

Ideal Theory in Commutative Γ - Semirings

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Abstract

In this paper, we study some results on the ideal theory of commutative Γ - semirings analogues to commutative semirings . In particular, Q-ideals , maximal ideals, primary ideals and radical ideals of commutative Γ - semiring are investigated . Furthermore we make an intensive examination of the notions of maximal ideal and local Γ - semirings. It is shown that the notion of primary ideals in Γ - semirings inherits most of essential properties of primary ideals of commutative semirings.

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1 Introduction

The notion of Γ - semirings was introduced by M. Murali Krishna Rao [14] as a generalization of Γ - rings ([3], [11]) , as well as semirings. Many mathematicians obtained interesting results on Γ - semiring (see for example [14], [15], [9] and [16]). The commutative Γ - semirings and sub Γ - semirings identified in [14]. Since ideals play a fundamental role in semiring theory (see for

example [8], [1], [2] and [10]), it is natural to consider them in the context of Γ -semirings theory. Some of topics related to ideals of semirings have been generalized and investigated for Γ -semirings. T. K. Dutta and S. K. Sardar established the notions of prime ideals and prime radicals of a Γ -semiring and studied them via its operator semiring [5]. Noetherian Γ -semirings, Cohen's theorem for a special class of Γ -semiring and weak primary decomposition theorem for a particular type of Γ -semirings were obtained by them [6]. S. K. Sardar and U. Dasgupta reviewed the notions of primitive Γ -semiring and primitive ideals of a Γ -semiring and studied them via operator semiring and obtained some results analogous to those of semiring theory [20]. S. K. Sardar introduced the notions of Jacobson radical of a Γ -semiring and semisimple Γ -semiring and characterize them via operator semirings [19]. In [9] the authors considered the congruences and ideals of a Γ -semiring, then constructed a new Γ -semiring and discussed the formation of ideals on this Γ -semiring. A study about the notion of k-ideal, m-k ideal, prime ideal, maximal ideal, irreducible ideal and strongly irreducible ideal in ordered Γ -semiring was introduced, also the properties of ideals in ordered Γ -semiring and the relations between them are studied [17].

In this paper we study some primitive operations of ideals in Γ -semiring we will use these properties in the paper, we introduce the notion of Q-ideal and maximal ideal in Γ -semiring. We obtain a number of results investigating maximal ideal and Q-ideal of a Γ -semiring. Similar to commutative semiring, we prove that any proper ideal (proper Q-ideal) I of a Γ -semiring, there exists a maximal ideal (k-ideal) M of S with $I \subseteq M$, and every prime k-ideal is a k-maximal ideal in the Artinian cancellative Γ -semiring. Moreover we study radical ideals and primary ideals of Γ -semiring and their properties. Finally, we show the uniqueness of reduced primary decomposition of k-ideals of a Γ -semiring as it was shown in commutative semiring [22].

2 Preliminaries

In this section we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1. *Let S and Γ be two additive commutative semigroups. Then S is called a Γ -semiring if there exists a mapping $S \times \Gamma \times S \mapsto S$ (images to be denoted by $a\gamma b$ for $a, b \in S$ and $\gamma \in \Gamma$ satisfying the following conditions:*

1. $a\alpha(b + c) = a\alpha b + a\alpha c$
2. $(a + b)\alpha c = a\alpha c + b\alpha c$
3. $a(\alpha + \beta)b = a\alpha b + a\beta b$

$$4. a\alpha(b\beta c) = (a\alpha b)\beta c$$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Examples of Γ -semiring are plenty. For example a semiring S can be considered as a Γ -semiring if we choose $\Gamma = S$ and the ternary operation $x\gamma y$ is the usual semiring multiplication.

Example 2.1. Let $S = (\mathbb{Z}^+, +)$ be the semigroup of positive integers and let $\Gamma = (2\mathbb{Z}^+, +)$ be the semigroup of even positive integers. Then S is a Γ -semiring.

Example 2.2. Let Q^+ denote the set of all positive rational numbers. Let Γ be the set of all positive integers. Then with respect to usual addition Q^+ and Γ are semigroups. Let $a \in Q^+, \gamma \in \Gamma$ and $b \in Q^+$ is defined by $(a\gamma b) \mapsto a\gamma b$ (usual multiplication). Then Q^+ is a Γ -semiring.

Definition 2.2. Let S be a Γ -semiring and $a, b \in S, \alpha \in \Gamma$. If $a\alpha b = b\alpha a$ then we say a, b are α -commutative. S is called multiplicatively commutative if $a\alpha b = b\alpha a$ for all $a, b \in S, \alpha \in \Gamma$ S is called a commutative Γ -semiring if $a\alpha b = b\alpha a$ and $a + b = b + a$ for all $a, b \in S, \alpha \in \Gamma$.

Example 2.3. Let S be the set of all even positive integers and Γ be set of all positive integers divisible by 3. Then with usual addition and multiplication of integers, S is a commutative Γ -semiring.

Definition 2.3. We say S is a Γ -semiring with zero if there exists a $0 \in S$ such that $0 + a = a + 0 = a$ and $0\alpha a = a\alpha 0 = 0$ for all $\alpha \in \Gamma, a \in S$.

Definition 2.4. Let S be a Γ -semiring. An element $1 \in S$ is said to be unity if for each $x \in S$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.5. In a Γ -semiring S with unity 1, an element $a \in S$ is said to be left invertible (right invertible) if there exist $b \in S, \alpha \in \Gamma$ such that $b\alpha a = 1$ ($a\alpha b = 1$).

Definition 2.6. A non-empty subset A of Γ -semiring S is called

- (i) a Γ -subsemiring of S if $(A, +)$ is a subsemigroup of $(S, +)$ and $A\Gamma A \subseteq A$.
- (ii) a left (right) ideal of S if A is a Γ -subsemiring of S and $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$).
- (iii) an ideal if A is a Γ -subsemiring of S , $A\Gamma S \subseteq A$ and $S\Gamma A \subseteq A$.

(iv) a k -ideal if A is a Γ - subsemiring of S , $A\Gamma S \subseteq A$, $S\Gamma A \subseteq A$ and $x \in S, x + y \in A, y \in A$ then $x \in A$.

Definition 2.7. An ideal I of a Γ - semiring S is called a proper ideal of the Γ - semiring S if $I \neq S$.

Definition 2.8. For each element a of a Γ - semiring S , the smallest right ideal containing a is called the principal right ideal generated by a and is denoted by $|a >$. Similarly we define $< a|$, $< a >$, respectively the principal left and principal two-sided ideals generated by a . In fact,

$$\begin{aligned} < a| &= \left\{ ma + \sum_{i=1}^n x_i \alpha_i a : m, n \in \mathbb{Z}^+ \cup \{0\}, x_i \in S, \alpha_i \in \Gamma \right\} \\ |a > &= \left\{ ma + \sum_{j=1}^n a \beta_j y_j : m, n \in \mathbb{Z}^+ \cup \{0\}, y_j \in S, \beta_j \in \Gamma \right\} \end{aligned}$$

and

$$\begin{aligned} < a > &= \left\{ na + \sum_{k=1}^p a \gamma_k z_k + \sum_{t=1}^s w_t \delta_t a + \sum_{j=1}^q u_j \lambda_j a \mu_j v_j : n, p, s, q \in \mathbb{Z}^+ \cup \{0\} \right. \\ &\quad \left. , z_k, w_t, u_j, v_j \in S, \gamma_k, \delta_t, \lambda_j, \mu_j \in \Gamma \right\} \end{aligned}$$

Definition 2.9. Let S be a Γ - semiring. An element $e \in S$ is said to be an idempotent S if there exists an $\alpha \in \Gamma$ such that $e = e\alpha e$. In this case we say that e is an idempotent.

Definition 2.10. let S be a Γ - semiring. If every element of S is an idempotent of S , then S is said to be idempotent Γ - semiring S .

Definition 2.11. A proper ideal I of a Γ - semiring S is said to be irreducible if for ideals H and K of S , $I = H \cap K$ implies that $I = H$ or $I = K$.

