Hypertableau and Path-Hypertableau Calculi for some Families of Intermediate Logics

Agata Ciabattoni and Mauro Ferrari

Dipartimento di Scienze dell'Informazione
Università degli Studi di Milano
Via Comelico 39, 20135 Milano–Italy
{ciabatto,ferram}@dsi.unimi.it

Abstract. In this paper we investigate the tableau systems corresponding to hypersequent calculi. We call these systems hypertableau calculi. We define hypertableau calculi for some propositional intermediate logics. We then introduce path-hypertableau calculi which are simply defined by imposing additional structure on hypertableaux. Using path-hypertableaux we define analytic calculi for the intermediate logics Bd_k, with $k \geq 1$, which are semantically characterized by Kripke models of depth $\leq k$. These calculi are obtained by adding one more structural rule to the path-hypertableau calculus for Intuitionistic Logic.

1 Introduction

Hypersequent calculi are a simple and natural generalization of Gentzen sequent calculi to *sets* of sequents (see [4] for an overview). Hypersequents allow to formalize logics of a different nature ranging from modal to many-valued logics.

In this paper we are concerned with intermediate logics, that is, logics between Intuitionistic and Classical Logic. In [4, 3, 9, 8] cut-free hypersequent calculi have been defined for:

1. the logics Bw_k, with $k \geq 1$, which are semantically characterized by Kripke models of width $\leq k$;
2. the logics Bc_k, with $k \geq 1$, which are semantically characterized by Kripke models of cardinality $\leq k$;
3. the logics G_{k+1}, with $k \geq 1$, which are semantically characterized by linearly ordered Kripke models of cardinality $\leq k$;
4. *LQ* logic (also known as Jankov logic [18]).

In particular Bw_1 coincides with *infinite-valued Gödel (Dummett) logic* G_∞, while for $k \geq 2$, G_k is k-valued Gödel logic. Bc_2 is identical with *Sm* logic [7].

In the literature, there do exist sequent or tableau calculi for some of these logics. For instance, in [1] duplication-free tableau calculi for *Sm*, *LQ* and G_∞ have been defined (see also [10] for a deterministic terminating sequent calculus for G_∞). Analytic calculi for finite-valued Gödel logics, based on their many-valued semantics, can be found, e.g., in [19, 14, 6]. Nevertheless all these calculi are so much tailored to their corresponding logic that they hardly give information on the existing connections with other logics. In particular, they cannot help to define analytic calculi for related logics.

Hypersequent calculi for all the above logics are simply obtained by adding just one structural rule to a common system, namely the hypersequent calculus for Intuitionistic Logic [4, 3, 9, 8]. This structural rule reflects in a natural way the characteristic semantical features of the corresponding logic.

In this paper we introduce the notion of hypertableau[1]. Hypertableau calculi stand to hypersequent calculi as tableau systems stand to sequent calculi.

By dualizing the hypersequent calculi of [4, 3, 9, 8] we define hypertableau calculi for the logics $\mathrm{Bw}_k, \mathrm{Bc}_k, \mathrm{G}_k$ and LQ. Then, by simply generalizing the peculiar rule of LQ logic, we obtain a hypertableau calculus for the family of intermediate logics semantically characterized by rooted posets with at most k final states. Although hypertableaux (hypersequents) turn out to be more expressive than tableaux (sequents), there do exist intermediate logics with a simple Kripke semantics for which such calculi seem to be hardly definable. An important example of a logic for which no hypertableau (hypersequent) systems have been devised so far is the logic Bd_2. This logic is semantically characterized by the class $\mathcal{F}_{d\leq 2}$ of all rooted posets whose depth is at most 2 (see [7]). As is well known, like LQ, Sm and G_∞, Bd_2 is one of the seven interpolable propositional logics [15]. More generally, for each $k > 2$, the intermediate logic Bd_k semantically characterized by Kripke models of depth $\leq k$ does not have any tableau (sequent)-style formalization yet.

In Section 4 we introduce a new hypertableau framework, called path-hypertableaux. The notion of path-hypertableau naturally arises by introducing additional structure on hypertableaux. Using path-hypertableaux we define uniform analytic calculi for the Bd_k logics, with $k \geq 1$. This is done by adding one more structural rule to the path-hypertableau calculus for intuitionisticlogic.

We assume familiarity with intermediate logics and Kripke models. Introductory material can be found, e.g., in [7]. Henceforth we shall denote by Int and Cl the set of valid well-formed formulas (*wffs* for short) of propositional intuitionistic logic and classical logic, respectively.

2 Hypersequent calculi

Hypersequent calculi [17, 2] are a simple and natural generalization of Gentzen calculi. See [4] for an overview.

Definition 1. *A* hypersequent *is an expression of the form*

$$\Gamma_1 \vdash \Delta_1 \mid \ \dots \ \mid \Gamma_n \vdash \Delta_n,$$

where, for all $i = 1, \dots n$, $\Gamma_i \vdash \Delta_i$ is an ordinary sequent. $\Gamma_i \vdash \Delta_i$ is called a component *of the hypersequent.*

[1] The name "hypertableau" was already used in [5] in the context of a different kind of calculus.

The intended meaning of the symbol | is disjunctive.

Like in ordinary sequent calculi, in a hypersequent calculus there are axioms and rules, which are divided into *logical* and *structural rules*. The logical rules are the same as in sequent calculi but for the presence of dummy contexts, denoted by G and G', that are called *side hypersequents*. For instance, in the hypersequent calculus for Intuitionistic Logic, the rules for the $\rightarrow$ connective are:

$$(\rightarrow, l) \quad \frac{G \mid \Gamma \vdash A \quad G' \mid \Gamma, B \vdash C}{G \mid G' \mid \Gamma, A \rightarrow B \vdash C} \qquad\qquad (\rightarrow, r) \quad \frac{G \mid \Gamma, A \vdash B}{G \mid \Gamma \vdash A \rightarrow B}$$

The structural rules can either be *internal* or *external*. The internal rules deal with wff's within components. They are the same as in ordinary sequent calculi. The external rules manipulate whole components within a hypersequent. They are external weakening (EW), exchange (EE) and contraction (EC):

$$(EW) \quad \frac{G}{G \mid G'} \qquad (EE) \quad \frac{G \mid G'}{G' \mid G} \qquad (EC) \quad \frac{G \mid G' \mid G'}{G \mid G'}$$

In hypersequent calculi it is possible to define new kind of structural rules which simultaneously act on several components of one or more hypersequents. It is this type of rule which increases the expressive power of hypersequent calculi with respect to ordinary sequent calculi. See [2–4, 9, 8] for some examples of hypersequent calculi ranging from modal to many-valued logics.

3 Hypertableau calculi

It is well known that tableau calculi can be easily obtained by dualizing sequent calculi (see, e.g., [20, 11]). Here we define the tableau systems corresponding to hypersequent calculi. We call these systems hypertableau calculi.

