) and (3.10) are satisfied.

We define the upper and lower height functions

$$\begin{aligned} &\varphi_+ : B^n_{\varrho_0}(0) \to [-\infty, \infty[ \quad \text{upper semicontinuous,} \\ &\varphi_- : B^n_{\varrho_0}(0) \to ]-\infty, \infty] \quad \text{lower semicontinuous,} \end{aligned}$$

by

$$\begin{aligned} \varphi_+(y) &:= \sup\{t \in ]-\infty, \varrho_0[ \,|\, (y,t) \in \Sigma \cap Z_{\varrho_0}\}, \\ \varphi_-(y) &:= \inf\{t \in ]-\varrho_0, \infty[ \,|\, (y,t) \in \Sigma \cap Z_{\varrho_0}\}. \end{aligned} \tag{3.11}$$

We observe that by our definition of height functions, we have

$$\begin{aligned} \varphi_-(y) \le \varphi_+(y) &\Longleftrightarrow \varphi_\pm(y) \in \mathbb{R} \\ &\Longleftrightarrow -\infty < \varphi_+(y) \text{ or } \varphi_-(y) < \infty \\ &\Longleftrightarrow (\{y\} \times ]-\varrho_0, \varrho_0[) \cap \Sigma \neq \emptyset. \end{aligned}$$

From the local Hausdorff-convergence in (2.17), we see that

$$-\omega(\varrho)\varrho \le \varphi_-(y) \le \varphi_+(y) \le \omega(\varrho)\varrho \tag{3.12}$$

for $y \in B^n_\varrho(0)$ if $\varphi_+(y) \in \mathbb{R}$ or $\varphi_-(y) \in \mathbb{R}$, and where $\omega(\varrho_0) < 1/2$. Further

$$\begin{aligned} &|(\omega_n \varrho^n)^{-1} \mu_V(B^{n+1}_\varrho(0)) - \theta_0| \le \omega(\varrho), \\ &\varrho^{-n} \mu_V([\theta^n(\mu_V) \neq \theta_0] \cap B^{n+1}_\varrho(0)) \le \omega(\varrho), \\ &\operatorname{lipapp}_V(0, \varrho, P) \le \omega(\varrho), \end{aligned} \tag{3.13}$$

by (3.1) and since $P$ is the tangent plane at 0, as assumed in the above Definition.

Since $\partial E \subseteq \Sigma$; see (2.13), we get for $\varrho_0$ small enough, that

$$B^n_{\varrho_0}(0) \times ] - \varrho_0, -\omega(\varrho_0)\varrho_0[ \subseteq E \text{ or } E^c \qquad \text{and}$$
$$B^n_{\varrho_0}(0) \times ]\omega(\varrho_0)\varrho_0, \varrho_0[ \subseteq E \text{ or } E^c.$$

As in the proof of Lemma 3.2, these are four cases. From now on, we will assume that on top there is the set $E$ that is

$$B^n_{\varrho_0}(0) \times ]\omega(\varrho_0)\varrho_0, \varrho_0[ \subseteq E. \tag{3.14}$$

Clearly, we can replace $E$ by the interior of its complement when we change $u$ to $-u$.

From Lemma 3.2, we infer that

$$\begin{aligned} &B^n_{\varrho_0}(0) \times ] - \varrho_0, -\omega(\varrho_0)\varrho_0[ \subseteq E^c \Longleftrightarrow \theta_0 \text{ is odd,} \\ &B^n_{\varrho_0}(0) \times ] - \varrho_0, -\omega(\varrho_0)\varrho_0[ \subseteq E \Longleftrightarrow \theta_0 \text{ is even.} \end{aligned} \tag{3.15}$$

In particular, we see that if $\theta_0$ is odd then

$$\varphi_\pm(y) \in \mathbb{R} \quad \text{for all } y \in B^n_{\varrho_0}(0). \tag{3.16}$$

We come to the second lemma in this section.

**Lemma 3.4.** *Let $0 \in \Sigma$ be generic and $\varrho_0$ as above. We put*

$$\begin{aligned} \Sigma_0 := \{x = (y, \varphi_\pm(y)) | y \in B^n_{\varrho_0}(0) \cap [\varphi_+ = \varphi_-], \\ x \in \Sigma \textit{ is generic}, \theta^n(\mu_V, x) = \theta_0\}. \end{aligned} \tag{3.17}$$

*Then*

$$\varrho^{-n}\mu_V(B^{n+1}_\varrho(0) - \Sigma_0) \leq \omega(\varrho), \tag{3.18}$$

$$\varrho^{-n}\mathcal{L}^n([\varphi_+ \neq \varphi_-] \cap B^n_\varrho(0)) \leq \omega(\varrho), \tag{3.19}$$

$$\varrho^{-n}\mathcal{L}^n([\varphi_+ = \varphi_-] \cap B^n_\varrho(0)) \geq \omega_n - \omega(\varrho). \tag{3.20}$$

*Proof.* First, we show that (3.19) and (3.20) follow from (3.18). Clearly, (3.19) and (3.20) are equivalent.

To prove (3.20), we apply the Co-Area formula; see [33] §12, to the projection $\pi$. We compute the Jacobian of $\pi$ for any $T \in G(n+1,n)$

$$J_T\pi = |\nu(T)e_{n+1}| = 1 - \frac{1}{2} \parallel \nu(T) - e_{n+1} \parallel^2 = \sqrt{1 - \frac{1}{2} \parallel T - P \parallel^2}, \tag{3.21}$$

where $\nu(T)$ is a normal at $T$ with $\nu(T)e_{n+1} \geq 0$, and where we use the inner product trace norm $\parallel A \parallel := \sqrt{\operatorname{tr}(A^T A)}$ for $A \in \mathbb{R}^{n+1,n+1}$.

This yields for any $\tau > 0$ that

$$\begin{aligned}\mathcal{L}^n([\varphi_+ = \varphi_-] \cap B^n_\varrho(0)) \geq & \int\limits_{\Sigma_0 \cap B^{n+1}_\varrho(0)} J_V\pi(x) d\mathcal{H}^n(x) \\ \geq & \sqrt{1-\tau}\theta_0{}^{-1}\mu_V(\Sigma_0 \cap B^{n+1}_\varrho(0)) \\ & - \mu_V(B^{n+1}_\varrho(0) \cap [\parallel TV - P \parallel^2 \geq 2\tau]) \\ \geq & \sqrt{1-\tau}\theta_0{}^{-1}\mu_V(B^{n+1}_\varrho(0)) \\ & - \mu_V(B^{n+1}_\varrho(0) - \Sigma_0) \\ & - \tau^{-1}\varrho^n \operatorname{tiltex}_V(0,\varrho,P),\end{aligned}$$

and using (3.13) and (3.18)

$$\lim_{\varrho \to 0} (\omega_n \varrho^n)^{-1} \mathcal{L}^n([\varphi_+ = \varphi_-] \cap B^n_\varrho(0)) \geq \sqrt{1-\tau}$$

which is (3.20), since $\tau > 0$ was arbitrary.

We turn to (3.18), rescale in $\varrho$ for $0 < \varrho < \varrho_0/7$ and put $V_\varrho := \zeta_{\varrho,\#}V$ for $\zeta_\varrho(x) := \varrho^{-1}x$. From (3.13), we see that (3.2) is satisfied with $\varepsilon = \omega(\varrho)$ for $\varrho$ small. Therefore, we can apply the Lipschitz-Approximation Theorem to $V_\varrho$ and get for the sets $Y, X$ defined in (3.3), (3.4) that

$$\mu_{V_\varrho}(Z_1 - X) + \mathcal{L}^n(B^n_1(0) - Y) \leq \omega(\varrho); \tag{3.22}$$

see (3.5).

Clearly for $y \in Y$, we see from (3.4) that

$$\operatorname{spt} V_\varrho \cap Z_1 \cap \pi^{-1}(y) = \{y\} \times \{f_1(y) \leq \ldots \leq f_{\theta_0}(y)\} \subseteq X,$$

hence

$$\varphi_-(\varrho y) = f_1(y) \leq f_{\theta_0}(y) = \varphi_+(\varrho y).$$

In particular, for $x = (y,t) \in X$ with $\theta^n(\mu_{V_\varrho}, x) = \theta_0$, we get $\varphi_-(\varrho y) = \varphi_+(\varrho y)$. This yields

$$X \cap [\theta^n(\mu_{V_\varrho}) = \theta_0] \cap [x \text{ generic}] \subseteq \varrho^{-1}\Sigma_0,$$

and we estimate using (3.13) and (3.22) that

$$\begin{aligned}\varrho^{-n}\mu_V(B_\varrho^{n+1}(0) - \Sigma_0) \leq & \mu_{V_\varrho}(Z_1 - X) \\ & + \varrho^{-n}\mu_V([\theta^n(\mu_V) \neq \theta_0] \cap B_\varrho^{n+1}(0)) \\ \leq & \omega(\varrho),\end{aligned}$$

concluding the proof. q.e.d.

