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HYPOTHESIS TESTS ON THE SCALE PARAMETER USING MEDIAN RANKED SET SAMPLING

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1. INTRODUCTION

Ranked set sampling (RSS) (see, for example, McIntyre, 1952 and Takahasi and Wakimoto, 1968) can be used in many ecological and agricultural studies where the measurement of each unit is laborious and expensive, but several units can easily be arranged in order of magnitude without requiring the actual measurement. The field of research related to the RSS has recently become increasingly important. In the last few years numerous developments have been made in this field. For example, we refer to Patil, Sinha and Taillie (1994), Ni Chuiv and Sinha (1998), Muttlak (1997), Hossain (1999), Abu-Dayyeh and Muttlak (1996) and Muttlak and Abu-Dayyeh (1998).

Muttlak (1997) suggested a median ranked set sampling (MRSS) method in which only the median observation from each of a number of randomly selected sets is considered for quantification. The main advantages of the method over the usual RSS method, as he stated, were that it could reduce the chance of errors in ranking and could increase the efficiency of the estimator both for perfect ranking and for erroneous ranking. Since for both RSS and MRSS, observations are ranked without measuring them, errors in ranking is a common issue encountered in such sampling. The hypothetical situation where there is no ranking error, the ranking can be termed as perfect ranking otherwise erroneous. The RSS method performs best when the rankingis perfect but looses efficiency for erroneous ranking.

It can be seen that the RSS and the MRSS methods utilize point estimation for gathering information on the population parameters like the usual simple random sampling (SRS) method. For proper statistical application of the estimators suggested for the RSS and the MRSS methods, some test procedures compatible with these methods are also required to be developed. Test procedures using the sample obtained by the RSS and MRSS method are not straight forward since the distribution properties of the sample observations are not simple. For the MRSS method with even sample size and for the RSS method, the sample observations are not identically distributed. Abu-Dayyeh and Muttlak (1996) and Muttlak and Abu-Dayyeh (1998) described some test procedures using the RSS data, which they showed to be more powerful than the respective tests using the usual simple random sample (SRS) data. They used empirical distribution of the sample statistic. Since the MRSS method was found to point estimate the population mean more precisely than the RSS method for most of the distribution, it is expected that the test procedures similar to those using RSS data would produce higher power with MRSS data. In this paper, some test procedures for scale parameters of exponential and rectangular distributions involving the MRSS data are discussed. The necessary tables for the critical regions for the suggested tests are supplied and the tests are compared in respect of the power function with those using RSS data.

2. SAMPLING METHODS

2.1. *Ranked Set Sampling (RSS)*

RSS method can be used to obtain a sample of size n from an infinite population. To obtain a sample of size n (see McIntyre, 1952), at first a set of n elements is drawn randomly and they are ranked according to their magnitude without measuring (for example by visual inspection), and the smallest observation is measured and selected in the sample. For notation, we denote this selected observation by $X_{(1,n)1}$ (first order statistic in the first group of n random observations).

Then an independent second set of n elements is drawn randomly and ranked, and the second smallest observation is selected in the sample. This observation can be denoted as $X_{(2,n)2}$ (second order statistic in the second group of n random observations).

Similarly from a third set the third smallest observation $X_{(3,n)3}$ is selected and the procedure is carried on until the n^{th} smallest observation from the n^{th} set $X_{(n,n)n}$ is selected. The resulting sample can be written as $\{X_{(1,n)1}, X_{(2,n)2}, ..., X_{(n,n)n}\}$. It can be shown that the sample mean from this method is an unbiased estimator of the population mean (see Takahasi and Wakimoto, 1968).

Since the ranking of n observations is required to be done without measurement, use of a large n is practically infeasible. That is why for obtaining larger sample repetition of the same procedure is made to obtain sample sizes which are multiple of n. m repetitions (can be termed as cycles) of the procedure gives a sample size of mn.

2.1. *Median Ranked Set Sampling (MRSS)*

The MRSS method (Muttlak, 1997) is a modification of the RSS method in which instead of selecting smallest observation from the first set of n random observations, second smallest observation from the second set of n random observations, etc., the median observation is selected from each of the sets. To adjust for even n, the method for obtaining a sample of size n can be described as follows:

i) If the sample size n is odd:

A random set of elements is drawn and ordered by visual inspection or some other means not requiring the actual measurement, then the $\left(\frac{n+1}{2}\right)^{th}$ smallest element (the median) is selected in the sample. This observation can be denoted as $X_{\left(\frac{n+1}{2}:n\right)}$. The process is repeated n times, the resulting sample can be written as $\left\{X_{\left(\frac{n+1}{2}:n\right)1}, X_{\left(\frac{n+1}{2}:n\right)2}, ..., X_{\left(\frac{n+1}{2}:n\right)n}\right\}$.

ii) If the sample size n is even:

