

Hysteresis in the Random-Field Ising Model and Bootstrap Percolation

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We study hysteresis in the random-field Ising model with an asymmetric distribution of quenched fields, in the limit of low disorder in two and three dimensions. We relate the spin flip process to bootstrap percolation, and show that the characteristic length for self-averaging L^* increases as $\exp[\exp(J/\Delta)]$ in 2D, and as $\exp\{\exp[\exp(J/\Delta)]\}$ in 3D, for disorder strength Δ much less than the exchange coupling J . For system size $1 \ll L < L^*$, the coercive field h_{coer} varies as $2J - \Delta \ln \ln L$ for the square lattice, and as $2J - \Delta \ln \ln \ln L$ on the cubic lattice. Its limiting value is 0 for $L \rightarrow \infty$ for both square and cubic lattices. For lattices with coordination number 3, the limiting magnetization shows no jump, and h_{coer} tends to J .

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In recent years, there has been a lot of interest in the study of hysteresis in magnetic systems, both theoretically [1] and in experiments [2]. Hysteresis in the random-field Ising model (RFIM) model was first discussed by Sethna *et al.* [3], who proposed it as a model of return point memory, and of Barkhausen noise [4]. Sethna *et al.* solved the model in the mean-field limit, and showed that, if the strength Δ of the quenched random field is large, the average magnetization per site is a continuous function of the external field, but, for small Δ , it shows a discontinuous jump as the external field is increased. Interestingly, the nonequilibrium hysteresis response in the RFIM can be determined exactly on a Bethe lattice [5,6], though the corresponding equilibrium problem has not been solved thus far, even in zero field. These calculations have been extended to determining the distribution of sizes of the Barkhausen jumps [7], and the calculation of minor hysteresis loops [8,9].

In this paper, we study the low disorder limit of the hysteresis loop in the RFIM on periodic lattices in two and three dimensions. We find that the behavior of hysteresis loops depends nontrivially on the coordination number z . For $z = 3$, for continuous unbounded distributions of random fields, the hysteresis loops show no jump discontinuity of magnetization even in the limit of small disorder, but for higher z they do. This is exactly as found in the exact solution on the Bethe lattice [6].

The analytical treatment of self-consistent equations on the Bethe lattice is immediately generalized to the asymmetrical case. However, we find that behavior of hysteresis loops in Euclidean lattices can be quite different from that on the Bethe lattice, for asymmetrical distributions. On hypercubical lattices in d dimensions, there is an instability related to bootstrap percolation, that is absent on the Bethe lattice. This reduces the value of the coercive field h_{coer} away from the Bethe lattice value $\mathcal{O}(J)$ to zero, where J is the exchange coupling. We note that the limit $\Delta \rightarrow 0$ is somewhat subtle, as the system size L^* required for self-averaging diverges very fast for small Δ , and the finite-size

corrections to the thermodynamic limit tend to zero very slowly.

In the RFIM, the Ising spins $\{s_i\}$ with nearest neighbor ferromagnetic interaction J are coupled to the on-site quenched random magnetic field h_i and the external field h . The Hamiltonian of the system is given by

$$H = -J \sum_{\langle i,j \rangle} s_i s_j - \sum_i h_i s_i - h \sum_i s_i. \quad (1)$$

We assume that $\{h_i\}$ are quenched independent identically distributed random variables with the probability that the value of the random field at site i lies between h_i and $h_i + dh_i$ being $\phi(h_i)dh_i$.

The system evolves under the zero-temperature Glauber single-spin-flip dynamics [10]: A spin flip is allowed only if the process lowers energy. We assume that the rate of spin flips is much larger than the rate at which h is changed, so that all flippable spins may be said to relax instantly, and any spin s_i always remains parallel to the net local field ℓ_i at the site:

$$s_i = \text{sgn}(\ell_i) = \text{sgn}\left(J \sum_{j=1}^z s_j + h_i + h\right). \quad (2)$$

Under this dynamics, for ferromagnetic coupling ($J > 0$), if we start with any stable configuration, and then increase the external field and allow the system to relax, the final stable configuration reached is independent of the order in which the unstable spins are flipped. Also, in the relaxation process, no spin flips more than once.