## 4. Differential properties of the height function

In this section, we will derive a differential equation for the height functions.

We start with a lemma that gives a Lipschitz condition from above and below for the upper and lower height function, respectively, at almost all points where the height functions are finite.

**Lemma 4.1.** *For $\mathcal{L}^n$-almost all $y \in B_{\varrho_0}^n(0)$ with $\varphi_\pm(y) \in \mathbb{R}$, there exists $C = C(y) < \infty$ such that*

$$\begin{aligned}\varphi_+(z) &\leq \varphi_+(y) + C|y-z| \quad \textit{for all } z \in B_{\varrho_0}^n(0), \\ \varphi_-(z) &\geq \varphi_-(y) - C|y-z| \quad \textit{for all } z \in B_{\varrho_0}^n(0).\end{aligned} \tag{4.1}$$

*Proof.* We put

$$X := \{x \in Z_{\varrho_0} \cap \Sigma | T_x V \text{ does not exist, or } \nu(T_x V) e_{n+1} = 0\}$$

and observe by Co-Area formula that

$$\mathcal{L}^n(\pi(X)) = 0.$$

For $y \notin \pi(X)$ with $\varphi_\pm(y) \in \mathbb{R}$, we put $x := (y, \varphi_+(y)) \in \Sigma$ and observe that its tangent plane $T_x V$ is not orthogonal to $P$. By local Hausdorff-convergence in (2.17), we see that there is a cone

$$\text{Cone} := \{(y,t) | |t| \leq C|y|\}$$

for some $C < \infty$ and $\varrho > 0$ such that

$$\Sigma \cap B^{n+1}_\varrho(x) \subseteq x + \text{Cone}.$$

Since $\varphi_+ \leq \omega(\varrho_0)\varrho_0$ is bounded from above and upper semicontinuous, (4.1) follows for $\varphi_+$, hence the lemma is proved by symmetry. q.e.d.

We will prove that $\varphi_\pm$ are viscosity sub- and supersolutions of the minimal surface equation with right-hand side in $L^s$. Since $u$ is not continuous, we need a refined definition of viscosity solutions which can be found in [5] or [7].

**Definition 4.2.** We consider $U \subseteq \mathbb{R}^n$ open and

$$F : U \times \mathbb{R}^n \times S(n) \to \mathbb{R}$$

which is degenerate elliptic that is

$$F(.,.,X) \leq F(.,.,Y) \quad \text{if } X \leq Y.$$

For $v \in L^r_{\text{loc}}(U), r > n/2, r \geq 1$, we call an upper semicontinuous function

$$\varphi : U \to [-\infty, \infty[$$

a $W^{2,r}$-viscosity subsolution of

$$-F(., \nabla\varphi, D^2\varphi) \leq v \quad \text{in } U,$$

if for all $\eta \in W^{2,r}(U'), U' \subset\subset U, \tau > 0$, such that

$$-F(., \nabla\eta, D^2\eta) \geq v + \tau \quad \text{pointwise almost everywhere in } U',$$

the function $\varphi - \eta$ has no interior maximum in $U'$, that is there is no $y \in U'$ with

$$\varphi - \eta \leq (\varphi - \eta)(y) \in \mathbb{R} \quad \text{in } U'.$$

The supersolutions are defined analogously. Solutions are functions which are both sub- and supersolutions.

Clearly, $\varphi - \eta$ having an interior maximum in $U'$ implies that there is $y \in U'$ and $\varrho > 0$ such that $B^n_\varrho(y) \subset\subset U'$ and

$$\varphi - \eta \leq (\varphi - \eta)(y) \in \mathbb{R} \quad \text{in } B^n_\varrho(y). \tag{4.2}$$

When $j \geq j_{\varrho_0}$ is large enough, we can define the upper- and lower height functions $\varphi_{\pm,j}$ for $\Sigma_j = \partial E_j \cap \Omega$ in $B^n_{\varrho_0}(0)$ as in Definition 3.3. The height functions $\varphi_{\pm,j}$ and $\varphi_\pm$ are connected through the following limit process which we recall from [10] §6.

**Definition 4.3.** We consider $U \subseteq \mathbb{R}^n$ open and a sequence of functions $\varphi_j : U \to [-\infty, \infty]$. Then

$$\lim_{j\to\infty}{}^{*}\varphi_j(y) := \sup\left\{ \limsup_{k\to\infty} \varphi_{j_k}(y_k) \,\middle|\, j_k \to \infty, y_k \to y \right\}.$$

$\lim_{j\to\infty *} \varphi_j$ is defined analogously.

By local Hausdorff-convergence (2.16), we get the following proposition.

**Proposition 4.4.**

$$\varphi_+ = \lim_{j\to\infty}{}^{*}\varphi_{+,j} \quad \textit{and} \quad \varphi_- = \lim_{j\to\infty *} \varphi_{-,j} \quad \textit{on} \quad B^n_{\varrho_0}(0). \tag{4.3}$$

When $\partial E_j$ are smooth, we obtain that the height functions $\varphi_{\pm,j}$ are viscosity sub- and supersolutions. For the non-smooth case, we have to invoke the maximum principle [32] Theorem 6.1.

**Proposition 4.5.** *For $j \geq j_{\varrho_0}$, $\varphi_{+,j}$ is a $W^{2,s}$-viscosity subsolution of*

$$-\nabla \left( \frac{\nabla \varphi_{+,j}}{\sqrt{1 + |\nabla \varphi_{+,j}|^2}} \right) \leq -u_j(.,\varphi_{+,j}) \quad \textit{in } B^n_{\varrho_0}(0), \tag{4.4}$$

*and $\varphi_{-,j}$ is a $W^{2,s}$-viscosity supersolution of*

$$-\nabla \left( \frac{\nabla \varphi_{-,j}}{\sqrt{1 + |\nabla \varphi_{-,j}|^2}} \right) \geq \begin{cases} -u_j(.,\varphi_{-,j}) & \textit{if } \theta_0 \textit{ is odd,} \\ u_j(.,\varphi_{-,j}) & \textit{if } \theta_0 \textit{ is even,} \end{cases} \quad \textit{in } B^n_{\varrho_0}(0). \tag{4.5}$$

We emphasize that the right-hand sides of these equations are considered as fixed functions $(y \mapsto u_j(y, \varphi_{\pm,j}(y)))$.

*Proof.* Clearly, it suffices to prove (4.4).

First, we know from (1.4) and Theorem 1.3 that

$$\vec{\mathbf{H}}_{V_j}, u_j \in L^s_{\mathrm{loc}}(\mu_{V_j})$$

and (1.3) holds, more precisely

$$\vec{\mathbf{H}}_{V_j} = u_j \nu_{E_j} \quad \text{on } \Sigma_j,$$

since $\mu_{V_j}(\Sigma_j - \partial^* E_j) = 0$ by (2.15).

By local Hausdorff-convergence (2.16) and (3.10), we see

$$\Sigma_j \cap B^{n+1}_{2\varrho_0}(0) \subseteq \{|x_{n+1}| < 3\varrho_0/4\}.$$

for $j$ large enough. Together with $s > n, s \geq 2$, the weak maximum principle [32] Theorem 6.1 implies that $\varphi_{+,j}$ is a $W^{2,s}$-viscosity subsolution of

$$-\nabla \left( \frac{\nabla \varphi_{+,j}}{\sqrt{1 + |\nabla \varphi_{+,j}|^2}} \right) \leq \vec{\mathbf{H}}_{V_j}(.,\varphi_{+,j}) \frac{(\nabla \varphi_{+,j}, -1)}{\sqrt{1 + |\nabla \varphi_{+,j}|^2}} \quad \text{in } B^n_{\varrho_0}(0),$$

where the right-hand side is extended arbitrarily on $B^n_{\varrho_0}(0) - [\varphi_{+,j} \in \mathbb{R}]$ to a function still in $L^s(B^n_{\varrho_0}(0))$. As $E_j$ lies at the top by assumption, we see that the inner normal is given by

$$\nu_{E_j}(.,\varphi_{+,j}) = \frac{(-\nabla \varphi_{+,j}, 1)}{\sqrt{1 + |\nabla \varphi_{+,j}|^2}},$$

hence

$$\vec{\mathbf{H}}_{V_j}(.,\varphi_{+,j}) \frac{(\nabla \varphi_{+,j}, -1)}{\sqrt{1 + |\nabla \varphi_{+,j}|^2}} = -u_j(.,\varphi_{+,j}),$$

which yields (4.4). q.e.d.