A random set of n elements is drawn and ordered by visual inspection or some other means not requiring the actual measurement, then the $\left(\frac{n}{2}\right)^{th}$ smallest element $X_{\left(\frac{n}{2}:n\right)1}$ is selected in the sample. The process is repeated $\frac{n}{2}$ times. Again a random set of n elements is drawn, ordered and the $\left(\frac{n}{2}+1\right)^{th}$ smallest element $X_{\left(\frac{n}{2}+1:n\right)2}$ is selected in the sample. This process is also repeated $\frac{n}{2}$ times. The resulting sample can be written as

$$\left\{X_{\left(\frac{n}{2}:n\right)1}, X_{\left(\frac{n}{2}+1:n\right)2}, ..., X_{\left(\frac{n}{2}:n\right)\frac{n}{2}}, X_{\left(\frac{n}{2}+1:n\right)\frac{n}{2}+1}, X_{\left(\frac{n}{2}+1:n\right)\frac{n}{2}+2}, ..., X_{\left(\frac{n}{2}:n\right)n}\right\}.$$

It can be seen that like the RSS method a m cycles of the procedure gives a sample size of mn. Note that in both methods m is the number of cycles used only to multiply the sample size. In this paper, we studied the test procedures for $m=1$ only.

3. TEST PROCEDURES USING RSS DATA

Let a sample s be drawn using the RSS method in one cycle (see section 2.1) and also let $s=\{X_{(1,n)1}, X_{(2,n)2}, ..., X_{(n,n)n}\}$, where $X_{(i,n)i}$ denotes the i^{th} smallest order statistics in the i^{th} random set of size n. It is clear from the sampling method given in section 2.1 that $X_{(1,n)1}, X_{(2,n)2}, ..., X_{(n,n)n}$ are independently distributed but they are not identically distribution. Abu-Dayyeh and Muttlak (1996) described the following test procedures:

Test 1:

Consider testing the hypotheses

$$\begin{aligned} H_0 &: \quad \theta = \theta_0 \\ H_a &: \quad \theta > \theta_0 \end{aligned} \tag{1}$$

where s is drawn from an exponential distribution with scale parameter θ. A test (assuming $\theta_0 = 1$) is given by

$$\phi_{r1}(x) = \begin{cases} 1 & \text{if} \sum_{i=1}^{n} X_{(i,n)i} > c_{r1} \\ 0 & \text{Otherwise} \end{cases} \tag{2}$$

Test 2:

Consider testing the hypotheses

$$\begin{aligned} H_0 &: \quad \theta = \theta_0 \\ H_a &: \quad \theta \neq \theta_0 \end{aligned} \tag{3}$$

where s is drawn from an exponential distribution with scale parameter θ. A two-sided test (assuming $\theta_0 = 1$) is given by

$$\phi_{r2}(x) = \begin{cases} 0 & \text{if } K_{rL} < \sum_{i=1}^{n} X_{(i,n)i} < K_{rU} \\ 1 & \text{Otherwise} \end{cases} \tag{4}$$

Test 3:

Consider testing the hypotheses

$$\begin{aligned} H_0 &: \quad \theta = \theta_0 \\ H_a &: \quad \theta \neq \theta_0 \end{aligned} \tag{5}$$

where s is drawn from a rectangular distribution with scale parameter θ using RSS method. Since for rectangular distribution the largest order statistic is complete minimal sufficient for the density, use of the RSS method is not as effective as it is for other distributions (the RSS mean for rectangular distribution is also not highly efficient, see Dell and Clutter (1972). A sampling method in which all the largest order statistics from each of the sets are selected is obviously a better option. However, considering the situation where a RSS data from a rectangular distribution is already in hand, Abu-Dayyeh and Muttlak (1996) suggested a test procedure. The test is given (assuming $\theta_0 = 1$) by

$$\phi_{r1}(x) = \begin{cases} 1 & \text{if } \max\{X_{(i,n)i}\} < c_{r3} \text{ or } \max\{X_{(i,n)i}\} > 1 \\ 0 & \text{Otherwise} \end{cases} \tag{6}$$

Abu-Dayyeh and Muttlak (1996) compared the tests (2, 4 and 6) with the corresponding tests conducted with the SRS data by studying the powers of the tests. They carried out the comparison for fixed $\alpha = 0.05$ and for $n = 3,4$ and 5. But they didn't supply the values of the critical regions for different α values. In this study these critical values are calculated and they are tabulated for different values of α and for $2 \le n \le 7$ in the Appendix.