As usual, a *signed formula* (*swff* for short) is an expression of the form $\mathbf{T}X$ or $\mathbf{F}X$ where X is any wff. The meaning of the signs $\mathbf{T}$ and $\mathbf{F}$ is as follows: Given a Kripke model $\underline{K} = \langle P, \leq, v \rangle$ and a swff H, we say that $\alpha \in P$ *realizes* H (in symbols $\alpha \rhd H$) if $H \equiv \mathbf{T}X$ and $\alpha \Vdash X$, or $H \equiv \mathbf{F}X$ and $\alpha \nVdash X$.

A set S of swff's is realized in $\underline{K}$ (in symbols $\alpha \rhd S$) if there exists an element α realizing all the swff's in S. S is *contradictory* if it contains either $\mathbf{T}\bot$ or both $\mathbf{T}X$ and $\mathbf{F}X$ for some wff X[2].

Definition 2. *An* h-set *is an expression of the form*

$$S_1 \mid \ldots \mid S_n$$

where, for all $i = 1, \ldots, n$, S_i *is a set of swff's.* S_i *is called a* component *of the h-set. An* h-configuration *is an expression of the form*

$$\Psi_1 \parallel \ldots \parallel \Psi_m$$

where, for all $i = 1, \ldots, n$, Ψ_i *is an h-set called* component *of the h-configuration.*

[2] This condition can be equivalently reformulated by requiring X to be a propositional variable.

As shown by the following definition the intended meaning of the symbol $|$ is conjunctive while the one of the symbol $\|$ is disjunctive.

Definition 3. *We say that an* h-set $S_1 \mid \ldots \mid S_n$ *is* realized *in* $\underline{K}$*, if all the sets* S_j*, with* $j = 1, \ldots, n$*, are realized in* $\underline{K}$*.*

An h-configuration $\Psi_1 \parallel \ldots \parallel \Psi_m$ *is* realized *in* $\underline{K}$*, if there exists* $j \in \{1, \ldots, m\}$ *such that* Ψ_j *is realized in* $\underline{K}$*.*

Definition 4. *An h-set is* contradictory *if at least one of the* S_i*, with* $i = 1, \ldots, n$*, is contradictory.*

Hypertableau calculi are defined by dualizing hypersequent calculi as follows: Given a hypersequent $\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$, each component $\Gamma_i \vdash \Delta_i$ translates into the set of swff's $\mathbf{T}(\Gamma_i) \cup \mathbf{F}(\Delta_i)$ where $\mathbf{T}(\Gamma_i) = \{\mathbf{T}A \mid A \in \Gamma_i\}$ and $\mathbf{F}(\Delta_i) = \{\mathbf{F}A \mid A \in \Delta_i\}$. Thus the above hypersequent translates into the h-set

$$\mathbf{T}(\Gamma_1) \cup \mathbf{F}(\Delta_1) \mid \ldots \mid \mathbf{T}(\Gamma_n) \cup \mathbf{F}(\Delta_n).$$

Clearly, the axioms of hypersequent calculi are translated into contradictory h-sets. As for the tableau rules, the hypertableau ones are obtained by simply reversing the corresponding hypersequent rules.

Definition 5. *A* hypertableau *for* $\Psi_1 \parallel \ldots \parallel \Psi_m$ *is a finite sequence of h-configurations obtained by applying the rules of the calculus to* $\Psi_1 \parallel \ldots \parallel \Psi_m$*.*

A hypertableau is said to be closed *if all the h-sets in its final configuration are contradictory.*

In Table 1 one can find the rules of the hypertableau calculus T-Int for Intuitionistic Logic coming from the above translation. The notation S, H with S set of swff's and H swff will denote the set $S \cup \{H\}$.

We remark that, as in the hypersequent calculus for Intuitionistic Logic, in T-Int the external structural rules are redundant.

Section 3.1. To simplify the notation, in the above rules we omitted the components of the h-configurations not involved in the derivation. E.g., the schema

$$\frac{\Psi_1 \parallel \ldots \parallel \Psi \mid \Phi \parallel \ldots \parallel \Psi_n}{\Psi_1 \parallel \ldots \parallel \Psi \parallel \ldots \parallel \Psi_n}\,\text{HEW}$$

illustrates the external weakening rule HEW spelt out in more detail, and similarly for the other rules.

Remark 1. In the above calculus the internal structural rules are internalized into logical rules, for the former are playing no semantical rôle.

Remark 2. By reversing the $(\vee, l), (\wedge, r), (\rightarrow, l)$ rules of the hypersequent calculus for Intuitionistic Logic one would obtain the following hypertableau rules:

$$\frac{\Psi \mid \Psi' \mid S, \mathbf{T}(A \vee B)}{\Psi \mid S, \mathbf{T}A \parallel \Psi' \mid S, \mathbf{T}B} \quad \frac{\Psi \mid \Psi' \mid S, \mathbf{F}(A \wedge B)}{\Psi \mid S, \mathbf{F}A \parallel \Psi' \mid S, \mathbf{F}B} \quad \frac{\Psi \mid \Psi' \mid S, \mathbf{T}(A \rightarrow B)}{\Psi \mid S, \mathbf{T}(A \rightarrow B), \mathbf{F}A \parallel \Psi' \mid S, \mathbf{T}B}$$

External Structural Rules

$$\frac{\Psi \mid \Phi}{\Psi}\,\text{HEW} \qquad \frac{\Psi \mid S}{\Psi \mid S \mid S}\,\text{HEC} \qquad \frac{\Psi \mid S_1 \mid S_2 \mid \Phi}{\Psi \mid S_2 \mid S_1 \mid \Phi}\,\text{HEE}$$

Logical Rules

$$\frac{\Psi \mid S, \mathbf{T}(A_1 \wedge A_2)}{\Psi \mid S, \mathbf{T}A_i}\,\mathbf{T}\wedge_i \quad \text{for } i = 1,2 \qquad \frac{\Psi \mid S, \mathbf{F}(A \wedge B)}{\Psi \mid S, \mathbf{F}A \parallel \Psi \mid S, \mathbf{F}B}\,\mathbf{F}\wedge$$

$$\frac{\Psi \mid S, \mathbf{T}(A \vee B)}{\Psi \mid S, \mathbf{T}A \parallel \Psi \mid S, \mathbf{T}B}\,\mathbf{T}\vee \qquad \frac{\Psi \mid S, \mathbf{F}(A_1 \vee A_2)}{\Psi \mid S, \mathbf{F}A_i}\,\mathbf{F}\vee_i \quad \text{for } i = 1,2$$

$$\frac{\Psi \mid S, \mathbf{T}(A \to B)}{\Psi \mid S, \mathbf{T}(A \to B), \mathbf{F}A \parallel \Psi \mid S, \mathbf{T}B}\,\mathbf{T}\to \qquad \frac{\Psi \mid S, \mathbf{F}(A \to B)}{\Psi \mid S^{\mathbf{T}}, \mathbf{T}A, \mathbf{F}B}\,\mathbf{F}\to$$

$$S^{\mathbf{T}} = \{\mathbf{T}X \mid \mathbf{T}X \in S\}$$

Table 1. Hypertableau calculus T-Int for Intuitionistic Logic

Nevertheless, these rules introduce non-determinism in proof search. It is easy to see that using the external rules, these rules are interderivable with the rules $\mathbf{T}\vee, \mathbf{F}\wedge$ and $\mathbf{T}\to$ of T-Int, respectively.