We can pass to the limits in (4.4) and (4.5) and obtain the following lemma.

**Lemma 4.6.** *$\varphi_+$ is a $W^{2,s}$-viscosity subsolution of*

$$-\nabla \left( \frac{\nabla \varphi_+}{\sqrt{1 + |\nabla \varphi_+|^2}} \right) \leq -v_+ \quad \textit{in } B^n_{\varrho_0}(0), \tag{4.6}$$

*and $\varphi_-$ is a $W^{2,s}$-viscosity supersolution of*

$$-\nabla \left( \frac{\nabla \varphi_-}{\sqrt{1 + |\nabla \varphi_-|^2}} \right) \geq \begin{cases} -v_- & \textit{if } \theta_0 \textit{ is odd,} \\ v_- & \textit{if } \theta_0 \textit{ is even,} \end{cases} \quad \textit{in } B^n_{\varrho_0}(0), \tag{4.7}$$

*where $v_\pm \in L^s(B^n_{\varrho_0}(0))$ satisfying*

$$v_\pm = u(.,\varphi_\pm) \quad \mathcal{L}^n\textit{-almost everywhere on } [\varphi_+ = \varphi_-] \cap B^n_{\varrho_0}(0). \tag{4.8}$$

*Proof.* First, we put

$$\hat{u}_j(y) := \sup_{|t| \leq \varrho_0} |u_j(y,t)|$$

and observe that

$$\| \hat{u}_j \|_{L^s(B^n_{\varrho_0}(0))} \leq C_{n,p}(\varrho_0)\Lambda, \tag{4.9}$$

since $\| u_j \|_{W^{1,p}(B^{n+1}_{2\varrho_0}(0))} \leq \Lambda$.

Hence for a subsequence

$$u_j(.,\varphi_{\pm,j}) \to v_\pm \quad \text{weakly in } L^s(B^n_{\varrho_0}(0)), \tag{4.10}$$

where we do an appropriate choice for $u_j(.,\pm\infty)$. By standard compactness argument for the trace mapping

$$W^{1,p}(B^{n+1}_{\varrho_0}(0)) \to L^p(B^n_{\varrho_0}(0) \times \{0\}),$$

we obtain

$$\| u_j(.,\varphi_{\pm,j})) - u(.,\varphi_{\pm,j}) \|_{L^p([\varphi_\pm \in \mathbb{R}] \cap B^n_{\varrho_0}(0))} \to 0,$$

and, putting

$$u_+(y) = \sup_{\varphi_-(y) \leq t \leq \varphi_+(y)} u(y,t) \quad \text{and} \quad u_-(y) = \inf_{\varphi_-(y) \leq t \leq \varphi_+(y)} u(y,t),$$

we conclude from Proposition 4.4 that

$$u_- \leq v_\pm \leq u_+,$$

which yields (4.8).

We proceed in proving (4.6) and consider $\psi \in W^{2,s}(U'), U' \subset\subset B^n_{\varrho_0}(0), \tau > 0$ such that

$$-\nabla \left( \frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} \right) = -\partial_{lk} A(\nabla \psi) \partial_{lk} \psi \geq -v_+ + 2\tau \quad \text{in } U', \tag{4.11}$$

where $A(p) := \sqrt{1 + |p|^2}$.

We have to show that $\varphi_+ - \psi$ has no interior maximum. Assume on contrary, that there is one, hence by (4.2)

$$\varphi_+ - \psi \leq (\varphi_+ - \psi)(y_0) \in \mathbb{R} \quad \text{in } B^n_{\varrho_1}(y_0) \tag{4.12}$$

for some $B^n_{\varrho_1}(y_0) \subset\subset U'$.

By (4.9), we fix an upper bound

$$\| \hat{u}_j + \tau \|_{L^s(B^n_{\varrho_0}(0))}, \ \| |v_+| + \tau \|_{L^s(B^n_{\varrho_0}(0))}, \ \| \psi \|_{(W^{2,s} \cap C^{1,\alpha})(B^n_{\varrho_1}(y_0))} \leq \Gamma.$$

For $0 < \varrho < \varrho_1$ small enough, we get by a perturbation argument that there are unique solutions $\psi_j, \tilde{\psi} \in W^{2,s}(B^n_\varrho(y_0))$ of

$$\begin{aligned} -\nabla \left( \frac{\nabla \tilde{\psi}}{\sqrt{1 + |\nabla \tilde{\psi}|^2}} \right) &= - v_+ + \tau \quad \text{in } B^n_\varrho(y_0), \\ -\nabla \left( \frac{\nabla \psi_j}{\sqrt{1 + |\nabla \psi_j|^2}} \right) &= - u_j(., \varphi_{+,j}) + \tau \quad \text{in } B^n_\varrho(y_0), \\ \tilde{\psi} = \psi_j &= \psi \quad \text{on } \partial B^n_\varrho(y_0), \end{aligned} \tag{4.13}$$

which moreover satisfy

$$\| \tilde{\psi} \|_{(W^{2,s} \cap C^{1,\alpha})(B^n_\varrho(y_0))}, \ \| \psi_j \|_{(W^{2,s} \cap C^{1,\alpha})(B^n_\varrho(y_0))} \leq C_{n,p}(\Gamma, \varrho_1). \tag{4.14}$$

Indeed, translating and rescaling $B^n_\varrho(y_0)$ to $B^n_1(0)$,

$$\psi_\varrho(y) := \varrho^{-1} \psi(y_0 + \varrho y), \quad v_\varrho(y) := \varrho(-v_+(y_0 + \varrho y) + \tau)$$

and likewise for $u_j(., \varphi_{+,j})$, we have to solve

$$\begin{aligned} -\partial_{kl} A(\nabla \xi) \partial_{kl} \xi &= v_\varrho \quad \text{in } B^n_1(0), \\ \xi &= \psi_\varrho \quad \text{on } \partial B^n_1(0), \end{aligned}$$

with

$$\begin{aligned} \| \nabla \psi_\varrho \|_{C^{0,\alpha}(B^n_1(0))} &\leq \Gamma, \\ \| \psi_\varrho - \psi_\varrho(0) \|_{C^{1,\alpha}(B^n_1(0))} &\leq 2\Gamma, \\ \| v_\varrho \|_{L^s(B^n_1(0))}, \| D^2 \psi_\varrho \|_{L^s(B^n_1(0))} &\leq \Gamma \varrho^\alpha. \end{aligned}$$

We fix $0 < \beta < \alpha$ and define for large $R > 0$ an operator $F : \overline{B_R(0)} \subseteq C^{1,\beta}(B^n_1(0)) \to W^{2,s}(B^n_1(0)) \hookrightarrow C^{1,\beta}(B^n_1(0))$ by putting $F(\eta) := \xi$ where $\xi$ solves the linear elliptic boundary value problem

$$\begin{aligned} -\partial_{kl} A(\nabla \eta) \partial_{kl} \xi &= v_\varrho \quad \text{in } B^n_1(0), \\ \xi &= \psi_\varrho - \psi_\varrho(0) \quad \text{on } \partial B^n_1(0). \end{aligned}$$

Since $\| D^2 A \| \leq 1$ and by Calderon-Zygmund estimates; see [15] Theorem 9.14, we get

$$\begin{aligned}\| \xi - \psi_\varrho + \psi_\varrho(0) \|_{W^{2,s}(B_1^n(0))} &\leq C_{n,p}(R) \| v_\varrho + \partial_{kl} A(\nabla \eta) \partial_{kl} \psi_\varrho \|_{L^s(B_1^n(0))} \\ &\leq C_{n,p}(R) \Gamma \varrho^\alpha,\end{aligned}$$

hence

$$\| \xi \|_{(W^{2,s} \cap C^{1,\alpha})(B_1^n(0))} \leq C_{n,p} \Gamma (1 + C_{n,p}(R) \varrho^\alpha).$$

Choosing $R = 2C_{n,p}\Gamma$ and $\varrho = \varrho(n,p,\Gamma)$ small enough, we get from Schauder's Fixed-point Theorem a fixed point $\xi$ of $F$ which moreover satisfies

$$\| \xi - \psi_\varrho + \psi_\varrho(0) \|_{(W^{2,s} \cap C^{1,\alpha})(B_1^n(0))} \leq C_{n,p}(\Gamma) \varrho^\alpha.$$

Putting $\tilde{\xi}(y) := \varrho \xi(\varrho^{-1}(y - y_0)) + \varrho \psi_\varrho(0)$, we see that $\tilde{\xi}$ solves (4.13) and satisfies

$$\| \xi - \psi \|_{(W^{2,s} \cap C^{1,\alpha})(B_\varrho^n(y_0))} \leq C_{n,p}(\Gamma, \varrho_1),$$

hence (4.14). If $\hat{\xi}$ is a further solution, we subtract $w := \hat{\xi} - \tilde{\xi}$ and get that $w$ is a local weak solution of

$$\begin{aligned}-\partial_l(a_{lk} \partial_k w) = 0 \quad \text{in } B_\varrho^n(y_0), \\ w = 0 \quad \text{on } \partial B_\varrho^n(y_0),\end{aligned}$$

where $a_{lk} := \int_0^1 \partial_{lk} A(\nabla \tilde{\xi} + t \nabla w) dt$. This equation is uniformly elliptic, since $\tilde{\xi}, \hat{\xi} \in C^{1,\alpha}(B_\varrho^n(y_0))$, and we conclude by strong maximum principle; see [15] Theorem 8.19, that $w = 0$, hence the solutions are unique.