4. TEST PROCEDURES USING MRSS DATA

4.1. *Test 1 using MRSS data*

Let s be a sample drawn using the MRSS method (see section 2.2) from an exponential distribution with *pdf*

$$f_\theta(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

The *cdf* is so given by

$$F_\theta(x) = \int_0^x f_\theta(x)dx = \begin{cases} 1 - e^{-\frac{x}{\theta}} & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Case I: If the sample size n is odd

We have $s = \left\{ X_{\left(\frac{n+1}{2}:n\right)1}, X_{\left(\frac{n+1}{2}:n\right)2}, ..., X_{\left(\frac{n+1}{2}:n\right)n} \right\}$ where $X_{(i,n)j}$ denotes the i^{th} smallest order statistics in the i^{th} random set of size n. Let, for simplicity, we denote $X_{\left(\frac{n+1}{2}:n\right)i}$ by Y_i. Unlike the sample obtained by RSS method, the components Y_i s of this sample are identically and independently distributed. The Test 1 with size α for testing the hypotheses (equation 1) can be obtained following equation (2) as

$$\phi_{m1}(x)=\begin{cases} 1 & if \ \dfrac{\sum_{i=1}^{n} Y_i}{\theta_0} > c_{m1} \\ 0 & Otherwise \end{cases} \tag{9}$$

Without loss of generality we can take $\theta_0 = 1$. For specified value of α, value of c_{m1} can be obtained by finding the sampling distribution of $\sum_{i=1}^{n} Y_i$. Since Y_i is the $\left(\frac{n}{2}+1\right)^{th}$ smallest order statistics from the i^{th} group, using equation (7), (8) and the sampling distribution of median (see for example Arnold, 1992), the following can be written as

$$f(y_i)=\begin{cases} \dfrac{n!}{\left\{\left(\frac{n-1}{2}\right)!\right\}^2}\left[\left(1-e^{-\frac{y_i}{\theta}}\right)e^{-\frac{y_i}{\theta}}\right]^{\frac{n-1}{2}}\dfrac{1}{\theta}e^{-\frac{y_i}{\theta}} & y_i \geq 0 \\ 0 & otherwise \end{cases} \tag{10}$$

Now, the joint distribution of $y_1, y_2, \ldots, y_n$ can be written as

$$g_\theta(y_1, y_2, \ldots, y_n)=\prod_{i=1}^{n} f(y_i)$$

$$=\left[\frac{n!}{\left\{\left(\frac{n-1}{2}\right)!\right\}^2 \theta}\right]^n e^{-\frac{n+1}{2\theta}\sum_{i=1}^{n} y_i}\prod_{i=1}^{n}\left(1-e^{-\frac{y_i}{\theta}}\right)^{\frac{n-1}{2}} \ ; \ y_i \geq 0 \tag{11}$$

Using the transformations $Z_i = \sum_{l=1}^{i} Y_l$, $i = 1,2,\ldots,n$, the joint probability distribution of $z_1, z_2, \ldots, z_n$ can be obtained as

$$h_\theta(z_1, z_2, \ldots, z_n)$$
$$= \left[\frac{n!}{\left\{ \left(\frac{n-1}{2} \right)! \right\}^2 \theta} \right]^n e^{-\frac{n+1}{2\theta} z_n} \left\{ \left(1 - e^{-\frac{z_1}{\theta}} \right) \prod_{i=1}^{n} \left(1 - e^{-\frac{z_i - z_{i-1}}{\theta}} \right) \right\}^{\frac{n-1}{2}} \; ; \; z_i \geq 0 \tag{12}$$

Now the probability distribution of $\sum_{i=1}^{n} Y_i$, ie. the *pdf* of z_n can be obtained by implementing direct integration over the joint distribution. Let $k_\theta(z_n)$ denote the marginal density function of z_n, hence

$$k_\theta(z_n) = \int_0^{z_n} \int_0^{z_{n-1}} .. \int_0^{z_2} h_\theta(z_1, z_2, \ldots, z_n) dz_1 dz_2 \ldots, dz_n \tag{13}$$

The size of the test (9) is given by

$$\alpha = P_{\theta=1}\left(\sum_{i=1}^{n} Y_i > c_{m1} \right)$$
$$= \left[\frac{n!}{\left\{ \left(\frac{n-1}{2} \right)! \right\}^2} \right]^n \int_{c_{m1}}^{\infty} \int_0^{z_n} \int_0^{z_{n-1}} .. \int_0^{z_2} e^{-\frac{n+1}{2\theta} z_n} (1 - e^{-z_1})^{\frac{n-1}{2}} \tag{14}$$
$$\times \prod_{i=1}^{n} (1 - e^{-\{z_i - z_{i-1}\}})^{\frac{n-1}{2}} dz_1 dz_2 \ldots, d; \; z_i \geq 0$$

The values of c_{m1} for fixed α can be obtained by solving equation (14).

Case II: If the sample size n is even

If the sample size n is even, we have

$$s = \left\{ X_{\left(\frac{n}{2}:n\right)1}, X_{\left(\frac{n}{2}+1:n\right)2}, \ldots, X_{\left(\frac{n}{2}:n\right)\frac{n}{2}}, X_{\left(\frac{n}{2}+1:n\right)\frac{n}{2}+1}, X_{\left(\frac{n}{2}+1:n\right)\frac{n}{2}+2}, \ldots, X_{\left(\frac{n}{2}:n\right)n} \right\}$$