Any hypertableau rule coming from the above translation is *correct*, i.e., when its premise is realized, so is the conclusion. This immediately follows from the correctness of the corresponding hypersequent rule. Thus, the proof of the soundness theorem for the hypersequent calculus can be translated into a proof of the soundness theorem for the corresponding hypertableau calculus. Accordingly, one can look at the proof of the completeness theorem given for the hypersequent calculus as a completeness proof for the corresponding hypertableau calculus. Indeed, each proof of a valid hypersequent directly translates into a closed hypertableau.

Theorem 1. *A wff A is intuitionistically valid iff there exists a closed hypertableau for* $\{\mathbf{F}A\}$ *in* T-Int.

3.1 On Hypertableaux for Intermediate Logics

The hypertableau framework is stronger than that of tableau. Intuitively, the former allows to formalize logics whose properties can be simply expressed in a disjunctive form. This section is devoted to define hypertableau calculi for some families of intermediate logics and to give some insights on the expressive power of hypertableaux (hypersequents).

In [4, 3, 9, 8] cut-free hypersequent calculi have been defined for the intermediate logics Bw_k, Bc_k, G_{k+1} and LQ, whose semantics are given by

Bw_k **:** the class of finite trees not containing $k+1$ pairwise incomparable nodes (for short, finite trees of *width* $\leq k$);
Bc_k **:** the class of trees containing at most k nodes;
G_{k+1} **:** the class of trees of width ≤ 1 and with at most k nodes;
$\boldsymbol{LQ}$ **:** the class of finite posets with a single final node.

By dualizing these calculi as described in the previous section, one gets sound and complete hypertableau calculi for $\mathrm{Bw}_k, \mathrm{Bc}_k, \mathrm{G}_{k+1}$ and LQ. All these logics share the property that their Kripke models $\underline{K}$ can be described by disjunctively combining "basic" conditions of the form:

(a) $\alpha_i \leq \alpha_j$ or $\alpha_j \leq \alpha_i$ $\quad (b)$ $\alpha_i = \alpha_j$ $\quad (c)$ $\exists \alpha_k \in \underline{K}:$ $\alpha_i \leq \alpha_k$ and $\alpha_j \leq \alpha_k$

Indeed, the Kripke models of the Bw_k logic can be characterized as follows: For every $k+1$ elements $\alpha_0, \ldots, \alpha_k$,

$$\bigvee_{i \neq j \in \{0,\ldots,k\}} \alpha_i \leq \alpha_j \quad \text{or} \quad \alpha_j \leq \alpha_i.$$

The Kripke models of the Bc_k logic have the following property: For every $k+1$ elements $\alpha_0, \ldots, \alpha_k$,

$$\bigvee_{i \neq j \in \{0,\ldots,k\}} \alpha_i = \alpha_j.$$

Finally, the Kripke models of the LQ logic satisfy (c).

The above conditions can be formalized in the hypertableau framework. Indeed, consider the following rule, originally defined in [4] in the context of cut-free hypersequent calculi

$$\frac{\Psi \mid \mathbf{T}(\Gamma_0), \mathbf{F}A_0 \mid \mathbf{T}(\Gamma_1), \mathbf{F}A_1}{\Psi \mid \mathbf{T}(\Gamma_0), \mathbf{T}(\Gamma_1), \mathbf{F}A_1 \parallel \Psi \mid \mathbf{T}(\Gamma_0), \mathbf{T}(\Gamma_1), \mathbf{F}A_0} (\leq)$$

By adding this rule to T-Int one gets a hypertableau calculus for infinite-valued Gödel logic.

Example 1. We display a proof of axiom $(q \to p) \vee (p \to q)$ in the above calculus.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{F}(q\to p)\vee(p\to q)}{\mathbf{F}(q\to p)\vee(p\to q) \mid \mathbf{F}(q\to p)\vee(p\to q)}\,\mathrm{HEC}}{\mathbf{F}(q\to p)\vee(p\to q) \mid \mathbf{F}(p\to q)}\,\mathbf{F}\vee}{\mathbf{F}(q\to p)\vee(p\to q) \mid \mathbf{T}p, \mathbf{F}q}\,\mathbf{F}\to}{\mathbf{T}p, \mathbf{F}q \mid \mathbf{F}(q\to p)\vee(p\to q)}\,\mathrm{HEE}}{\mathbf{T}p, \mathbf{F}q \mid \mathbf{F}(q\to p)}\,\mathbf{F}\vee}{\mathbf{T}p, \mathbf{F}q \mid \mathbf{T}q, \mathbf{F}p}\,\mathbf{F}\to}{\mathbf{T}p, \boxed{\mathbf{T}\mathrm{q}\ ,\ \mathbf{F}\mathrm{q}} \parallel \mathbf{T}q, \boxed{\mathbf{T}\mathrm{p}\ ,\ \mathbf{F}\mathrm{p}}}\,(\leq)$$

As is well known, in every Kripke model $\underline{K}$ of G_∞ for all $\alpha_i, \alpha_j \in \underline{K}$, either we have $\alpha_i \leq \alpha_j$ or $\alpha_j \leq \alpha_i$. Thus a hypertableau calculus for the Bw_k logic is simply defined by adding to T-Int the following generalization of the ($\leq$) rule (see [8])

$$\frac{\Psi \mid \mathbf{T}(\Gamma_0), \mathbf{F}A_0 \mid \ \ldots \ \mid \mathbf{T}(\Gamma_k), \mathbf{F}A_k}{\ldots \ \| \ \Psi \mid \mathbf{T}(\Gamma_i), \mathbf{T}(\Gamma_j), \mathbf{F}A_i \ \|_{\text{ for every } 0 \leq i \neq j \leq k} \cdots}\,(\mathrm{Bw_k})$$

As an example we show a proof of axiom $(p_0 \to p_1 \vee p_2) \vee (p_1 \to p_0 \vee p_2) \vee (p_2 \to p_0 \vee p_1)$ in the above calculus for Bw_2.

