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IDEAL CONVERGENCE AND DIVERGENCE OF NETS IN (ℓ)-GROUPS

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Abstract. In this paper we introduce the $\mathcal{I}$- and $\mathcal{I}^*$-convergence and divergence of nets in (ℓ)-groups. We prove some theorems relating different types of convergence/divergence for nets in (ℓ)-group setting, in relation with ideals. We consider both order and (D)-convergence.

By using basic properties of order sequences, some fundamental properties, Cauchy-type characterizations and comparison results are derived.

We prove that $\mathcal{I}^*$-convergence/divergence implies $\mathcal{I}$-convergence/divergence for every ideal, admissible for the set of indexes with respect to which the net involved is directed, and we investigate a class of ideals for which the converse implication holds. Finally we pose some open problems.

Keywords: net, (ℓ)-group, ideal, ideal order, (D)-convergence, ideal divergence

MSC 2010: 28B15, 54A20

1. Introduction

The theory of ideal convergence for sequences was introduced in [24] and independently in [30] under the name "cofilter convergence". This notion was further investigated in [13], [19], [23], [26], [32], [33], where some basic properties are studied. The $\mathcal{I}$-Cauchy notion was introduced and examined, in the context of metric spaces, in [15], [20], [29]. The concept of $\mathcal{I}$-convergence was investigated in the setting of normed spaces in [31] and was extended to the class of topological spaces in [14], [17], [27], and ideal convergence for nets was presented in [17].

Several applications of ideal convergence have been obtained in the context of Riesz spaces and lattice groups. In [5] a Bochner-type integral was investigated, and

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in [6] the problem of defining limits by means of suitable integrals was studied in this setting. In [10] we introduced the concept of ideal convergence in (ℓ)-groups, we dealt with the main basic properties and in [8], [10] we proved some versions of basic matrix theorems. Moreover, in [9] we obtained some limit theorems for ideal pointwise convergent measures taking values in an (ℓ)-group R. Some other applications of $\mathcal{I}$-convergence in the real case (that is, when $R = \mathbb{R}$) can be found in [1], [7], concerning weak compactness, Ascoli-type theorems, ideal exhaustiveness and uniform convergence on compact sets. In [7], [11] some results contained in [21] were extended to the ideal setting. In [7] the notion of ideal exhaustiveness was used also to obtain some further versions of limit theorems for ideal pointwise convergent measures.

In this paper we deal with ideal convergence and ideal divergence of nets in the (ℓ)-group context. Motivated by earlier results proved in [10] on $\mathcal{I}$-convergence for sequences, we prove some technical theorems relating various convergences/divergences of nets in Dedekind complete (ℓ)-groups. We derive basic properties, some completeness characterizations and comparison results. Moreover, we deal with $\mathcal{I}^*$-convergence/divergence for nets in the (ℓ)-group setting and prove that these two properties imply ideal convergence/divergence for every fixed ideal, admissible with respect to the index set which form the net involved. Furthermore, we give a class of ideals for which the converse implications are true. In the proofs we essentially involve fundamental properties of order sequences. Finally, we pose some open problems.

2. The main results

We begin with the following

Definitions 2.1. (a) An abelian group $(R, +)$ is called *(ℓ)-group* if it is a lattice and the following implication holds: $a \leqslant b \Longrightarrow a + c \leqslant b + c$ for all a, b, $c \in R$.

If not stated otherwise, in what follows R denotes an (ℓ)-group.

(b) An (ℓ)-group R is said to be *Dedekind complete* iff every nonempty subset of R, bounded from above, has supremum in R. A Dedekind complete (ℓ)-group is said to be *super Dedekind complete* iff every subset $R_1 \subset R$, $R_1 \neq \emptyset$ bounded from above contains a countable subset having the same supremum as R_1.

(c) A pair $\Lambda = (\Lambda, \geqslant)$ is a *directed set* iff Λ it a nonempty set and $\geqslant$ is a reflexive and transitive binary relation on Λ, such that for any two elements λ_1, $\lambda_2 \in \Lambda$ there is a $\lambda_0 \in \Lambda$ with $\lambda_0 \geqslant \lambda_1$ and $\lambda_0 \geqslant \lambda_2$. From now on, let us denote by M_λ the set $\{\zeta \in \Lambda \colon \zeta \geqslant \lambda\}$, $\lambda \in \Lambda$.

(d) We say that a sequence $(p_n)_n$ of positive elements of R is an *(O)-sequence* iff it is decreasing and $\wedge_n p_n = 0$. A net $(x_\lambda)_{\lambda \in \Lambda}$ in R (an indexed system of elements of R such that the index set Λ is directed) is said to be *order-convergent* (or (O)-*convergent*) to $x \in R$ iff there exists an (O)-sequence $(p_l)_l$ in R such that to every $l \in \mathbb{N}$ there corresponds an element $\lambda \in \Lambda$ with $|x_\zeta - x| \leqslant p_l$ for all $\zeta \in \Lambda$, $\zeta \geqslant \lambda$, and in this case we will write $(O) \lim\limits_{\lambda \in \Lambda} x_\lambda = x$.