The uniqueness of the solutions yields together with (4.10) and (4.14) imply that

$$\psi_j \to \tilde{\psi} \quad \text{weakly in } W^{2,s}(B_\varrho^n(y_0)) \text{ and uniformly on } \overline{B_\varrho^n(y_0)}.$$

Further, we get from (4.11), (4.13) and the strong maximum principle; see [15] Theorem 8.19, that

$$\psi > \tilde{\psi} \quad \text{in } B_\varrho^n(y_0).$$

This yields with (4.12) that

$$\sup_{\partial B^n_\varrho(y_0)} (\varphi_+ - \tilde{\psi}) < (\varphi_+ - \tilde{\psi})(y_0) \leq \sup_{B^n_\varrho(y_0)} (\varphi_+ - \tilde{\psi}).$$

From Proposition 4.4 and the uniform convergence $\psi_j \to \tilde{\psi}$, we get $\varphi_+ - \tilde{\psi} = \lim_{j\to\infty}{}^*(\varphi_{+,j} - \psi_j)$, hence for large $j$ that

$$\sup_{\partial B^n_\varrho(y_0)} (\varphi_{+,j} - \psi_j) < \sup_{B^n_\varrho(y_0)} (\varphi_{+,j} - \psi_j).$$

and $\varphi_{+,j} - \psi_j$ has an interior maximum in $B^n_\varrho(y_0)$. By (4.13), this contradicts (4.4), and (4.6) is proved.

Equation (4.7) follows in the same way from (4.5). q.e.d.

## 5. Quadratic tilt-excess decay

In this section, we will establish the crucial quadratic tilt-excess decay, see (1.16), that

(5.1) $\quad \text{tiltex}_V(x, \varrho, T_x V) = O_x(\varrho^2) \quad \mu_V - \text{almost everywhere on } \Sigma.$

Actually, we will prove a quadratic decay of the height-excess which implies (5.1) by standard estimates, see Lemma 5.4.

We have already pointed out in the introduction that the height-excess decay estimates in [1] and [3] do not seem sufficient to infer (5.1). Instead taking into account that the height functions are viscosity sub- and supersolutions of the minimal surface equation with right-hand side in $L^s$ by Lemma 4.6, we will apply tools from fully-nonlinear elliptic equations. We start with a definition; see [6, §1.2].

**Definition 5.1.** We call a function P with

$$P(y) := a + by \pm M|y|^2,$$

where $a \in \mathbb{R}, b \in \mathbb{R}^n, M \geq 0$, a paraboloid of opening $M$. We call $P$ convex when we have the positive sign, else we call it concave.

For $\varphi : U \to [-\infty, \infty], Q \subseteq U \subseteq \mathbb{R}^n, y \in Q$, we define

$$\overline{\theta}(\varphi, Q)(y)$$

to be the infimum of all positive constants $M$ for which there is a convex paraboloid $P$ of opening $M$ that touches $\varphi$ at $y$ from above in $Q$, that is

$$\varphi(y) = P(y) \quad \text{and} \quad \varphi \leq P \text{ on } Q.$$

Likewise, we define $\underline{\theta}(\varphi, Q)$ and put

$$\theta(\varphi, Q) := \max(\overline{\theta}(\varphi, Q), \underline{\theta}(\varphi, Q)).$$

We remark that our definition of opening of a paraboloid differs from that in [6] by a factor 2.

We will apply the following theorem of Caffarelli and Trudinger which was established for getting $W^{2,s}$-interior regularity for solutions of fully-nonlinear elliptic equations; see [5], [35], [6] Lemma 7.8 and [7].

**Theorem** ([5], [35]). *Let $\mathcal{M}$ be a uniformly elliptic operator and $\varphi$ be a bounded $W^{2,n}$-viscosity subsolution of*

$$-\mathcal{M}(D^2\varphi) \leq v \quad \textit{in } B_1^n(0)$$

*for some $v \in L^n(B_1^n(0))$.*

*Then*

$$\overline{\theta}(\varphi, B_1^n(0)) < \infty \quad \mathcal{L}^n\textit{-almost everywhere in } B_1^n(0). \tag{5.2}$$

We will apply this theorem when $\varphi$ is a Lipschitz-continuous $C^2$-viscosity subsolution and $v$ is constant, in particular bounded; see (5.24).

As already pointed out, the minimal surface equation is not uniformly elliptic, and we cannot immediately apply Caffarelli's and Trudinger's theorem. We will consider a sup-convolution of order 1 of $\varphi_+$.

**Lemma 5.2.** *For an upper semicontinuous function $\varphi : B_\varrho^n(0) \to [-\infty, \infty[$ which is bounded from above and $\varphi \not\equiv -\infty$, we define the sup-convolution of order 1 by*

$$\varphi^\varepsilon(y) := \sup_{z \in B_\varrho^n(0)} \left( \varphi(z) - \frac{1}{\varepsilon}|y - z| \right). \tag{5.3}$$

*Then*

$$\operatorname{Lip} \varphi^\varepsilon \leq \frac{1}{\varepsilon}. \tag{5.4}$$

*If there is $y_0$ satisfying*

$$\varepsilon \left( \sup_{B_\varrho^n(0)} \varphi - \varphi(y_0) \right) + |y_0| \leq \overline{\varrho} := \varrho/4, \tag{5.5}$$

*then for any* $y \in B^n_{\overline{\varrho}}(0)$ *the supremum in* (5.3) *is attained in the interior of* $B^n_{3\overline{\varrho}}(0)$.

*Proof.* (5.4) is immediate.

For $z \in B^n_\varrho(0)$ with

$$\varphi(z) - \frac{1}{\varepsilon}|y - z| \geq \varphi(y_0) - \frac{1}{\varepsilon}|y - y_0|,$$

we get

$$|y - z| \leq \varepsilon(\varphi(z) - \varphi(y_0)) + |y - y_0| \leq \overline{\varrho} + |y|,$$

hence

$$|z| \leq \overline{\varrho} + 2|y| < 3\overline{\varrho},$$

and the supremum is attained since $\varphi$ is upper semicontinuous. q.e.d.

Now for Lipschitz-continuous functions, and in particular for sup-convolutions of order 1, the minimal surface equation (4.6) is uniformly elliptic. On the other hand, the right-hand side in (4.6) will be replaced in the equation for the sup-convolution $\varphi^\varepsilon_+$ by

$$v^\varepsilon_+(y) := \sup_{z \in B^n_{\omega(\varepsilon)}(y)} (-v_+(z)).$$

As $v_+$ is only $L^s$-integrable, $v^\varepsilon_+$ may not even be integrable any longer. Therefore, we first have to subtract a solution of a certain elliptic equation so that the right-hand side is bounded. Then, to get the conclusion for $\varphi_+$ itself, we combine this argument with Lemma 4.1 which yields that $\mathcal{L}^n([\varphi^\varepsilon_+ = \varphi_+]) \nearrow \mathcal{L}^n([\varphi_\pm \in \mathbb{R}])$, hence $\varphi^\varepsilon_+$ and $\varphi_+$ coincide on a large set.

As in Lemma 5.2, we abbreviate

$$\varrho_1 := \varrho_0/4 \tag{5.6}$$

and get the following lemma.

**Lemma 5.3.**

$$\begin{aligned}&\overline{\theta}(\varphi_+, B^n_{\varrho_1}(0)), \underline{\theta}(\varphi_-, B^n_{\varrho_1}(0)) < \infty \\ &\qquad \mathcal{L}^n\text{-almost everywhere on } [\varphi_\pm \in \mathbb{R}] \cap B^n_{\varrho_1}(0),\end{aligned} \tag{5.7}$$

$$\begin{aligned}&\theta(\varphi_\pm, [\varphi_\pm \in \mathbb{R}] \cap B^n_{\varrho_1}(0)) < \infty \\ &\qquad \mathcal{L}^n\text{-almost everywhere on } [\varphi_+ = \varphi_-] \cap B^n_{\varrho_1}(0).\end{aligned} \tag{5.8}$$

*Proof.* By (3.12) and symmetry, it suffices to prove

$$\overline{\theta}(\varphi_+, B^n_{\varrho_1}(0)) < \infty \quad \mathcal{L}^n\text{-almost everywhere on } [\varphi_\pm \in \mathbb{R}] \cap B^n_{\varrho_1}(0). \tag{5.9}$$

We fix $0 < \varepsilon < 1$ and choose $v \in (C^0 \cap L^\infty)(B^n_{\varrho_0}(0))$ such that

$$\| v_+ - v \|_{L^s(B^n_{\varrho_0}(0))} \leq \delta \tag{5.10}$$

for some $0 < \delta \ll \varepsilon$ to be specified later.