Example 2. Let $H = (p_0 \to p_1 \vee p_2) \vee (p_1 \to p_0 \vee p_2) \vee (p_2 \to p_0 \vee p_1)$. The derivation proceeds as follows:

$$\dfrac{\dfrac{\mathbf{F}H}{\mathbf{F}H \mid \mathbf{F}H}\,\mathrm{HEC}}{\mathbf{F}H \mid \mathbf{F}H \mid \mathbf{F}H}\,\mathrm{HEC}$$

$$\vdots \ \text{by several applications of } \mathbf{F}\vee \text{ and HEE}$$

$$\mathbf{F}(p_0 \to p_1 \vee p_2) \mid \mathbf{F}(p_1 \to p_0 \vee p_2) \mid \mathbf{F}(p_2 \to p_0 \vee p_1)$$

$$\vdots \ \text{by several applications of } \mathbf{F}\to \text{ and HEE}$$

$$\frac{\mathbf{T}p_0, \mathbf{F}(p_1 \vee p_2) \mid \mathbf{T}p_1, \mathbf{F}(p_0 \vee p_2) \mid \mathbf{T}p_2, \mathbf{F}(p_0 \vee p_1)}{\begin{array}{c}\mathbf{T}p_0, \mathbf{T}p_1, \mathbf{F}(p_1 \vee p_2) \ \| \ \mathbf{T}p_0, \mathbf{T}p_2, \mathbf{F}(p_1 \vee p_2) \ \| \ \mathbf{T}p_1, \mathbf{T}p_0, \mathbf{F}(p_0 \vee p_2) \ \| \\ \mathbf{T}p_1, \mathbf{T}p_2, \mathbf{F}(p_0 \vee p_2) \ \| \ \mathbf{T}p_2, \mathbf{T}p_0, \mathbf{F}(p_0 \vee p_1) \ \| \ \mathbf{T}p_2, \mathbf{T}p_1, \mathbf{F}(p_0 \vee p_1)\end{array}}\,(\mathrm{Bw}_2)$$

$$\vdots \ \text{by several applications of } \mathbf{F}\vee \text{ and HEE}$$

$$\begin{array}{c}\mathbf{T}p_0, \boxed{\mathbf{T}p_1, \mathbf{F}p_1} \ \| \ \mathbf{T}p_0, \boxed{\mathbf{T}p_2, \mathbf{F}p_2} \ \| \ \mathbf{T}p_1, \boxed{\mathbf{T}p_0, \mathbf{F}p_0} \ \| \\ \mathbf{T}p_1, \boxed{\mathbf{T}p_2, \mathbf{F}p_2} \ \| \ \mathbf{T}p_2, \boxed{\mathbf{T}p_0, \mathbf{F}p_0} \ \| \ \mathbf{T}p_2, \boxed{\mathbf{T}p_1, \mathbf{F}p_1}\end{array}$$

By extending the hypertableau calculus for Intuitionistic Logic with the following rule

$$\frac{\Psi \mid \mathbf{T}(\Gamma_0), \mathbf{F}A_0 \mid \mathbf{T}(\Gamma_1)}{\Psi \mid \mathbf{T}(\Gamma_0), \mathbf{T}(\Gamma_1), \mathbf{F}A_0}\,(=)$$

one gets a calculus for Classical Logic (see [9]). As is well known, in this logic for every two states $\alpha_i, \alpha_j \in \underline{K}$, $\alpha_i = \alpha_j$. A hypertableau calculus for the Bc_k logic is simply defined by adding to T-Int the following generalization of the $(=)$ rule (see [8])

$$\frac{\Psi \mid \mathbf{T}(\Gamma_0), \mathbf{F}A_0 \mid \ \ldots \ \mid \mathbf{T}(\Gamma_k)}{\ldots \ \| \ \Psi \mid \mathbf{T}(\Gamma_i), \mathbf{T}(\Gamma_j), \mathbf{F}A_i \ \|_{\text{ for every } 0 \leq i \leq k-1 \text{ and } i+1 \leq j \leq k} \cdots}$$

To obtain a hypertableau calculus for G_{k+1} it suffices to extend T-Int with both the ($\leq$) rule and the above rule. However, an alternative hypertableau calculus for G_{k+1} can be defined by replacing these two rules with the following one (see [8])

$$\frac{\Psi \mid \mathbf{T}(\Gamma_0), \mathbf{F}A_0 \mid \ \ldots \ \mid \mathbf{T}(\Gamma_k)}{\ldots \ \parallel \Psi \mid \mathbf{T}(\Gamma_i), \mathbf{T}(\Gamma_{i+1}), \mathbf{F}A_i \ \parallel_{\text{ for every } 0 \leq i \leq k-1} \ldots}$$

Finally, the (c) condition exactly corresponds to the following rule defined for LQ logic (see [9])

$$\frac{\Psi \mid \mathbf{T}(\Gamma_0) \mid \mathbf{T}(\Gamma_1)}{\Psi \mid \mathbf{T}(\Gamma_0), \mathbf{T}(\Gamma_1)} (\exists\leq)$$

thus, for each $k \geq 1$, adding to T-Int the following generalization of the $(\exists \leq)$ rule

$$\frac{\Psi \mid \mathbf{T}(\Gamma_0) \mid \ \ldots \ \mid \mathbf{T}(\Gamma_k)}{\ldots \ \parallel \Psi \mid \mathbf{T}(\Gamma_i), \mathbf{T}(\Gamma_j) \parallel_{\text{ for every } 0 \leq i \neq j \leq k} \ldots}$$

one gets a calculus for the logic which is semantically characterized by Kripke models with at most k final states.

However, there do exists intermediate logics characterized by simple semantical conditions that cannot be described only combining the (a)-(c) conditions above. Let us consider, for instance, the Bd_2 logic which is semantically characterized by the class $\mathcal{F}_{d\leq 2}$ of all rooted posets with depth at most 2 (see [7]). To describe its models one needs to express the condition: for all $\alpha_i, \alpha_j, \alpha_k$

$$\alpha_i \leq \alpha_j \leq \alpha_k \implies \alpha_i = \alpha_j \quad \text{or} \quad \alpha_j = \alpha_k.$$

We believe that this condition is hardly formalizable in the hypertableau (hypersequent) framework.

In the next section we will show how to suitably modify hypertableaux in order to define an analytic calculus for this logic.

4 Path-Hypertableau Calculi

In this section we introduce path-hypertableaux. Whereas hypertableaux are based on h-sets with explicit external rules for manipulating the order and the number of their components, the idea behind path-hypertableaux is to consider the components of h-sets as suitable ordered sequences. This eliminates the external exchange rule. Thus path-hypertableaux can be seen as substructural hypertableaux.

Let us call a *path* of a Kripke model $\underline{K}$ any sequence $\underline{\alpha} = \alpha_1, \ldots, \alpha_n \in \underline{K}$ such that $\alpha_1 \leq \ldots \leq \alpha_n$.