(e) A bounded double sequence $(a_{t,r})_{t,r}$ in R is called (D)-*sequence* or *regulator* iff for all $t \in \mathbb{N}$ the sequence $(a_{t,r})_r$ is an (O)-sequence. A net $(x_\lambda)_{\lambda \in \Lambda}$ in R is said to be (D)-*convergent* to $x \in R$ (and we write $(D) \lim\limits_{\lambda \in \Lambda} x_\lambda = x$) iff there exists a (D)-sequence $(a_{t,r})_{t,r}$ in R, such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a $\lambda_0 \in \Lambda$ such that $|x_\lambda - x| \leqslant \bigvee\limits_{t=1}^{\infty} a_{t,\varphi(t)}$ for all $\lambda \in \Lambda$, $\lambda \geqslant \lambda_0$.

(f) An (ℓ)-group R is said to be *weakly σ-distributive* iff for every (D)-sequence $(a_{t,r})_{t,r}$ we have:

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \right) = 0.$$

Observe that, if $\mathcal{B}$ is the σ-algebra of all Borel subsets of $[0,1]$ and ν is the Lebesgue measure, then the space $L^0([0,1], \mathcal{B}, \nu)$ is weakly σ-distributive (see [3, Example 2.17]).

(g) A Dedekind complete Riesz space R is said to have *property* (PR) (*positive regularity*) iff there exists an increasing sequence $(h_k)_k$ of positive elements of R such that for every $r \in R$ there are a $t > 0$ and a natural number k such that $|r| \leqslant t h_k$. Note that, if R is a Riesz space having property (PR), then for each $r \in R$ there is $k \in \mathbb{N}$ such that $|r| \leqslant k h_k$. Observe that, if R has an order unit u (that is a positive element such that for all $r \in R$ we get $|r| \leqslant tu$ for some suitable positive real number t), then R has (PR), while the converse is in general not true: for instance the space c_{00} of all real-valued sequences which are eventually 0, with the coordinatewise order, has property (PR) but does not admit an order unit. The space c_0 of all real-valued sequences convergent to 0, ordered coordinatewise, does not have property (PR) (see [34]).

(h) A sequence $(t_k)_k$ of positive elements of R is said to be *bounding* iff $2t_k \leqslant t_{k+1}$ for each $k \in \mathbb{N}$.

(i) A subset S of R is (PR)-*bounded* by a bounding sequence $(t_k)_k$ iff for every $s \in S$ there exists a positive integer $\overline{k}$ such that $|s| \leqslant t_{\overline{k}}$. Since $(t_k)_k$ is bounding, this implies that $|s| \leqslant t_k$ whenever $k \geqslant \overline{k}$. Note that, if R has property (PR), then R is (PR)-bounded by some bounding sequence, but there are many situations in which we deal with (PR)-bounded sets in a space R without property (PR) (for example $R = L^0(X, \Sigma, \mu)$, where $\mu\colon \Sigma \to \mathbb{R} \cup \{+\infty\}$ is a positive, σ-additive and σ-finite measure, see for instance [4, Example 4.7]).

(j) Let $(\Lambda, \geqslant)$ be a directed set. A family of sets $\mathcal{I} \subset \mathcal{P}(\Lambda)$ is called an *ideal* of Λ iff $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$. An ideal is said to be *non-trivial* iff $\mathcal{I} \neq \emptyset$ and $\Lambda \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is said to be (Λ)-*admissible* iff $\Lambda \setminus M_\lambda \in \mathcal{I}$ for all $\lambda \in \Lambda$.

(k) A (Λ)-admissible ideal $\mathcal{I}$ is said to be a (ΛP)-*ideal* iff for any disjoint sequence $(A_j)_j$ in $\mathcal{I}$ there are sets $B_j \subset \Lambda$ and elements $p_j \in \Lambda$, $j \in \mathbb{N}$, such that the symmetric difference $A_j \Delta B_j \subset \Lambda \setminus M_{p_j}$ for all $j \in \mathbb{N}$ and $\bigcup\limits_{j=1}^{\infty} B_j \in \mathcal{I}$ (see also [17]).

(l) Given a *fixed* (Λ)-admissible ideal $\mathcal{I}$, its *dual filter* is the set

$$\mathcal{F} = \mathcal{F}(\mathcal{I}) := \{X \setminus I \colon\ I \in \mathcal{I}\}.$$

When we deal with an ideal $\mathcal{I}$, we always suppose that $\mathcal{I}$ is (Λ)-admissible, without saying it explicitly.

(m) If $\mathcal{I}$ is an ideal of Λ, we say that a net $(x_\lambda)_{\lambda\in\Lambda}$ in R $(O\mathcal{I})$-*converges to* $x \in R$ iff there exists an (O)-sequence $(\sigma_l)_l$ with the property that

$$\{\lambda \in \Lambda \colon\ |x_\lambda - x| \leqslant \sigma_l\} \in \mathcal{F} \tag{1}$$

for all $l \in \mathbb{N}$.

