IDEAL LATTICES AND THE STRUCTURE OF RINGS(1)

BY

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It is well known that the set of all ideals(2) of a ring forms a complete modular lattice with respect to set inclusion. The same is true of the set of all right ideals. Our purpose in this paper is to consider the consequences of imposing certain additional restrictions on these ideal lattices. In particular, we discuss the case in which one or both of these lattices is complemented, and the case in which one or both is distributive. In §1 two strictly lattice-theoretic results are noted for the sake of their application to the complemented case. In §2 rings which have a complemented ideal lattice are considered. Such rings are characterized as discrete direct sums of simple rings. The structure space of primitive ideals of such rings is also discussed. In §3 corresponding results are obtained for rings whose lattice of right ideals is complemented. In particular, it is shown that a ring has a complemented right ideal lattice if and only if it is isomorphic with a discrete direct sum of quasi-simple rings. The socle [7](3) and the maximal regular ideal [5] are discussed in connection with such rings. The effect of an identity element is considered in §4. In §5 rings with distributive ideal lattices are considered and still another variant of regularity [20] is introduced. It is shown that a semi-simple ring with a distributive right ideal lattice is isomorphic with a subdirect sum of division rings. In the concluding section a type of ideal, introduced by L. Fuchs [9] in connection with commutative rings with distributive ideal lattice, and which we call *strongly irreducible*, is considered. Some properties of these ideals, analogous to corresponding ones for prime ideals [19], are developed. Finally, it is observed that a topology may be introduced in the set of all proper strongly irreducible ideals in such a way that the resulting space contains the spaces of prime [19] and primitive [13] ideals as subspaces.

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1. Some lattice-theoretic preliminaries. In this section we state two results of a strictly lattice-theoretic nature with a view toward subsequent applications to rings. Our notation and terminology is that of Birkhoff [3]. In

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(2) Unless otherwise stated, the term "ideal" will mean "two-sided ideal."

(3) Numbers in brackets refer to the references at the end of the paper.

particular, if $\{y_\alpha\}$ is an ascending chain of elements of a lattice L, where α ranges over a set of ordinals, we write $y_\alpha \uparrow$. Furthermore, if $\{y_\alpha\}$ has a least upper bound y in L, we write $y_\alpha \uparrow y$. The following result is well known (see [3, p. 129]):

LEMMA 1. *Let L be a complete modular lattice in which $y_\alpha \uparrow y$ implies $x \cap y_\alpha \uparrow x \cap y$. If the unit I of L is a join of points, then each element of L is a join of points and L is complemented.*

We remark in passing that the ideal lattices of a ring satisfy the conditions of the first statement of Lemma 1. We seek next a class of complete modular lattices in which complementation implies that I is a join of points(4).

LEMMA 2. *If L is a complete complemented modular lattice with at least two elements, and if each element of L is a meet of meet-irreducible elements, then I is a join of points.*

Proof. We show first that L contains at least one point. Since 0 is a meet of meet-irreducible elements, there is a meet-irreducible element a in L such that $a < I$. If $a < b \leqq I$, then $b \cup c = I$ and $b \cap c = a$ for some element c in L since L is relatively complemented. Then $c = a$ so that $b = I$, and we see that I covers a. It is then easy to show that any complement of a is a point. Thus the join s of all points of L is not 0. If $s \neq I$, let s' be a complement of s. Since the interval sublattice $[0, s']$ of L is isomorphic with $[s, I]$, it is clear that $[0, s']$ satisfies the hypotheses of the lemma, and therefore, as in the first part of this proof, contains a point p. Then p is a point of L so that $p \leqq s \cap s' = 0$, contrary to $p \neq 0$. We conclude that $s = I$, and the proof is complete.

The preceding observations are applicable in settings more general than that of ring theory. For example, the lattice of all normal subloops of a loop satisfies the conditions of the first statement of Lemma 1. Moreover, each element of this lattice is a meet of meet-irreducible elements(5). Thus the lattice of all normal subloops of a loop G (consisting of more than one element) is complemented if and only if G is the sum of its minimal normal subloops. However, since our primary interest here is in rings, we shall not pursue such questions further.

2. **Rings with complemented ideal lattice.** In this section we consider the

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(4) In this connection we point out that there exist complemented lattices which satisfy the requirements of the first statement of Lemma 1, but in which I is not a join of points. The continuous geometries of J. von Neumann are examples of this type.

(5) See for example [16] or [9]. Since the proof is short, we reproduce that of [9] here. An ideal B of a ring A is surely contained in the intersection X of all meet-irreducible ideals which contain B. On the other hand, if $a \notin B$, then by Zorn's lemma there is an ideal M which is maximal in the family of ideals containing B but not a. Then M is meet-irreducible so that $a \notin X$. The same type of proof is clearly applicable to the lattice of right ideals of a ring, the lattice of normal subloops of a loop, etc.

role which complementation in the lattice of ideals of a ring plays in the structure of that ring. For brevity, we say that a ring A satisfies *condition* C in case the lattice of ideals of A is complemented. We call an ideal of A *meet-irreducible* in case it is meet-irreducible in the ideal lattice of A. Using Zorn's lemma it is possible to show that each ideal of a ring is the intersection of all those meet-irreducible ideals which contain it(5). Thus as an immediate consequence of Lemma 2 we see that a ring which satisfies condition C is the sum of its minimal two-sided ideals, and this sum may be refined to a direct sum in the usual way(6). Conversely, Lemma 1 shows that a ring which is a direct sum of minimal two-sided ideals must satisfy condition C. We therefore obtain the following result.

THEOREM 1. *A ring A satisfies condition* C *if and only if A is a direct sum of minimal two-sided ideals.*

We consider next a connection between condition C and the notion of the *discrete direct sum* [18] of rings. We prove first the following lemma.

LEMMA 3. *If an ideal B is a direct summand of a ring A, then every* (*right*) *ideal of B is a* (*right*) *ideal of A.*

Proof. If I is a right ideal of B, let $x \in I$ and $a \in A$. Since $B \dot{+} B' = A$ for some ideal B' of A, we have $a = b + b'$ with $b \in B$ and $b' \in B'$ so that $xa = xb + xb'$. Then $xa \in I$ since xb' is in $B \cap B' = 0$. Similarly, if I is a left ideal of B, then $ax \in I$.