For a given distribution $\phi(h_i)$, we define $p_m(h)$ with $0 \leq m \leq z$ as the conditional probability that the local field at any site i will be large enough so that it will flip up, if m of its neighbors are up, when the uniform external field is h . Clearly,

$$p_m(h) = \int_{(z-2m)J-h}^{\infty} \phi(h_i) dh_i. \quad (3)$$

For any given value of h , the magnetization depends on the distribution $\phi(h_i)$ only through $p_m(h)$.

Historically, RFIM was first studied in the context of possible destruction of long range order by arbitrarily weak quenched disorder in equilibrium systems. Accordingly, the distribution of random field was assumed to be symmetrical. However, in the hysteresis problem, the symmetry between up and down spin states is already broken by the specially prepared initial state (all down in our case), and the symmetry of the distribution plays no special role. In the following, we shall assume that the distribution has an asymmetrical shape, given by

$$\phi(h_i) = \frac{1}{\Delta} \exp(-h_i/\Delta)\theta(h_i), \quad (4)$$

where θ is the step function. The mean value of h_i can be made zero by a shift in the value of the external uniform field. Our treatment is easily extended to other continuous unimodal distributions. The exact form of $\phi(x)$ is not important, and other forms such as $\exp(-x - e^{-x})$ which fall sharply for negative x have the same behavior.

Consider first the case of the two-dimensional hexagonal lattice with $z = 3$. For periodic boundary conditions (PBC), if $\Delta = 0$, starting with a configuration with all spins down, clearly one has $h_{\text{coer}} = 3J$. For $\Delta \neq 0$, the site with the largest local field flips first, and then if $h > J$, $p_1(h) = 1$, this causes neighbors of the flipped spin to flip, and their neighbors, and so on. Thus, so long as there is at least one flipped spin, all other spins also flip, and the magnetization is 1. The largest local field in a system of L^2 spins is of order $2\Delta \ln L$. Once this spin turns up, other spins will flip also up, causing a jump in magnetization from a value ≈ -1 to a value $+1$ in each sample. Hence, the coercive field (the value of h where magnetization changes sign) to lowest order in Δ , is given by

$$h_{\text{coer}} = 3J - 2\Delta \ln L, \quad \text{for } 1 \ll \ln L \ll J/\Delta. \quad (5)$$

Sample to sample fluctuations in the position of the jump are of order Δ . On averaging over disorder, the magnetization will become a smooth function of h , with the width of the transition region being of order Δ .

For a fixed $\Delta \ll J$, if L is increased to a value near $\exp(J/\Delta) \equiv L_{\text{hex}}^*$, h_{coer} decreases to a value near J . For $h \approx J$, $p_1(h)$ is no longer nearly 1, but $p_0(h) \approx 0$, $p_2(h) \approx p_3(h) \approx 1$. The value of magnetization depends only on $p_1(h)$, which is a function of $\tilde{h} = (h - J)/\Delta$. As \tilde{h} increased from $-\infty$, $p_1(h)$ increases continuously from 0 to 1.

In Fig. 1, curve A shows the result of a simulation on the hexagonal lattice with $L = 4096$, and PBC. To avoid the problem of probability of nucleation being very small for h near J , we made the local field at a small fraction of randomly chosen sites very large, so that these spins are up at any h . The number of such spins is of order L , so that their effect on the average magnetization is negligible. Introduction of these ‘‘nucleation centers’’ makes $L^* \approx \mathcal{O}(\sqrt{L})$ (the average separation between centers),

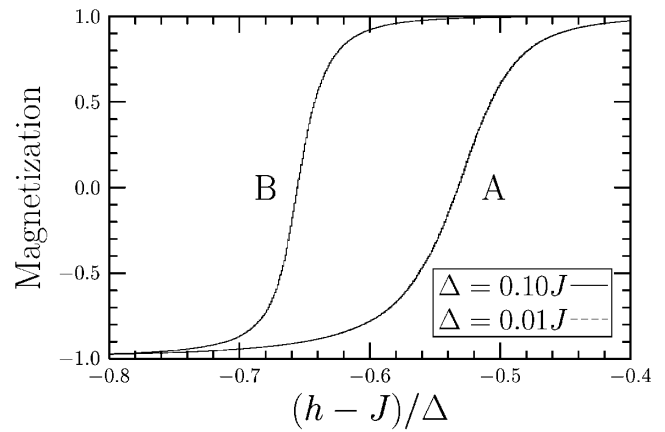


FIG. 1. Magnetization in the increasing field. The curves for the two values of Δ coincide. Curves A and B are for 2D and 3D lattice with $z = 3$.

and h_{coer} drops to a value near J , so that we can study the large L limit with available computers. For $L > L_{\text{hex}}^*$, the behavior of hysteresis loops becomes independent of L .