Definition 2.12. A proper ideal I of a Γ - semiring S is said to be strongly irreducible ideal if for ideals J and K of S , $J \cap K \subseteq I$ then $J \cap I$ or $K \cap I$.

Definition 2.13. A Γ - semiring S is said to be right (k -Noetherian) Noetherian if for any ascending chain $A_1 \subset A_2 \subset A_3 \dots$ of right(k - ideals) ideals of S there exists a positive integer n such that $A_i = A_n$ for all $i \geq n$. Similarly we define left (k -Noetherian) Noetherian and (k -Noetherian) Noetherian Γ - semiring.

Definition 2.14. A Γ - semiring is said to be Artinian (k -Artinian) if for every descending chain $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ of ideals (respectively k -ideals) in S there exists a positive integer n such that $A_i = A_n$ for all $i \geq n$.

Definition 2.15. Let S be a Γ - semiring and I be any ideal of S . Then the k -closure of I is denoted by $cl(I)$ and dened by $cl(I) = \{x \in S : x + i \in I, \text{ for some } i \in I\}$.

Note that $cl(I)$ is the smallest k -ideal containing I . and If A and B are two ideals of the Γ - semiring S with $A \subset B$, then $cl(A) \subset cl(B)$ [21]

Definition 2.16. A Γ - semiring S is said to be right (left) multiplicatively cancellable if $x\gamma y = z\gamma y$; (resp. $x\gamma y = x\gamma z$) for all $x, y, z \in S$ and for all $\gamma \in \Gamma$ implies that $x = z$ (resp. $y = z$).

Definition 2.17. Let S be a Γ - semiring. A proper ideal P of S is said to be prime if for any two ideals H and K of S , $H\Gamma K \subseteq P$ implies that either $H \subseteq P$ or $K \subseteq P$.

The set of all prime ideals of a semiring S is called the **spectrum** of S and will be denoted by **spec**(S) .

Throughout this paper the Γ - semiring S is assumed to be commutative and have a zero .

3 Operations on ideals of Γ - semiring

First, we investigate ideal theoretic basic results of semiring for ideals of a Γ - semiring.

Proposition 3.1. Let S be a commutative Γ - semiring with unity 1 and zero element 0 . Let I, T and D be ideals of S . If we define the addition and multiplications as follows :

$$I + T := \{a + b : a \in I, b \in T\} \text{ and}$$

$$I.T := \left\{ \sum_{i \leq m} a_i \gamma_i b_i : a \in I, b \in T, \gamma_i \in \Gamma \text{ and } m \in \mathbb{N} \right\},$$

then the following statements hold :

1. The set $I + T$ and $I.T$ are ideals of S .
2. $I + (T + D) = (I + T) + D$ and $I(TD) = (IT)D$.

3. $I + T = T + I$ and $IT = TI$.
4. $I(T + D) = IT + ID$.
5. $I + I = I, I + (0) = I, ITS = I$ and $I\Gamma(0) = (0)$.
6. If $I + T = (0)$ then $I = T = (0)$.
7. $IT \subseteq I \cap T$ and if $I + T = S$, then $IT = I \cap T$.
8. $(I + T)(I \cap T) \subseteq IT$.

Proof. We only prove numbers 2, 4, 7 and 8 and the proofs of the other results is routine and we hence omit the proofs :

2- a) Let $a + b \in I + (T + D)$, where $a \in I, b \in (T + D)$, thus $b = x + y$ such that $x \in T$ and $y \in D$.

$$\Leftrightarrow a + (x + y) \in I + (T + D).$$

$$\Leftrightarrow (a + x) + y \in (I + T) + D, \text{ where } (a + x) \in I + T \text{ and } y \in D.$$

$$\Leftrightarrow I + (T + D) = (I + T) + D.$$

2- b) Let $x \in I(TD)$. So we can write $x = \sum_{i \leq m_1} a_i \gamma_i b_i$, where $a_i \in I, \gamma_i \in \Gamma, b_i \in TD$ and $m_1 \in \mathbb{N}$, since $b_i \in TD$ then $b_i = \sum_{i \leq m_2} x_i \alpha_i y_i$, where $x_i \in T, \alpha_i \in \Gamma, y_i \in D$ and $m_2 \in \mathbb{N}$, thus :

$$\begin{aligned} x &= \sum_{i \leq m_1} a_i \gamma_i \sum_{i \leq m_2} x_i \alpha_i y_i \\ &= \sum_{i \leq m_1} a_i \gamma_i (x_1 \alpha_1 y_1 + x_2 \alpha_2 y_2 + \dots + x_{m_1} \alpha_{m_1} y_{m_1}) \\ &= \sum_{i \leq m_1} a_i \gamma_i (x_1 \alpha_1 y_1) + a_i \gamma_i (x_2 \alpha_2 y_2) + \dots + a_i \gamma_i (x_{m_1} \alpha_{m_1} y_{m_1}) \\ &= \sum_{i \leq m_1} \sum_{j \leq m_2} a_i \gamma_i (x_j \alpha_j y_j) \\ &= \sum_{i \leq m_1} \sum_{j \leq m_2} (a_i \gamma_i x_j) \alpha_j y_j, \end{aligned}$$

since $a_i \gamma_j x_j \in IT$ and $y_j \in D$, then $x \in (IT)D$, it follows that $I(TD) \subseteq (IT)D$, similarly we get $(IT)D \subseteq I(TD)$. Hence, $I(TD) = (IT)D$.

4- Let $z = \sum_{i \leq m} a_i \alpha_i b_i \in I(T + D)$, where $a_i \in I, b_i \in (T + D)$ and $\alpha_i \in \Gamma$, then $\sum_{i \leq m} a_i \alpha_i (x_i + y_i) \in I(T + D)$, where $x_i \in T$ and $y_i \in D$. So $\sum_{i \leq m} a_i \alpha_i x_i + \sum_{i \leq m} a_i \alpha_i y_i \in IT + ID$, it follows that $I(T + D) \subseteq IT + ID$.

Conversely, let $z = \sum_{i \leq m_1} a_i \alpha_i b_i + \sum_{i \leq m_2} x_i \gamma_i y_i \in IT + ID$, where $a_i, x_i \in I, \alpha_i, \gamma_i \in \Gamma, b_i \in T$ and $y_i \in D$. then :

$$z = \sum_{i \leq m_1} (a_i \alpha_i b_i) + (x_i \gamma_i y_i) = \sum_{i \leq m_1, j \leq 2m_1} c_j \beta_j (b_i + y_i)$$

$$\text{where } c_j = \begin{cases} a_i & \text{if } 1 \leq i \leq m_1, \\ x_i & \text{if } m_1 < i \leq 2m_1. \end{cases} \quad \beta_j = \begin{cases} \alpha_i & \text{if } 1 \leq i \leq m_1, \\ \gamma_i & \text{if } m_1 < i \leq 2m_1. \end{cases}$$

It follows that $IT + ID \subseteq I(T + D)$. Hence $I(T + D) = IT + ID$.