Incidentally, Lemma 3 shows that condition C is preserved under homomorphism. For suppose that A satisfies condition C and that B' is an ideal of A. Then $B \dot{+} B' = A$ for some ideal B of A, and $A - B' \cong B$. Any ideal I of B is an ideal of A and hence, since the ideal lattice of A is relatively complemented, $I \dot{+} I' = B$ for some ideal I' of A. It follows that B, and hence $A - B'$, satisfies condition C.

It is easily shown that a ring A is isomorphic with the discrete direct sum of rings S_α if and only if A is the direct sum of two-sided ideals A_α such that, for each α, $A_\alpha \cong S_\alpha$. If each S_α is a simple ring, then clearly each A_α is a minimal two-sided ideal of A so that A satisfies condition C by Theorem 1. On the other hand, if A satisfies condition C, then A is the direct sum of minimal two-sided ideals A_α, and by Lemma 3 each A_α is a simple ring. We may therefore reformulate Theorem 1 as follows:

THEOREM 2. *A ring A satisfies condition* C *if and only if A is isomorphic with a discrete direct sum of simple rings.*

(6) A minimal two-sided ideal of a ring with more than one element means a minimal *nonzero* two-sided ideal. However, we adopt the convention that a one-element ring is a minimal two-sided ideal of itself. A similar convention is adopted for minimal right ideals. A ring A is the *direct sum* of (right) ideals A_α of A in case $A = \sum_\alpha A_\alpha$ and $A_\beta \cap \sum_{\alpha \neq \beta} A_\alpha = 0$. We also remark that by a *simple* ring we mean a ring whose only ideals are itself and the zero ideal. Thus in our terminology a simple ring may be a radical ring.

We now make some remarks concerning a semi-simple ring A which satisfies condition C. Since an ideal of a semi-simple ring is itself semi-simple, a minimal two-sided ideal of A is semi-simple as well as simple; its complement is therefore a primitive ideal [13]. Conversely, a primitive ideal is meet-irreducible by [12, Lemma 4] and hence maximal, as in the proof of Lemma 2; its complement is then a minimal two-sided ideal. Thus the minimal two-sided ideals of a ring of this type are the complements of primitive ideals. It is clear also that every difference ring of A is semi-simple, and this implies that a proper ideal of A must be contained in some primitive ideal of A.

In [13] Jacobson has shown that a Stone topology may be introduced in the set $\mathfrak{S}$ of all primitive ideals of a ring A. In fact, the closure of a subset $\mathfrak{S}_1$ of $\mathfrak{S}$ is defined to be the set of all primitive ideals of A which contain the intersection $\bigcap \mathfrak{S}_1$ of all primitive ideals in $\mathfrak{S}_1$. The resulting space is the *structure space* of A. In the case of a semi-simple ring A which satisfies condition C, the lattice of all open sets of $\mathfrak{S}$ bears a close relationship to the ideal lattice of A, as we shall now show. If A is such a ring, consider the mapping $\mathfrak{J} \rightarrow \bigcap \mathfrak{J}$ of the lattice of closed sets of $\mathfrak{S}$ into the ideal lattice of A. If I is an ideal of A, let $\mathfrak{J}$ be the set of all primitive ideals which contain I. Then $I = \bigcap \mathfrak{J}$ since $A - I$ is semi-simple, and therefore, since $\mathfrak{J}$ is clearly a closed subset of $\mathfrak{S}$, the mapping is exhaustive.

Now let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be closed subsets of $\mathfrak{S}$. If $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$, then obviously $\bigcap \mathfrak{J}_2 \subseteq \bigcap \mathfrak{J}_1$. Conversely, if $\bigcap \mathfrak{J}_2 \subseteq \bigcap \mathfrak{J}_1$ and if $B \in \mathfrak{J}_1$, then $B \supseteq \bigcap \mathfrak{J}_1 \supseteq \bigcap \mathfrak{J}_2$ so that B is in the closure of $\mathfrak{J}_2$; hence $B \in \mathfrak{J}_2$. It follows that the mapping is a lattice anti-isomorphism between the lattice of closed sets of $\mathfrak{S}$ and the ideal lattice of A. The following theorem expresses this result in terms of the lattice of open sets of $\mathfrak{S}$.

THEOREM 3. *If A is a semi-simple ring which satisfies condition* C, *then the ideal lattice of A is isomorphic with the lattice of open sets of the structure space of A.*

Incidentally, since the lattice of open sets of any topological space is distributive, Theorem 3 shows that the ideal lattice of a semi-simple ring which satisfies condition C is distributive, and hence a Boolean algebra. In particular, complementation is unique in the ideal lattice of such a ring. A somewhat sharper result of the same type will be given in Theorem 9 of the next section. We conclude this section with some additional results concerning the structure space of a ring which satisfies condition C.

THEOREM 4. *If A is a ring which satisfies condition* C, *then the structure space of A is discrete. Moreover, the structure space of A is compact if and only if there is an element in A which is in no primitive ideal of A.*

Proof. If J is the Jacobson radical of A, we first observe that an element x of A is in no primitive ideal of A if and only if $x+J$ is in no primitive ideal of the semi-simple ring $A-J$. Thus, since the structure space of A is homeomorphic with the structure space of $A-J$ [13], we may assume that A is semi-simple, and the remarks following Theorem 2 are therefore applicable. Then every primitive ideal of A is maximal so that the structure space $\mathfrak{S}$ of A is T_1. Furthermore, Theorem 3 shows that the lattice of open sets of $\mathfrak{S}$ is complemented; it follows that $\mathfrak{S}$ is discrete.