(n) A net $(x_\lambda)_{\lambda\in\Lambda}$ is said to be $(O\mathcal{I})$-*Cauchy* iff there is an (O)-sequence $(\sigma_l)_l$ such that for each $l \in \mathbb{N}$ there is a $\tau \in \Lambda$ with

$$\{\lambda \in \Lambda \colon\ |x_\lambda - x_\tau| \leqslant \sigma_l\} \in \mathcal{F}.$$

(o) A net $(x_\lambda)_\lambda$ in R $(D\mathcal{I})$-*converges to* $x \in R$ iff there exists a (D)-sequence $(a_{t,r})_{t,r}$ with the property that

$$\left\{\lambda \in \Lambda \colon\ |x_\lambda - x| \leqslant \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\right\} \in \mathcal{F} \tag{2}$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$; $(x_\lambda)_\lambda$ is $(D\mathcal{I})$-*Cauchy* iff there exists a (D)-sequence $(a_{t,r})_{t,r}$ with the property that for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is an element $\tau \in \Lambda$ such that

$$\left\{\lambda \in \Lambda \colon\ |x_\lambda - x_\tau| \leqslant \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\right\} \in \mathcal{F}.$$

(p) A net $(x_\lambda)_\lambda$ in R $(O\mathcal{I}^*)$- $[(D\mathcal{I}^*)]$-*converges to* $x \in R$ if there exists a set $M \in \mathcal{F}(\mathcal{I})$ with $(O) \lim\limits_{\lambda\in M} x_\lambda = x$ $[(D) \lim\limits_{\lambda\in M} x_\lambda = x]$.

(q) We say that a net $(x_\lambda)_\lambda$ is *$\mathcal{I}$-divergent* iff there are an element $x \in R$ and a bounding sequence $(t_l)_l$ in R with the property that $\{\lambda \in \Lambda\colon |x_\lambda - x| \geqslant t_l\} \in \mathcal{F}$ for every $l \in \mathbb{N}$. In this case we say also that $(x_\lambda)_\lambda$ is *$\mathcal{I}$-divergent at x.*

(r) A net $(x_\lambda)_{\lambda\in\Lambda}$ is said to be *$\mathcal{I}^*$-divergent at* $x \in R$ iff there are a set $M \in \mathcal{F}$ and a bounding sequence $(t_l)_l$ in R such that for all $l \in \mathbb{N}$ there is a $\lambda \in M$ with $|x_\zeta - x| \geqslant t_l$ whenever $\zeta \in M$, $\zeta \geqslant \lambda$. We say that $(x_\lambda)_\lambda$ is *$\mathcal{I}^*$-divergent* iff it is $\mathcal{I}^*$-divergent at some $x \in R$.

Remark 2.2. It is not difficult to check that, if R is endowed with property (PR), then a net $(x_\lambda)_\lambda$ is $\mathcal{I}$-divergent if and only if there is $x \in R$ such that $\{\lambda \in \Lambda\colon |x_\lambda - x| \geqslant r\} \in \mathcal{F}$ for every $r \in R$, $r \geqslant 0$.

Indeed, let r be an arbitrary positive element of R. By property (PR) there exist an increasing sequence $(t_l)_l$ in R and an integer $\bar{l}$ such that $r \leqslant 2^l h_{2^l}$ for every $l \geqslant \bar{l}$. Setting $t_l := 2^l h_{2^l}$, $l \in \mathbb{N}$, we get that the sequence $(t_l)_l$ is bounding, and thus

$$\{\lambda \in \Lambda\colon |x_\lambda - x| \geqslant t_l\} \subset \{\lambda \in \Lambda\colon |x_\lambda - x| \geqslant r\}$$

for all $l \geqslant \bar{l}$. From this one can easily deduce the "only if" part. The "if" part is straightforward.

Analogously it is easy to see that, if R has property (PR), then $(x_\lambda)_\lambda$ is $\mathcal{I}^*$-divergent at $x \in R$ iff there is a set $M \in \mathcal{F}$ such that for every $r \in R$, $r \geqslant 0$, there is a $\lambda \in M$ with $|x_\zeta - x| \geqslant r$ whenever $\zeta \in M$, $\zeta \geqslant \lambda$.

The following result holds.

Proposition 2.3. *Let $(\Lambda, \geqslant)$ be a directed set, $\mathcal{I}$ be a (Λ)-admissible ideal on Λ and M be an element of the dual filter $\mathcal{F}$. Then $(M, \geqslant)$ is a directed set, where the order is the one induced by Λ.*

Proof. It is readily seen that $(\Lambda, \geqslant)$ induces a reflexive and transitive order on M, see $\geqslant$ again.