7- Let $x \in IT$. Then we can write $x = \sum_{i \leq m} a_i \gamma_i b_i$ where $a_i \in I, \gamma_i \in \Gamma$ and $b_i \in T$. Since I and T are ideals, then $\sum_{i \leq m} a_i \gamma_i b_i \in I \cap T$. It follows that $IT \subseteq I \cap T$. Now we will prove if $I + T = S$ then $IT = I \cap T$. Let $x \in I \cap T \Rightarrow x \in I$ and $x \in T$

$$\Rightarrow x = x\alpha 1 = x\alpha(x_1 + y_1) \text{ where } x_1 \in I, y_1 \in T \text{ since } I + T = S.$$

$$\Rightarrow x = x\alpha x_1 + x\alpha y_1,$$

$$\Rightarrow x = x_1 \alpha x + x\alpha y_1 \in IT,$$

$$\Rightarrow I \cap T = I + T.$$

8- Let $x \in (I + T)(I \cap T)$ then :

$$\begin{aligned} x &= \sum_{i \leq m} (a_i + b_i) \gamma_i z_i, \text{ where } a_i + b_i \in I + T \text{ and } z_i \in I \cap T \\ &= \sum_{i \leq m} a_i \gamma_i z_i + b_i \gamma_i z_i \\ &= \sum_{i \leq m} a_i \gamma_i z_i + z_i \gamma_i b_i, \end{aligned}$$

then $x \in IT$, it follows that $(I + T)(I \cap T) \subseteq IT$. □

Proposition 3.2. *Let S be a Γ - semiring where S and Γ are additive abelian semigroups with identity elements 0 and $\{0\}'$ respectively. If we denote the set of all ideals of S by $Id(S)$, then the following statement holds : $(Id(S), +, \cdot)$ is an additively-idempotent Γ - semiring.*

Proof. Let $Id(S) = \{I_{\lambda_i}\}_i$, then there exist a map $: Id(S) \times \Gamma \times Id(S) \mapsto Id(S)$, since $I_i \gamma_i I_i = I_i \in Id(S) \forall i \in \mathbb{N}$, and it satisfies the following axioms for all $I_{\lambda_1}, I_{\lambda_2}, I_{\lambda_3} \in Id(S)$ and $\gamma, \beta \in \Gamma$:

$$1- I_{\lambda_1} \gamma (I_{\lambda_2} + I_{\lambda_3}) = I_{\lambda_1} \gamma I_{\lambda_2} + I_{\lambda_1} \gamma I_{\lambda_3} :$$

$$x \gamma y \in I_{\lambda_1} \gamma (I_{\lambda_2} + I_{\lambda_3}) \text{ where } x \in I_{\lambda_1}, y \in I_{\lambda_2} + I_{\lambda_3} \text{ and } \gamma \in \Gamma,$$

$$\begin{aligned} &\Leftrightarrow x\gamma(a+b) \in I_{\lambda_1}\gamma(I_{\lambda_2} + I_{\lambda_3}) \text{ where } a+b=y, a \in I_{\lambda_2} \text{ and } b \in I_{\lambda_3}. \\ &\Leftrightarrow (x\gamma a) + (x\gamma b) \in I_{\lambda_1}\gamma I_{\lambda_2} + I_{\lambda_1}\gamma I_{\lambda_3}. \end{aligned}$$

2- we can prove that $(I_{\lambda_1} + I_{\lambda_2})\gamma I_{\lambda_3} = I_{\lambda_1}\gamma I_{\lambda_3} + I_{\lambda_2}\gamma I_{\lambda_3}$ similarly.

$$\begin{aligned} 3- & I_{\lambda_1}(\gamma + \beta)I_{\lambda_2} = I_{\lambda_1}\gamma I_{\lambda_2} + I_{\lambda_1}\beta I_{\lambda_2} : \\ & x(\gamma + \beta)y \in I_{\lambda_1}(\gamma + \beta)I_{\lambda_2} \text{ where } x \in I_{\lambda_1}, y \in I_{\lambda_2} \text{ and } \gamma, \beta \in \Gamma. \\ & \Leftrightarrow x\gamma y + x\beta y \in I_{\lambda_1}\gamma I_{\lambda_2} + I_{\lambda_1}\beta I_{\lambda_2}. \end{aligned}$$

$$\begin{aligned} 4- & (I_{\lambda_1}\gamma I_{\lambda_2})\beta I_{\lambda_3} = I_{\lambda_1}\gamma(I_{\lambda_2}\beta I_{\lambda_3}) : \\ & x\beta y \in (I_{\lambda_1}\gamma I_{\lambda_2})\beta I_{\lambda_3} \text{ where } x \in I_{\lambda_1}\gamma I_{\lambda_2}, y \in I_{\lambda_3} \text{ and } x = a\gamma b \text{ where } a \in I_{\lambda_1}, b \in \\ & I_{\lambda_2} \text{ and } \gamma \in \Gamma \\ & \Leftrightarrow (a\gamma b)\beta y \in (I_{\lambda_1}\gamma I_{\lambda_2})\beta I_{\lambda_3} \\ & \Leftrightarrow a\gamma(b\beta y) \in (I_{\lambda_1}\gamma I_{\lambda_2})\beta I_{\lambda_3} \\ & \Leftrightarrow a\gamma(b\beta y) \in I_{\lambda_1}\gamma(I_{\lambda_2}\beta I_{\lambda_3}). \end{aligned}$$

Finally, since $I_\lambda = I_\lambda + I_\lambda$ by proposition 3.1(5), $\text{Id}(S)$ is additively idempotent. □

Next, we present the following definition corresponding to the definition in semiring see [10]

Definition 3.1. For ideals I and T of a Γ - semiring S , it is defined that $[I : T] = \{s \in S, \gamma \in \Gamma : s\gamma T \subseteq I\}$.

Let us define a new notation: For each ideal I of S and any element $x \in S$, we define $[I : x] := \{s \in S, \gamma \in \Gamma : s\gamma x \in I\}$.

Proposition 3.3. Let I, T, D, I_α and T_α be ideals of a commutative Γ - semirings S with a zero element $0 \in S$. The following statements hold:

1. $I \subseteq [I : T]$.
2. $[I : T]T \subseteq I$.
3. $[[I : T] : D] = [I : TTD] = [[I : D] : T]$.
4. $[\bigcap_\alpha I_\alpha : T] = \bigcap_\alpha [I_\alpha : T]$.
5. $[I : \sum_\alpha T_\alpha] = \bigcap_\alpha [I : T_\alpha]$.
6. $[I : T] = [I : I + T]$.

Proof.

1- Obvious.

2- Let $z \in [I : T]T$. So we can write $z = \sum_{i \leq m} x_i \delta_i y_i$ where $x_i \in [I : T]$ and $y_i \in T$, since $x_i \in [I : T]$ we have $x_i \gamma T \subseteq I$, for all $\gamma \in \Gamma$, then $x_i \delta_i y_i \in I$ implies that $z \in I$. Therefore, $[I : T]T \subseteq I$.

3- a)

$$\begin{aligned} \text{Let } x \in [[I : T] : D] &\Leftrightarrow x\gamma D \subseteq [I : T] \\ &\Leftrightarrow (x\gamma D)\alpha T \subseteq I \\ &\Leftrightarrow x\gamma(D\alpha T) \subseteq I \\ &\Leftrightarrow x\gamma(T\alpha D) \subseteq I \\ &\Leftrightarrow x \in [I : T\Gamma D]. \end{aligned}$$

It follows that $[[I : T] : D] = [I : T\Gamma D]$.

3- b)

$$\begin{aligned} \text{Let } x \in [I : T\Gamma D] &\Leftrightarrow x\gamma(T\Gamma D) \subseteq I \\ &\Leftrightarrow x\gamma \sum_{i \leq m} t_i \alpha_i d_i \in I, \text{ where } t_i \in T_i \text{ and } d_i \in D_i \\ &\Leftrightarrow \sum_{i \leq m} (x\gamma t_i) \alpha_i d_i \in I \\ &\Leftrightarrow (x\gamma T_i) \in [I : D] \\ &\Leftrightarrow x \in [[I : D] : T]. \end{aligned}$$

It follows that $[I : T\Gamma D] = [[I : D] : T]$. Moreover from (a) and (b) we conclude that $[[I : T] : D] = [[I : D] : T]$.

4-

$$\begin{aligned} \text{Let } x \in \left[\bigcap_{\alpha} I_{\alpha} : T \right] &\Leftrightarrow x\gamma T \in \bigcap_{\alpha} I_{\alpha}, \text{ since } x\gamma T \subseteq \bigcap_{\alpha} I_{\alpha}, \\ &\Leftrightarrow x\gamma T \in I_{\alpha}, \text{ for all } \alpha \\ &\Leftrightarrow x \in [I_{\alpha} : T], \text{ for all } \alpha \\ &\Leftrightarrow x \in \bigcap_{\alpha} [I_{\alpha} : T], \text{ for all } \alpha. \end{aligned}$$

It follows that $\left[\bigcap_{\alpha} I_{\alpha} : T \right] = \bigcap_{\alpha} [I_{\alpha} : T]$.