Now let x be an element of A which is in no primitive ideal of A, and let $\mathfrak{F}$ be a family of closed sets of $\mathfrak{S}$ whose intersection is empty. Then, by [13, Lemma 1], the sum $\sum_{\mathcal{J}\in\mathfrak{F}}\bigcap\mathcal{J}$ is contained in no primitive ideal of A, and hence is equal to A. Then x is in $\sum_{\mathcal{J}\in\mathfrak{F}_1}\bigcap\mathcal{J}$ for some finite subset $\mathfrak{F}_1$ of $\mathfrak{F}$, and therefore $\sum_{\mathcal{J}\in\mathfrak{F}_1}\bigcap\mathcal{J}$ is contained in no primitive ideal. Again using [13, Lemma 1], the intersection of the sets in $\mathfrak{F}_1$ is empty, and we conclude that $\mathfrak{S}$ is compact. To prove the remaining part of the second statement of the theorem, we first observe that a compact discrete space is finite. Thus let B_i $(i=1, \cdots, n)$ denote the primitive ideals of A, and let B_i' be the (unique) complement of B_i. Let x_i be a nonzero element of B_i'. Since $\sum_{i\neq j} B_i' = B_j$, it follows readily that $x=x_1+\cdots+x_n$ is in no primitive ideal of A, and the proof is complete.

Since the minimal two-sided ideals of a semi-simple ring which satisfies condition C are the unique complements of primitive ideals, and since a discrete space is compact if and only if it is finite, Theorem 4, together with Theorem 2, yields the following corollary.

COROLLARY. *A semi-simple ring A satisfies condition* C *and contains an element which is in no primitive ideal of A if and only if A is isomorphic with the direct sum of a finite number of simple rings.*

The following result is a partial converse of Theorem 4.

THEOREM 5. *If a semi-simple ring A has a discrete structure space, and if each proper ideal of A is contained in a primitive ideal, then A satisfies condition* C.

Proof. Let I be an ideal of A and $\mathcal{J}$ the (closed) set of all primitive ideals of A which contain I. Since the structure space of A is discrete, the set $\mathcal{J}'$ of all primitive ideals which do not contain I is also closed. Then $I+\bigcap\mathcal{J}'=A$ since otherwise $I+\bigcap\mathcal{J}'$ is contained in a primitive ideal, and this would imply that $\mathcal{J}\cap\mathcal{J}'$ is not empty. Moreover, the intersection of I with $\bigcap\mathcal{J}'$ is zero since A is semi-simple. Thus $I\dotplus\bigcap\mathcal{J}'=A$, and A satisfies condition C.

The requirement that each proper ideal be contained in a primitive ideal is satisfied, for example, by a ring with an identity element [13]. We remark also that the preceding theorem is not true when the requirement of semi-simplicity is omitted. For example, the ring of integers modulo 8 has a discrete structure space, but does not satisfy condition C.

3. Rings with complemented right ideal lattice. We consider next those rings A which are such that the lattice of right ideals of A is complemented. We shall say that a ring of this type satisfies *condition* C_r. This condition is somewhat stronger than condition C discussed in the preceding section; in fact, we shall show that in the semi-simple case condition C_r implies condition C. As with two-sided ideals, a right ideal of a ring A is *meet-irreducible* in case it is meet-irreducible in the lattice of right ideals of A, and, as before, any right ideal of A is the intersection of all those meet-irreducible right ideals which contain it. Thus as a consequence of Lemmas 1 and 2 we have the following result.

THEOREM 6. *A ring A satisfies condition C_r if and only if A is a direct sum of minimal right ideals.*

The next lemma implies that condition C_r is preserved under homomorphism; its proof is similar to that of Lemma 3.

LEMMA 4. *If B is a two-sided ideal of a ring A such that $B\dotplus B'=A$ for some right ideal B' of A, then every right ideal of B' is a right ideal of A.*

Following Dieudonné [7], we call the sum S of all minimal right ideals of a ring A the (right) *socle* of A. For convenience we list a few of the results of [7]:

(a) If I is a minimal right ideal of A, then the sum of all minimal right ideals of A which are A-isomorphic with I (i.e., isomorphic as A-modules) is a two-sided ideal of A, and is called a *foot* of S. The socle is itself a two-sided ideal of A and is the direct sum of its feet.

(b) If F is a foot of the socle of A, then $F=B\dotplus C$, where B is a two-sided ideal of A (the sum of all nilpotent minimal right ideals contained in F), and C is a right ideal of A (the sum of all idempotent minimal right ideals contained in F). The ring C is simple and $B^2=0$.

We shall call a ring A *quasi-simple* in case it is the sum of A-isomorphic minimal right ideals(7). In this connection we mention also that Jacobson [12] has called a ring *atomic* in case it is the sum of its minimal right ideals.

THEOREM 7. *The following conditions are equivalent for a ring A:*

(i) *A satisfies condition C_r.*

(ii) *A is atomic.*

(iii) *A is isomorphic with a discrete direct sum of quasi-simple rings.*

Proof. It is clear from Theorem 6 that (i) and (ii) are equivalent. If A satisfies (ii), then A coincides with its socle and is therefore the direct sum of its feet. Lemma 4 then shows that each foot is a quasi-simple ring, and hence A satisfies (iii). If on the other hand A is isomorphic with the discrete direct sum of quasi-simple rings S_α, then A is the direct sum of two-sided

(7) This definition is that given by Dieudonné [7] except that we do not require the presence of an idempotent minimal right ideal.

ideals A_α which are such that $A_\alpha \cong S_\alpha$ for each α. By Lemma 4 each minimal right ideal of A_α is a minimal right ideal of A, and hence A satisfies (ii).

Using the first conclusion of Lemma 1, one may verify that the Jacobson radical J of a ring A which satisfies condition C_r is just the sum of the nilpotent minimal right ideals of A. For later use we note also that from (a) and (b) and the preceding remark it follows that J is a direct sum of nilpotent ideals of index 2; hence $J^2=0$. Now if A is semi-simple and satisfies condition C_r, each foot of A is a simple ring which contains a minimal right ideal, and A is the direct sum of its feet. On the other hand, a simple ring which contains a minimal right ideal is surely quasi-simple, and therefore a discrete direct sum of such rings must satisfy condition C_r. These statements are summarized in the next theorem.