Let now λ_1, $\lambda_2 \in M$. There exists $\tau \in \Lambda$ such that $\tau \geqslant \lambda_1$, $\tau \geqslant \lambda_2$. Since $\mathcal{I}$ is by hypothesis (Λ)-admissible, then $M_\tau := \{\lambda \in \Lambda\colon \lambda \geqslant \tau\} \in \mathcal{F}$. Since M, $M_\tau \in \mathcal{F}$, we get $M \cap M_\tau \in \mathcal{F}$ too. Pick an element $\lambda_0 \in M$, $\lambda_0 \geqslant \tau$: we get that $\lambda_0 \geqslant \lambda_1$, $\lambda_0 \geqslant \lambda_2$. □

We now prove the following relation between order and (D)-convergence of nets in (ℓ)-groups with respect to ideals.

Proposition 2.4. *Every* $(O\mathcal{I})$*-convergent net is* $(D\mathcal{I})$*-convergent to the same limit. Moreover, if* R *is a super Dedekind complete and weakly* σ*-distributive* (ℓ)*-group, then the converse implication holds too.*

P r o o f. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net, $(O\mathcal{I})$-convergent to x, and $(\sigma_l)_l$ be an associated (O)-sequence. For every $t, r \in \mathbb{N}$ set $a_{t,r} := \sigma_{t+r}$. It is easy to check that the double sequence $(a_{t,r})_{t,r}$ is a regulator.

Pick arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$. By hypothesis, in correspondence with $l = 1 + \varphi(1)$ we get: $\{\lambda \in \Lambda\colon |x_\lambda - x| \leqslant \sigma_{1+\varphi(1)} := a_{1,\varphi(1)}\} \in \mathcal{F}$, and a fortiori

$$\Big\{\lambda \in \Lambda\colon |x_\lambda - x| \leqslant \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big\} \in \mathcal{F}.$$

This concludes the first part.

We now turn to the second part. Let $(x_\lambda)_{\lambda\in\Lambda}$ be $(O\mathcal{I})$-convergent to x, and $(a_{t,r})_{t,r}$ be a regulator, satisfying the condition of $(D\mathcal{I})$-convergence. Since R is super Dedekind complete and weakly σ-distributive, by [2, Theorem 3.1] there exists an (O)-sequence $(\sigma_l)_l$ such that for every $l \in \mathbb{N}$ there is $\varphi_l \in \mathbb{N}^{\mathbb{N}}$ with $\bigvee\limits_{t=1}^{\infty} a_{t,\varphi_l(t)} \leqslant \sigma_l$. Thus, since by hypothesis

$$\Big\{\lambda \in \Lambda\colon |x_\lambda - x| \leqslant \bigvee_{t=1}^{\infty} a_{t,\varphi_l(t)}\Big\} \in \mathcal{F}$$

for all $l \in \mathbb{N}$, then a fortiori $\{\lambda \in \Lambda\colon |x_\lambda - x| \leqslant \sigma_l\} \in \mathcal{F}$ for every $l \in \mathbb{N}$. This ends the proof. □

Analogously as in Proposition 2.4 it is possible to prove the following:

Proposition 2.5. *Every* $(O\mathcal{I})$*-Cauchy net is* $(D\mathcal{I})$*-Cauchy too. Moreover, if* R *is a super Dedekind complete and weakly* σ*-distributive* (ℓ)*-group, then the converse is true.*

Proposition 2.6. *Let* $\mathcal{I}$ *be any fixed* (Λ)*-admissible ideal of* Λ*. If* $(D)\lim\limits_{\lambda} x_\lambda = x$*, then* $(D\mathcal{I})\lim\limits_{\lambda} x_\lambda = x$*.*

Moreover, if $(x_\lambda)_\lambda$ *is an increasing net in* R *and* $x \in R$*, then* $(D\mathcal{I})\lim\limits_{\lambda} x_\lambda = x$ *if and only if* $(D)\lim\limits_{\lambda} x_\lambda = x$*.*

P r o o f. The first part is straightforward.

We now turn to the final part. It is enough to prove the "only if" implication. By hypothesis there is a (O)-sequence $(\sigma_l)_l$ such that for all $l \in \mathbb{N}$ an element $\lambda^* \in \Lambda$ can be found, with

$$0 \leqslant x - x_{\lambda^*} \leqslant \sigma_l.$$

By monotonicity we get:

$$0 \leqslant x - x_\lambda \leqslant x - x_{\lambda^*} \leqslant \sigma_l$$

whenever $\lambda \geqslant \lambda^*$. So the net $(x_\lambda)_\lambda$ (D)-converges monotonically to x, according to the usual notion of (D)-convergence. This concludes the proof. □

We now prove a Cauchy-type condition, which extends [10, Proposition 2.13]. We recall that a sequence $(x_n)_n$ is (O)-*Cauchy* iff there exists an (O)-sequence $(p_n)_n$ such that $|x_n - x_q| \leqslant p_n$ for all $n \in \mathbb{N}$ and $q \geqslant n$.