Let us denote $X_{\left(\frac{n}{2}:n\right)i}$ by Y'_i and $X_{\left(\frac{n}{2}+1:n\right)\frac{n}{2}+i}$ by Y''_i. Again, it can be seen that the sample observations are independently distributed but they are not identically distributed. The Test 1, $\phi_{m1}(x)$ in (9) with size α for testing the hypotheses (1) can be re-written as

$$\phi_{m1}(x)=\begin{cases}1 & if \ \dfrac{\sum_{i=1}^{n}(Y'_i+Y''_i)}{\theta_0}>c_{m1}\\ 0 & Otherwise\end{cases} \tag{15}$$

Again using Arnold *et.al.* (1992) and equations (7) and (8), we can write the distribution functions $f(y'_i)$ and $f(y''_i)$ of Y'_i and Y''_i, the $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ smallest order statistics, respectively, i.e.

$$f(y'_i)=\begin{cases}\dfrac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n-2}{2}\right)!}\left(1-e^{-\frac{y'_i}{\theta}}\right)^{\frac{n-2}{2}}\left(e^{-\frac{y'_i}{\theta}}\right)^{\frac{n}{2}}\dfrac{1}{\theta}e^{-\frac{y'_i}{\theta}} & y'_i\geq 0\\ 0 & otherwise\end{cases} \tag{16}$$

and

$$f(y''_i)=\begin{cases}\dfrac{n!}{\left(\frac{n-2}{2}\right)!\left(\frac{n}{2}\right)!}\left(1-e^{-\frac{y''_i}{\theta}}\right)^{\frac{n}{2}}\left(e^{-\frac{y''_i}{\theta}}\right)^{\frac{n-2}{2}}\dfrac{1}{\theta}e^{-\frac{y''_i}{\theta}} & y''_i\geq 0\\ 0 & otherwise\end{cases} \tag{17}$$

Considering the transformations $Z'_i=\sum_{l=1}^{i}Y'_l$ and $Z_i=\sum_{l=1}^{i}Y_l$; $i=1,2,...,\frac{n}{2}$, we can obtain

$$k_{1\theta}\left(z'_{\frac{n}{2}}\right)=\int_0^{z'_{\frac{n}{2}}}\int_0^{z'_{\frac{n}{2}-1}}..\int_0^{z'_2}g_{1\theta}(z'_1,z'_2-z'_1,z'_3-z'_2\dots,z'_{\frac{n}{2}}-z'_{\frac{n}{2}-1})dz'_1\,dz'_2\dots,dz'_{\frac{n}{2}} \tag{18}$$

$$k_{2\theta}\left(z''_{\frac{n}{2}}\right)=\int_0^{z''_{\frac{n}{2}}}\int_0^{z''_{\frac{n}{2}-1}}..\int_0^{z''_2}g_{1\theta}(z''_1,z''_2-z''_1,\dots,z''_{\frac{n}{2}}-z''_{\frac{n}{2}-1})dz''_1\,dz''_2\dots dz''_{\frac{n}{2}} \tag{19}$$

where

$$g_{1\theta}(y'_1, y'_2, \ldots, y'_{\frac{n}{2}}) = \prod_{i=1}^{\frac{n}{2}} f(y'_i) \tag{20}$$

$$g_{2\theta}(y''_1, y''_2, \ldots, y''_n) = \prod_{i=1}^{\frac{n}{2}} f(y''_i) \tag{21}$$

Now, the *pdf* of $\sum_{i=1}^{\frac{n}{2}} \left(Y'_i + Y''_i\right) = Z'_{\frac{n}{2}} + Z''_{\frac{n}{2}} = W$ (say) can be obtained as

$$k_{3\theta}(w) = \int_0^\infty k_{1\theta}(z'_{\frac{n}{2}}) \times k_{2\theta}(w - z''_{\frac{n}{2}}) dz'_{\frac{n}{2}} \,. \tag{22}$$

The values of c_{m1} for fixed α can be obtained by solving equation (taking $\theta_0 = 1$)

$$\alpha = \beta_{\phi_{m1}}(1) = \int_0^\infty k_{3\theta(\theta=1)}(w) dw \tag{23}$$

The values of c_{m1} for different values of α and for $2 \le n \le 7$ are computed by numerical integration using computer software Maple V and they are tabulated in Table 1.

TABLE 1

Critical Values for Test 1 with MRSS data from Exponential distribution

	α						
n	.001	.005	.010	.025	.050	.100	.250
2	8.2928	6.6794	5.9819	5.0540	4.3433	3.6177	2.6085
3	7.0311	6.0119	5.5546	4.9256	4.4241	3.8897	3.0968
4	7.8617	6.8739	6.4285	5.8125	5.3176	4.7853	3.9825
5	7.9414	7.1108	6.7301	6.1963	5.7609	5.2854	4.5520
6	8.7041	7.8943	7.5214	6.9966	6.5663	6.0938	5.3581
7	9.0933	8.3524	8.0059	7.5229	7.1335	6.6587	5.9998

4.2. *Test 2 for scale parameter of an exponential distribution*

A two sided Test 2 also can be designed for testing the hypotheses (3) where the sample s is drawn by the MRSS method from an exponential distribution.