Next, we recall the definition of the Pucci-extremal operator; see [6] §2.2,

$$\mathcal{M}^-_\lambda(X) := \lambda \sum_{\varsigma_i > 0} \varsigma_i + \sum_{\varsigma_i < 0} \varsigma_i$$

for $0 < \lambda \leq 1$ and $X \in S(n)$ with eigenvalues $\varsigma_i$ counted according to their multiplicity. The minimal surface operator $\partial_{lk} A(\nabla \varphi) \partial_{lk} \varphi$, we recall $A(p) := \sqrt{1 + |p|^2}$, is uniformly elliptic for bounded gradients, more precisely

$$c_0(R) I_n \leq (\partial_{lk} A(p))_{lk} \leq I_n \quad \text{for } |p| \leq R,$$

hence for $\lambda = c_0(R)$ that

$$\mathcal{M}^-_\lambda(X) \leq \partial_{lk} A(p) X_{lk} \quad \text{for } |p| \leq R. \tag{5.11}$$

We will choose

$$\lambda := c_0(3/\varepsilon). \tag{5.12}$$

Approximating $v_+ - v$ by smooth functions, using Perron's method (see [10] Theorem 4.1) and combining this with the ABP-estimate, Evans-Krylov Theorem, as $\mathcal{M}^-_\lambda$ is concave, and the $W^{2,s}$-interior estimates due to Caffarelli (see [5] and [6] Theorems 3.2, 6.6, 7.1 and 7.4) we get a function $w \in C^0(\overline{B^n_{\varrho_0}(0)}) \cap W^{2,s}_{\text{loc}}(B^n_{\varrho_0}(0))$ satisfying

$$\begin{aligned} -\mathcal{M}^-_\lambda(D^2 w) = v_+ - v \quad & \text{almost everywhere in } B^n_{\varrho_0}(0), \\ w = 0 \quad & \text{on } \partial B^n_{\varrho_0}(0). \end{aligned} \tag{5.13}$$

We get the bounds

$$\| w \|_{L^\infty(B^n_{\varrho_0}(0))}, \| w \|_{W^{2,s} \cap C^{1,\alpha}(B^n_{3\varrho_1}(0))} \leq C_{n,p}(\varepsilon, \varrho_0) \delta. \tag{5.14}$$

Choosing $\delta$ small such that

$$C_{n,p}(\varepsilon,\varrho_0)\delta \le \varepsilon, \tag{5.15}$$

we get

$$|\nabla w| \le \varepsilon \quad \text{in } B^n_{3\varrho_1}(0). \tag{5.16}$$

Next, we put

$$\gamma := \varphi_+ + w \tag{5.17}$$

and compute formally from (4.6), (5.11), (5.12) and (5.13) that

$$\begin{aligned} -\partial_{lk}A(\nabla\gamma - \nabla w(y))\partial_{lk}\gamma = & - \partial_{lk}A(\nabla\varphi_+)\partial_{lk}\varphi_+ \\ & - \partial_{lk}A(\nabla\gamma - \nabla w(y))\partial_{lk}w \\ \le & - v_+ - \mathcal{M}^-_\lambda(D^2 w) \\ & + 2|D^2w|^2\chi_{[|\nabla\gamma-\nabla w(y)|\ge\frac{3}{\varepsilon}]} \\ \le & - v + g\chi_{[|\nabla\gamma-\nabla w(y)|\ge\frac{3}{\varepsilon}]} \end{aligned} \tag{5.18}$$

where $g := 2|D^2w|^2$ and

$$\| \, g \, \|_{L^s(B^n_{3\varrho_1}(0))} \le C_{n,p}(\varepsilon,\varrho_0)\delta \tag{5.19}$$

with (5.14).

We claim from (5.18) that $\gamma$ is a $W^{2,s}$-viscosity subsolution of

$$-\partial_{lk}A(\nabla\gamma - \nabla w(y))\partial_{lk}\gamma \le -v + g\chi_{[|\nabla\gamma-\nabla w(y)|\ge\frac{3}{\varepsilon}]} \quad \text{in } B^n_{3\varrho_1}(0). \tag{5.20}$$

Indeed recalling Definition 5.1, we have to consider test functions $\eta, \xi := \eta - w \in W^{2,s}(U'), U' \subset\subset B^n_{3\varrho_1}(0)$. Clearly, $\gamma - \eta = \varphi_+ - \xi$, and (5.20) follows from (4.6) when we replace $\gamma$ and $\varphi_+$ by $\eta$ and $\xi$, respectively, in (5.18).

Putting

$$F_r(p,X) := \sup_{|q|\le r}(\partial_{lk}A(p+q)X_{lk}),$$

we simplify (5.20) and obtain with (5.16) that

$$-F_\varepsilon(\nabla\gamma, D^2\gamma) \le -v + g\chi_{[|\nabla\gamma|\ge\frac{2}{\varepsilon}]} \quad \text{in } B^n_{3\varrho_1}(0). \tag{5.21}$$

Now, we consider the sup-convolution for $\gamma$ given by

$$\gamma^\varepsilon(y) := \sup_{z \in B^n_{\varrho_0}(0)} \left( \gamma(z) - \frac{1}{\varepsilon}|y-z| \right) \quad \text{for } y \in B^n_{\varrho_1}(0). \tag{5.22}$$

Since $\varphi_+$ is bounded from above by (3.12) and $\varphi_+ \not\equiv -\infty$ in $B^n_{\varrho_1}(0)$ by Lemma 3.4, we observe from Lemma 5.2 that the supremum is attained in the interior of $B^n_{3\varrho_1}(0)$ if $\varepsilon$ is small enough, still under the assumption of (5.15).

Then by standard procedure for sup-convolutions; see [6] §5.1, we get that $\gamma^\varepsilon$ is a $W^{2,s}$-viscosity subsolution of

$$-F_\varepsilon(\nabla\gamma^\varepsilon, D^2\gamma^\varepsilon) \leq \sup_{B^n_{3\varrho_1}(0)} -v \quad \text{in } B^n_{\varrho_1}(0). \tag{5.23}$$

We observe that the term $g\chi_{[|\nabla\gamma^\varepsilon| \geq \frac{2}{\varepsilon}]}$ drops out since $|\nabla\gamma^\varepsilon| \leq \frac{1}{\varepsilon}$. Further, the equation (5.23) is uniformly elliptic since $\gamma^\varepsilon$ is Lipschitz-continuous.

Now, we can apply Caffarelli's and Trudinger's theorem to conclude that

$$\overline{\theta}(\gamma^\varepsilon, B^n_{\varrho_1}(0)) < \infty \quad \text{almost everywhere in } B^n_{\varrho_1}(0), \tag{5.24}$$

and as $\gamma \leq \gamma^\varepsilon$ that

$$\overline{\theta}(\gamma, B^n_{\varrho_1}(0)) < \infty \quad \text{almost everywhere on } [\gamma^\varepsilon = \gamma] \cap B^n_{\varrho_1}(0).$$

On the other hand,

$$\theta(w, B^n_{3\varrho_1}(0)) < \infty \quad \text{almost everywhere on } B^n_{3\varrho_1}(0),$$

since $w \in W^{2,s}(B^n_{3\varrho_1}(0))$, and (5.17) yields

$$\overline{\theta}(\varphi_+, B^n_{\varrho_1}(0)) < \infty \quad \text{almost everywhere on } [\gamma^\varepsilon = \gamma] \cap B^n_{\varrho_1}(0).$$

Finally, we observe from (5.16) that

$$[\varphi_+^{2\varepsilon} = \varphi_+] \cap B^n_{\varrho_1}(0) \subseteq [\gamma^\varepsilon = \gamma],$$

and (5.9) follows observing

$$\mathcal{L}^n([\varphi_+^\varepsilon = \varphi_+] \cap B^n_{\varrho_1}(0)) \nearrow \mathcal{L}^n([\varphi_\pm \in \mathbb{R}] \cap B^n_{\varrho_1}(0))$$

by Lemma 4.1. q.e.d.

With this lemma, we are able to prove the desired height-excess and tilt-excess decays mentioned at the beginning of this section.