We see that magnetization no longer undergoes a single large jump, but many small jumps. In the figure, we also show the plot of magnetization when the random field at each site is decreased by a factor of 10. This changes the value Δ from $0.1J$ to $0.01J$. However, plotted as a function of \tilde{h} , the magnetization for these two different values (for small Δ) fall on top of each other for the same realization of disorder (except for the overall scale Δ). Thus, we can decrease Δ further to arbitrarily small values, and the limit of $\Delta \rightarrow 0$ is straightforward for each realization of disorder. Then, averaging over disorder, for a fixed Δ , we see that h_{coer} tends to the value J as Δ tends to 0. Also, we see that there is no macroscopic jump discontinuity for any nonzero Δ .

We also show, in Fig. 1 (curve B), the results of simulation of a three-dimensional lattice with $z = 3$ of size 256^3 with PBC. The behavior is qualitatively the same as that in two dimensions. The value of $h_{\text{coer}} = J$ in the limit $\Delta \rightarrow 0$ is the same for symmetrical distribution, and also is the same as predicted by the Bethe approximation.

On the square lattice also, the value of h_{coer} is determined by the need to create a nucleation event. Arguing as before, we see that h_{coer} to lowest order in Δ is given by $h_{\text{coer}} \approx 4J - 2\Delta \ln L$, for $1 \ll \ln L \ll J/\Delta$. Adding a small number of nucleation sites suppresses this slow transient, and lowers h_{coer} from $4J$ to a value near $2J$. However, in this case, even after adding the nucleation centers, the system shows a large single jump in magnetization, indicating the existence of another instability. We observed in the simulation that at low Δ , as h is increased, the domains of up spins grow in rectangular clusters (see Fig. 2) and, at a critical value of h_{coer} , one of them suddenly fills the entire lattice. This value h_{coer} fluctuates a bit from sample to sample. In Fig. 3 we have plotted the distribution of the scaled variable $\tilde{h}_c = (h_{\text{coer}} - 2J)/\Delta$ for different system sizes L , for $\Delta = 0.001J$. The number of

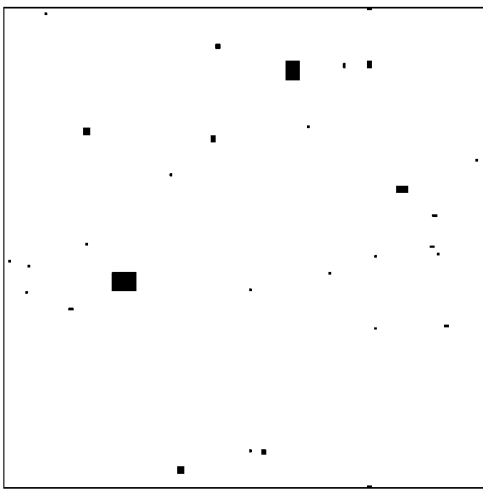


FIG. 2. A snapshot of the up spins just before the jump ($h = 1.998243J$). The lattice size is 200×200 and $\Delta = 0.001J$. Initial configuration is prepared with 0.05% up spins.

different realizations varies from 10^4 (for the largest L) to 10^5 (for the smallest L). Note that the distribution shifts to the left with the increasing system size, and becomes narrower.

This instability can be understood as follows: On a square lattice, for the asymmetric distribution [Eq. (4)] for $h > 0$, $p_m = 1$ for $m \geq 2$, and any spins with more than one up neighbor flips up. Therefore, stable clusters of up spins are rectangular in shape. The growth of domains of up spins is the same as in the bootstrap percolation process BP_m with $m = 2$ [11–13]. In the process BP_m , the initial configuration is prepared by occupying lattice sites independently with a probability p and the resulting configuration is evolved by the rules: The occupied sites remain occupied forever, while an unoccupied site having at least m occupied neighbors, becomes occupied. For $m = 2$, on a square lattice, in the final configuration, the sites which are occupied form disjoint rectangles, such as the cluster of up spins in Fig. 2. It has been proven that, in the thermodynamic limit of large L , for any initial concentration