Case I: If the sample size n is odd

Following (4), a Test 2 for testing the hypotheses (3) can be given as

$$\phi_{m2}(x)=\begin{cases} 0 & if\ K_{mL}<\dfrac{\sum_{i=1}^{n}Y_i}{\theta_0}<K_{mU} \\ 1 & Otherwise \end{cases}$$

where Y_i is defined the same as was defined in section 4.1.

The values of K_{mL} and K_{mU} are determined by solving (again without loss of generality we take $\theta_0=1$) the size of the test

$$\begin{aligned}\alpha=\beta_{\phi_{m2}}&=1-P_{\theta(\theta=1)}\left(K_{mL}<\sum_{i=1}^{n}Y_i<K_{mU}\right)\\ &=\int_{K_{mL}}^{K_{mU}}k_{\theta(\theta=1)}(z_n)dz_n\end{aligned} \tag{26}$$

Case II : If the sample size n is even

If the sample size n is even, a Test 2 with size α for testing the hypotheses (3) can be given as

$$\phi_{e3l}(x)=\begin{cases} 0 & if\ K_{mL}<\dfrac{\sum_{i=1}^{\frac{n}{2}}(Y'_i+Y''_i)}{\theta_0}>K_{mU} \\ 1 & Otherwise \end{cases} \tag{26}$$

where Y'_i and Y''_i are defined the same as was defined in section 4.1.

Similar to Case I, the values of K_{mL} and K_{mU} are determined by solving (again without loss of generality we take $\theta_0=1$) the following equation

$$\begin{aligned}\alpha=\beta_{\phi_{e3l}}&=1-P_{\theta(\theta=1)}\left(K_{mL}<\sum_{i=1}^{\frac{n}{2}}(Y'_i+Y''_i)<K_{mU}\right)\\ &=\int_{K_{mL}}^{K_{mU}}k_{3\theta}(w)dw\end{aligned}$$

where $k_{3\theta}(w)$ is the distribution of $w=z'_{\frac{n}{2}}+z''_{\frac{n}{2}}=\sum_{i=1}^{\frac{n}{2}}(Y'_i+Y''_i)$, which was given in equation (22). For both the cases I and II, we consider

$$\frac{\alpha}{2} = \beta_{\phi_{e3l}} = \int_0^{K_{mU}} k_{3(\theta=1)}(w)dw \text{ and } 1-\frac{\alpha}{2} = \beta_{\phi_{e3l}} = \int_0^{K_{mL}} k_{3(\theta=1)}(w)dw$$

The values of K_{mL} and K_{mU} for different α values are calculated tabulated for $2 \le n \le 7$ in Table 2 (Computations done by numerical integration using computer software Maple V).

TABLE 2

Critical Values for Test 2 with MRSS data from Exponential distribution

	α					
	0.001		0.01		0.025	
n	K_{mL}	K_{mU}	K_{mL}	K_{mU}	K_{mL}	K_{mU}
2	0.0945	8.9865	0.2138	6.6795	0.3005	5.7567
3	0.3960	7.4570	0.7703	5.4043	0.6303	6.0119
4	0.8765	8.2725	1.2106	6.8739	1.3957	6.2816
5	1.4544	8.2818	1.8000	7.1108	1.9700	6.6037
6	2.1716	9.0347	2.4210	7.8943	2.4985	7.3974
7	2.9692	9.3945	3.1120	8.3524	3.1920	7.8955

	α			
	0.05		0.1	
n	K_{mL}	K_{mU}	K_{mL}	K_{mU}
2	0.3929	5.0541	0.5206	4.3433
3	0.9048	4.9256	1.0751	4.4241
4	1.5669	5.8125	1.7761	5.3176
5	2.1600	6.1963	2.3951	5.7609
6	2.6260	6.9966	3.0120	6.5663
7	3.3000	7.5229	3.7710	7.1335

4.3. *Test 3 using MRSS data*

Let S be drawn from a rectangular distribution with probability density function (*pdf*)

$$f_\theta(x) = \begin{cases} \dfrac{1}{\theta} & 0 \le x \le \theta \\ 0 & otherwise \end{cases} \tag{28}$$

The corresponding cumulative distribution function (*cdf*) is given as

$$F_\theta(x) = \int_0^x f_\theta(x)dx = \begin{cases} \dfrac{x}{\theta} & 0 \le x \le \theta \\ 0 & otherwise \end{cases}. \tag{29}$$

Again, it can be seen that for rectangular population, the MRSS method does not perform well even for estimating the sample mean (see Muttlak 1997). Hence a test procedure using MRSS data is not expected to perform well, theoretically a sample containing all the largest order statistics is absolutely a better option. But, in this paper the following test is studied to see the comparison between the RSS and MRSS method.