5-

$$\begin{aligned}
\text{Let } x \in [I : \sum_{\alpha} T_{\alpha}] &\Leftrightarrow x\gamma \sum_{\alpha} T_{\alpha} \subseteq I \\
&\Leftrightarrow x\gamma \sum_{\alpha} t_{\alpha} \in I, \text{ for all } t_i \in T_i \\
&\Leftrightarrow x\gamma t_{\alpha} = x\gamma(t_{\alpha} + 0 + 0 \dots \dots + 0) \in I_{\alpha}, \text{ for all } \alpha \\
&\Leftrightarrow x \in [I : T_{\alpha}], \text{ for all } \alpha \\
&\Leftrightarrow x \in \cap [I : T_{\alpha}], \text{ for all } \alpha.
\end{aligned}$$

Hence, $[I : \sum_{\alpha} T_{\alpha}] = \cap_{\alpha} [I : T_{\alpha}]$.

6- Let $x \in [I : T]$ then we have $x\gamma T \subseteq I$, thus $x\gamma a \subseteq I$ where $a \in T$.

$$\begin{aligned}
\text{Let } b \in I &\Rightarrow x\gamma b \in I \\
&\Rightarrow x\gamma b + x\gamma a \in I \\
&\Rightarrow x\gamma(b + a) \in I \text{ and } (b + a) \in I + T \\
&\Rightarrow x \in [I : I + T] \\
&\Rightarrow [I : T] \subseteq [I : I + T].
\end{aligned}$$

Conversely,

$$\begin{aligned}
\text{let } x \in [I : I + T] &\Rightarrow x\gamma(a + b) \in I \\
&\Rightarrow x\gamma a + x\gamma b \in I, \forall a \in I, b \in T, \text{ when } a = 0, \text{ then } x\gamma 0 = 0 \\
&\Rightarrow x\gamma b \in I, \forall b \in T \\
&\Rightarrow x \in [I : T] \\
&\Rightarrow [I : I + T] \subseteq [I : T].
\end{aligned}$$

Therefore $[I : T] = [I : I + T]$.

□

Proposition 3.4. *Let S be a Γ -semiring and I be a nonzero ideal of S . Then the following statements are equivalent:*

1. I is a cancellation ideal of S ,
2. $[I\Gamma J : I] = J$ for any ideal J of S ,
3. $I\Gamma J \subseteq I\Gamma K$ implies $J \subseteq K$ for all ideals J and K of S .

Proof. (1) \implies (2) :

Let $[I\Gamma J : I] = \{s \in S, \gamma \in \Gamma : s\gamma I \subseteq I\Gamma J\}$, since Γ -semiring is commutative then $[I\Gamma J : I] = \{s \in S, \gamma \in \Gamma : I\gamma s \subseteq I\Gamma J\}$ $[I\Gamma J : I] = \{s \in S : s \in J\} = J$ since I is cancellation ideal of S .

(2) \implies (3) :

Let $I\Gamma J \subseteq I\Gamma K$, so $[I\Gamma J : I] \subseteq [I\Gamma K : I]$, it follows that $J \subseteq K$ from (2).

(3) \implies (1):

Obvious.

□

Next, we introduce the notion of a quotient Γ -semiring and study the properties of ideals of quotient Γ -semiring. For more details see [9]

Definition 3.2. An ideal I of Γ -semiring S is called a *partitioning ideal* ($=Q$ -ideal) if there exist a subset Q of S such that:

1. $S = \cup\{a + I : a \in Q\}$
2. if $a_1, a_2 \in Q$, then $(a_1 + I) \cap (a_2 + I) \neq \emptyset \Leftrightarrow a_1 = a_2$.

Let I be a Q -ideal of Γ -semiring S and let $S/I = \{a + I : a \in Q\}$, then S/I form a Γ -semiring under the binary operations \oplus, \odot define as follows:

$$(a_1 + I) \oplus (a_2 + I) = a_3 + I,$$

where $a_3 \in Q$ is the unique element such that $a_1 + a_2 + I \subseteq a_3 + I$.

$$(a_1 + I) \odot \gamma \odot (a_2 + I) = a_4 + I,$$

where $a_4 \in Q$ is the unique element such that $a_1\gamma a_2 + I \subseteq a_4 + I \forall \gamma \in \Gamma$. This Γ -semiring S/I is called the **quotient Γ -semiring** of S by I . By definition of Q -ideal, there exists a unique $a_0 \in Q$ such that $0 + I \subseteq a_0 + I$. Then $a_0 + I$ is a zero element of S/I .

The following results can easily be proved for a Γ -semiring as proved in the case of a semiring in [7].

lemma 3.1. Let S be a Γ -semiring with zero and commutative addition, and let P be a Q -ideal in S . If $a \in Q$ and $a + P$ is the zero in S/P , then $a + P = P$.

Theorem 3.2. let I be a Q -ideal of a Γ -semiring S . If L is a k -ideal of S/I . Then $L = J/I$ for some k -ideal J of S .

lemma 3.3. Assume that I is a Q -ideal of Γ -semiring S and let J, L be k -ideals of S . Then the following holds:

$$\text{If } I \subseteq J \text{ and } I \subseteq L, \text{ then } \frac{J}{I} = \frac{L}{I} \text{ if and only if } J = L.$$

lemma 3.4. *Let I be a Q -ideal of a Γ - semiring S . If J, K and L are k -ideals of S containing I , then $\left(\frac{J}{I}\right) \cap \left(\frac{K}{I}\right) = \frac{L}{I}$ if and only if $J \cap K = L$.*

Proof. Assume that $\left(\frac{J}{I}\right) \cap \left(\frac{K}{I}\right) = \frac{L}{I}$, we show that $J \cap K = L$. Let $z \in J \cap K$. Then $z = q + i$, for some $q \in Q$ and $i \in I$, so $q \in Q \cap J$ and $q \in Q \cap K$, since J and K are k -ideals, hence $q + I \in \left(\frac{J}{I}\right) \cap \left(\frac{K}{I}\right) = \frac{L}{I}$ by theorem 3.2. Therefore $q \in L$, thus $z \in L$ since L is a k -ideal. So $J \cap K \subseteq L$. Conversely, assume that $z \in L$. Then $z = \dot{q} + \dot{i}$ for some $\dot{q} \in Q$ and $\dot{i} \in I$. It follows that $\dot{q} + I \in \frac{L}{I} = \left(\frac{J}{I}\right) \cap \left(\frac{K}{I}\right)$, so $\dot{q} \in K \cap J$, hence $z \in K \cap J$. Thus $L = J \cap K$. The other implication is similar. \square

4 Maximal ideals of a Γ - semiring S and local Γ - semirings .

In this section we study maximal ideals of Γ - semiring and local Γ - semirings. We obtain some results .These results should be compared with [10] and [8] .