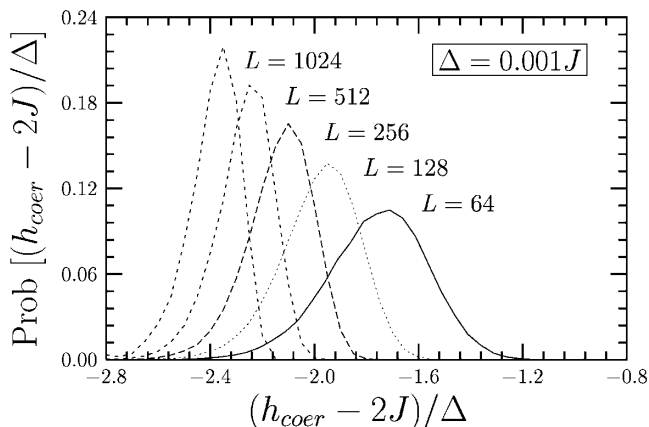


FIG. 3. Distribution of the scaled coercive field on a square lattice for different lattice size L^2 .

$p > 0$, in the final configuration all sites are occupied with probability 1 [12].

Now consider a rectangular cluster of up spins, of length l and width m . Let $P(l, m)$ be the probability that, if this rectangle is put in a randomly prepared background of density $p_1(h)$, this rectangle will grow by the BP_2 process to fill the entire space. The probability that the random fields at any sites neighboring this rectangle will be large enough to cause it to flip up is $p_1(h)$. The probability that there is at least one such site along each of two adjacent sides of length l and m of the rectangle is $(1 - q^l)(1 - q^m)$, where $q = 1 - p_1(h)$. Once these spins flip up, this induces all the other spins along the boundary side to flip up and the size of the rectangle grows to $(l + 1) \times (m + 1)$. Therefore,

$$P(l, m) \geq (1 - q^l)(1 - q^m)P(l + 1, m + 1). \quad (6)$$

Thus, the probability of occurrence of a nucleation which finally grows to fill the entire lattice is

$$P_{\text{nuc}} \geq p_0(h) \prod_{j=1}^{\infty} (1 - q^j)^2. \quad (7)$$

The right-hand side can be shown to vary as $p_0(h) \times \exp(-\frac{\pi^2}{3p_1(h)})$ for small $p_1(h)$. The condition to determine h_{coer} is that, for this value of h , P_{nuc} becomes of order $1/L^2$, so that we get

$$p_0(h_{\text{coer}}) \exp\left(-\frac{\pi^2}{3p_1(h_{\text{coer}})}\right) \approx \frac{1}{L^2}. \quad (8)$$

This equation can be solved for h_{coer} for any given L . For the distribution given by Eq. (4), this becomes

$$\exp\left(\frac{h_{\text{coer}} - 4J}{\Delta}\right) \exp\left[-\frac{2\pi^2}{3} \exp\left(\frac{-h_{\text{coer}} + 2J}{\Delta}\right)\right] \approx \frac{1}{L^2}. \quad (9)$$

It is easy to see from this equation that, for $1 \ll \ln L \ll J/\Delta$, the leading L dependence of h_{coer} , to lowest order in Δ is given by

$$h_{\text{coer}} \approx 4J - 2\Delta \ln L, \quad (10)$$

and for $J/\Delta \ll \ln L \ll \exp(2J/\Delta)$,

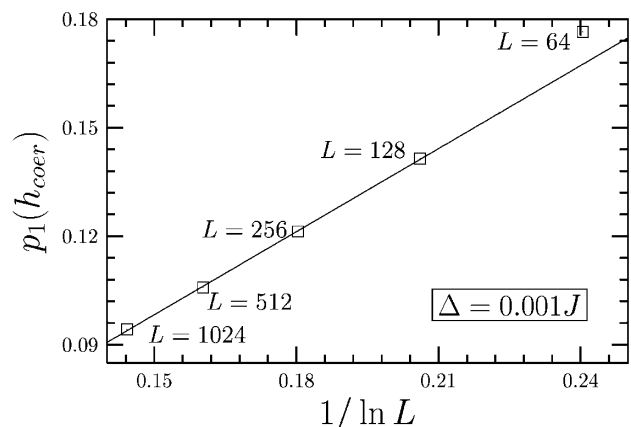


FIG. 4. $p_1(h_{\text{coer}})$ vs $1/\ln L$ for square lattice.

$$h_{\text{coer}} \approx 2J - \Delta \ln \left[\frac{3}{\pi^2} (\ln L - J/\Delta) \right]. \quad (11)$$

This agrees with our observation that the scaled critical field \tilde{h}_c shifts to the left with increasing system size.