Case I: If the sample size n is odd

We have $s=\{Y_1, Y_2, \ldots, Y_n\}$. Now the Test 3 with size α for testing the hypotheses (5) can be obtained using (6), and it is given as follows

$$\phi_{m3}(x)=\begin{cases} 1 & if \ \dfrac{\max\{Y_i\}}{\theta}<c_{m3} \ or \ \dfrac{\max\{Y_i\}}{\theta}>1 \\ 0 & Otherwise \end{cases} \tag{30}$$

Since Y_i is the $\left(\frac{n+1}{2}\right)^{th}$ smallest order statistics, the sampling distribution can be written as

$$f(y_i)=\begin{cases} \dfrac{n!}{\left\{\left(\frac{n-1}{2}\right)!\right\}^2}\left[\dfrac{y_i}{\theta}\left(1-\dfrac{y_i}{\theta}\right)\right]^{\frac{n-1}{2}}\dfrac{1}{\theta} & 0\le y_i\le\theta \\ 0 & otherwise \end{cases} \tag{31}$$

We can obtain the values of c_{m3} for specified value of α by taking $\theta_0=1$ and solving the size of the test

$$\alpha=P_{(\theta=1)}(\max\{Y_i\}<c_{m3})$$

$$=\left\{\int_0^{c_{m3}}\frac{n!}{\left\{\left(\frac{n-1}{2}\right)!\right\}^2}\left[\frac{y_i}{\theta}\left(1-\frac{y_i}{\theta}\right)\right]^{\frac{n-1}{2}}\frac{1}{\theta}dy_i\right\}^n \tag{32}$$

$$0\le y_i\le\theta$$

Case II: If the sample size n is even

If the sample size n is even, we have $s=\left\{Y'_1, Y'_2, \ldots, Y'_{\frac{n}{2}}, Y''_1, Y''_2, \ldots, Y''_{\frac{n}{2}}\right\}$

Now the Test 3, $\phi_{m3}(x)$ in (30) with size α for testing the hypotheses (5) can be given as follows

$$\phi_{m3}(x)=\begin{cases}1 & if \ \dfrac{\max\{\{Y'_i\}\cup\{Y''_i\}\}}{\theta}<c_{m3} \ or \ \dfrac{\max\{\{Y'_i\}\cup\{Y''_i\}\}}{\theta}>1 \\ 0 & Otherwise\end{cases} \tag{33}$$

Again taking $\theta_0 = 1$ values of c_{m3} can be obtained by solving the size of the test

$$\begin{aligned}\alpha &= P_{(\theta=1)}(\max\{\{Y'_i\}\cup\{Y''_i\}\}<c_{m3}) \\ &= \prod_{i=1}^{\frac{n}{2}} P_{(\theta=1)}(Y'_i<c_{m3})\times\prod_{i=1}^{\frac{n}{2}} P_{(\theta=1)}(Y''_i<c_{m3}) \\ &= \left(\frac{n!}{\left\{\left(\frac{n-2}{2}\right)!\left(\frac{n}{2}\right)!\right\}}\right)^n \\ &\quad \times\left\{\int_0^{c_{m3}}(y'_i)^{\frac{n-2}{2}}(1-y'_i)^{\frac{n}{2}}dy'_i\times\int_0^{c_{m3}}(y''_i)^{\frac{n-2}{2}}(1-y''_i)^{\frac{n}{2}}dy''_i\right\}^{\frac{n}{2}} \\ &\qquad 0\le y'_i, y''_i\le\theta\end{aligned} \tag{34}$$

The values of c_{m3} for different values of α and for $2\le n\le 7$ are given in Table 3 (Computations done by numerical integration using computer software Maple V).

TABLE 3

Critical Values for TEST 3 with MRSS data from rectangular distribution

	α						
n	.001	.005	.010	.025	.050	.100	.250
2	0.0805	0.1390	0.1764	0.2423	0.3093	0.3966	0.5576
3	0.1959	0.2629	0.2996	0.3578	0.4114	0.4761	0.5876
4	0.3059	0.3721	0.4064	0.4590	0.5058	0.5610	0.6540
5	0.3602	0.4167	0.4453	0.4884	0.5264	0.5708	0.6456
6	0.4171	0.4687	0.4946	0.5334	0.5674	0.6071	0.6740
7	0.4411	0.4859	0.5083	0.5417	0.5709	0.6050	0.6626

5. POWER COMPARISON

For each of the tests Test 1, Test 2 and Test 3 using the MRSS data, the power functions can be evaluated as follows. The power functions of the test $\phi_{m1}(x)$ (given in equations (9) and (15)) are given by

For odd n:

$$1-\beta_{\phi_{m1}}(\theta)=P_\theta\left(\sum_{i=1}^{n}Y_i>c_{m1}\right)=\int_{c_{m1}}^{\infty}k_\theta(z_n)dz_n \tag{35}$$

where $k_\theta(z_n)$ is the distribution of $z_n=\sum_{i=1}^{n}Y_i$ which was given in (13).

