Ideals in rings of continuous functions *

by

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1. Introduction. Let X be a completely regular Hausdorff space. In this paper, we study several problems about ideals in the ring $C(X)$ of all continuous real-valued functions on X, and in the ring $C^*(X)$ of all bounded continuous real-valued functions on X.

Familiarity with the main results of [3], [5] and [8] will be assumed. However, a brief review of some of the concepts needed is given in section 2. We also make use of the results of the preceding paper [13], which will be referred to throughout as [K].

In sections 3 and 4, we are concerned with certain ideals of $C^*(X)$, namely, the ring of functions "vanishing at ∞", the subring of functions with compact supports, and the ideals of $C^*(X)$ which are contained in the first ring and contain the second. In section 4, we obtain an algebraic characterization of a certain subclass of this collection. In these sections, we usually assume that X is a locally compact Hausdorff space.

Section 5 is devoted to an investigation of the ideals contained in a given maximal ideal, and the quotient rings obtained from some of these ideals, under the hypothesis that the prime ideals in this family intersect in a prime ideal. The study of rings of functions satisfying this requirement was initiated in [4].

The last section contains miscellaneous results connected with some algebraic questions raised in [K], and with the concept of P-space introduced in [3].

2. Preliminary remarks. Throughout this paper, X denotes a completely regular Hausdorff space. The letter R is reserved for the field of real numbers. We are primarily concerned with the following rings: $C(X)$, the ring of all continuous real-valued functions on X; $C^*(X)$, the subring of all bounded functions of $C(X)$; $C_s(X)$, the subring of all

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functions of $C(X)$ with compact supports; and $C_\infty(X)$, the subring of all functions of $C(X)$ which "vanish at ∞": $f \in C_\infty(X)$ if and only if $f \in C(X)$ and for each $\varepsilon > 0$, the set $\{x \in X: |f(x)| \geqslant \varepsilon\}$ is compact. This last concept can be generalized. We define the subring of $C(X)$ of functions which "approach a limit at ∞" to be all functions of the form $f + r \cdot 1$, where $f \in C_\infty(X)$, $r \in R$ and 1 is the identity of $C(X)$.

As is well known, with each space X there is associated a compact Hausdorff space βX, the Čech compactification of X, having the properties: (1) X is (homeomorphic to) a dense subspace of βX; (2) every $f \in C^*(X)$ has a continuous extension f^β over βX. The space βX is unique (up to homeomorphism). The closure in X of any set $A \subseteq X$ will be written as $\bar{A}$; and in βX, as A^β. The space vX is the largest subspace of βX over which *every* function in $C(X)$ (whether bounded or not) has a continuous extension. Furthermore, if $f \in C(X)$ is regarded as a function from X to the one-point compactification of R, designated by $R \cup \{\infty\}$, then f may be extended to a continuous function $\hat{f}$ from βX to $R \cup \{\infty\}$. As observed in [5], this follows from a theorem of Stone. (See [5] for further discussion of the function $\hat{f}$).

For every $f \in C(X)$, the set $Z(f) = \{x \in X: f(x) = 0\}$ is called the zero-set of f. For any subset I of $C(X)$, we let $\mathcal{Z}(I) = \{Z(f): f \in I\}$.

Let A be a commutative ring. The set of primitive (*i. e.*, prime maximal) ideals of A is denoted by $\mathfrak{M}(A)$. If this set is given the Stone topology (cf. [K], § 2), it will be written $\mathfrak{M}_s(A)$; and if it is given some other topology T, this will be indicated as $\mathfrak{M}_T(A)$. When A is a subring of B, and the mapping γ defined by $\gamma(M) = M \cap A$, $M \in \mathfrak{M}_s(B)$, is into $\mathfrak{M}_s(A)$, then it is continuous. This statement follows from the discussion in [9], § 3.

It is well known that $\mathfrak{M}_s(C^*(X))$ and $\mathfrak{M}_s(C(X))$ are both homeomorphic to βX. In the first case, $M \in \mathfrak{M}_s(C^*(X))$ if and only if $M = M^{*p} = \{f \in C^*(X): f^\beta(p) = 0\}$ for some $p \in \beta X$. The correspondence in the second case is given explicitly in:

LEMMA 2.1 (Gelfand-Kolmogoroff). *For every point p in βX, the set $M^p = \{f \in C(X): p \in Z(f)^\beta\}$ is a maximal ideal of $C(X)$. Conversely, for every maximal ideal M of $C(X)$ there is a unique $p \in \beta X$ such that $M = M^p$.*

For a proof, see [5].

Furthermore, in either ring, the subspace of all fixed ideals (*i. e.*, such that $p \in X$) is homeomorphic to X.