**Lemma 5.4.** *Then for $\mu_V$-almost all $x \in \Sigma$, the height-excess and the tilt-excess decay quadratically that is*

$$\text{heightex}_V(x, \varrho, T_xV), \text{tiltex}_V(x, \varrho, T_xV) = O_x(\varrho^2). \tag{5.25}$$

*Proof.* First, we consider 0 to be a generic point and $\varrho_0 > 0$ as above. Clearly, (5.8) implies the estimate for the height-excess for $x = (y, \varphi_\pm(y))$ and $\mathcal{L}^n$-almost all $y \in [\varphi_+ = \varphi_-] \cap B^n_{\varrho_1}(0)$.

Since the tilt-excess is controlled by the height-excess and the mean curvature through the following estimate; see [3] Theorem 5.5 or [33] Lemma 22.2,

$$\text{tiltex}_V(x, \varrho/2, T) \leq C\,\text{heightex}_V(x, \varrho, T) + C\varrho^{2-n} \int\limits_{B^{n+1}_\varrho(x)} |\vec{\mathbf{H}}_V|^2 d\mu_V,$$

we obtain a quadratic tilt-excess decay

$$\text{tiltex}_V(x, \varrho, T_xV) = O_x(\varrho^2)$$

when $x$ is a Lebesguepoint of $\vec{\mathbf{H}}_V \in L^2_{\text{loc}}(\mu_V)$ and $\theta^n(\mu_V, x) < \infty$, hence for $\mathcal{L}^n$-almost all $y \in [\varphi_+ = \varphi_-] \cap B^n_{\varrho_1}(0)$.

Putting $Q := \{x \in \Sigma | x \text{ satisfies } (5.25)\}$, this yields

$$\mu_V(B^{n+1}_{\varrho_1}(0) \cap \Sigma_0 - Q) = 0,$$

and by (3.18) and since $\theta^n(\mu_V, 0) \geq 1$ that

$$\theta(\mu_V, \Omega - Q, 0) = 0.$$

On the other hand, this density is equal to 1 almost everywhere with respect to $\mu_V$; see, for example, [33] Theorem 4.7 or consider Lebesguepoints of $\chi_{\Omega - Q} \in L^1_{\text{loc}}(\mu_V)$. Therefore $\mu_V(\Omega - Q) = 0$, and the lemma is proved. q.e.d.

We conclude this section by converting (4.6) and (4.7) into pointwise estimates. To this end, we have to know that $\varphi_\pm$ have second order derivatives in some sense. We fix the following notion; see [14] 2.9.12 and 3.1.2.

**Definition 5.5.** A function $\varphi : U \to [-\infty, \infty]$, with $U \subseteq \mathbb{R}^n$ open, is called twice approximately differentiable at $y \in U$ if $\varphi(y) \in \mathbb{R}$ and there exist $b \in \mathbb{R}^n, X \in S(n)$ satisfying

$$\text{ap} - \lim_{z \to y} \frac{|\varphi(z) - \varphi(y) - b(z-y) - \frac{1}{2}(z-y)^T X (z-y)|}{|z-y|^2} = 0.$$

In this case, we set the approximate differentials to be

$$\nabla\varphi(y) := b \quad \text{and} \quad D^2\varphi(y) := X.$$

Clearly, two functions $\varphi$ and $\gamma$ are twice approximately differentiable at a point $y$ which has full density in the set where $\varphi$ and $\gamma$ coincide if and only if the other function is twice approximately differentiable at $y$ too, and in this case the approximate differentials are the same.

The proof of the following lemma is standard; see [7] Propositions 3.4, 3.5 and [36] Theorem 4.20.

**Lemma 5.6.** *$\varphi_\pm$ are twice approximately differentiable $\mathcal{L}^n$-almost everywhere on $[\varphi_\pm \in \mathbb{R}] \cap B^n_{\varrho_1}(0)$, and the approximate differentials satisfy*

$$-\nabla\left(\frac{\nabla\varphi_+}{\sqrt{1+|\nabla\varphi_+|^2}}\right)(y) \leq -u(y,\varphi_\pm(y)) \tag{5.26}$$

*and*

$$-\nabla\left(\frac{\nabla\varphi_-}{\sqrt{1+|\nabla\varphi_-|^2}}\right)(y) \geq \begin{cases} -u(y,\varphi_\pm(y)) & \text{if } \theta_0 \text{ is odd,} \\ u(y,\varphi_\pm(y)) & \text{if } \theta_0 \text{ is even.} \end{cases} \tag{5.27}$$

*for $\mathcal{L}^n$-almost all $y \in [\varphi_+ = \varphi_-] \cap B^n_{\varrho_1}(0)$.*

*Proof.* We use the notion of the proof of Lemma 5.3.

First we establish the twice approximate differentiability of $\varphi_\pm$. Since $w \in W^{2,s}(B^n_{3\varrho_1}(0))$ is almost everywhere twice differentiable; see [13] Theorem 6.2.1, and $\mathcal{L}^n([\varphi^\varepsilon_+ = \varphi_+] \cap B^n_{\varrho_1}(0)) \nearrow \mathcal{L}^n([\varphi_\pm \in \mathbb{R}] \cap B^n_{\varrho_1}(0))$ by Lemma 4.1, we see from the remark above that it suffices to prove that $\gamma^\varepsilon$ is twice approximately differentiable almost everywhere on $B^n_{\varrho_1}(0)$.

We define the function

$$\gamma^\varepsilon_\sigma(y) := \gamma^\varepsilon(y) - \sigma|y|^2$$

and consider its concave envelope $\Gamma_\sigma := \operatorname{conc}\gamma^\varepsilon_\sigma \geq \gamma^\varepsilon_\sigma$. Now $\Gamma_\sigma$ is twice differentiable almost everywhere by Alexandroff's Theorem; see [13, Theorem 6.4.1], and by the remark above, we see that $\gamma^\varepsilon$ is twice approximately differentiable almost everywhere on $[\gamma^\varepsilon_\sigma = \Gamma_\sigma]$.

Clearly,

$$[\overline{\theta}(\gamma^\varepsilon, B^n_{\varrho_1}(0)) \leq \sigma] = [\gamma^\varepsilon_\sigma = \Gamma_\sigma].$$

Further by (5.24), we get

$$\mathcal{L}^n([\overline{\theta}(\gamma^\varepsilon, B^n_{\varrho_1}(0)) \le \sigma]) \nearrow \mathcal{L}^n(B^n_{\varrho_1}(0)),$$

and $\gamma^\varepsilon$ is twice approximately differentiable almost everywhere on $B^n_{\varrho_1}(0)$.

Since $\Gamma_\sigma \ge \gamma^\varepsilon_\sigma$, the approximate differentials just obtained are also superdifferentials that is

$$\limsup_{z\to y} \frac{\gamma^\varepsilon(z) - \gamma^\varepsilon(y) - \nabla\gamma^\varepsilon(y)(z-y) - \frac{1}{2}(z-y)^T D^2\gamma^\varepsilon(y)(z-y)}{|z-y|^2} \le 0$$

for almost all $y \in B^n_{\varrho_1}(0)$.

Therefore, we get from (5.23) that

$$-F_\varepsilon(\nabla\gamma^\varepsilon, D^2\gamma^\varepsilon)(y) \le \sup_{B^n_{3\varrho_1}(0)} -v$$

for almost all $y \in B^n_{\varrho_1}(0)$.

Now choosing $v$ constant, more precisely choosing $v \equiv v_+(y_0)$ in a neighbourhood $B^n_\varrho(y_0)$ of a Lebesgue-point $y_0$ of $v_+$ such that (5.10) is replaced by

$$\varrho^{-\frac{n}{s}} \parallel v_+ - v_+(y_0) \parallel_{L^s(B^n_\varrho(y_0))} \le \delta$$

and rescaling in $\varrho$, we get

$$-F_\varepsilon(\nabla\gamma^\varepsilon, D^2\gamma^\varepsilon)(y) \le -v_+(y_0)$$

for almost all $y \in B^n_{\varrho/4}(y_0)$. We observe that (5.5) is satisfied for the sup-convolution of the rescaled $\gamma$ in $B^n_1(0)$ if $y_0 \in [\varphi^{5\varepsilon}_+ = \varphi_+]$ and $C_{n,p}(\varepsilon)\delta \le \varepsilon \le \sqrt{1/40}$.