Definition 6. *We say that an h-set* $S_1 \mid \ldots \mid S_n$ *is* path-realized *(*p-realized *for short) in a Kripke model* $\underline{K}$*, if there exists a path* $\underline{\alpha} = \alpha_1, \ldots, \alpha_n \in \underline{K}$ *such that* $\alpha_i \rhd S_i$*, for every* $1 \leq i \leq n$*. In this case we say that* the path $\underline{\alpha}$ realizes the h-set $S_1 \mid \ldots \mid S_n$*. An* h-configuration $\Psi_1 \parallel \ldots \parallel \Psi_m$ *is* p-realized *in* $\underline{K}$*, if there exists* $j \in \{1, \ldots, m\}$ *such that* Ψ_j *is p-realized in* $\underline{K}$*.*

A *p-hypertableau* is a finite sequence of h-configurations obtained by applying the rules of the p-hypertableau calculus. The closure condition for a p-hypertableau is the same as that we gave for a hypertableau (see Definition 5).

Intuitively, each component of an h-set describes a particular state of a Kripke model. Thus path-hypertableaux allow to simultaneously explore all the states constituting an ascending chain.

It is immediate to verify that the following proposition holds:

Proposition 1. *If an h-set is contradictory then it is not p-realizable.*

In Table 2 we display the p-hypertableau calculus PT-Int for Intuitionistic Logic.

External Structural Rules

$$\frac{\Psi \mid \Phi}{\Psi}\,\text{HEW}_\text{r} \qquad \frac{\Psi \mid \Phi}{\Phi}\,\text{HEW}_\text{l} \qquad \frac{\Psi \mid \Phi}{\Psi \mid \Phi \mid \Phi}\,\text{HEC}_\text{r} \qquad \frac{\Psi \mid \Phi}{\Psi \mid \Psi \mid \Phi}\,\text{HEC}_\text{l}$$

Logical Rules

$$\frac{\Psi \mid S, \mathbf{T}(A_1 \land A_2) \mid \Psi'}{\Psi \mid S, \mathbf{T}A_i \mid \Psi'}\,\text{T}\land_\text{i} \quad \text{for } i = 1, 2 \qquad \frac{\Psi \mid S, \mathbf{F}(A \land B) \mid \Psi'}{\Psi \mid S, \mathbf{F}A \mid \Psi' \parallel \Psi \mid S, \mathbf{F}B \mid \Psi'}\,\text{F}\land$$

$$\frac{\Psi \mid S, \mathbf{T}(A \lor B) \mid \Psi'}{\Psi \mid S, \mathbf{T}A \mid \Psi' \parallel \Psi, S, \mathbf{T}B \mid \Psi'}\,\text{T}\lor \qquad \frac{\Psi \mid S, \mathbf{F}(A_1 \lor A_2) \mid \Psi'}{\Psi \mid S, \mathbf{F}A_i \mid \Psi'}\,\text{F}\lor_\text{i} \quad \text{for } i = 1, 2$$

$$\frac{\Psi \mid S, \mathbf{T}(A \to B) \mid \Psi'}{\Psi \mid S, \mathbf{F}A, \mathbf{T}(A \to B) \mid \Psi' \parallel \Psi \mid S, \mathbf{T}B \mid \Psi'}\,\text{T}\to \qquad \frac{\Psi \mid S, \mathbf{F}(A \to B) \mid \Psi'}{\Psi \mid S^{\mathbf{T}}, \mathbf{T}A, \mathbf{F}B}\,\text{F}\to$$

$$S^{\mathbf{T}} = \{\mathbf{T}X \mid \mathbf{T}X \in S\}$$

Table 2. Path-hypertableau calculus PT-Int for Intuitionistic Logic

Remark 3. In the new interpretation of h-sets the external exchange rule does not hold. This entails the splitting of the external rules of weakening and contraction into left and right rules, according to the part of the h-set they modify.

As usual, the main step of the Soundness Theorem is to prove that the rules of the calculus preserve p-realizability.

Lemma 1. *The rules of* PT-Int *preserve p-realizability.*

Proof. As an example, we shall give the proof for the $\mathbf{F}\rightarrow$ rule. Suppose that $\alpha_1,\ldots,\alpha_n,\beta,\gamma_1,\ldots,\gamma_m$ is a path of a model $\underline{K}$ p-realizing $S_1 \mid \ldots \mid S_n \mid S, \mathbf{F}(A\rightarrow B) \mid S'_1 \mid \ldots \mid S'_m$. Thus, there exists an element δ in $\underline{K}$ such that $\beta \leq \delta$ and $\delta \rhd \mathbf{T}A, \mathbf{F}B$. Hence the path $\alpha_1,\ldots,\alpha_n,\delta$ p-realizes $S_1 \mid \ldots \mid S_n \mid S^{\mathbf{T}}, \mathbf{T}A, \mathbf{F}B$.

Remark 4. The absence of the right context Ψ' in the consequent of the $\mathbf{F}\rightarrow$ rule is essential to preserve p-realizability of $\mathbf{F}\rightarrow$. Indeed the p-realizability of $\mathbf{F}(A\rightarrow B)$ in a state β requires the existence in the model of a state $\delta \geq \beta$ p-realizing $\mathbf{T}A$ and $\mathbf{F}B$. However nothing is known about the relationship between δ and the path p-realizing Ψ'.

Theorem 2 (Soundness). *If there exists a closed p-hypertableau for* $\{\mathbf{F}A\}$ *in* PT-Int*, then* A *is valid in Intuitionistic Logic.*

Proof. Let us suppose, by way of contradiction, that there are a Kripke model $\underline{K} = \langle P, \leq, v\rangle$ and $\alpha \in P$ such that $\alpha \rhd \mathbf{F}A$. By the previous lemma the final h-configuration of the closed p-hypertableau for $\{\mathbf{F}A\}$ is p-realizable. This means that a contradictory h-set is p-realizable, contradicting Proposition 1.

Theorem 3 (Completeness). *If a wff* A *is valid in Intuitionistic Logic, then there exists a closed p-hypertableau for* $\{\mathbf{F}A\}$ *in* PT-Int.

Proof. Straightforward since the logical rules of PT-Int are essentially the same as in ordinary tableau calculi for Intuitionistic Logic (see, e.g., [11]) with in addition the contexts Ψ and Ψ'.

4.1 Logics of Bounded Depth Kripke Models

In the following we define path-hypertableau calculi for the intermediate logics of bounded depth Kripke models Bd_k, with $k \geq 1$. Our calculi are uniform, and are simply obtained by adding a suitable structural rule to PT-Int (see Table 2).