THEOREM 8. *A semi-simple ring A satisfies condition C_r if and only if A is isomorphic with a discrete direct sum of simple rings, each of which contains a minimal right ideal.*

In view of Theorems 2 and 8 we see that a semi-simple ring which satisfies condition C_r also satisfies condition C. (The converse of this statement is not true since there exist simple rings without minimal right ideals [11].) We shall now examine this situation in somewhat greater detail. An element x of a ring A is a *left annihilator* of A in case $xA=0$. If B is an ideal of A, then the *annihilator* of B (in A) is the two-sided ideal of A consisting of all $x\in A$ such that $xB=Bx=0$. We denote the annihilator of B by B^*. We prove first the following lemma(8).

LEMMA 5. *Let A_1 be an ideal of a ring A, and assume that the ring A_1 contains no nonzero left annihilator. If B_1 and B_2 are right ideals of A such that $A_1\cap B_1=0$ and $A=A_1\dotplus B_2$, then $B_1\subseteq B_2$.*

Proof. Let $b\in B_1$ and write $b=x+y$ with $x\in A_1$ and $y\in B_2$. Then $xA_1=0$ since $A_1\cap B_1=0$ and $A_1\cap B_2=0$. It follows that $x=0$, and hence $b\in B_2$.

Under the hypotheses of the preceding lemma, it follows immediately that if $A=A_1\dotplus B_1=A_1\dotplus B_2$, then $B_1=B_2$.

LEMMA 6. *Let B be an ideal of a ring A, and assume that the ring B contains no nonzero left annihilator. If $A=B\dotplus B'$ for some right ideal B' of A, then $B'=B^*$.*

Proof. The set C of all $x\in A$ such that $xB=0$ is clearly an ideal of A, and, since B contains no nonzero left annihilator, $B\cap C=0$. Then $BC=0$ as well as $CB=0$ so that $C\subseteq B^*$, and hence $C=B^*$. Moreover, $B'B=0$ because $B'\cap B=0$, and this implies that $B'\subseteq C$. Then $A=B\dotplus B^*$ and, by the remark following Lemma 5, $B'=B^*$.

If A is a semi-simple ring, then any ideal of A is semi-simple and therefore

(8) See also Lemma 1 of [18].

contains no nonzero left annihilator. Thus Lemma 6 shows directly that in the semi-simple case condition C_r implies condition C. In fact, in this last statement condition C_r may be replaced by the following condition: *every ideal of A has a right ideal complement.* We have also the following related result.

THEOREM 9. *Let A be a ring which contains no nonzero left annihilator. If A satisfies condition* C, *then the ideal lattice of A is a Boolean algebra.*

Proof. Since A satisfies condition C, a left annihilator of an ideal of A is also a left annihilator of A. Lemma 6 then shows that complementation is unique in the ideal lattice of A. The result then follows from the fact that a complete atomic lattice with unique complements is a Boolean algebra [3, p. 170].

A simple example considered by McCoy [18, p. 872] in another connection shows that the preceding theorem is not true when A contains a nonzero left annihilator. The ring considered in this example is a zero ring which satisfies condition C, but complementation in its ideal lattice is not unique; it is well known, however, that complementation is unique in a Boolean algebra.

The preceding remarks enable us to describe the structure space of a ring which satisfies condition C_r:

THEOREM 10. *If A is a ring which satisfies condition* C_r, *then the structure space of A is discrete. Moreover, the structure space of A is compact if and only if there is an element in A which is in no primitive ideal of A.*

Proof. If J is the Jacobson radical of A, then the semi-simple ring $A-J$ satisfies condition C_r, and hence condition C. We may therefore assume that A satisfies condition C, and the result then follows from Theorem 4.

An element a of a ring A is said to be *regular* [20] in case $a=axa$ for some element $x\in A$, and an ideal of A is regular in case each of its elements is regular. We consider next some connections between condition C_r and the *maximal regular ideal* [5] of A. We use the following lemma.

LEMMA 7. *If A is a dense ring of finite-valued linear transformations of a vector space R over a division ring D, then A is a regular ring.*

Proof. Let a be an element of A, and let $x_1, \cdots, x_n$ be a basis for Ra. For each x_i, let $y_i\in R$ such that $y_ia=x_i$. Since A is dense, there is an element $b\in A$ such that $x_ib=y_i$ $(i=1, \cdots, n)$. If $z\in R$, let $\alpha_i\in D$ such that $za=\sum x_i\alpha_i$; then $z(aba)=\sum(x_iba)\alpha_i=\sum x_i\alpha_i$. Hence $a=aba$ and A is regular.

THEOREM 11. *Let J be the Jacobson radical of a ring A, and let S be the socle of A. Then S is contained in the maximal regular ideal of A if and only if* $J\cap S=0$.

Proof. Let M be the maximal regular ideal of A. Since $J\cap M=0$ [5],

$S\subseteq M$ implies that $J\cap S=0$. Conversely, if $J\cap S=0$, then S contains no nilpotent minimal right ideal. It follows then from (a) and (b) that S is the direct sum of two-sided ideals C_α of A, and each C_α is a simple ring. Moreover, by [7, Proposition 4], every right ideal of C_α is a right ideal of A so that each ring C_α contains a minimal right ideal. Now a simple ring, not a zero ring, which contains a minimal right ideal is isomorphic with a dense ring of finite-valued linear transformations of a vector space over a division ring [11], and we conclude from Lemma 7 that each C_α, and hence S, is a regular ring. Since S is an ideal of A, it follows that $S\subseteq M$.

COROLLARY. *If A is a semi-simple ring which satisfies condition C_r, then A is a regular ring.*

We formulate our next result in terms of a variant of regularity which is due to Brown and McCoy [6]. An element a of a ring A is *weakly regular* in case $a\in a(a)$, where (a) is the ideal generated by a. An ideal of A is weakly regular in case each of its elements is weakly regular. If W is a weakly regular ideal of A, and if J is the Jacobson radical of A, then one may verify that $W\cap J=0$ so that $WJ=JW=0$. The proof of the following theorem is analogous to that of Theorem 4.3 of [2].