Definition 4.1. [13] *A proper ideal I of S is said to be maximal (resp- k -maximal) if J is an ideal (resp- k -ideal) in S such that $I \subset J$ then $J = S$. We denote the set of all maximal ideals of S by $Max(S)$*

Theorem 4.1. *Any proper ideal of S is a subset of a maximal ideal of S .*

Proof. let $\{I_i\}$ be all proper ideals of S that containing I . Since $\{I_i\}$ are sub-semigroups of S containing I . Then $\{I_i\}$ has an upper bound (the union of all those ideals) . Zorn's lemma implies that the proper ideals containing I have at least one maximal element that is, in fact, a maximal ideal of S . This means that any proper ideal I of S is a subset of a maximal ideal of S . \square

lemma 4.2. *Let S be a Γ -semiring with $1 \neq 0$. Then S has at least one k -maximal ideal.*

Proof. Since $\{0\}$ is a proper k -ideal of S then we can proof this lemma easily. \square

Theorem 4.3. *Let S be a Γ - semiring, I be a Q -ideal of S and J be a k -ideal of S with $I \subseteq J$. Then J is a k -maximal ideal of S if and only if J/I is a k -maximal ideal of S/I .*

Proof. This follows from Theorem 3.2 and lemma 3.4 . \square

Theorem 4.4. *Let I be a proper Q -ideal of a Γ - semiring S . Then there exists a maximal k -ideal M of S with $I \subseteq M$.*

Proof. Since S/I is non-trivial, and so by Lemma 4.2 has a k -maximal ideal L which, by Theorem 3.2 will have to have the form M/I for some k -ideal M of S with $I \subset M$. It now follows from Theorem 4.3 that M is a k -maximal ideal of S . \square

An element $u \in S$ is said to be unit if there exist $a \in S$ and $\alpha \in \Gamma$ such that $a\alpha u = 1 = u\alpha a$. The set of all invertible elements of S is denoted by $U(S)$. It is obvious that $U(S)$ is an abelian multiplicative group and is called the group of units of S because it satisfied the following axioms :

- **closure** : Let $u, v \in U(S)$ and $\beta \in \Gamma$, then we will prove $u\beta v \in U(S)$:
Since $u \in U(S)$ then there exist $s_1 \in S$ and $\beta_1 \in \Gamma$ such that

$$u\beta_1 s_1 = 1 \quad (1)$$

and $v \in U(S)$ then there exist $s_2 \in S$ and $\beta_2 \in \Gamma$ such that

$$v\beta_2 s_2 = 1 \quad (2)$$

So by (2) we get :

$$u = u\alpha 1 = u\alpha(v\beta_2 s_2) \quad (3)$$

Then from (1) and (3):

$$\begin{aligned} 1 &= u\beta_1 s_1 \\ &= u\alpha(v\beta_2 s_2)\beta_1 s_1 \\ &= (u\alpha v)\beta_2 s_2 \beta_1 s_1 \end{aligned}$$

it follows $1 = (u\alpha v)\beta_2 s_2 \beta_1 s_1$. Hence $u\alpha v \in U(S)$.

- **Associative** : Let u, v and $w \in U(S)$ and $\beta \in \Gamma$, since $U(S) \subseteq S$ then $(u\beta v)\beta w = u\beta(v\beta w)$ since this is one of the axioms for the Γ -semiring S .
- **Identity** : Since S has an identity element 1 , we have $1\alpha x = x$ for every $x \neq 0$ and in particular $1\alpha u = u$ for every unit element u , thus the set of units has an identity element under multiplication.
- **Inverse** : Let u be a unit, thus u^{-1} is also a unit, and $u\alpha u^{-1} = 1$. Thus every unit has a multiplicative inverse in the set of unit.

Thus , the four properties above show that the set of units is a group under multiplication.

Obviously , I is a proper ideal of S if and only if it contains no invertible element of S . Since I is a proper ideal of S then $1 \notin I$ then contains no invertible element of S . On the other hand I contains no invertible element of S then $1 \notin I$,then I is a proper ideal of S .

Proposition 4.1. *Let S be a Γ - semiring. Then*

$$U(S) = S - \left(\cup_{m \in \text{Max}(S)} m \right),$$

where by $\left(\cup_{m \in \text{Max}(S)} m \right)$ we mean the union of all maximal ideals of S .

Proof. The proof of this proposition is similar to the proof of proposition 3.12 in [10]. \square

Now , we will introduce the following definitions.

Definition 4.2. *Let S be a Γ -semiring with non-zero identity. A non zero element a of S is said to be semi unit in S if there exist $r, s \in S$ such that $1 + r\gamma a = s\beta a$ where $\gamma, \beta \in \Gamma$.*

Definition 4.3. *(S, m) is a local Γ - semiring if S is a Γ -semiring and m is its unique maximal ideal. A Γ - semiring S is semi-local if it possesses a finite number of maximal ideals, i.e., $|\text{Max}(S)| < \infty$.*

lemma 4.5. *Let I be a k -ideal of a Γ - semiring S .Then the following hold :*

- i. If a is a semi-unit element of S with $a \in I$, then $I = S$.*
- ii. If $x \in S$ and $\gamma \in \Gamma$, then $cl(S\gamma x)$ is a k -ideal of S .*

Proof.

i- Clearly $I \subseteq S$. Now we well prove that $S \subseteq I$. Since a is a semi-unit then $1 + r\alpha a = s\beta a$ where $r, s \in S$ and $\alpha, \beta \in \Gamma$, then $1 \in I$ since I is k -ideal , then $x = x\gamma 1 \in S\Gamma I \subset I$, it follows that $S \subseteq I$.Therefor $I = S$.

ii- Let x and $x + y \in cl(S\gamma x)$, then we will show that $y \in cl(S\gamma x)$. Let $a, b \in S\gamma x$ then

$$x + y + a = b \tag{4}$$

Now , let $a', b' \in S\gamma x$ where

$$x + a' = b' \tag{5}$$

then by adding a' to (4) we get:

$$x + a' + y + a = b + a' \tag{6}$$

but from (5)

$$x + a' + y + a = b' + y + a \quad (7)$$

so ,

$$b + a' = b' + a + y. \quad (8)$$

Since $b + a' \in S\gamma x$ and $b' + a \in S\gamma x$,it follows $y \in cl(S\gamma x)$, this implies $cl(S\gamma x)$ is a k-ideal.

□

Theorem 4.6. *Let S be an Artinian cancellative Γ - semiring. Then every prime k-ideal of S is k-maximal.*

Proof. Assume that I is a prime k-ideal of S and let $I \subsetneq J$ for some k-ideal J of S ; we will show that $J = S$. There is an element $x \in J$ with $x \notin I$. Then by Lemma 4.5 (ii) it follows that $cl(S\Gamma x) \supseteq cl(S\Gamma x\beta x) \supseteq cl(S\Gamma x\beta x\beta x) = cl(S\Gamma(x\beta)^2x) \supseteq cl(S\Gamma(x\beta)^3x) \supseteq \dots$ is a descending chain of k-ideals of S , since S be an Artinian so $cl(S\Gamma(x\beta)^n x) = cl(S\Gamma(x\beta)^{n+1}x)$ for some n ; hence $(x\beta)^n x + r\gamma(x\beta)^{n+1}x = s\gamma'(x\beta)^{n+1}x$ for some $r, s \in S, \gamma, \beta$ and $\gamma' \in \Gamma$. Then we have :

$$\begin{aligned} x(\beta x)^n + r\gamma x(\beta x)^{n+1} &= s\gamma'x(\beta x)^{n+1} \\ x(\beta x)^n + r\gamma(x(\beta x)^n\beta x) &= s\gamma'x(\beta x)^{n+1} \\ x(\beta x)^n + (r\gamma x(\beta x)^n)\beta x &= s\gamma'x(\beta x)^{n+1} \\ x(\beta x)^n + (x(\beta x)^n\gamma r)\beta x &= s\gamma'x(\beta x)^{n+1} \\ x(\beta x)^n\gamma 1 + x(\beta x)^n\gamma(r\beta x) &= s\gamma'x(\beta x)^{n+1} \\ x(\beta x)^n\gamma(1 + r\beta x) &= s\gamma'x(\beta x)^n\beta x \\ x(\beta x)^n\gamma(1 + r\beta x) &= x(\beta x)^n\gamma's\beta x. \end{aligned}$$

Since S is a cancellative and $(x\beta)x \neq 0$, it follows that we may cancel $x(\beta x)^n$, hence $1 + r\beta x = s\beta x$. Hence x is a semi-unit in J , and therefore $J = S$ by Lemma 4.5 .

□

The proofs of theorem 4.7 and theorem 4.8 are similar to the proofs of theorem 6 and theorem 2.12 in [8] and [6] respectively.

Theorem 4.7. *An Artinian cancellative Γ - semiring has only a finite number of maximal k-ideals.*

Theorem 4.8. *Let S be a Γ - semiring such that $S = (b_1, \dots, b_n)$ is a finitely generated ideal of S . Then each proper k-ideal B of S is contained in a maximal k-ideal of S .*

Let S be a Γ - semiring with identity (1), then each proper k-ideal of S is contained in maximal k-ideal of S .