For even n:

$$1-\beta_{\phi_{m1}}(\theta)=P_\theta\left(\sum_{i=1}^{\frac{n}{2}}(Y'_i+Y''_i)>c_{m1}\right)=\int_{c_{m1}}^{\infty}k_{3\theta}(w)dw \tag{36}$$

where $k_{3\theta}(w)$ is the distribution of $w=z'_{\frac{n}{2}}+z''_{\frac{n}{2}}=\sum_{i=1}^{\frac{n}{2}}(Y'_i+Y''_i)$, which was given in equation (22).

The power functions of the test $\phi_{m2}(x)$ (equations 24 and 26) are given in the following:

For odd n:

$$1-\beta_{\phi_{m2}}(\theta)=1-\int_{K_{mL}}^{K_{mU}}k_\theta(z_n)dz_n \tag{37}$$

For even n:

$$1-\beta_{\phi_{e3l}}(\theta)=1-\int_{K_{mL}}^{K_{mU}}k_{3\theta}(w)dw\,. \tag{38}$$

The power functions of the test $\phi_{m3}(x)$ (equations 30 and 33) are given by

For odd n:

$$1-\beta_{\phi_{m3}}(\theta)=\begin{cases}1 & \text{if} \quad \theta\le c_{m3}\\ \prod_{i=1}^{n}P_\theta(Y_i<c_{m3}) & \text{if} \quad c_{m3}<\theta\le 1\\ \prod_{i=1}^{n}P_\theta(Y_i<c_{m3})+1-\prod_{i=1}^{n}P_\theta(Y_i<1) & \text{if} \quad \theta>1\end{cases} \tag{39}$$

For even n:

$$1-\beta_{\phi_{m3}}(\theta)=\begin{cases} 1 & \text{if} \quad \theta \le c_{m3} \\ \prod_{i=1}^{\frac{n}{2}} P_{(\theta=1)}(Y'_i < c_{m3}) \times \prod_{i=1}^{\frac{n}{2}} P_{(\theta=1)}(Y''_i < c_{m3}) & \text{if} \quad c_{m3} < \theta \le 1 \\ \prod_{i=1}^{\frac{n}{2}} P_{(\theta=1)}(Y'_i < c_{m3}) \times \prod_{i=1}^{\frac{n}{2}} P_{(\theta=1)}(Y''_i < c_{m3}) & \\ \quad +1-\prod_{i=1}^{\frac{n}{2}} P_{(\theta=1)}(Y'_i < 1) \times \prod_{i=1}^{\frac{n}{2}} P_{(\theta=1)}(Y''_i < 1) & \text{if} \quad \theta > 1 \end{cases} \tag{40}$$

For fixed size $\alpha = 0.05$, each of these power functions are computed for different values of the parameter θ and for different values of n using computer software Maple V. The power of the same tests using RSS data are also computed for the same values of θ and n, then comparative power curves are drawn for each of the tests and for each values of $2 \le n \le 7$. The power curves are presented in Figure 1.

In Figure 1, for each of the tests and each of the values of n, the values of the scale parameter θ vary along horizontal axis and the power of the test vary along the vertical axis. The power curves for the tests using RSS data are shown by broken lines.

6. CONCLUDING REMARKS

Looking at the power curves in Figure 1, it is seen that the Test 1 using the MRSS data have greater power than those using the RSS data when the sample is drawn from a exponential distribution for each n considered in this study. Also, it can be seen that the Test 2 with the sample from exponential distribution has higher power when the sample is drawn by the MRSS method. In case of a rectangular distribution the Test 3 with RSS data possess greater power than that with MRSS data, this supports the facts that (see Muttlak 1997) the RSS method produce more efficient estimator of mean than the MRSS method for rectangular distribution. This seems to be due to the reason that the MRSS method totally disregards the tail information (the largest order statistic is a minimal sufficient statistic for rectangular distribution) on the distribution.