Proposition 2.7. *Let $(\Lambda, \geqslant)$ be as above, R be a Dedekind complete (ℓ)-group, $\mathcal{I}$ be any (ΛP)-ideal of Λ, $\mathcal{F}$ be its dual filter, and $(x_\lambda)_\lambda$ be a net in R. Then the following are equivalent:*

(i) $(x_\lambda)_\lambda$ *is $(O\mathcal{I})$-convergent;*
(ii) $(x_\lambda)_\lambda$ *is $(O\mathcal{I})$-Cauchy;*
(iii) *there exists an (O)-sequence $(\sigma_l)_l$ with the property that to every $l \in \mathbb{N}$ an element $D \in \mathcal{F}$ can be found, such that $|x_\lambda - x_\tau| \leqslant \sigma_l$ whenever $\lambda, \tau \in D$.*

Proof. (ii) $\Longrightarrow$ (i). Let $(x_\lambda)_\lambda$ be an $(O\mathcal{I})$-Cauchy sequence. Then, by Proposition 2.5, $(x_\lambda)_\lambda$ is $(O\mathcal{I})$-Cauchy. Let $(\varepsilon_p)_p$ be an (O)-sequence, related with the $(O\mathcal{I})$-Cauchy condition: then a sequence $(n_p)_p$ in $\mathbb{N}$ can be found, with

$$\{\lambda \in \Lambda\colon\ |x_\lambda - x_{n_p}| \leqslant \varepsilon_p\} \in \mathcal{F}(\mathcal{I}) \tag{3}$$

for all $p \in \mathbb{N}$. Let now $p, q \in \mathbb{N}$, $p \neq q$. Since $\mathcal{F}$ is a filter in Λ, we get

$$\{\lambda \in \Lambda\colon\ |x_\lambda - x_{n_p}| \leqslant \varepsilon_p\} \cap \{\lambda \in \Lambda\colon\ |x_\lambda - x_{n_q}| \leqslant \varepsilon_q\} \in \mathcal{F}.$$

Thus for every $p, q \in \mathbb{N}$ with $p \neq q$ there is $(i_{p,q}, j_{p,q}) \in \mathbb{N}^2$ with $|x_{i_{p,q}, j_{p,q}} - x_{m_p,n_p}| \leqslant \varepsilon_p$ and $|x_{i_{p,q}, j_{p,q}} - x_{m_q,n_q}| \leqslant \varepsilon_q$, and hence $|x_{m_p,n_p} - x_{m_q,n_q}| \leqslant \varepsilon_p + \varepsilon_q$. As $(\varepsilon_p)_p$ is an (O)-sequence, then $(x_{m_p,n_p})_p$ is an (O)-Cauchy sequence (in the classical sense). Since every Dedekind complete (ℓ)-group is (O)-complete (see [12]), there exists an element $x \in R$ with $x = (O)\lim\limits_p x_{m_p,n_p}$. Thanks to (3) and the main properties of filters, for every $p \in \mathbb{N}$ we get:

$$\begin{aligned}\{\lambda \in \Lambda\colon\ |x_\lambda - x| \leqslant 2\varepsilon_p\} &\supset \{\lambda \in \Lambda\colon\ |x_\lambda - x_{n_p}| + |x_{n_p} - x| \leqslant 2\varepsilon_p\} \\ &\supset \{\lambda \in \Lambda\colon\ |x_{n_p} - x| \leqslant \varepsilon_p\} \cap \{\lambda \in \Lambda\colon\ |x_\lambda - x_{n_p}| \leqslant \varepsilon_p\} \in \mathcal{F}.\end{aligned}$$

This concludes the proof.

(i) $\Longrightarrow$ (iii). Since, by hypothesis, $(x_\lambda)_\lambda$ is (O)-convergent to an element $x \in R$, there is an (O)-sequence $(\sigma_l)_l$ with the property that for every $l \in \mathbb{N}$ there is $D \in \mathcal{F}$ with $|x_\lambda - x| \leqslant \sigma_l$ whenever $\lambda \in D$. Let now λ, $\tau \in D$: we get

$$|x_\lambda - x_\tau| \leqslant |x_\lambda - x| + |x_\tau - x| \leqslant 2\sigma_l,$$

and hence the implication is proved.