To test the validity of Eq. (8) in simulations, we put $p_0(h) = 0.005$ independent of h . Equation (8) then simplifies to

$$p_1(h_{\text{coer}}) \approx \frac{\pi^2}{6 \ln L}. \quad (12)$$

In Fig. 4, we have plotted p_1 for the mean h_{coer} from Fig. 3 versus $1/\ln L$. The graph is approximately a straight line, which agrees with Eq. (12). The slope of the line is less than in Eq. (12), which gives only an upper bound to h_{coer} .

If $h > 0$, we will have $p_2 = 1$, and bootstrapping ensures that, as long as $p_0 > 0$, we will have all spins up in the limit of large L . This implies that $h_{\text{coer}} = 0$ in this limit.

If there are sites with large negative quenched fields, the bootstrap growth stops at such sites. Hence, the bootstrap instability cannot be seen for symmetric distributions. Even if the quenched fields are only positive, the instability does not occur on lattices with $z = 3$. On such lattices, if the unoccupied sites percolate, there are infinitely extended lines of unoccupied sites in the lattice. These cannot become occupied by bootstrapping under BP_2 . Thus, the critical threshold for BP_2 on such lattices is not 0.

The above analysis is easily extended to higher dimensions. In $d = 3$, if $h > 0$, then $p_m(h) = 1$ for $m \geq 3$; therefore the spin flip process is similar to the spanning process of three-dimensional BP_3 [14]. But in this case, it is known that for any initial nonzero density, in the thermodynamical limit, the final configuration has all sites occupied with probability 1. The clusters of up spins grow as cuboids, and at each surface of the cluster, the nucleation process is similar to that in two dimensions. Let ϵ be the probability that a nucleation occurs at a given point of a surface of the clusters of up spins which sweeps the entire two-dimensional plane at h .

$$\epsilon \approx p_1(h) \exp\left(-\frac{\pi^2}{3p_2(h)}\right). \quad (13)$$

The probability that there exists at least one nucleation which sweeps the entire plane of size $l \times l$ is $1 - (1 - \epsilon)^{l^2}$. Therefore, the probability P_{nuc} that a nucleation sweeps the entire three-dimensional lattice at h satisfies

$$P_{\text{nuc}} \geq p_0(h) \prod_{l=1}^{\infty} [1 - (1 - \epsilon)^{l^2}]^3. \quad (14)$$

For small ϵ , the infinite product can be shown to vary as $\exp(-A/\sqrt{\epsilon})$, with $A = \frac{3}{2}\sqrt{\pi} \zeta(3/2)$.

h_{coer} is determined by the condition that P_{nuc} must be of the order of $1/L^3$:

$$p_0(h_{\text{coer}}) \exp\left[-\frac{A}{\sqrt{p_1(h_{\text{coer}})}} \exp\left(\frac{\pi^2}{6p_2(h_{\text{coer}})}\right)\right] \approx 1/L^3. \quad (15)$$

The leading L dependence of h_{coer} is different in different ranges of h_{coer} , depending on whether the strongest dependence of the left-hand side comes from variation of $p_0(h)$, $p_1(h)$, or $p_2(h)$. We find that $h_{\text{coer}} \approx 6J - 3\Delta \ln L$, for $4J < h_{\text{coer}} < 6J$. It is $\approx 4J - 2\Delta \ln[\ln L - (2J)/(3\Delta)]$, for $2J < h_{\text{coer}} < 4J$; and $\approx 2J - \Delta \ln \ln[\ln L - (2J)/(3\Delta)]$, for $\Delta \ll h_{\text{coer}} < 2J$. It is straightforward to determine the corresponding ranges of L for the validity of these equations.

In the limit $L \gg L_{\text{cub}}^* = \exp\{\exp[\exp(2J/\Delta)]\}$, the loop becomes independent of L , with $h_{\text{coer}} \rightarrow 0$. We have also verified the existence of jump in numerical simulation for $z = 4$ (diamond lattice) in three dimensions.

In brief, we have shown that the hysteresis loops on lattices with coordination number three are qualitatively different from those with $z > 3$. For the square and cubic lattices, h_{coer} decreases to 0 very slowly for large L . In general, it is true for lattices where the corresponding bootstrap percolation problem has an instability.

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