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(I,γ)-GENERALIZED SEMI-CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract

In this paper we introduce (I,γ)-generalized semi-closed sets in topological spaces and also introduce $\gamma S - T_I$-spaces and investigate some of their properties.

1. Introduction

Recently Julian Dontchev et. al. [1] introduce (I,γ)-generalized closed sets via topological ideals. In this paper we introduce (I,γ)-generalized semi-closed sets and investigate some of their properties.

An ideal I on a topological space (X,τ) is a non-empty collection of subsets of X satisfying the following two properties:
(1) $A \in I$ and $B \subset A$ implies $B \in I$
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$

For a subset $A \subset X$, $A^*(I) = \{x \in X/U \cap A \notin I$ for each neighbourhood U of x $\}$ is called the local function of A with respect to I and τ. Recall that $A \subseteq (X,\tau,I)$ is called τ^*-closed [2] if $A^* \subseteq A$. It is well known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology $\tau^*(I)$, finer than τ. An operation γ [3,6] on the topology τ on a given topological space (X,τ) is a function from the topology itself into the power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V.

The following operators are examples of the operation γ: the closure operator γ_{cl} defined by $\gamma(U) =$ cl (U), the identity operator γ_{id} defined by $\gamma(U) = U$. Another example of the operation γ is the γ_f-operator defined by $U^{\gamma_f} = (FrU)^c = X/FrU$[7]. Two operators γ_1 and γ_2 are called mutually dual [7] if $U^{\gamma_1} \cap U^{\gamma_2} = U$ for each $U \in \tau$. For example the identity operator is mutually dual to any other operator, while the γ_f-operator is mutually dual to the closure operator [7].
Definition: A subset A of a space (X,τ) is called

(a) an α-open set [5] if $A \subseteq int(cl(int(A)))$.

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(b) a generalized closed(briefly g-closed) set [4] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

(c) a (I, γ)- generalized closed (briefly (I, γ)- g-closed) set [1] if $A^* \subseteq U^\gamma$ whenever $A \subseteq U$ and U is open in (X, τ).

We denote the family of all (I, γ)- generalized semi-closed subsets (briefly (I, γ)-gs-closed) of a space (X, τ, I, γ) by $IGS(X)$ and simply write I-gs-closed in case when γ is an identity operator. Throughout this paper the operator γ is defined as $\gamma : \tau^s \to P(X)$, where τ^s denotes the set of all semi-open sets of (X, τ).

2. Basic properties of (I, γ)- generalized semi-closed sets

Definition 2.1. A subset A of a topological space (X, τ) is called (I, γ)-generalized semi closed (briefly (I, γ)-gs-closed) if $A^* \subseteq U^\gamma$, whenever $A \subseteq U$ and U is semi-open in (X, τ).
Example 2.2. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, and $I = \{\{a\}, \{a, b\}\}$. Here (I, γ)-gs-closed sets are $X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}$.
Theorem 2.3. Every (I, γ)-gs-closed set is (I, γ)-g-closed set.
Proof. Let $A \subseteq U$, U is open and hence it is semi-open. Since A is (I, γ)-gs-closed, $A^* \subseteq U^\gamma$. Hence A is (I, γ)-g-closed.
Remark 2.4. The converse of the above theorem need not be true by the following example.
Example 2.5 Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{a\}$. Let $\gamma_1 : \tau^s \to P(X)$ and $\gamma_2 : \tau \to P(X)$ be defined by $U^{\gamma_1} = \text{cl } U$ and $U^{\gamma_2} = \text{cl } U$ respectively. Therefore $A = \{b, c\}$ is (I, γ)-g-closed but not (I, γ)-gs-closed.
Theorem 2.6. If A is I-gs-closed and semi-open, then A is τ^*-closed.
Proof: Since A is I-gs-closed, then $A^* \subseteq U$, U is semi-open. It is given that A is semi-open implies $A^* \subseteq U = A$, this implies that $A^* \subseteq A$. Hence A is τ^*-closed.
Theorem 2.7. Let (X, τ, I, γ) be a topological space.

(i) If $(A_i)_{i \in I}$ is a locally finite family of sets and each $A_i \in IGS(X)$, then $\cup_{i \in I} A_i \in IGS(X)$

(ii) Finite intersection of (I, γ)-gs-closed sets need not be (I, γ)-gs-closed.