For any ideal I of $C(X)$, we define $\Delta(I) = \bigcap_{f \in I} Z(f)^\beta = \bigcap_{Z \in \mathcal{Z}(I)} Z^\beta$. Equivalently, $\Delta(I) = \{p \in \beta X: M^p \supseteq I\}$. As noted in [5], p. 453, the equivalence is a consequence of the Gelfand-Kolmogoroff lemma. It is evident that $\Delta(I)$ is a closed subset of βX.

An ideal of $C(X)$ of particular interest to us, which was introduced in [3], is N^p $(p \in \beta X)$. This is defined to be all $f \in C(X)$ such that $Z(f)$ contains the intersection of X with a neighborhood of p in βX.

In the terminology of [K], for any commutative ring A, and any $a \in A$, the set $\mathfrak{M}(a) = \{M \in \mathfrak{M}(A)\colon a \in M\}$ is called the $\mathfrak{M}$-set of a. Let I be an ideal of A. We shall say that I is a $\mathfrak{Z}$*-ideal* if whenever $\mathfrak{M}(a) = \mathfrak{M}(b)$ and $b \in I$, then $a \in I$. It is useful to examine this definition for the rings $C^*(X)$ and $C(X)$. If $f \in C^*(X)$, then $\mathfrak{M}(f)$ is the zero-set of f^β, regarded as an element of $C(\beta X)$. Thus, an ideal I of $C^*(X)$ is a $\mathfrak{Z}$-ideal if whenever $Z(f^\beta) = Z(g^\beta)$ (in βX), and $g \in I$, then $f \in I$. On the other hand, if $f \in C(X)$, then $\mathfrak{M}(f)$ is the set $Z(f)^\beta$. Now it is easily shown that $Z(f)^\beta = Z(g)^\beta$ if and only if $Z(f) = Z(g)$. Thus an ideal I of $C(X)$ is a $\mathfrak{Z}$-ideal if whenever $Z(f) = Z(g)$ and $g \in I$, then $f \in I$.

Lemma 2.2. *Every $\mathfrak{Z}$-ideal of $C(X)$ is an intersection of prime ideals.*

The proof is almost identical with the first part of [4], Theorem 1.4.

It was shown in [8], that if $C(X)/M^p$ is not isomorphic to R, then it is isomorphic to a non-Archimedean ordered field containing R. In section 5, we obtain a similar result for other quotient rings.

We conclude these remarks with a lemma of McKnight [16] about topologies on $\mathfrak{M}(A)$:

Lemma 2.3. *If T is a topology on $\mathfrak{M}(A)$ such that each $a \in A$ is a continuous function from $\mathfrak{M}(A)$ to a (T_1) topological ring, then T is at least as strong as the Stone topology.*

Proof. The inverse images of zero by elements of A are closed; these are precisely the $\mathfrak{M}$-sets. But the $\mathfrak{M}$-sets form a base for the closed sets of the Stone topology on $\mathfrak{M}(A)$ (see [K], § 2).

3. $C_s(X)$, $C_\infty(X)$ **and related ideals of** $C^*(X)$**.** As is well known, if X is compact, the space $\mathfrak{M}_s(C(X))$ is homeomorphic to X. The first part of the section is devoted to a study of a generalization of this statement.

The following lemma and proof are taken from [16], with some minor expository modifications, and a slight generalization.

Lemma 3.1 (McKnight). *Let X be a completely regular Hausdorff space. Let Δ be any closed subset of βX. Then the set I consisting of all $f \in C(X)$ for which $Z(f)^\beta$ contains a neighborhood of Δ, is the smallest ideal of $C(X)$ such that $\Delta(I) = \Delta$.*

Proof. If $f, g \in I$, then there exist open subsets V, W of βX such that $\Delta \subseteq V \subseteq Z(f)^\beta$, $\Delta \subseteq W \subseteq Z(g)^\beta$; thus

$$\Delta \subseteq V \cap W \subseteq Z(f)^\beta \cap Z(g)^\beta \subseteq Z(f-g)^\beta.$$

And for any $h \in C(X)$, we have $\Delta \subseteq V \subseteq Z(hf)^\beta$. Thus, I is an ideal.

Obviously, $\Delta(I) \supseteq \Delta$. For any $p \notin \Delta$, there is an $f \in C^*(X)$ such that $f^\beta(p) = 1$, $f^\beta(\Delta) = -1$. Let $g = \max\{f, 0\}$. Then $g \in I$, and $p \notin Z(g)^\beta$. It follows that $\Delta(I) = \Delta$.