(iii) $\Longrightarrow$ (ii). Let $(\sigma_l)_l$ be an (O)-sequence, satisfying (iii) by hypothesis, pick arbitrarily $l \in \mathbb{N}$ and let $D \in \mathcal{F}$ be as in (iii). Since $D \in \mathcal{F}$, there is an element $\bar{\lambda} \in D$. By (iii), for all $\lambda \in D$ we get: $|x_\lambda - x_{\bar{\lambda}}| \leqslant \sigma_l$. So (ii) follows. □

Proposition 2.8. *Let R be any Dedekind complete (ℓ)-group, and $(x_\lambda)_{\lambda\in\Lambda}$ be a net in R. Suppose that $(O\mathcal{I}^*)\lim\limits_{\lambda\in\Lambda} x_\lambda = x$. Then $(O\mathcal{I})\lim\limits_{\lambda\in\Lambda} x_\lambda = x$.*

Proof. By hypothesis, there is $M \in \mathcal{F}(\mathcal{I})$ such that $(O)\lim\limits_{\lambda\in M} x_\lambda = x$ with respect to a suitable (O)-sequence $(\sigma_l)_l$. Then for every $l \in \mathbb{N}$ there exists $\lambda \in M$ such that

$$|x_\zeta - x| \leqslant \sigma_l$$

whenever $\zeta \in M$, $\zeta \geqslant \lambda$. Therefore, we get

$$\{\zeta \in \Lambda\colon\ |x_\zeta - x| \nleqslant \sigma_l\} \subset \Lambda \setminus (M \cap M_\lambda) \in \mathcal{I}.$$

This ends the proof. □

Proposition 2.9. *Let R be a Dedekind complete (ℓ)-group, $(x_\lambda)_\lambda$ be a net in R, $(O\mathcal{I})$-convergent to $x \in R$. If $\mathcal{I}$ is a (ΛP)-ideal, then $(x_\lambda)_\lambda$ $(O\mathcal{I}^*)$-converges to x.*

Proof. Let $(\sigma_j)_j$ be an (O)-sequence related to $(O\mathcal{I})$-convergence of the net $(x_\lambda)_{\lambda\in\Lambda}$ to x. Set $O_j := [-\sigma_j, \sigma_j]$, $j \in \mathbb{N}$; $A_1 := \{\lambda \in \Lambda\colon\ x_\lambda - x \notin O_1\}$ and $A_j\colon \{\lambda \in \Lambda\colon\ x_\lambda - x \in O_{j-1} \setminus O_j\}$, $j \geqslant 2$. It is easy to check that $(A_j)_j$ is a disjoint sequence of elements of $\mathcal{I}$. Since $\mathcal{I}$ is a (ΛP)-ideal, in correspondence with $(A_j)_j$ there exist a sequence $(B_j)_j$ in $\mathcal{P}(\Lambda)$ and a sequence $(p_j)_j$ in Λ such that the symmetric difference $A_j \Delta B_j$ is contained in $\Lambda \setminus M_{p_j}$ for every $j \in \mathbb{N}$ and $B := \bigcup\limits_{j=1}^{\infty} B_j \in \mathcal{I}$. Set $M := \Lambda \setminus B$: then $M \in \mathcal{F}(\mathcal{I})$.

Fix arbitrarily $l \in \mathbb{N}$. There is $\lambda_0 \in \Lambda$ such that $\lambda_0 \geqslant p_j$ for all $j = 1, \ldots, l$. By construction we get:

$$\left(\bigcup_{j=1}^{l} B_j\right) \cap M_{\lambda_0} = \left(\bigcup_{j=1}^{l} A_j\right) \cap M_{\lambda_0}. \tag{4}$$

As $\mathcal{I}$ is (Λ)-admissible, $M_{\lambda_0} \in \mathcal{F}(\mathcal{I})$. Hence, $M \cap M_{\lambda_0} \in \mathcal{F}(\mathcal{I})$. Pick now $\lambda \in M$ with $\lambda \geqslant \lambda_0$. Choose arbitrarily $\zeta \in M$, $\zeta \geqslant \lambda$. Then $\zeta \notin B$, and in particular $\zeta \notin \bigcup\limits_{j=1}^{l} B_j$,

$\zeta \in M_{\lambda_0}$. From this and (4) it follows that $\zeta \notin \bigcup_{j=1}^{l} A_j$. Thus $x_\zeta - x \in O_l$, namely $-\sigma_l \leqslant x_\zeta - x \leqslant \sigma_l$. So we have proved the existence of an element M of $\mathcal{F}(\mathcal{I})$ and of an (O)-sequence $(\sigma_l)_l$ with the property that: to every $l \in \mathbb{N}$ there corresponds $\lambda \in M$ such that $|x_\zeta - x| \leqslant \sigma_l$ whenever $\zeta \in M$, $\zeta \geqslant \lambda$, that is $(O) \lim_{\lambda \in M} x_\lambda = x$. This completes the proof. □

We now prove the following:

Proposition 2.10. *Let R be a Dedekind complete (ℓ)-group, $(x_\lambda)_\lambda$ be a net in R, $\mathcal{I}$-divergent at $x \in R$. If $\mathcal{I}$ is a (ΛP)-ideal, then $(x_\lambda)_\lambda$ $\mathcal{I}^*$-diverges at x.*

Conversely, if $\mathcal{I}$ is any (Λ)-admissible ideal and $(x_\lambda)_\lambda$ $\mathcal{I}^*$-diverges at $x \in R$, then $(x_\lambda)_\lambda$ $\mathcal{I}$-diverges at x.