Proof:

(i) Let $\cup_{i \in I} A_i \subseteq U$, where $U \in \tau^s$. Since $A_i \in IGS(X)$ for each $i \in I$, then $A_i^* \subseteq U^\gamma$. Hence $\cup_{i \in I} A_i^* \subseteq U^\gamma$, But we know that $(\cup_{i \in I} A_i)^* = \cup_{i \in I} A_i^*$, Therefore $(\cup_{i \in I} A_i)^* \subseteq U^\gamma$. Hence $\cup_{i \in I} A_i \in IGS(x)$

(ii) Let $X = \{a, b, c\}$,$\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\{a\}, \{a, b\}\}$. Set $A = \{a, b\}$ and $B = \{b, c\}$, clearly $A, B \in IGS(X)$ but $A \cap B = \{b\} \notin IGS(X)$.

Theorem 2.8. Let $(X, \tau, I, \gamma_{id})$ be a space. If $A \subseteq X$ is I-gs-closed and B is closed and τ^*-closed, then $A \cap B$ is I-gs-closed.
Proof: Let $U \in \tau^s$ be such that $A \cap B \subseteq U$. Then $A \subseteq U \cup (X/B)$. Since A is I-gs-closed, then $A^* \subseteq U \cup (X/B)$. Hence $B \cap A^* \subseteq U \cap B \subseteq U$, But we know that

$B^* \subseteq B$, Therefore $(A \cap B)^* \subseteq A^* \cap B^* \subseteq A^* \cap B \subseteq U$, Since B is τ^*-closed. Hence $A \cap B$ is I-gs-closed.

Result 2.9. A subset S of a space (X, τ, I) is a topological space with an ideal $I_s = \{I \cap S : I \in I\}$ on S.

Theorem 2.10. Let $A \subseteq S \subseteq (X, \tau, I, \gamma_{id})$. If A is I_s-gs-closed in $(S, \tau/s, I_s, \gamma_{id})$ and S is closed in (X, τ), then A is I-gs-closed in $(X, \tau, I, \gamma_{id})$.

Proof: Let $A \subseteq U$, where $U \in \tau^s$. Let $x \notin U$. We consider the following two cases. Case(i) $x \in S$. By assumption, $A^*(I_s, \tau/s) \subseteq U \cap S \subset U$, We show that $A^*(I) \subseteq A^*(I_s, \tau/s)$. Let $x \notin A^*(I_s, \tau/s)$. Since $x \in S$, then for some open subset V_s of $(S, \tau/s)$ containing x, we have $V_s \cap A \in I_s$; since $V_s = V \cap S$ for some $V \in \tau$, then $(S \cap V) \cap A \in I_s \subseteq I$, that is $V \cap A \in I$ for some $V \in \tau$ containing x. This shows that $x \notin A^*(I)$. Hence $A^*(I) \subseteq U$.

Case(ii) $x \notin S$. Then X/S is an open neighbourhood of x disjoint from A. Hence $x \notin A^*(I)$. Consequently $A^*(I) \subseteq U$.

Both cases we show that the local function of A with respect to I and τ is in U. Hence A is I-gs-closed in $(X, \tau, I, \gamma_{id})$.

Theorem 2.11. Let $A \subseteq S \subseteq (X, \tau, I, \gamma)$. If $A \in (IGS(X))$ and $S \in \tau$, then $A \in IGS(S)$.

Proof: Let U be a semi-open subset of $(S, \tau/s)$ such that $A \subseteq U$. Since $S \in \tau$, then $U \in \tau^s$. Then $A^*(I) \subseteq U^\gamma$, since $A \in IGS(X)$. We show that $A^*(I_s, \tau/s) \subseteq A^*(I)$. Let $x \notin A^*(I)$. We assume that $x \in S$, since otherwise we are done. Now, for some $V \in \tau$ containing x, $V \cap S \in I$. Moreover, $V \cap A \in I_s$, since $A \subseteq S$. Then $V \cap S$ is an open neighbourhood of x in $(S, \tau/s)$ such that $(V \cap S) \cap A = V \cap A \in I_s$. This shows that $x \notin A^*(I_s, \tau/s)$. Hence $A^*(I_s, \tau/s) \subseteq U^{\gamma/s}$, where $U^{\gamma/s}$ means the image of the operation $\gamma/s : \tau^s/S \to P(S)$ defined by, $(\gamma/s)(U) = \gamma(U) \cap S$ for each $U \in \tau^s/S$. Hence $A \in IGS(S)$.