Bd_k is characterized by the class $\mathcal{F}_{d\leq k}$ of rooted posets with depth $\leq k$. In other words, every chain of its models has at most k elements. Thus Bd_1 coincides with Classical Logic. A Hilbert style axiomatization of Bd_k is obtained by extending the axioms of Intuitionistic Logic with the axiom scheme (Bd_k) recursively defined as follows (see [12]):

$$(\mathrm{Bd}_1)\ A_1 \vee \neg A_1$$
$$(\mathrm{Bd}_{i+1})\ A_{i+1} \vee (A_{i+1} \rightarrow (\mathrm{Bd}_i))$$

For $k \geq 1$, the p-hypertableau calculus PT-Bd_k is simply obtained by adding to PT-Int the following structural rule:

$$\frac{\Psi \mid S_0 \mid \ldots \mid S_k \mid \Psi'}{\Psi \mid S_0, S_1 \mid \Psi' \parallel \Psi \mid S_0 \mid S_1, S_2 \mid \Psi' \parallel \ldots \parallel \Psi \mid S_0 \mid \ldots \mid S_{k-2} \mid S_{k-1}, S_k \mid \Psi'}\,(\leq\mathrm{k})$$

Remark 5. The ($\leq k$) rule resembles the n-Shifting restart rule introduced in [13] in order to define goal-oriented deduction methods for Bd_k.

We prove that PT-Bd_k is sound and complete with respect to the Bd_k logic.

Lemma 2. *The rules of* PT-Bd_k *preserve p-realizability.*

Proof. By Lemma 1 we only have to show that the ($\leq k$) rule preserves p-realizability over the frames for Bd_k. Indeed, if its premise is realized in a model $\underline{K}$ built on a frame of $\mathcal{F}_{d\leq k}$, then there exists a path $\beta_0 \leq \ldots \leq \beta_p \leq \alpha_0 \leq \ldots \leq \alpha_k \leq \gamma_0 \leq \ldots \leq \gamma_q$ in $\underline{K}$ such that $\underline{\beta} = \beta_0, \ldots, \beta_p$ realizes the h-set Ψ, $\alpha_i \rhd S_i$ for every $i = 0, \ldots, k$, and $\underline{\gamma} = \gamma_0, \ldots, \gamma_q$ realizes the h-set Ψ'. Since any path in $\underline{K}$ has depth at most k, there exists $h \in \{0, \ldots, k\}$ such that $\alpha_h = \alpha_{h+1}$. Thus the sequence $\underline{\beta}, \alpha_0, \ldots, \alpha_h, \underline{\gamma}$ realizes the h-set $\Psi \mid S_0 \mid \ldots \mid S_h, S_{h+1} \mid \Psi'$.

Theorem 4 (Soundness). *If there exists a closed p-hypertableau for* $\{\mathbf{F}A\}$ *in* PT-Bd_k*, then A is valid in* Bd_k.

Proof. The proof proceeds as in Theorem 2.

Definition 7. *An h-set Φ is* Bd_k-consistent *if it has no closed p-hypertableau in* PT-Bd_k.

The completeness theorem has the following form: *If a wff A is valid in every Kripke model of depth $\leq k$, then there is a closed p-hypertableau for $\{\mathbf{F}A\}$ in* PT-Bd_k. According to the semantical interpretation of the swff's, it suffices to prove that: *If an h-set S of swff's is Bd_k-consistent then there is a Kripke model $\underline{K}$ built on a poset in $\mathcal{F}_{d\leq k}$ realizing it.*

Our proof is based on a general method allowing to built up a rooted Kripke model $\underline{K}(S)$ realizing each Bd_k-consistent h-set S (see, e.g., [1, 16]). We start by defining the basic notions of *node set* and *successor set*.

Let $\Phi \mid S$ be any Bd_k-consistent h-set with $S = \{A_1, \ldots, A_n\}$. We define the sequence $\{S_i\}_{i\in\omega}$ of sets of swff's as follows:

- $S_0 = S$;
- Let $S_i = \{H_1, \ldots, H_q\}$; then $S_{i+1} = \bigcup_{H_j \in S_i} \mathcal{U}(H_j, i)$

where, setting $S'_j = \bigcup_{k=1}^{j-1} \mathcal{U}(H_k, i) \cup \bigcup_{k=j+1}^{q} H_k$, $\mathcal{U}(H_j, i)$ is defined as follows:

1. If $H_j \equiv \mathbf{T}(A \vee B)$ and $\Phi \mid (S'_j \cup \{\mathbf{T}A\})$ is Bd_k-consistent, then $\mathcal{U}(H_j, i) = \{\mathbf{T}A\}$, otherwise $\mathcal{U}(H_j, i) = \{\mathbf{T}B\}$;
2. If $H_j \equiv \mathbf{F}(A \wedge B)$ and $\Phi \mid (S'_j \cup \{\mathbf{F}A\})$ is Bd_k-consistent, then $\mathcal{U}(H_j, i) = \{\mathbf{F}A\}$, otherwise $\mathcal{U}(H_j, i) = \{\mathbf{F}B\}$;
3. If $H_j \equiv \mathbf{T}(A \to B)$ and $\Phi \mid (S'_j \cup \{\mathbf{T}B\})$ is Bd_k-consistent, then $\mathcal{U}(H_j, i) = \{\mathbf{T}B\}$, otherwise $\mathcal{U}(H_j, i) = \{\mathbf{F}A, \mathbf{T}(A \to B)\}$;
4. If $H_j \equiv \mathbf{F}(A \to B)$, then $\mathcal{U}(H_j, i) = \{\mathbf{F}(A \to B)\}$.
5. Otherwise, $\mathcal{U}(H_j, i)$ is the set of h-configurations obtained by applying the rule in PT-Bd_k for H_j to the h-configuration $\{H_j\}$.

Since each step in the construction of S_{i+1} corresponds to the application of a rule of PT-Bd$_k$, it is easy to see (by induction on i) that every $\Phi \mid S_i$ is Bd$_k$-consistent. The finiteness of each S_i directly follows for S being finite. Thus there exists $k \geq 0$ such that $S_k = S_{k+1}$. We call S_k the *node set of S w.r.t. Φ* and we denote it with $\overline{S}$. We call the set $S^* = \bigcup_{i\geq 0} S_i$ the *saturated set related to $\overline{S}$.*

Given an h-set $\Phi \mid S$, for every $\mathbf{F}(A \to B) \in S$ we call *path-extension of $\Phi \mid S$ (w.r.t. $\mathbf{F}(A \to B)$)* the path $\Phi \mid S \mid (S^{\mathbf{T}} \cup \{\mathbf{T}A, \mathbf{F}B\})$. Moreover, we call $S^{\mathbf{T}} \cup \{\mathbf{T}A, \mathbf{F}B\}$ the *successor set of S (w.r.t. $\mathbf{F}(A \to B)$).*

Using the rules HEC$_r$ and $\mathbf{F}\!\to$, one can easily check that every path-extension of a Bd$_k$-consistent h-set is Bd$_k$-consistent.