Using (5.14) and (5.16), we get

$$-F_{2\varepsilon}(\nabla\varphi_+, D^2\varphi_+)(y) \le -v_+(y) + g_{y_0}(y)$$

for almost all $y \in [\varphi^{2\varepsilon}_+ = \varphi_+] \cap B^n_{\varrho/4}(y_0)$ with

$$\varrho^{-n} \parallel g_{y_0} \parallel_{L^1(B^n_{\varrho/4}(y_0))} \le C\varrho^{-\frac{n}{s}} \parallel g_{y_0} \parallel_{L^s(B^n_{\varrho/4}(y_0))} \le C_{n,p}(\varepsilon)\delta.$$

Using Vitali's or Besicovitch's Covering Theorem; see [13] Theorem 1.5.1 or 1.5.2., we get

$$-F_{2\varepsilon}(\nabla\varphi_+, D^2\varphi_+)(y) \le -v_+(y) + g_\delta(y)$$

for almost all $y \in [\varphi_+^{5\varepsilon} = \varphi_+] \cap B^n_{\varrho_1}(0)$ with

$$\varrho_1^{-n} \parallel g_\delta \parallel_{L^1(B^n_{\varrho_1}(0))} \leq C_{n,p}(\varepsilon)\delta.$$

Letting $\delta \to 0$ and then $\varepsilon \to 0$, we arrive at (5.26), concluding the proof. q.e.d.

## 6. Approximation with $C^{1,1}$-manifolds

We assume as in the previous sections that 0 is a generic point, and we consider the upper and lower height functions $\varphi_\pm$.

In this section, we assume additionally that $u$ is approximately continuous in 0 with respect to $\mu_V$ that is

$$(\mu_V)\mathrm{ap} - \lim_{x \to 0} u(x) = u(0). \tag{6.1}$$

Since $\varrho^{-n}\mu_V(B^{n+1}_\varrho(0)) \to \theta_0\omega_n$, we get for any $\delta > 0$ that

$$\varrho^{-n}\mu_V([|u - u(0)| > \delta] \cap B^{n+1}_\varrho(0)) \leq \omega_\delta(\varrho). \tag{6.2}$$

From [14] Theorem 2.9.13, we know that $u$ is approximately continuous with respect to $\mu_V$ at $\mu_V$-almost all points $x \in \Sigma = \operatorname{spt} V$.

We are already able to prove the case for even multiplicity.

**Lemma 6.1.** *If $\theta_0$ is even then*

$$u(0) \leq 0. \tag{6.3}$$

*Proof.* From Lemma 5.6, we know

$$\begin{aligned} u(y, \varphi_\pm(y)) &\leq -\nabla\left(\frac{\nabla\varphi_-}{\sqrt{1 + |\nabla\varphi_-|^2}}\right)(y) \\ &= -\nabla\left(\frac{\nabla\varphi_+}{\sqrt{1 + |\nabla\varphi_+|^2}}\right)(y) \\ &\leq -u(y, \varphi_\pm(y)) \end{aligned}$$

for almost all $y \in [\varphi_+ = \varphi_-] \cap B^n_{\varrho_1}(0)$. This yields $u(y, \varphi_\pm(y)) \leq 0$, and

$$\varrho^{-n}\mu_V(B^{n+1}_{2\varrho}(0) \cap [u \leq 0]) \geq \varrho^{-n}\mathcal{L}^n(B^n_\varrho(0) \cap [\varphi_+ = \varphi_-]) \geq \omega_n - \omega(\varrho)$$

by Lemma 3.4. Together with (6.2), this yields

$$u(0) \leq 0. \qquad \text{q.e.d.}$$

For odd multiplicity, we approximate $V$ in the sequel by a $C^{1,1}$-graph.

When $\theta_0$ is odd, we know form (3.16) that the height functions take only finite values that is $\varphi_\pm \in \mathbb{R}$. In particular, we get $\varphi_- \leq \varphi_+$ and Lemma 5.3 strengthens to

$$\begin{aligned} &\theta(\varphi_\pm, B^n_{\varrho_1}(0)) < \infty \\ &\qquad \mathcal{L}^n\text{-almost everywhere on } [\varphi_+ = \varphi_-] \cap B^n_{\varrho_1}(0). \end{aligned} \tag{6.4}$$

Therefore for almost all $y \in [\varphi_+ = \varphi_-] \cap B^n_{\varrho_1}(0)$, there exists an affine function $l_y$ such that

$$\| \varphi_\pm - l_y \|_{L^\infty(B^n_\varrho(y))} \leq C_y \varrho^2 \tag{6.5}$$

which is

$$\varphi_\pm \in T^{2,\infty}(y)$$

in the sense of [37] 3.5.4.

Next we choose $0 < \varrho_2 < \varrho_1$ and get from Lusin-type Theorems; see [37] Theorem 3.6.2 and Lemma 3.7.1, that there is a set $Q \subseteq B^n_{\varrho_2}(0)$ and a function $\psi \in C^{1,1}(B^n_{\varrho_2}(0))$ satisfying

$$Q \subseteq [\varphi_+ = \varphi_-] \cap B^n_{\varrho_2}(0) \cap \pi(\Sigma_0), \tag{6.6}$$

$$\varrho_2^{-n} \mathcal{L}^n(Q \cap B^n_{\varrho_2}(0)) \geq \omega_n - 3\omega(\varrho_2), \tag{6.7}$$

$$D^\beta \varphi_\pm = D^\beta \psi \quad 0 \leq |\beta| \leq 2 \text{ on } Q. \tag{6.8}$$

The estimate (6.7) is obtained from Lemma 3.4 when we observe from the definition of $\Sigma_0 \subseteq \pi^{-1}([\varphi_+ = \varphi_-])$ that

$$\pi(\Sigma_0) \cap \pi(B^{n+1}_{\varrho_0}(0) - \Sigma_0) = \emptyset.$$

(6.8) equates the differentials of $\psi$ which exist almost everywhere by Alexandroff's Theorem; see [13] Theorem 6.4.1, and the approximate differentials of $\varphi_\pm$ which exist almost everywhere on $[\varphi_+ = \varphi_-] \cap B^{n+1}_{\varrho_0}(0)$ by Lemma 5.6.

We fix $\Gamma \geq \Lambda$ such that

$$\| \psi \|_{C^{1,1}(B^n_{\varrho_2}(0))} \leq \Gamma$$

and may assume that

$$\| \psi \|_{L^\infty(B^n_{\varrho_2}(0))} \leq 2\omega(\varrho_2)\varrho_2,$$

as $\| \varphi_\pm \|_{L^\infty(B^n_{\varrho_2}(0))} \leq \omega(\varrho_2)\varrho_2$.

We define the approximating $C^{1,1}$-manifold

$$M := \operatorname{graph} \psi | B^n_{\varrho_2}(0) \subseteq Z_{\varrho_2} \quad \text{and} \quad W := \mathbf{v}(M, \theta_0). \tag{6.9}$$

For $y \in B^n_{\varrho_2}(0)$, we see that

$$\nu(y) := \frac{(-\nabla\psi(y), 1)}{\sqrt{1 + |\nabla\psi(y)|^2}}$$

is the normal of $M$ pointing upwards. Abbreviating the plane whose normal is a given unit vector $\nu$ by putting

$$T_{\pm\nu} := \{z \in \mathbb{R}^{n+1} | \nu z = 0\},$$

we get

$$T_{\nu(y)} = T_{(y,\psi(y))} W. \tag{6.10}$$

From (6.8), we see that

$$\nu(y) = \frac{(-\nabla\psi(y), 1)}{\sqrt{1 + |\nabla\psi(y)|^2}} = \frac{(-\nabla\varphi_\pm(y), 1)}{\sqrt{1 + |\nabla\varphi_\pm(y)|^2}} \quad \text{for } y \in Q. \tag{6.11}$$

In the sequel, we will abbreviate

$$x = (y, \varphi_\pm(y)) = (y, \psi(y)) \quad \text{for } y \in Q.$$

The following proposition summarizes the approximation properties of $M$ which are of first order.

**Proposition 6.2.** *$W$ approximates $V$ in the sense that*

$$\mu_V \lfloor Z_{\varrho_2} \cap \pi^{-1}(Q) = \mu_W \lfloor Z_{\varrho_2} \cap \pi^{-1}(Q), \tag{6.12}$$

$$V \lfloor G(Z_{\varrho_2} \cap \pi^{-1}(Q)) = W \lfloor G(Z_{\varrho_2} \cap \pi^{-1}(Q)), \tag{6.13}$$

*and for $\mathcal{L}^n$-almost all $y \in Q$*

(6.14)
$$\varrho^{-n} \Big( \mu_V(B^{n+1}_\varrho(x) - \pi^{-1}(Q)) + \mu_W(B^{n+1}_\varrho(x) - \pi^{-1}(Q)) \Big) \leq \omega_y(\varrho).$$

*Proof.* We see for $y \in Q$ that

$$x \in \Sigma_0 \cap M \cap Z_{\varrho_2}$$

and

$$\theta^n(\mu_V, x) = \theta_0 = \theta^n(\mu_W, x).$$

This yields (6.12).