Theorem 2.12. Let A be a subset of $(X, \tau, I, \gamma_{id})$. If A is I-gs-closed, then A^*/A does not contain any non-empty semi-closed subset.

Proof: Assume that F is semi-closed subset of A^*/A. Clearly $A \subseteq X/F$, where A is I-gs-closed and $X/F \in \tau^s$. This $A^* \subseteq X/F$, that is $F \subseteq X/A^*$. Since due to our assumption $F \subseteq A^*$, $F \subseteq (X/A^*) \cap A^* = \phi$.

Theorem 2.13. If the set $A \subseteq (X, \tau, I)$ is both (I, γ_1)-gs-closed and $(I, \gamma_2) - gs$-closed, then it is I-gs-closed, granted the operators γ_1 and γ_2 are mutually dual.

Proof: Let $A \subseteq U$, where $U \in \tau^s$. Since $A^* \subseteq U^{\gamma_1}$ and $A^* \subseteq U^{\gamma_2}$, then $A^* \subseteq U^{\gamma_1} \cap U^{\gamma_2} = U$. Since γ_1 and γ_2 are mutually dual. Hence A is I-gs-closed.

Theorem 2.14. Every set $A \subseteq (X, \tau, I)$ is (I, γ_{cl})-gs-closed.

Proof: Let $A \subseteq U, U$ is semi-open. We know that $A \cup A^* = cl^*(A) \subseteq cl(A) \subseteq cl(U)$. This implies that $A^* \subseteq cl(U)$. Hence A is (I, γ_{cl})-gs-closed.

Corollary 2.15. For a set $A \subseteq (X, \tau, I)$, the following conditions are equivalent.

(i) A is (I, γ_f) - gs-closed.

(ii) A is I-gs-closed.

Proof:

(i) $\Rightarrow$ (ii), By the above theorem, A is (I, γ_{cl}) - gs-closed. Since γ_f and γ_{cl} are mutually dual due to [7], then $\gamma_f(U) \cap \gamma_{cl}(U) = U$. This implies that $A^* \subseteq U$, that is, A is I-gs-closed.

(ii) $\Rightarrow$ (i), Let $A \subseteq U$, U is semi-open. Since A is I-gs-closed, $A^* \subseteq U$. But we know that $U \subseteq U^{\gamma_f}$, we have $A^* \subseteq U \subseteq U^{\gamma_f}$, this implies that $A^* \subseteq U^{\gamma_f}$. Therefore A is (I, γ_f) - gs- closed.

3. $\gamma S - T_I$-space

Definition 3.1. A space (X, τ, I, γ) is called an $\gamma S - T_I$-space if every (I, γ)-gs-closed subset of X is τ^*-closed. We use the simple notation ST_I-space, in case γ is the identity operator.

Theorem 3.2. For a space (X, τ, I), the following conditions are equivalent.

(i) X is a ST_I-space

(ii) Each singleton of X is either semi-closed or τ^*-open.

Proof: (i) $\Rightarrow$ (ii), Let $x \in X$. If $\{x\}$ is not semi-closed, then $A = X \setminus \{x\} \notin \tau^s$ and then A is trivially I-gs-closed. By (i) A is τ^*-closed and $\{x\}$ is τ^*-open.
(ii) $\Rightarrow$ (i), Let A be I-gs-closed and let $x \in cl^*(A)$. We have the following two cases.
case(i): $\{x\}$ is semi-closed. By theorem 2.12, A^*/A does not contain a non-empty semi-closed subset. This shows that $x \in A$.
case(ii): $\{x\}$ is τ^*-open. Then $\{x\} \cap A \neq \phi$. Hence $x \in A$. Thus in both cases x is in A and so $A = cl^*A$. that is A is τ^*-closed, which shows that X is a ST_I-space.

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