Henceforth, we shall consider trees $\mathcal{T}$ whose nodes are sets of swff's and we shall identify each path $\Gamma_0, \ldots, \Gamma_m$ of these trees with the h-set $\Gamma_0 \mid \ldots \mid \Gamma_m$.

To help the reader, we first give an overview on the construction of $\underline{K}(S)$. Let $\overline{\mathcal{T}}_0$ be the tree only containing a node set of S (w.r.t. the empty h-set) as root. Our aim is to build up a sequence of trees $\overline{\mathcal{T}}_1, \ldots, \overline{\mathcal{T}}_t$ having depth $\leq k$. This shall be done in the following steps:

Step 1: Expansion. For every leaf $\overline{\Delta}$ of depth j in $\overline{\mathcal{T}}_i$, with $j < k+1$, the nodes of depth $j+1$ in $\overline{\mathcal{T}}_{i+1}$ are obtained by adding, as the immediate descendent of $\overline{\Delta}$, the node sets of its successor sets. If $\overline{\mathcal{T}}_{i+1}$ has depth $\leq k$ go to Step 3. Otherwise, go to Step 2.

Step 2: Contraction. The contraction of $\overline{\mathcal{T}}_i$ results in a tree of depth $\leq k$ obtained by deleting some subtrees of $\overline{\mathcal{T}}_i$. This shall be essentially done by using the $(\leq k)$ rule. Go to Step 1.

Step 3: Model construction. $\underline{K}(S)$ is built on the last tree $\overline{\mathcal{T}}_{last}$ of the sequence and for every element α of $\overline{\mathcal{T}}_{last}$, $v(\alpha)$ is defined as $\{p : \mathbf{T}p \in \alpha\}$.

Expansion step

First we define an invariant on the trees $\mathcal{T}$ that we shall consider hereafter

T.1 For each internal node $\overline{\Gamma}$, the path $\overline{\Gamma}_0 \mid \ldots \mid \overline{\Gamma}$ is Bd$_k$-consistent and each set occurring in it is a node set;
T.2 For every leaf Δ, the path $\overline{\Gamma}_0 \mid \ldots \mid \overline{\Gamma}_m \mid \Delta$ is Bd$_k$-consistent (possibly, Δ is not a node set);
T.3 The depth of $\mathcal{T}$ is at most $k+1$.

Let $\mathcal{T}$ be a tree satisfying the above conditions. The sequence $\{\mathcal{S}_i\}^{\mathcal{T}}_{i\leq k+1}$ generated by $\mathcal{T}$ is defined as follows:

- $\mathcal{S}_0 = \mathcal{T}$;
- Given $\mathcal{S}_i$ with $i \leq k+1$, let $V_1, \ldots, V_q$ be the leaves of depth $\leq k$. For every $j = 1, \ldots, q$ let $\overline{V}_j$ be the node set with respect to the h-set Φ_j (corresponding to the path from the root of $\mathcal{S}_i$ to the parent of V_j). Let $U^j_1, \ldots, U^j_{l_j}$ be the successor sets of $\overline{V}_j$. For every U^j_g, let $\overline{U}^j_g$ be its node set w.r.t. $\Phi_j \mid \overline{V}_j$. $\mathcal{S}_{i+1}$ is the tree obtained by substituting the node V_j in $\mathcal{S}_i$ with the subtree of depth 2 having $\overline{V}_j$ as root and as leaves the node sets $\overline{U}^j_1, \ldots, \overline{U}^j_{l_j}$.

It is easy to check that $\mathcal{S}_{i+1}$ satisfies the conditions (T.1)-(T.3).

Contraction

We call *cut of the sequence* $\{\mathcal{S}_i\}^{\mathcal{T}}_{i\leq k+1}$ the tree $\overline{\mathcal{T}}$ defined as follows:

- If the depth of $\mathcal{S}_{k+1}$ is $\leq k+1$, then $\overline{\mathcal{T}} = \mathcal{S}_{k+1}$;
- Otherwise, let $\pi_0, \ldots, \pi_r$ be the paths in $\mathcal{S}_{k+1}$ with length $k+1$. Since any $\pi_j = \overline{\Gamma}_0 \mid \ldots \mid \overline{\Gamma}_k$ is Bd$_k$-consistent, the application of the ($\leq k$) rule to this h-set yields an h-configuration

$$\overline{\Gamma}_0, \overline{\Gamma}_1 \parallel \overline{\Gamma}_0 \mid \overline{\Gamma}_1, \overline{\Gamma}_2 \parallel \ldots \parallel \overline{\Gamma}_0 \mid \ldots \mid \overline{\Gamma}_{k-2} \mid \overline{\Gamma}_{k-1}, \overline{\Gamma}_k$$

  that contains at least a Bd$_k$-consistent h-set. Let π_j^+ be the first Bd$_k$-consistent h-set in this h-configuration, i.e., the one corresponding to the shortest path. Consider the paths $\pi_0^+, \ldots, \pi_r^+$ together with the ordering induced on this set from the sub-path relation. Let us denote with $\tilde{\pi}_0, \ldots, \tilde{\pi}_z$ the minimal elements of this ordered set. For every $j = 0, \ldots, z$, if $\tilde{\pi}_j = \overline{\Gamma}^j_0 \mid \ldots \mid \overline{\Gamma}^j_{p-1} \mid \overline{\Gamma}^j_p, \overline{\Gamma}^j_{p+1}$ and H_j is the swff used to build up the successor node $\overline{\Gamma}^j_{p+1}$ of $\overline{\Gamma}^j_p$, we define $\overline{\pi}_j = \overline{\Gamma}^j_0 \mid \ldots \mid \overline{\Gamma}^j_{p-1} \mid \overline{\Gamma}^j_p, \overline{\Gamma}^j_{p+1} \setminus \{H_j\}$. We remark that the Bd$_k$-consistency of $\tilde{\pi}_j$ immediately implies the Bd$_k$-consistency of $\overline{\pi}_j$. Let $\overline{\Delta}_j$ be the node set of $(\overline{\Gamma}^j_p \setminus \{H_j\}) \cup \overline{\Gamma}^j_{p+1}$ with respect to $\overline{\Gamma}^j_0 \mid \ldots \mid \overline{\Gamma}^j_{p-1}$. We define $\overline{\mathcal{T}}$ as the tree obtained by replacing the subtree of root $\overline{\Gamma}^j_p$ in $\mathcal{S}_{i+1}$ with the subtree only consisting of the node set $\overline{\Delta}_j$. Moreover, we say that $\overline{\Delta}_j$ is obtained by a *cut on the set* $(\overline{\Gamma}^j_p \setminus \{H_j\}) \cup \overline{\Gamma}^j_{p+1}$.