Proof. For every $j \in \mathbb{N}$, let $H_j := \{\lambda \in \Lambda\colon\ |x_\lambda - x| \ngeqslant t_j\}$; moreover, set $A_1 := H_1$; $A_j := H_j \setminus H_{j-1}$, $j \geqslant 2$. It is easy to check that $(H_j)_j$ is an increasing sequence and $(A_j)_j$ is a disjoint sequence of elements of $\mathcal{I}$. Since $\mathcal{I}$ is a (ΛP)-ideal, in correspondence with $(A_j)_j$ there exist a sequence $(B_j)_j$ in $\mathcal{P}(\Lambda)$ and a sequence $(p_j)_j$ in Λ such that $A_j \Delta B_j \subset \Lambda \setminus M_{p_j}$ for every $j \in \mathbb{N}$ and $B := \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. Set $M := \Lambda \setminus B$.

Fix arbitrarily $l \in \mathbb{N}$. There is $\lambda_0 \in \Lambda$ such that $\lambda_0 \geqslant p_j$ for all $j = 1, \ldots, l$. By construction we get:

$$\left(\bigcup_{j=1}^{l} B_j\right) \cap M_{\lambda_0} = \left(\bigcup_{j=1}^{l} A_j\right) \cap M_{\lambda_0}. \tag{5}$$

As $\mathcal{I}$ is (Λ)-admissible, $M_{\lambda_0} \in \mathcal{F}(\mathcal{I})$. Hence, $M \cap M_{\lambda_0} \in \mathcal{F}(\mathcal{I})$. Pick now $\lambda \in M$ with $\lambda \geqslant \lambda_0$. Choose arbitrarily $\zeta \in M$, $\zeta \geqslant \lambda$. Then $\zeta \notin B$, and in particular $\zeta \notin \bigcup_{j=1}^{l} B_j$, $\zeta \in M_{\lambda_0}$. From this and (5) it follows that $\zeta \notin \bigcup_{j=1}^{l} A_j$. Thus $\zeta \notin H_l$, namely $|x_\zeta - x| \geqslant t_l$. So we have proved that there exist a set $M \in \mathcal{F}$ and a bounding sequence $(t_l)_l$ in R with the property that: to every $l \in \mathbb{N}$ there corresponds $\lambda \in M$ such that $|x_\zeta - x| \geqslant t_l$ whenever $\zeta \in M$, $\zeta \geqslant \lambda$; that is $\mathcal{I}^*$-divergence of the net $(x_\lambda)_\lambda$. This ends the proof of the first part.

We now turn to the last part. By hypothesis, there are a set $M \in \mathcal{F}(\mathcal{I})$ and a bounding sequence $(t_l)_l$ in R such that to any $l \in \mathbb{N}$ there corresponds an element $\lambda \in M$ with $|x_\zeta - x| \geqslant t_l$ for all $\zeta \in M$, $\zeta \geqslant \lambda$. Thus we get:

$$\{\zeta \in \Lambda\colon\ |x_\zeta - x| \ngeqslant t_l\} \subset \Lambda \setminus (M \cap M_\lambda) \in \mathcal{I}.$$

This concludes the proof. □

Open problems: (a) Extend the definition of the Kurzweil-Henstock integral found in [12] to the ideal context.

(b) Explore similar results considering $\mathcal{I}$-convergence of double sequences introduced in [35] and further studied in [16], [18], [22], [25].

(c) Investigate analogous results using the notion of $\mathcal{I}$-convergence to a set introduced in [28].

(d) If the three conditions formulated in Proposition 2.7 are equivalent in an arbitrary (ℓ)-group R, does this imply that R should be Dedekind complete or that $\mathcal{I}$ should be a (ΛP)-ideal?

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References

[1] *E. Athanassiadou, A. Boccuto, X. Dimitriou, N. Papanastassiou*: Ascoli-type theorems and ideal (α)-convergence. Filomat *26* (2012), 397–405.

[2] *A. Boccuto*: Egorov property and weak σ-distributivity in Riesz spaces. Acta Math. (Nitra) *6* (2003), 61–66.

[3] *A. Boccuto, D. Candeloro*: Uniform s-boundedness and convergence results for measures with values in complete l-groups. J. Math. Anal. Appl. *265* (2002), 170–194.

[4] *A. Boccuto, D. Candeloro*: Uniform boundedness theorems in Riesz spaces. Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia *52* (2004), 369–382.

[5] *A. Boccuto, D. Candeloro*: Integral and ideals in Riesz spaces. Inf. Sci. *179* (2009), 2891–2902.

[6] *A. Boccuto, D. Candeloro*: Defining limits by means of integrals. Oper. Theory Adv. Appl. *201* (2010), 79–87.

[7] *A. Boccuto, P. Das, X. Dimitriou, N. Papanastassiou*: Ideal exhaustiveness, weak convergence and weak compactness in Banach spaces. Real Anal. Exch. *37* (2012), 409–430.