THEOREM 12. *Let W be a weakly regular ideal of a ring A, and let S be the socle of A. Then $W\subseteq S$ if and only if $\overline{W}=(W+J)-J$ is contained in the socle of $A-J$.*

Proof. Assume that $W\subseteq S$, and let $\bar{w}\in\overline{W}$. Then $\bar{w}\in\sum\overline{I}_i$, where each I_i is a minimal right ideal of A. Regarding I_i and $\overline{I}_i$ as A-modules, $\overline{I}_i$ is an A-homomorphic image of I_i and is therefore either zero or a minimal right ideal of $A-J$. Hence $\bar{w}$ is in the socle of $A-J$. (In this part of the proof no use has been made of the special properties of the ideals W and J.) Now assume that $\overline{W}$ is contained in the socle of $A-J$, and let $a\in W$ so that $a=ay$ with $y\in(a)\subseteq W$. Then $\bar{y}\in\sum\overline{I}_i$ with each $\overline{I}_i$ a minimal right ideal of $A-J$. If I_i is the inverse image of $\overline{I}_i$, then $z+J\rightarrow az$ is an A-homomorphism of $\overline{I}_i$ onto aI_i since $aJ=0$. Then $aI_i=0$ or aI_i is a minimal right ideal of A. Furthermore, $aJ=0$ implies that $ay\in\sum aI_i$. Hence $a\in S$ so that $W\subseteq S$, and the proof is complete.

COROLLARY. *If J is the Jacobson radical of a ring A, and if $A-J$ satisfies condition C_r, then the maximal weakly regular ideal [6] of A is contained in the socle of A.*

Now let J be the Jacobson radical of a ring A which satisfies condition C_r. Then $J\dotplus J'=A$ for some right ideal J' of A. If M is the maximal regular ideal of A, and if $a\in M$, then $a=axa$ for some $x\in A$, and we have $a=j+j'$ with $j\in J$ and $j'\in J'$. Then $a=(j+j')xa=j'xa$ since $J\cap M=0$, and hence $a\in J'$ and $M\subseteq J'$. Since the lattice of right ideals of A is relatively complemented, we then have $M\dot{+}N=J'$ for some right ideal N of A, and then $A=J\dot{+}M\dot{+}N$. The ring $M+N\cong A-J$ is semi-simple and satisfies condition C_r. The corollary to Theorem 11 then shows that $M+N$ is a regular ring, from which we conclude that M is the largest two-sided ideal of A which is contained in J'. By Lemma 6, N is a two-sided ideal of $M+N$; in fact, N is the annihilator of M in $M+N$. Since N is a two-sided ideal of a regular ring, N is itself a regular ring.

We now consider the decomposition $A=J\dot{+}M\dot{+}N$ from a slightly different point of view. By Lemma 6, $J+N=M^*$ and therefore $A=M\dot{+}M^*$. Since N is a regular ring, so is $M^*-J\cong N$. Furthermore, the maximal regular ideal of $M^*\cong A-M$ is zero [5]. It follows then from [5, Theorem 6] that the ring M^* is *bound to its radical* in the sense of M. Hall [10]. We therefore have the following result(9).

THEOREM 13. *If A is a ring which satisfies condition C_r, and if M is the maximal regular ideal of A, then $A=M\dot{+}M^*$. The ring M is semi-simple and satisfies condition C_r; the ring M^* is bound to its radical and satisfies condition C_r.*

REMARK 1. Using the same notation as before, we note that $NJ\subseteq N\cap J=0$. Thus if also $JN=0$, then $N\subseteq J^*$. Since M^* is bound to its radical this implies that $N\subseteq J$, and then $J=M^*$. Conversely, $J=M^*$ implies that $JN=0$. Thus $J=M^*$ if and only if $JN=0$.

REMARK 2. In the proof of [5, Theorem 6] Brown and McCoy show that if $A-J$ is regular, then the ideal J^{*2} is regular, and this implies that $J^{*2}=M$. Thus if $A-J$ satisfies condition C_r, then $J^{*2}=M$.

4. **Rings with identity element.** We now consider some consequences of assuming an identity element in rings which satisfy condition C or C_r. We call a right ideal I of a ring A *right quasi-regular* in case each of its elements is right quasi-regular [12]. A right ideal I of A is *modular*(10) in case there is an element $e\in A$ such that $ex-x\in I$ for every $x\in A$. The following lemma generalizes Theorem 7 of [12]:

LEMMA 8. *If I is a right quasi-regular right ideal of a ring A, and if $I\dot{+}I'=A$ for some modular right ideal I' of A, then $I=0$.*

Proof. Since I' is modular, there is an element $e\in A$ such that $ex-x\in I'$ for every $x\in A$, and we may write $e=u+v$ with $u\in I$ and $v\in I'$. Since I is right quasi-regular, we have $u+w-uw=0$ for some $w\in A$. Then $vw+u=(e-u)w+u=ew-w$ so that u is in $I\cap I'=0$. Then $e\in I'$ which implies that $I'=A$, and then $I=0$.

Since the Jacobson radical of a ring is a right quasi-regular ideal, we ob-

(9) Theorem 13 is closely related to Theorem 7 of [5].

(10) The notion of a modular right ideal is due to I. Segal; the term "modular" is due to Jacobson.

tain the following result as an immediate consequence of Lemma 8.

THEOREM 14. *If a ring A contains a (left) identity element and satisfies condition* C *or* C_r, *then A is semi-simple.*

Since a left identity element of a ring A cannot be contained in a primitive ideal of A, Theorem 14, together with the corollary to Theorem 4, yields the following result: *A ring A satisfies condition* C *and contains a left identity element if and only if A is isomorphic with the direct sum of a finite number of simple rings, each of which contains a left identity element.* As a second application of Theorem 14, let A be a ring with a (two-sided) identity element, and let A satisfy condition C_r. Then A is semi-simple and therefore, by Theorem 8, is isomorphic with a discrete direct sum of simple rings S_i, each of which contains a minimal right ideal. The presence of the identity element insures that there are but a finite number of components in this direct sum, and that each S_i contains an identity element. Each S_i is then isomorphic with a dense ring of finite-valued linear transformations of a vector space R_i over a division ring, and each R_i is clearly finite-dimensional. It follows that each S_i satisfies the descending chain condition for right ideals [11]. On the other hand, the direct sum of a finite number of simple rings of this type surely contains an identity element and satisfies condition C_r.