By the above construction we immediately get that, if $\mathcal{T}$ satisfies conditions (T.1)-(T.3), $\overline{\mathcal{T}}$ satisfies these conditions too. Moreover $\overline{\mathcal{T}}$ has depth $\leq k$.

The sequence of trees

Given a Bd$_k$-consistent h-set S, we define the sequence of trees $\{\overline{\mathcal{T}}_i\}_{i\in\omega}$ generated by S, as follows:

– $\overline{\mathcal{T}}_1$ is the cut of the sequence $\{\mathcal{S}_i\}^{\{S\}}_{i\leq k+1}$ where $\{S\}$ is the tree only consisting of the root S.
– $\overline{\mathcal{T}}_{i+1}$ is the cut of $\{\mathcal{S}\}^{\overline{\mathcal{T}}_i}_{i\leq k+1}$.

For every tree $\overline{\mathcal{T}}_i$ in the above sequence, we define a function ρ_i associating with each node of $\overline{\mathcal{T}}_i$ a saturated set as follows: For every node $\overline{\Gamma}$ in $\overline{\mathcal{T}}_i$ ($i \geq 1$):

1. If $\overline{\Gamma}$ is a node of the tree $\mathcal{S}_{k+1}$ in the sequence $\{\mathcal{S}_i\}^{\mathcal{T}_i}_{i\leq k+1}$, then

$$\rho_i(\overline{\Gamma}) = \begin{cases} \text{the saturated set related to } \overline{\Gamma} & \text{if } i = 1 \\ \rho_{i-1}(\overline{\Gamma}) & \text{otherwise} \end{cases}$$

2. If $\overline{\Gamma}$ is not a node of $\mathcal{S}_{k+1}$, then it is obtained by a cut on a set $\Gamma = (\overline{\Gamma}_p \setminus \{H\}) \cup \overline{\Gamma}_{p+1})$, thus

$$\rho_i(\overline{\Gamma}) = \begin{cases} \Gamma_p^* \cup \Gamma_{p+1}^* \cup \Gamma^* & \text{if } i = 1 \\ \rho_{i-1}(\overline{\Gamma}_p) \cup \rho_{i-1}(\overline{\Gamma}_{p+1}) \cup \Gamma^* & \text{otherwise} \end{cases}$$

where Γ_p^*, Γ_{p+1}^* and Γ^* are the saturated sets related to $\overline{\Gamma}_p$, $\overline{\Gamma}_{p+1}$ and $\overline{\Gamma}$, respectively.

An inspection on the construction of the trees $\overline{\mathcal{T}}_i$, easily shows that for any node $\overline{\Gamma}$ of $\overline{\mathcal{T}}_i$, the set $\rho_i(\overline{\Gamma})$ has the usual properties of a saturated set.

Model Construction

Being S a finite set of swff's, one can easily prove that there exists an integer t such that, for every $i \geq t$, $\overline{\mathcal{T}}_i = \overline{\mathcal{T}}_t$. The Kripke model $\underline{K}(S)$ is defined thus:

1. $\langle P, \leq \rangle$ is the poset where P contains all the nodes of $\overline{\mathcal{T}}_t$ and $\leq$ is the reflexive and transitive closure of the immediate descendent relation of $\overline{\mathcal{T}}_t$.
2. For every $\overline{\Gamma} \in P$ and for every propositional variable p, $p \in v(\overline{\Gamma})$ iff $\mathbf{T}p \in \overline{\Gamma}$.

We associate with $\underline{K}(S)$ the function $\rho = \rho_t$. Using the construction of the trees in $\{\overline{\mathcal{T}}_i\}_{i\in\omega}$, it is easy to check that $\underline{K}(S)$ is a Kripke model built on $\mathcal{F}_{d\leq k}$.

Remark 6. The above construction shows that in PT-Bd$_k$ the HEC$_l$ rule is actually redundant.

Now, it is only a matter of technicality to prove the main lemma:

Lemma 3. *For each $\overline{\Gamma} \in \underline{K}(S)$ and swff $H \in \rho(\Gamma)$, one has $\overline{\Gamma} \rhd H$ in $\underline{K}(S)$.*

Proof. By induction on the structure of H. The base cases $H \equiv \mathbf{T}p$ and $H \equiv \mathbf{F}p$ immediately follow by definition of $\underline{K}(S)$. As for the induction step consider the case $H \equiv \mathbf{F}(A \to B)$. Let $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_{t-1}, \overline{\Gamma}_t = \overline{\Gamma}$ be the "history" of $\overline{\Gamma}$ in the sequence $\{\overline{\mathcal{T}}_i\}_{i\leq t}$. That is, for every $j = t-1, \ldots, 1$, if $\overline{\Gamma}_j$ is obtained by a cut on the set $(\overline{\Gamma}_p \setminus \{H\}) \cup \overline{\Gamma}_{p+1}$ then $\overline{\Gamma}_j$ is $\overline{\Gamma}_p$, otherwise $\overline{\Gamma}_j$ is $\overline{\Gamma}_{j+1}$. Let $\overline{\Gamma}_h$ be the first set in this sequence not containing $\mathbf{F}(A \to B)$. Thus there exists a node set $\overline{\Delta}_h$ in $\overline{\mathcal{T}}_h$ such that $\overline{\Delta}_h$ is an immediate descendent of $\overline{\Gamma}_h$ and $\mathbf{T}A, \mathbf{F}B$ belongs to $\rho_h(\overline{\Delta}_h)$. By definition of the sequence $\{\overline{\mathcal{T}}_i\}_{i\leq t}$ and of the model $\underline{K}(S)$ there exists an element $\overline{\Delta} \in P$ such that $\overline{\Gamma} \leq \overline{\Delta}$ and $\{\mathbf{T}A, \mathbf{F}B\} \subseteq \rho(\Delta)$. By induction hypothesis one has $\overline{\Delta} \rhd \mathbf{T}A, \mathbf{F}B$. Therefore, $\overline{\Gamma} \rhd \mathbf{F}(A \to B)$.

This immediately yields the completeness theorem:

Theorem 5 (Completeness). *If a wff A is valid in $\mathcal{F}_{d\leq k}$, then there exists a closed p-hypertableau for $\{\mathbf{F}A\}$ in* PT-Bd$_k$.

Proof. Let us suppose by way of contradiction that $\{\mathbf{F}A\}$ is Bd$_k$-consistent. By the above lemma, this implies that $\mathbf{F}A$ is p-realizable in $\underline{K}(S)$, contradicting the assumed validity of A in $\mathcal{F}_{d\leq k}$.

Remark 7. By dualizing PT-Bd_k, with $k \geq 1$, one can easily define path-hypersequent calculi for the Bd_k logics. However, while in path-hypertableau calculi there is an immediate relation between the semantics of these logics and the interpretation of h-sets, this relation is not so obvious in the corresponding path-hypersequent calculi.

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