The proof is immediate by $S = (1)$.

lemma 4.9. *Let S be a Γ -semiring and let $a \in S$. Then a is a semi-unit of S if and only if a lies outside each k -maximal ideal of S .*

Proof. By lemma 4.5 (ii), if $a \in S$ then $cl(S\gamma a)$ is k -ideal of S . Then from lemma 4.5 (i) a is a semi-unit of S if and only if $S = cl(S\Gamma a)$. First, suppose that a is a semi-unit of S and let $a \in M$ for some maximal k -ideal M of S . Since $S\gamma a$ is k -ideal then $S\gamma a = cl(S\gamma a)$. Then we should have $S\gamma a \subseteq M \subsetneq S$, so that a could not be a semi-unit of S . Conversely, if a were not a semi-unit of S , then $1 + r\alpha a = s\beta a$ holds for no $r, s \in S, \alpha, \beta \in \Gamma$. Hence, $1 \notin cl(S\gamma a)$ yields that $cl(S\gamma a)$ is a proper k -ideal of S by Lemma 4.5 (ii). By Corollary 4, $cl(S\gamma a) \subseteq J$ for some maximal k -ideal J of S ; but this would contradict the fact that a lies outside each maximal k -ideal of S . \square

Theorem 4.10. *Let S be a Γ - semiring. Then S is a local Γ -semiring if and only if the set of non-semi-unit elements of S is k -ideal.*

Proof. Assume that S is a local Γ - semiring with unique maximal k -ideal P . By Lemma 4.9, P is precisely the set of non-semi-unit elements of S . Conversely, assume that the set of non-semi-units of S is a k -ideal I of S (so $I \neq S$ since 1 is a semi-unit of S). Since S is not trivial, it has at least one maximal k -ideal: let J be one such. By Lemma 4.9, J consists of non-semi-units of S , and so $J \subseteq I \subset S$. Thus $I = J$ since J is k -maximal. We have thus shown that S has at least one maximal k -ideal, and for any maximal k -ideal of S must be equal to I . \square

Definition 4.4. *In a Γ - semiring . Two ideals I and T of S are called comaximal if $I + T = S$. The ideals $\{I_k\}_{k=1}^n$ of S are said to be pairwise comaximal if $I_k + I_j = S$ for any $1 \leq k < j \leq n$.*

The following two results can be easily proved for a Γ - semiring as proved in the case of a semiring [10] .

Proposition 4.2. *Let S be a Γ -semiring. Then the following statements hold:*

1. *If the ideals I, T of S are comaximal, then $I \cap T = IT$.*
2. *If the ideals $\{I_k\}_{k=1}^n$ of S are pairwise comaximal, then $\cap_{k=1}^n I_k = \prod_{k=1}^n I_k$.*
3. *If $\{W_k\}_{k=1}^n$ is a set of n distinct maximal ideals of S , then they are pairwise comaximal and $\cap_{k=1}^n W_k = \prod_{k=1}^n W_k$.*

Theorem 4.11. *If S is an Artinian Γ - semiring, then S is semi-local.*

Now , in the next proposition we can represented each k -ideal in a Γ -semiring as an intersection of a finite number of irreducible k -ideals

Proposition 4.3. *Let S be a Noetherian Γ -semiring. Then every k -ideal of S can be represented as an intersection of a finite number of irreducible k -ideals of S .*

Proof. Let M be the set of all k -ideals of S which are not a finite intersection of irreducible k -ideals of S . We claim that $M = \phi$. On the contrary, assume that $M \neq \phi$. Since S is Noetherian, M has a maximal element N . Since $N \in M$, it is not a finite intersection of irreducible k -ideals of S . Especially it is not irreducible, which means that there are k -ideals A and B properly containing N with $N = A \cap B$. Since N is a maximal element of M , $A, B \notin M$. Therefore A and B are a finite intersection of irreducible k -ideals of S . But, then $N = A \cap B$ is a finite intersection of irreducible k -ideals of S , a contradiction. \square

5 Decomposition of ideals

In this section we introduced the notion of radical ideal, primary ideal and discuss their properties and study the relations between them.

Radical Ideals

First we recall the following definition from [18].

Definition 5.1. *Let I be an ideal of a Γ -semiring S . Then radical of I is defined as the set of all elements $x \in S$ such that $(x\alpha)^n x \in I$ for some $n \in \mathbb{Z}^+$ for all $\alpha \in \Gamma$ and it is denoted by \sqrt{I} .*

Proposition 5.1. *Let S be a Γ -semiring and I, J be ideals of S . Then the following statements hold:*

1. $I \subseteq J$ implies that $\sqrt{I} \subseteq \sqrt{J}$.
2. $I \subseteq \sqrt{I}$ and $\sqrt{I} = \sqrt{\sqrt{I}}$.
3. $\sqrt{I\Gamma J} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
4. $\sqrt{I} = S \iff I = S$.
5. $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.

Proof. 1- Straightforward.

- 2- let $x \in I$ then $x\Gamma x \in I$, so for all $\alpha \in \Gamma$ and $n=1$ $x\alpha x \in I$, so $x \in \sqrt{I}$. Hence $I \subseteq \sqrt{I}$.

Next we will prove that $\sqrt{I} = \sqrt{\sqrt{I}}$. Take $x \in \sqrt{\sqrt{I}}$ this implies

$$\begin{aligned}
& (x\beta)^n x \in \sqrt{I} \\
& \Rightarrow (((x\beta)^n x) \beta)^m (x\beta)^n x \in I, \\
& \Rightarrow (x\beta x\beta x\beta \dots x\beta)^m (x\beta)^n x \in I, \\
& \Rightarrow (x\beta(x\beta x\beta \dots x\beta))^m (x\beta)^n x \in I, \\
& \Rightarrow ((x\beta)^{n+1})^m (x\beta)^n x \in I, \\
& \Rightarrow (x\beta)^{m(n+1)} (x\beta)^n x \in I, \\
& \Rightarrow (x\beta)^{mn+m+n} x \in I, \\
& \Rightarrow x \in \sqrt{I}, \\
& \Rightarrow \sqrt{\sqrt{I}} \subseteq \sqrt{I} \text{ and from (2) } \sqrt{I} \subseteq \sqrt{\sqrt{I}}, \\
& \Rightarrow \sqrt{\sqrt{I}} = \sqrt{I}.
\end{aligned}$$

3- since $I\Gamma J \subseteq I \cap J \subseteq I$ and J , then from (1) we have:

$$\sqrt{I\Gamma J} \subseteq \sqrt{I \cap J}, \quad (9)$$

since $I \cap J \subseteq I$ and $I \cap J \subseteq J$, then $\sqrt{I \cap J} \subseteq \sqrt{I}$ and $\sqrt{I \cap J} \subseteq \sqrt{J}$, it follows that $\sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$. Conversely, let $x \in \sqrt{I} \cap \sqrt{J}$ then $x \in \sqrt{I}$ and $x \in \sqrt{J}$, then for each $\alpha \in \Gamma$, there exist positive integers n and m satisfying: $(x\alpha)^n x \in I$ and $(x\alpha)^m x \in J$

$$\begin{aligned}
& \Rightarrow ((x\alpha)^n x)\alpha((x\alpha)^m x) \in I\Gamma J \\
& \Rightarrow (x\alpha x\alpha x\alpha \dots x\alpha)x\alpha((x\alpha)^m x) \in I\Gamma J \\
& \Rightarrow x\alpha(x\alpha x\alpha x\alpha \dots x\alpha)((x\alpha)^m x) \in I\Gamma J \\
& \Rightarrow (x\alpha)^{n+1}((x\alpha)^m x) \in I\Gamma J \\
& \Rightarrow (x\alpha)^{n+m+1} x \in I\Gamma J \\
& \Rightarrow x \in \sqrt{I\Gamma J} \\
& \Rightarrow \sqrt{I} \cap \sqrt{J} \subseteq \sqrt{I\Gamma J} \\
& \Rightarrow \sqrt{I} \cap \sqrt{J} \subseteq \sqrt{I\Gamma J} \subseteq \sqrt{I \cap J} \text{ from (1)}.
\end{aligned}$$

Hence, $\sqrt{I\Gamma J} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

4- If $\sqrt{I} = S$ then we will prove $I = S$. Suppose that $I \subsetneq S$, so $\sqrt{I} \subsetneq \sqrt{S} = S$ from (1), which is a contradiction with assumption. Hence $I = S$

If $I = S$ then we will prove $\sqrt{I} = S$. Since $I \subseteq \sqrt{I}$ and $I = S$ then,

$$S \subseteq \sqrt{I} \quad (10)$$

since $I \subseteq S$ then by (1) $\sqrt{I} \subseteq \sqrt{S}$, but $\sqrt{S} = S$ then

$$\sqrt{I} \subseteq S \quad (11)$$

Therefore from (10) and (11) , $\sqrt{I} = S$.