Next, we see from (6.5) that

$$|\varphi_\pm(z) - \varphi_\pm(y) - \nabla\varphi_\pm(y)(z-y)| \leq C_y|z-y|^2 \quad \text{for } z \in B^{n+1}_{\varrho_2}(0).$$

Recalling (6.10) and (6.11), we observe that

$$\operatorname{spt}(\zeta_{x,\varrho,\#}V) \to T_{\nu(y)} = T_xW,$$

where $\zeta_{x,\varrho}(\xi) := \varrho^{-1}(\xi - x)$.

On the other hand, $x \in \Sigma_0$ and $T_xV$ exists and

$$T_xV = T_xW,$$

which proves (6.13).

To prove (6.14), we calculate for $\varrho$ small and using (6.12) and (6.13) that

$$\begin{aligned}
&(\omega_n\varrho^n)^{-1}\mu_{V,W}(B^{n+1}_\varrho(x) \cap \pi^{-1}(Q)) \\
&\quad = (\omega_n\varrho^n)^{-1}\theta_0 \int_Q \chi_{B^{n+1}_\varrho(x)}(z,\psi(z))\sqrt{1+|\nabla\psi(z)|^2}dz \\
&\quad \geq (\omega_n\varrho^n)^{-1}\mu_W(B^{n+1}_\varrho(x) \cap M) \\
&\qquad - C\Gamma\theta_0\varrho^{-n}\mathcal{L}^n(B^n_\varrho(y) - Q),
\end{aligned}$$

and if $\theta(\mathcal{L}^n, D, y) = 1$ that

$$\begin{aligned}
&\lim_{\varrho\to 0}(\omega_n\varrho^n)^{-1}\mu_{V,W}(B^{n+1}_\varrho(x) \cap \pi^{-1}(Q)) \\
&\qquad \geq \theta_0 = \lim_{\varrho\to 0}(\omega_n\varrho^n)^{-1}\mu_{V,W}(B^{n+1}_\varrho(x)).
\end{aligned}$$

This yields

$$\lim_{\varrho\to 0}\varrho^{-n}\Big(\mu_V(B^{n+1}_\varrho(x) - \pi^{-1}(Q)) + \mu_W(B^{n+1}_\varrho(x) - \pi^{-1}(Q))\Big) = 0$$

which is (6.14). q.e.d.

We are now ready to prove the case for odd multiplicity.

**Lemma 6.3.** *If $\theta_0$ is odd and $\theta_0 \neq 1$ then*

$$u(0) = 0. \tag{6.15}$$

*Proof.* We state three claims:

$$\vec{\mathbf{H}}_W(x) = \nabla \left( \frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} \right)(y) \nu(y) = u(x)\nu(y). \tag{6.16}$$

$$\theta_0 \vec{\mathbf{H}}_V(x) = u(x)\nu(y). \tag{6.17}$$

$$\vec{\mathbf{H}}_V(x) = \vec{\mathbf{H}}_W(x) \tag{6.18}$$

for almost all $y \in Q$.

(6.18) shows that the approximation $W$ of $V$ is of second order; see (1.15) and the remark there.

Since $M$ is $C^{1,1}$, (6.16) is immediate from (6.11), Lemma 5.6 and (6.8).

Next, we use (1.14) which was justified in Conclusion (i) of Section 2. Since $y = \pi(x) \in Q \subseteq \pi(\Sigma_0)$, we see that $x$ is generic and, using (6.10), (6.11) and (6.13), we get that

$$T_x V = T_{\nu(y)}$$

for almost all $y \in Q$. Secondly, since $\theta_0$ is odd, Lemma 3.2 yields that $x \in \partial_* E$ and, since $\mathcal{H}^n(\partial_* E - \partial^* E) = 0$; see [13] Lemma 5.8.1,

$$x \in \partial^* E$$

for almost all $y \in Q$.

By DeGiorgi's Theorem; see [33] Theorem 14.3, we see that $\nu_E(x)$ is normal to the tangent plane $T_x V$, hence $\nu_E(x) = \pm\nu(y)$ and

$$\nu_E(x) = \nu(y),$$

since $\nu(y)$ is pointing upwards to the inside of $E$ by (3.14), and $\nu_E(x)$ is the inner normal. Then (1.14) yields (6.17).

We turn to (6.18). We choose $\chi \in C_0^\infty(B_1^{n+1}(0))$ rotationally symmetric with

$$0 \leq \chi \leq 1 \quad \text{and} \quad \chi \equiv 1 \text{ on } B_{\frac{1}{2}}^{n+1}(0).$$

and put $\chi_\varrho(\xi) := \chi(\varrho^{-1}(\xi - x))$.

We calculate for $U = V, W$ that

$$\begin{aligned}\lim_{\varrho\to 0}(\omega_n\varrho^n)^{-1}\delta U(\chi_\varrho) =& -\lim_{\varrho\to 0}(\omega_n\varrho^n)^{-1}\int\limits_{B_\varrho^{n+1}(x)}\chi_\varrho\vec{\mathbf{H}}_U d\mu_U\\ =& -\omega_n^{-1}\theta_0\vec{\mathbf{H}}_U(x)\int\limits_{T_xU\cap B_1^{n+1}(0)}\chi d\mathcal{L}^n\end{aligned}$$

if $x$ is a Lebesgue-point of $\vec{\mathbf{H}}_U$, hence for almost all $y \in Q$.

Since we already know from (6.16) and (6.17) that $\vec{\mathbf{H}}_V(x), \vec{\mathbf{H}}_W(x) \in \operatorname{span}\{\nu(y)\}$, in order to get (6.18), it suffices to prove that

$$I_\varrho := \varrho^{-n}(\delta V(\chi_\varrho) - \delta W(\chi_\varrho))\nu(y) \to 0 \quad \text{when } \varrho \downarrow 0. \tag{6.19}$$

We recall for $U = V, W$ that

$$\delta U(\chi_\varrho)\nu(y) = \int\limits_{B_\varrho^{n+1}(x)} D\chi_\varrho(\xi)T_\xi U\nu(y)d\mu_U(\xi)$$

and abbreviate $T := T_xV = T_xW$ and

$$R_{\varrho,U} := \varrho^{-n}\int\limits_{B_\varrho^{n+1}(x)-\pi^{-1}(Q)} D\chi_\varrho(\xi)(T_\xi U - T)\nu(y)d\mu_U(\xi).$$

Using (6.13) and $T\nu(y) = 0$, as $\nu(y)$ is normal to $T$, we obtain that

$$I_\varrho = R_{\varrho,V} - R_{\varrho,W}.$$

We estimate

$$\begin{aligned}|R_{\varrho,U}| \le& C\varrho^{-n-1}\int\limits_{B_\varrho^{n+1}(x)-p^{-1}(Q)} \| T_\xi U - T \| \, d\mu_U(\xi)\\ \le& C\varrho^{-1}\Big(\varrho^{-n}\mu_U(B_\varrho^{n+1}(x) - p^{-1}(Q))\Big)^{\frac{1}{2}}\\ &\cdot\Big(\varrho^{-n}\int\limits_{B_\varrho^{n+1}(x)} \| T_\xi U - T \|^2 \, d\mu_U(\xi)\Big)^{\frac{1}{2}}\\ \le& C\varrho^{-1}\omega_y(\varrho)^{\frac{1}{2}}\operatorname{tiltex}_U(x,\varrho,T)^{\frac{1}{2}},\end{aligned}$$

where we have used (6.14).

Now for $U = V$, we have quadratic decay of the tilt-excess for $\mathcal{L}^n$-almost all $y \in Q$ by Lemma 5.4, whereas such decay is immediate for $U = W$, since $|D^2\psi| \leq \Gamma$. Therefore

$$|R_{\varrho,U}| \leq C_{y,\Gamma}\omega_y(\varrho)^{\frac{1}{2}}$$

which proves (6.19), hence (6.18).

Combining (6.16), (6.17), (6.18) with $\theta_0 \neq 1$, we conclude that

$$u(x) = 0$$

for almost all $y \in Q$.

Using (6.7), we obtain

$$\varrho_2{}^{-n}\mu_V(B^{n+1}_{2\varrho_2}(0) \cap [u = 0]) \geq \varrho_2{}^{-n}\mathcal{L}^n(Q \cap B^n_{\varrho_2}(0)) \geq \omega_n - 3\omega(\varrho_2).$$

Together with (6.2), this yields

$$u(0) = 0.$$

q.e.d.

Finally, we infer Theorems 1.1 and 1.2 from (1.14), Lemmas 3.2, 6.1, 6.3 and since $\mathcal{H}^n(\partial_* E - \partial^* E) = 0$ by [13] Lemma 5.8.1.

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