THEOREM 15. *A ring A contains an identity element and satisfies condition* C_r *if and only if A is isomorphic with the direct sum of a finite number of simple rings, each of which satisfies the descending chain condition for right ideals.*

A fundamental Wedderburn-Artin structure theorem states that a ring A is semi-simple and satisfies the descending chain condition for right ideals if and only if A is isomorphic with the direct sum of a finite number of simple rings, each of which satisfies this chain condition. Thus Theorem 15 shows that condition C_r, together with an identity element, is equivalent to semi-simplicity together with the descending chain condition for right ideals.

5. **Rings with distributive ideal lattices.** We turn now to a consideration of distributivity in the ideal lattices of a ring. We say that a ring A satisfies *condition* D (D_r) in case the lattice of ideals (right ideals) of A is distributive. As an example of a (commutative) ring which satisfies condition D we mention the ring of integers; in fact, the ring of algebraic integers of any extension of the rational field of finite degree satisfies condition D [3, p. 135].

We observe first that condition D is preserved under homomorphism. To see this, let B be an ideal of a ring A which satisfies condition D. The sublattice of the ideal lattice of A which consists of all ideals between B and A is then distributive, and this sublattice is isomorphic with the ideal lattice of $A-B$. Similar observations show that condition D_r is also preserved under homomorphism.

We point out next a rather obvious sufficient condition for condition D.

In fact, if $BC = B \cap C$ for all ideals B and C of a ring A, then A surely satisfies condition D. In this connection we introduce another variant of the notion of regularity: an element a of A will be called *f-regular* in case $a \in (a)^2$, where (a) is the principal ideal generated by a. An ideal of A is f-regular in case each of its elements is f-regular, and A is *f-primitive* in case there is a nonzero element $e \in A$ which is contained in every nonzero ideal of A and which is such that $(e)^2 = 0$.

Now let G be the additive group of A and let Ω be the set of all left multiplications and all right multiplications, together with the identity automorphism, of A. For each $a \in A$, set $F(a) = (a)^2$. It is then easily verified that G is an (F, Ω, Ω)*-group* in the sense of Brown and McCoy [6], and the results of [6] are therefore applicable. In particular, the set N of all $a \in A$ such that (a) is f-regular is the *maximal f-regular ideal* of A. Furthermore, N is the intersection of ideals M_α, and each $A - M_\alpha$ is f-primitive. We obtain then the following result:

LEMMA 9. *If a ring A satisfies any of the following equivalent conditions, then A satisfies condition* D:

(i) *$BC = B \cap C$ for all ideals B and C of A.*

(ii) *For each ideal B of A, $A - B$ contains no nonzero nilpotent ideal.*

(iii) *A is f-regular*(11).

Proof. As remarked above, (i) implies that A satisfies condition D. Thus we need only show that (i)–(iii) are equivalent. The verification that (i) implies (ii) is straightforward. If A is not f-regular, then, by the remarks above, A contains an ideal M such that $A - M$ is f-primitive and therefore (ii) does not hold; hence (ii) implies (iii). Finally, (iii) implies that $(a) = (a)^2$ for every principal ideal (a) of A, from which (i) readily follows.

It is clear that any weakly regular ring is f-regular. Furthermore, any regular, *strongly regular* [1], or *biregular* [1] ring is weakly regular. Thus we have the corollary:

COROLLARY. *If A is a weakly regular (regular, strongly regular, biregular) ring, then A satisfies condition* D.

We point out that none of the above mentioned conditions is a necessary condition for condition D. For example, the ring of integers satisfies condition D but is not f-regular.

LEMMA 10. *If A is a dense ring of linear transformations of a vector space R over a division ring* D, *then A is a division ring or A contains three distinct*

(11) Levitzki [15] has observed that (ii) holds if and only if each ideal of A is the intersection of all prime ideals [19] which contain it. Thus Lemma 9 yields the following result: *A ring A is f-regular if and only if each ideal of A is the intersection of all prime ideals which contain it.* Since f-regularity reduces to regularity in a commutative ring with identity element, this is a direct generalization of Theorem 9 of [16].

maximal right ideals I_1, I_2, *and* I_3 *such that* $I_1 \cap I_2 \subseteq I_3$.

Proof. We introduce the following notation. If $z \in R$, then $I[z]$ denotes the right ideal of A consisting of all $a \in A$ such that $za=0$. If $z \neq 0$, then $I[z]$ is a maximal right ideal of A [11, p. 231]. Now if R is one-dimensional over D, then A is a division ring. Otherwise, R contains two D-linearly independent elements x and y. Then $x+y \neq 0$ so that $I[x]$, $I[y]$, and $I[x+y]$ are maximal right ideals of A, and clearly $I[x] \cap I[y] \subseteq I[x+y]$. Since A is a dense ring, we have $xa=0$ and $ya \neq 0$ for some $a \in A$; hence $I[x] \neq I[y]$. Again, $xb=x$ and $yb=-x$ for some $b \in A$. Then $b \in I[x+y]$ while $b \notin I[x]$ and $b \notin I[y]$. Thus $I[x+y]$ is distinct from both $I[x]$ and $I[y]$, and the proof is complete.

LEMMA 11. *If A is a primitive ring which satisfies condition* D_r, *then A is a division ring.*

Proof. We first represent A as a dense ring of linear transformations of a vector space over a division ring. Then by Lemma 10, A is a division ring or A contains three distinct maximal right ideals I_1, I_2, and I_3 such that $I_1 \cap I_2 \subseteq I_3$. In the latter case, $I_3 = I_3 + (I_1 \cap I_2) = (I_3 + I_1) \cap (I_3 + I_2) = A$, which is a contradiction.

We point out now that neither condition C_r nor condition D implies condition D_r. As an example, the (semi-simple) ring of all 2×2 matrices with elements in a division ring satisfies conditions C_r and D. However, Lemma 11 shows that this ring does not satisfy condition D_r. We remark further that condition D (D_r) does not imply condition C (C_r). For example, the ring of integers satisfies condition D (D_r) but not condition C (C_r).