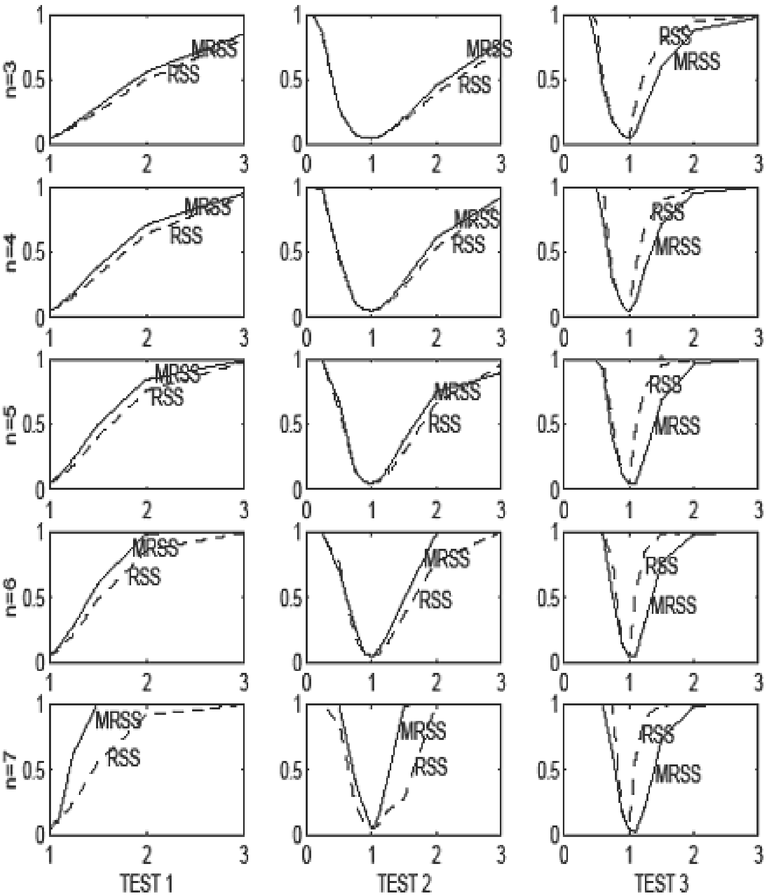

Figure 1 – Power curves for Test 1, Test 2 and Test 3 using RSS and MRSS data for $n = 3, 4, 5, 6$ and 7.

Finally, the power functions of the tests and the corresponding values of the critical regions can be used as a guideline for flexible use and application of the MRSS method.

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RIASSUNTO

Test di ipotesi per parametri di scala mediante median ranked set sampling

Il lavoro analizza test per parametri di scala di distribuzioni esponenziali e rettangolari che coinvolgono il *median ranked set sampling* (MRSS). Vengono inoltre fornite tavole di valori critici. Le funzioni di potenza dei test sono confrontate con le funzioni di potenza di test analoghi proposti da Abu-Dayyeh e Muttlak (1996) basati sul *ranked set sampling* (RSS). Si mostra che i test basati su MRSS hanno potenza maggiore di quelli basati su RSS quando il campione proviene da una distribuzione esponenziale.

SUMMARY

Hypothesis tests on the scale parameter using median ranked set sampling

In this paper, some test procedures for scale parameters of exponential and rectangular distributions involving the median ranked set sampling (MRSS) method are discussed. The necessary tables for the critical regions for the suggested tests are supplied. The power functions of the tests are compared with the power functions of the same tests suggested by Abu-Dayyeh and Muttlak (1996) using ranked set sampling (RSS) data. It is seen that the test procedures using the MRSS data have greater power than those using the RSS data when the sample is drawn from an exponential distribution.

APPENDIX

TABLES OF CRITICAL VALUES FOR TESTS WITH RSS DATA

TABLE A1

Critical Values for TEST 1 with RSS data from exponential distribution

	α						
n	.001	.005	.010	.025	.050	.100	.250
2	8.2928	6.6794	5.9819	5.0540	4.3433	3.6177	2.6085
3	9.5080	7.8913	7.1905	6.2541	5.5316	4.7859	3.7247
4	10.6577	9.0382	8.3346	7.3915	6.6600	5.8996	4.8018
5	11.7725	10.1505	9.4445	8.4958	7.7571	6.9849	5.8589
6	12.8655	11.2413	10.5332	9.5797	8.8348	8.0529	6.9038
7	13.9438	12.3175	11.6075	10.6498	9.8995	9.1091	7.9408

TABLE A2

Critical Values for Test 2 with RSS data from exponential distribution

	α					
	0.001		0.01		0.025	
n	K_{mL}	K_{mU}	K_{mL}	K_{mU}	K_{mL}	K_{mU}
2	0.0945	8.9865	0.2138	6.6794	0.3005	5.7567
3	0.4479	10.2023	0.7156	7.8913	0.8769	6.9637
4	0.9908	11.3524	1.3760	9.0382	1.5914	8.1066
5	1.6400	12.4675	2.1240	10.1505	2.3803	9.2155
6	2.3985	13.5610	2.9290	11.2413	3.2154	10.3040
7	3.1516	14.6396	3.8650	12.3175	4.1111	11.3768

	α			
	0.050		0.100	
	K_{mL}	K_{mU}	K_{mL}	K_{mU}
2	0.3929	5.0540	0.5206	4.3433
3	1.0328	6.2541	1.2313	5.5316
4	1.7920	7.3915	2.0390	6.6600
5	2.6141	8.4959	2.8970	7.7571
6	3.4746	9.5797	3.7853	8.8348
7	4.3656	10.6498	4.6947	9.8995

TABLE A3

Critical Values for Test 3 with RSS data from exponential distribution

	α						
n	.001	.005	.010	.025	.050	.100	.250
2	0.0805	0.1390	0.1764	0.2423	0.3093	0.3966	0.5576
3	0.2350	0.3152	0.3587	0.4279	0.4911	0.5665	0.6927
4	0.3662	0.4449	0.4855	0.5471	0.6013	0.6640	0.7652
5	0.4644	0.5366	0.5727	0.6267	0.6732	0.7263	0.8103
6	0.5381	0.6033	0.6355	0.6830	0.7235	0.7692	0.8409
7	0.5946	0.6536	0.6824	0.7247	0.7604	0.8006	0.8630