[8] *A. Boccuto, X. Dimitriou, N. Papanastassiou*: Unconditional convergence in lattice groups with respect to ideals. Commentat. Math. *50* (2010), 161–174.

[9] *A. Boccuto, X. Dimitriou, N. Papanastassiou*: Brooks-Jewett-type theorems for the pointwise ideal convergence of measures with values in (l)-groups. Tatra Mt. Math. Publ. *49* (2011), 17-26.

[10] *A. Boccuto, X. Dimitriou, N. Papanastassiou*: Basic matrix theorems for $\mathcal{I}$-convergence in (ℓ)-groups. Math. Slovaca *62* (2012), 1–23.

[11] *A. Boccuto, X. Dimitriou, N. Papanastassiou, W. Wilczyński*: Ideal exhaustiveness, continuity and (α)-convergence for lattice group-valued functions. Int. J. Pure Appl. Math. *70* (2011), 211–227.

[12] *A. Boccuto, B. Riečan, M. Vrábelová*: Kurzweil-Henstock Integral in Riesz Spaces. Bentham Science Publ., Singapore, 2009.

[13] *J. Činčura, T. Šalát, M. Sleziak, V. Toma*: Sets of statistical cluster points and $\mathcal{I}$-cluster points. Real Anal. Exch. *30* (2004–2005), 565–580.

[14] *P. Das, S. K. Ghosal*: On I-Cauchy nets and completeness. Topology Appl. *157* (2010), 1152–1156.

[15] *P. Das, S. K. Ghosal*: Some further results on $\mathcal{I}$-Cauchy sequences and condition (AP). Comput. Math. Appl. *59* (2010), 2597–2600.

[16] *P. Das, P. Kostyrko, W. Wilczyński, P. Malik*: I and I^*-convergence of double sequences. Math. Slovaca *58* (2008), 605–620.
[17] *P. Das, B. K. Lahiri*: $\mathcal{I}$ and $\mathcal{I}^*$-convergence of nets. Real Anal. Exch. *33* (2007–2008), 431–442.
[18] *P. Das, P. Malik*: On extremal $\mathcal{I}$-limit points of double sequences. Tatra Mt. Math. Publ. *40* (2008), 91–102.
[19] *K. Demirci*: I-limit superior and limit inferior. Math. Commun. *6* (2001), 165–172.
[20] *K. Dems*: On $\mathcal{I}$-Cauchy sequences. Real Anal. Exch. *30* (2004–2005), 123–128.
[21] *V. Gregoriades, N. Papanastassiou*: The notion of exhaustiveness and Ascoli-type theorems. Topology Appl. *155* (2008), 1111–1128.
[22] *M. Gürdal, A. Şahiner*: Extremal $\mathcal{I}$-limit points of double sequences. Appl. Math. E-Notes *8* (2008), 131–137.
[23] *P. Kostyrko, M. Mačaj, T. Šalát, M. Sleziak*: $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points. Math. Slovaca *55* (2005), 443–464.
[24] *P. Kostyrko, T. Šalát, W. Wilczyński*: I-convergence. Real Anal. Exch. *26* (2000–2001), 669–685.
[25] *V. Kumar*: On I and I^*-convergence of double sequences. Math. Commun. *12* (2007), 171–181.
[26] *B. K. Lahiri, P. Das*: Further results on $\mathcal{I}$-limit superior and limit inferior. Math. Commun. *8* (2003), 151–156.
[27] *B. K. Lahiri, P. Das*: I and I^*-convergence in topological spaces. Math. Bohem. *130* (2005), 153–160.
[28] *P. Letavaj*: $\mathcal{I}$-convergence to a set. Acta Math. Univ. Comen., New Ser. *80* (2011), 103–106.
[29] *A. Nabiev, S. Pehlivan, M. Gürdal*: On $\mathcal{I}$-Cauchy sequences. Taiwanese J. Math. *11* (2007), 569–576.
[30] *F. Nuray, W. H. Ruckle*: Generalized statistical convergence and convergence free spaces. J. Math. Anal. Appl. *245* (2000), 513–527.
[31] *S. Pehlivan, C. Şençimen, Z. H. Yaman*: On weak ideal convergence in normed spaces. J. Interdiscip. Math. *13* (2010), 153–162.
[32] *T. Šalát, B. C. Tripathy, M. Ziman*: On some properties of $\mathcal{I}$-convergence. Tatra Mt. Math. Publ. *28* (2004), 274–286.
[33] *T. Šalát, B. C. Tripathy, M. Ziman*: On $\mathcal{I}$-convergence field. Ital. J. Pure Appl. Math. *17* (2005), 45–54.
[34] *C. Swartz*: The Nikodým boundedness theorem for lattice-valued measures. Arch. Math. *53* (1989), 390–393.
[35] *B. Tripathy, B. C. Tripathy*: On I-convergent double sequences. Soochow J. Math. *31* (2005), 549–560.

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