Now let A be a semi-simple ring (with more than one element) which satisfies condition D_r. Then A is isomorphic with a subdirect sum of primitive rings, each of which satisfies condition D_r. This observation, together with Lemma 11, yields our principal result concerning rings which satisfy condition D_r:

THEOREM 16. *If A is a semi-simple ring (with more than one element) which satisfies condition* D_r, *then A is isomorphic with a subdirect sum of division rings.*

The converse of Theorem 16 is not true. For example, the ring of polynomials in two indeterminants over the rational field is commutative and semi-simple and is therefore isomorphic with a subdirect sum of fields. However, the ideal lattice of this ring is not distributive([12]). Theorem 16 generalizes the following well known result of Forsythe and McCoy [8]: *A regular ring of more than one element and without nonzero nilpotent elements is isomorphic with a subdirect sum of division rings.* For a regular ring without non-

([12]) See for example [9].

zero nilpotent elements is strongly regular [8] and hence, by the corollary to Lemma 9, satisfies condition D; moreover, any right ideal of a strongly regular ring is a two-sided ideal [1].

We now combine the results just obtained with those of §3 and consider rings whose lattice of right ideals is a Boolean algebra; that is, rings which satisfy both conditions C_r and D_r. Let A be a semi-simple ring (with more than one element) whose lattice of right ideals is a Boolean algebra. By Theorem 8, A is isomorphic with a discrete direct sum of simple rings, each of which is a division ring by Lemma 11. Conversely, if A is isomorphic with a discrete direct sum of division rings, then A is strongly regular and therefore satisfies condition D. As remarked above, any right ideal of a strongly regular ring is two-sided and hence A satisfies condition D_r as well as C_r. This proves the following theorem.

THEOREM 17. *A ring A (with more than one element) is semi-simple and has a right ideal lattice which is a Boolean algebra if and only if A is isomorphic with a discrete direct sum of division rings.*

In view of Theorem 13 it is perhaps of some interest to investigate the structure of a ring which is bound to its radical and whose lattice of right ideals is a Boolean algebra. In this connection we have the following result.

THEOREM 18. *If A is a ring which is bound to its radical, and if the lattice of right ideals of A is a Boolean algebra, then A is a zero ring.*

Proof. By Theorem 6, A is a direct sum of its minimal right ideals. If A is not a radical ring, it follows that A must contain a minimal right ideal whose square is not zero. Thus the sum C of all minimal right ideals with nonzero square is a nonzero right ideal of A. We shall show that C is a two-sided ideal. Let I be any minimal right ideal of A which satisfies $I^2 \neq 0$. Then $I = eA$ for some nonzero idempotent element e, and $I \dotplus B = A$ for some right ideal B of A. Moreover, since complementation is unique in the lattice of right ideals, we may easily show that B is a maximal right ideal. Let $(B:A)$ denote the ideal of A which consists of all $x \in A$ such that $Ax \subseteq B$. If $b \in I \cap (B:A)$, then $b = eb$ is in $I \cap B = 0$ so that $I \cap (B:A) = 0$. Then $(B:A) = (B:A) \cap (I+B) = (B:A) \cap B$, and hence $(B:A) \subseteq B$. It follows that $A - (B:A)$ is a primitive ring, and hence a division ring by Lemma 11. Then $(B:A) = B$ and $I \cong A - B$ is a division ring with identity element e. Now let a be any element of A and consider the right ideal aI. If $aI \neq 0$, then aI is a minimal right ideal and $ae \neq 0$. Now since $I \dotplus B = A$ we have $a = x + y$ with $x \in I$ and $y \in B$. Then $(ae)^2 = aexe + aeye$, and $ey \in I \cap B = 0$ so that $(ae)^2 = aexe$. Since I is a division ring there is an element $u \in I$ such that $xeu = e$. Then $(ae)^2 u = ae$ and hence $(ae)^2 \neq 0$. This implies that $(aI)^2 \neq 0$ and therefore $aI \subseteq C$. It follows now that C is a two-sided ideal. Now let J be the Jacobson radical of A. Either $J \cap C = 0$ or $J \cap C$ contains a minimal right ideal of A whose square is not zero, and the latter is impossible since J contains no nonzero idempotent element. Then $JC=CJ=0$ so that $C\subseteq J$ since A is bound to its radical. This is a contradiction and we conclude that $J=A$. Since $J^2=0$ in a ring which satisfies condition C_r, the proof is complete.

If J is the Jacobson radical of a ring A whose lattice of right ideals is a Boolean algebra, and if M is the maximal regular ideal of A, then we conclude from Theorems 13 and 18 that $A=J\dot{+}M$. Moreover, A is π-regular [17] in the following strong sense: for each $a\in A$, $a^2=a^2xa^2$ for some element $x\in A$.

We consider now some further consequences of conditions D and D_r. Following Birkhoff [3] we call an element c of a lattice L the *pseudo-complement* of an element a relative to an element b in case $a\cap x\leqq b$ if and only if $x\leqq c$. A lattice is *relatively pseudo-complemented* in case a pseudo-complement of a relative to b exists for each a and b in L. In [9] Fuchs has shown that the lattice of ideals of a (commutative) ring which satisfies the ascending chain condition for ideals is relatively pseudo-complemented, provided that this lattice is distributive. We now point out that only distributivity is required; the following proof is essentially that of Fuchs, Zorn's lemma being used in place of the chain condition. Let B and C be ideals of a ring A which satisfies condition D. By Zorn's lemma there is an ideal M of A which is maximal in the family of ideals I which satisfy $B\cap I\subseteq C$. Moreover, if I is any ideal of A for which $B\cap I\subseteq C$, then $B\cap(I+M)=(B\cap I)+(B\cap M)$ is contained in C so that $I+M=M$, and then $I\subseteq M$. It is now clear that M is the pseudo-complement of B relative to C. Thus a ring which satisfies condition D has a relatively pseudo-complemented ideal lattice, and an analogous proof shows that the corresponding result holds for right ideals. Furthermore, it is known [3, p. 147] that if L is any complete relatively pseudo-complemented lattice, than L satisfies the infinite distributive law $x\cap\bigcup_\alpha x_\alpha=\bigcup_\alpha(x\cap x_\alpha)$. This proves the following theorem.

THEOREM 19. *The following conditions are equivalent for a ring A:*

(i) *A satisfies condition* D (D_r).