□

A nonempty subset W of a Γ - semiring S is said to be a multiplicatively closed set (for short an \mathcal{MC} - set) if $1 \in W$ and for all $w_1, w_2 \in W, \gamma \in \Gamma$ we have $w_1\gamma w_2 \in W$. In other words, W is an \mathcal{MC} -set if and only if it is a submonoid of (S, \cdot) . It is clear that an ideal P of S is a prime ideal of S if and only if $S - P$ is an \mathcal{MC} -set.

Theorem 5.1. *Let S be a Γ - semiring and I an ideal of S . Then the following statements hold:*

1. $\sqrt{I} = \bigcap_{P \in V(I)} P$, where $V(I) = \{P \in \text{space}(S) : I \subset P\}$.
2. \sqrt{I} is an ideal of S .

Proof. 1- let $x \in \sqrt{I}$, then for each $\beta \in \Gamma$ there exists positive integer n such that $(x\beta)^n x \in I$, but $I \subseteq P$ then $(x\beta)^n x \in P$, this implies $(x\beta)^n x \in \bigcap P$. Hence $\sqrt{I} \subseteq \bigcap_{P \in V(I)} P$. Now let $a \in \bigcap_{P \in V(I)} P$ but $a \notin \sqrt{I}$. It is clear that $W_a = \{(a\beta)^n a : n > 0\}$ is an \mathcal{MC} - set of S disjoint from I . since $a \notin \sqrt{I}$ then $(a\beta)^n a \notin I$ and by proposition 5.1 $(a\beta)^n a \notin \sqrt{I}$, it follows \sqrt{I} disjoint from W_a . Hence \sqrt{I} is prime ideal containing I and not containing a , which is a contradiction with assumption . So $\bigcap_{P \in V(I)} P \subseteq \sqrt{I}$. Therefore $\sqrt{I} = \bigcap_{P \in V(I)} P$

2- Since \sqrt{I} is an intersection of some ideals, it is an ideal and this completes the proof.

□

Primary ideals

In this section , we introduce primary ideals and D-primary ideal , then we prove the uniqueness of the reduced primary decomposition of k -ideal of a Γ -semiring.

Definition 5.2. [6] *An ideal P of Γ - semiring S is said to be primary ideal of S if $x\alpha y \in P, \alpha \in \Gamma, x, y \in S$ then $x \in P$ or $(y\beta)^n y \in P$, for all $\beta \in \Gamma$ for a positive integer n .*

Theorem 5.2. *Let S be a Noetherian Γ - semiring and I a k - ideal of S . If I is irreducible, then it is primary.*

Proof. Let I be a non-primary ideal of S . This means that there are $s, t \in S$, $\gamma \in \Gamma$ such that $s\gamma t \in I$ but $t \notin I$ and $(s\beta)^n s \notin I$ for all $n \in \mathbb{N}$. Since $s\gamma t \in I$, $t \in [I : s]$. Now by proposition 3.3, we have that $[I : s]$ and $[I : (s\beta)^n] \subset$

$[[I : (s\beta)^n] : s] \subset [I : (s\beta)^{n+1}]$, which gives us the following ascending chain of ideals: $I \subset [I : s] \subset [I : s\beta s] \subset [I : s\beta s\beta s] \subset [I : s\beta s\beta s\beta s] \subset \dots [I : (s\beta)^n s]$. Since S is Noetherian, this chain must stop somewhere, which means that there is some $m \in \mathbb{N}$ such that $[I : (s\beta)^m s] = [I : (s\beta)^{m+i} s]$ for any $i \geq 0$. Our claim is that $I = [I : (s\beta)^m s] \cap (I + ((s\beta)^m s))$. Obviously, $[I : (s\beta)^m s]$ and $[I + ((s\beta)^m s)]$ contain I .

Now let $x \in [I : (s\beta)^m s] \cap (I + ((s\beta)^m s))$. Since $x \in I + (s\beta)^m s$, there are some $y \in I, z \in S$ such that $x = y + z\beta(s\beta)^m s$. But $x \in [I : (s\beta)^m s]$, which means that $y\beta(s\beta)^m s + z\beta(s\beta)^m s\beta(s\beta)^m s = x\beta(s\beta)^m s \in I$. Since I is a k -ideal of S , we have $z\beta(s\beta)^{2m+1} s \in I$, which means that $z \in [I : (s\beta)^{2m+1} s]$. But $[I : (s\beta)^{2m+1} s] = [I : (s\beta)^m s]$, so $z\beta(s\beta)^m s \in I$ and this finally causes $x \in I$. This means that I is reducible, the thing it was required to have shown. \square

Theorem 5.3. *If S is a Noetherian Γ - semiring, then every proper k -ideal is a finite intersection of primary k -ideals.*

Proof. This follows from theorem 3.2 and theorem 5.2. \square

Now, we define the concept of cyclic ideals in a Γ - semiring as the following :

Definition 5.3. *A proper ideals $S\Gamma a$ and $S\Gamma b$ of a Γ - semiring S is said to be cyclic ideals if $S\Gamma a \cap S\Gamma b \subseteq I$, then either $a \in I$ or $b \in I$.*

Proposition 5.2. *Let I be an ideal of Γ - semiring S . Then the following holds:*

I is a strongly irreducible ideal, then $S\Gamma a$ and $S\Gamma b$ are cyclic ideals of S .

Proof. Let J and K be ideals of S such that $J \cap K \subseteq I$; we show that either $J \subseteq I$ or $K \subseteq I$. Suppose $J \not\subseteq I$. Then there exists $a \in J$ such that $a \notin I$. if $a \in J$ then $S\Gamma J \subseteq J \Rightarrow S\Gamma a \subseteq J$. and if $b \in K$ then $S\Gamma b \subseteq K \Rightarrow S\Gamma a \cap S\Gamma b \subseteq J \cap K \subseteq I$, so $b \in I$, as required. \square

Definition 5.4. *If P is a primary ideal of S and $\sqrt{P} = D$, then P is said to be D -primary.*

Proposition 5.3. *If P is an ideal of a Γ - semiring S such that $\sqrt{P} \in \text{Max}(S)$, then P is a primary ideal of S . In particular, any power of a maximal ideal is a primary ideal.*

Proof. Let P be an ideal of a Γ - semiring S and $\sqrt{P} = m$ such that $m \in \text{Max}(S)$. Take $x\alpha y \in P, \alpha \in \Gamma$ such that $(y\beta)^n y \notin P$, then $y \notin P$. Since

$\sqrt{P} = m$ is a maximal ideal of S , $\sqrt{P} + \langle y \rangle = S$. This implies that

$$\begin{aligned} \sqrt{P} + \sqrt{\langle y \rangle} &= S \\ \Rightarrow \sqrt{\sqrt{P} + \sqrt{\langle y \rangle}} &= S \\ \Rightarrow \sqrt{P + \langle y \rangle} &= S \\ \Rightarrow P + \langle y \rangle &= S, \text{ by proposition 5.1} \end{aligned}$$

which means that there are $a \in P$ and $c \in S$ such that $a + c\beta y = 1$. From this, we get that

$$\begin{aligned} a\alpha x + c\beta y\alpha x &= 1\alpha x. \\ a\alpha x + c\beta x\alpha y &= x. \end{aligned}$$