(ii) *The lattice of ideals (right ideals) of A is relatively pseudo-complemented.*

(iii) *The lattice of ideals (right ideals) of A satisfies the infinite distributive law $I\cap\sum_\alpha I_\alpha=\sum_\alpha(I\cap I_\alpha)$.*

6. **Strongly irreducible ideals.** We make the following definition: An ideal (right ideal) I of a ring A is called *strongly irreducibile*(13) in case $B\cap C\subseteq I$

(13) This type of ideal was introduced by Fuchs in [9]; our terminology, however, differs from that of Fuchs. It is clear that "strongly irreducible" elements may be defined in any lattice, and that many of the results of this section are applicable to systems more general than rings (cf. [4]). For simplicity of exposition we have chosen to retain the ring terminology. We might also remark that the term *strongly irreducible* has recently been used in a different sense by C. W. Curtis (*On additive ideal theory in general rings*, Amer. J. Math. vol. 74 (1952) pp. 687–700).

implies that $B \subseteq I$ or $C \subseteq I$, where B and C are any ideals (right ideals) of A. In [9] Fuchs has considered strongly irreducible ideals in a commutative ring and the role which they play in the distributivity of the ideal lattice of that ring. In this last section we consider some further properties of these ideals, and obtain some additional results concerning conditions D and D_r.

An ideal P of a ring A is *prime* [14] in case $BC \subseteq P$ implies that $B \subseteq P$ or $C \subseteq P$, where B and C are any ideals of A. It is clear that any prime ideal (and hence any primitive ideal) is strongly irreducible, but, as we shall presently point out, the converse is not true. However, results can be obtained for strongly irreducible ideals which are analogous to corresponding results for prime ideals, as we shall now show.

In analogy with the notions of *multiplicative system* and *m-system* [19], we shall call a subset M of a ring A an *i-system* (intersection system) of A in case $b \in M$ and $c \in M$ implies that $(b) \cap (c) \cap M$ is not empty. (Here (b) and (c) denote the principal two-sided ideals generated by b and c, respectively.) A *right* i-system of A is defined in the obvious way.

THEOREM 20. *If I is an ideal (right ideal) of a ring A, then the following conditions are equivalent:*

(i) *I is strongly irreducible.*

(ii) *$(a) \cap (b) \subseteq I$ implies that $(a) \subseteq I$ or $(b) \subseteq I$, where (a) and (b) are any principal ideals (right ideals) of A.*

(iii) *The set-theoretic complement of I in A is an i-system (right i-system) of A.*

Proof. We give the proof for ideals, that for right ideals requiring only a trivial change of terminology. Clearly (i) implies (ii). Thus assume (ii) and let I' be the set-theoretic complement of I in A. If $a \in I'$ and $b \in I'$, then $(a) \not\subseteq I$ and $(b) \not\subseteq I$ so that $(a) \cap (b) \not\subseteq I$. Hence I' is an i-system and (ii) implies (iii). Now assume that I is not strongly irreducible so that there exist ideals B and C such that $B \cap C \subseteq I$, while $B \not\subseteq I$ and $C \not\subseteq I$. Then there are elements $b \in B \cap I'$ and $c \in C \cap I'$. But $(b) \cap (c) \subseteq I$ so that I' is not an i-system, and the proof is complete.

In earlier sections we have had occasion to refer to the topology introduced by Jacobson in the set of primitive ideals of a ring. In [19] McCoy points out that the set of prime ideals of a ring may be similarly topologized. We now observe that *the set of all (proper) strongly irreducible ideals may be topologized in precisely the same way*. To be specific, if $\mathfrak{S}$ denotes the set of all proper strongly irreducible ideals of A, and if $\mathfrak{S}_1 \subseteq \mathfrak{S}$, then the closure of $\mathfrak{S}_1$ is defined to be the set of all ideals in $\mathfrak{S}$ which contain $\bigcap \mathfrak{S}_1$. Since any prime ideal is strongly irreducible, it is clear that the space $\mathfrak{S}$ contains the spaces of prime and primitive ideals as subspaces. We point out also that the set $\mathfrak{S}_r$ of all proper strongly irreducible right ideals of A may be topologized by using a similar definition of closure. Finally, we remark that the notion of a strongly irreducible ideal, together with the associated topological space defined above, can be considerably generalized. This generalization will be discussed elsewhere [4], and for this reason we omit further comment on the spaces $\mathfrak{S}$ and $\mathfrak{S}_r$.

For convenience, we state the following result due to Fuchs [9]:

LEMMA 12. *A ring A satisfies condition* D (D_r) *if and only if every meet-irreducible ideal* (*right ideal*) *of A is strongly irreducible.*

Now let B be an ideal of a ring A which satisfies condition D. Since B is the intersection of all meet-irreducible ideals which contain it, it follows from Lemma 12 that B is the intersection of all strongly irreducible ideals which contain it. Conversely, assume that A is a ring in which every ideal is the intersection of all strongly irreducible ideals which contain it. A proof similar to that of Theorem 3 then shows that the lattice of ideals of A is isomorphic with the lattice of open sets of the space of proper strongly irreducible ideals of A. It follows that A satisfies condition D. Corresponding remarks can be made when condition D is replaced by condition D_r. Thus we have the following result.

THEOREM 21. *A ring A satisfies condition* D (D_r) *if and only if each ideal* (*right ideal*) *of A is the intersection of all strongly irreducible ideals* (*right ideals*) *which contain it.*

If B is an ideal of a ring A, let $R(B)$ denote the intersection of all strongly irreducible ideals of A which contain B. Then Theorem 21 states that A satisfies condition D if and only if $R(B)=B$ for every ideal B of A. Thus a convenient characterization of $R(B)$ should be of some interest. Although no such characterization has been obtained, we remark in this connection that, since every prime ideal is strongly irreducible, $R(B)$ is contained in the intersection $P(B)$ of all prime ideals which contain B. Since $P(B)-B$ is a nil ideal of $A-B$ [19], it follows that $R(B)-B$ is a nil ideal of $A-B$. We note finally that in the ring of integers modulo 8, $R(0)=0$ since this ring satisfies condition D, while $P(0)\neq 0$. This shows, in particular, that not every strongly irreducible ideal is prime.

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