Since $a, x\alpha y \in P$, we get that $x \in P$ and this finishes the proof. \square

Theorem 5.4. *Let $P_1, P_2, P_3, \dots, P_n$. be primary ideals in an Γ - semiring S . If $\sqrt{P_i} = P$ for each $i = 1, 2, \dots, n$, then $\cap_{i=1}^n P_i$ is primary and $\sqrt{\cap_{i=1}^n P_i} = P$.*

Proof. Let $a\alpha b \in \cap_{i=1}^n P_i$ where $a, b \in S$ and $\alpha \in \Gamma$. If $a \notin \cap_{i=1}^n P_i$ then there exist $P_s \in \cap_{i=1}^n P_i$ such that $a \notin P_s$ where $1 \leq s \leq n$, then $(b\beta)^n b \in P_s$ for some $n \geq 1$. So $b \in \sqrt{P_s} = P$, hence $(b\beta)^s b \in P_i$ where $1 \leq i \leq n-1$, since P_i is P-Primary. Let $K = \max(n, s)$ then $(b\beta)^K b \in \cap_{i=1}^n P_i$. Now $z \in P$ implies $(z\beta)^t z \in \cap_{i=1}^n P_i, t \geq 1$. Hence $P \subseteq \sqrt{\cap_{i=1}^n P_i}$. Conversely, Proposition 5.1 implies that $\sqrt{\cap_{i=1}^n P_i} \subseteq \sqrt{P_i} = P$. Thus

$$\sqrt{\cap_{i=1}^n P_i} = P.$$

The following argument will show that $\cap_{i=1}^n P_i$ is primary. Let $a, b \in S$ and $\alpha \in \Gamma$ such that $a\alpha b \in \cap_{i=1}^n P_i$ and $b \notin P$. Since each P_i is primary, $a\alpha b \in P_i$, and $b \notin P = \sqrt{P_i}$, then $(b\alpha)^n b \notin P_i$ for each i , it follows that $a \in P_i$ for each $i = 1, 2, \dots, n$. Thus, $a \in \cap_{i=1}^n P_i$. \square

The following proposition will be used to prove the uniqueness of reduced primary decomposition of k -ideals in Γ - semirings.

Proposition 5.4. *Let S be a Γ - semiring, x an element of S and P be a D -primary ideal. The following statements hold:*

1. *If $x \in P$, then $[P : x] = S$.*
2. *If $x \notin P$, then $[P : x]$ is a D -Primary and $\sqrt{[P : x]} = D$.*

3. If $x \notin D$, then $[P : x] = P$

Proof. The proof of the statements (1) and (3) is straightforward. We only prove (2) : it is obvious that $P \subseteq [P : x]$. Now take $y \in [P : x]$. So $y\alpha x \in P$, since $x \notin P$, then $(y\beta)^n y \in P$, hence $y \in \sqrt{P} = D$. This means that $P \subseteq [P : x] \subseteq D$ and therefore by taking radical, we get $\sqrt{P} \subseteq \sqrt{[P : x]} \subseteq \sqrt{\sqrt{P}} = \sqrt{P}$ implies that $\sqrt{[P : x]} = \sqrt{P} = D$. Now, we show that $[P : x]$ is a primary ideal of S . Assume that for each $\alpha \in \Gamma$, $a\alpha b \in [P : x]$ and $b \notin \sqrt{[P : x]}$, then $a\alpha b\beta x \in P$ and P is D-Primary ideal of S which implies that either $a\alpha x \in P$ or $(b\beta)^n b \in P$, so $b \in \sqrt{P} = D$, then $b \in \sqrt{[P : x]}$, thus $a \in [P : x]$.

□

The following theorem is an immediate consequence of definition 5.1.

Theorem 5.5. *If P is a proper ideal in a Γ - semiring S , then the following statements are equivalent:*

1. P is Primary ;
2. if $a, b \in S, \alpha \in \Gamma$ such that $a \notin P$ and $a\alpha b \in P$, then $b \in \sqrt{P}$;
3. if $a, b \in S, \alpha \in \Gamma$ such that $a\alpha b \in P$ and $b \notin \sqrt{P}$, then $a \in P$.

We now present the notion of primary decomposition of k-ideals as follows:

Definition 5.5. *Let I be a k-ideal of a Γ - semiring S . Then I is said to have a primary decomposition if I can be expressed as $I = \bigcap_{i=1}^n P_i$, where each P_i is a primary ideal of S .*

A primary decomposition of the type $I = \bigcap_{i=1}^n P_i$, with $\sqrt{P_i} = Q_i$ is called a **reduced primary decomposition** of I , if Q_i 's are distinct and I cannot be expressed as an intersection of a proper subset of ideals P_i in the primary decomposition of I . A reduced primary decomposition can be obtained from any primary decomposition by deleting those P_j that contains $\bigcap_{i=1, i \neq j}^n P_i$ and grouping together all distinct $\sqrt{P_i}$'s.

Now, we prove the uniqueness of the reduced primary decomposition of a k-ideals of a Γ -semiring as follows:

Theorem 5.6. *[Uniqueness of Primary Decomposition] Let S be a commutative Noetherian Γ - semiring and I a k-ideal of S . If $I = \bigcap_{i=1}^n P_i$ is a reduced primary decomposition of I with $\sqrt{P_i} = Q_i$ for $i = 1, 2, \dots, n$, then $\{Q_1, Q_2, \dots, Q_n\} = \{\text{Prime ideals } Q \mid \exists x \in S \text{ such that } Q = \sqrt{[I : x]}\}$. The set $\{Q_1, Q_2, \dots, Q_n\}$ is independent of the particular reduced primary decomposition chosen for I .*

Proof. Let $x \in S$. Then $\sqrt{[I : x]} = \sqrt{(\cap_{i=1}^n P_i : x)} = \cap_{i=1}^n \sqrt{(P_i : x)} = \cap_{i=1, x \notin P_i}^n \sqrt{(P_i : x)}$ by proposition 5.4 (2) and therefore, $\sqrt{[I : x]} \subseteq Q_i$ for all $i = 1, 2, \dots, n$. Also, if $\sqrt{[I : x]}$ is prime, then $\prod_{i=1}^n Q_i \subseteq \cap_{i=1, x \notin P_i}^n Q_i = \sqrt{[I : x]}$ implies that $Q_i \subseteq \sqrt{[I : x]}$ for some $i = 1, 2, \dots, n$. Thus, we have $\{ \text{Prime ideals } Q | \exists x \in S \text{ such that } Q = \sqrt{[I : x]} \} \subseteq \{Q_1, Q_2, \dots, Q_n\}$.

On the other hand, for $i \in \{1, 2, \dots, n\}$, we have $\cap_{j=1, j \neq i}^n P_j \not\subseteq P_i$, as the primary decomposition is reduced. So there exists some $x_i \in \cap_{j=1, j \neq i}^n P_j$ and $x_i \notin P_i$. If $y \in [P_i : x_i]$, then for each $\alpha \in \Gamma$, $y\alpha x_i \in P_i$, and $y\alpha x_i \in (\cap_{j=1, j \neq i}^n P_j) \cap P_i = I$ which implies that $y \in [I : x_i]$. Thus, $[P_i : x_i] \subseteq [I : x_i]$, as $I \subseteq P_i$. So $[P_i : x_i] = [I : x_i]$ implies that $\sqrt{[P_i : x_i]} = \sqrt{[I : x_i]} = Q_i$ by proposition 5.4 (2). Hence $\{Q_1, Q_2, \dots, Q_n\} = \{ \text{Prime ideals } Q | \exists x \in S \text{ such that } Q = \sqrt{[I : x_i]} \}$ \square

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