Finally, let J be any ideal satisfying $\Delta(J) = \Delta$. Given $f \in I$, there is an open subset U of βX such that $\Delta \subseteq U \subseteq Z(f)^\beta$. For each point $p \in \beta X - U$, there is a non-negative function $g_p \in J \cap C^*(X)$ such that $g_p^\beta(p) > 1$. The open sets $U_p = \{q\colon g_p^\beta(q) > 1\}$ cover $\beta X - U$; by compactness, there is a finite subcover, say $\{U_1, \ldots, U_n\}$. The sum $g = g_1 + \ldots + g_n$, where g_i is the defining function of U_i, is in $J \cap C^*(X)$; and $g^\beta(p) > 1$ for all $p \in \beta X - U$. Since βX is normal, there is an $h \in C^*(X)$ such that $h^\beta(\Delta) = 0$, $h^\beta(\beta X - U) = 1$. Define $m \in C(X)$ by: $m(x) = h(x)/g(x)$ if $g(x) \geqslant 1$, and $m(x) = h(x)$ if $g(x) < 1$. Let $e = mg$. Then $e \in J$; and $e(x) = 1$ for $x \in X - Z(f)$; so $ef = f$, which implies that $f \in J$. Thus, $I \subseteq J$.

We recall that an ideal I of any subring of $C(X)$ is said to be *free* if for each $p \in X$, there is an $f \in I$ such that $p \notin Z(f)$.

Lemma 3.2. *Let X be a completely regular Hausdorff space. The ring $C_\infty(X)$ is the intersection of the free maximal ideals of $C^*(X)$.*

Proof. The intersection of the free maximal ideals of $C^*(X)$ coincides with the set $I = \{f \in C^*(X)\colon f^\beta(\beta X - X) = 0\}$. Now it is easily seen that the following statements are equivalent: $f \in C_\infty(X)$, *i. e.*, for every $\varepsilon > 0$, the set $F_\varepsilon = \{x \in X\colon |f(x)| \geqslant \varepsilon\}$ is compact; for every $\varepsilon > 0$, $F_\varepsilon^\beta = F_\varepsilon$; for every $\varepsilon > 0$, $|f^\beta(p)| < \varepsilon$ for all $p \in \beta X - X$; and $f \in I$.

Theorem 3.3 [1]. *Let X be a locally compact Hausdorff space, and let A be an ideal of $C^*(X)$. Then the following statements are equivalent.*

(a) $C_s(X) \subseteq A \subseteq C_\infty(X)$.

(b) *For all* $M \in \mathfrak{M}_s(C^*(X))$, $M \supseteq A$ *if and only if M is a free ideal.*

(c) *The mapping* $p \to M^{*p} \cap A$ $(p \in X)$ *is a homeomorphism from X to* $\mathfrak{M}_s(A)$.

Proof. (b)$\leftrightarrow$(c). Since the mapping $p \to M^{*p}$ $(p \in X)$ is a homeomorphism from X to the space of fixed maximal ideals of $C^*(X)$, this follows without difficulty from [K], Theorem 5.2 (for the spaces $\mathfrak{M}_s(C^*(X))$ and $\mathfrak{M}_s(A)$).

(a)$\to$(b). Let $M \in \mathfrak{M}_s(C^*(X))$ be a free ideal. Then by Lemma 3.2, $A \subseteq C_\infty(X) \subseteq M$.

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[1] The statement (a)$\to$(c) has been obtained independently by J. G. Horne, using the concept of 0-ideal. For $A = C_s(X)$, this result was announced by M. E. Shanks in [17]. His proof (which has not been published) is also based on a viewpoint which is different from ours. For $A = C_\infty(X)$, (and Lemma 3.4), cf. Loomis [14], p. 60. (Added in proof: See also Théorèm 1 of K. Fujiwara, *Sur les anneaux des fonctions continues à support compact*, Math. J. Okayama Univ. 3 (1954)- p. 175-184.)

Now suppose $M \in \mathfrak{M}_s(C^*(X))$ is a fixed ideal, *i. e.* $M = M^p$ for some $p \in X$. Since X is locally compact, there is a neighborhood V of p with compact closure. And since X is completely regular, there is an $f \in C^*(X)$ such that $f(p) = 1$, $f(X - V) = 0$. Thus $f \in C_s(X)$, and $f \notin M^p$, so $C_s(X) \not\subset M^p$. Hence $A \not\subset M^p$.

Before concluding the proof of Theorem 3.3, we state and prove a lemma.

The one-point compactification $X \cup \{\infty\}$ of a locally compact Hausdorff space X will be denoted by X^*.

LEMMA 3.4. *Let X be a locally compact Hausdorff space. Then $(C_\infty(X); R)$ (notation as in [K], § 6) is isomorphic to $C(X^*)$.*

Proof. By Theorem 3.3, (a)→(c), whose proof has been completed, the mapping $p \to M^{*p} \cap C_\infty(X)$ $(p \in X)$ is a homeomorphism from X to $\mathfrak{M}_s(C_\infty(X))$. Thus, $C_\infty(X)$ may be regarded as the ring $C_\infty(\mathfrak{M}_s(C_\infty(X)))$. By [K], Theorem 6.3, $(C_\infty(X); R)$ is a subring of $C(X^*)$. But it is evident that in this case, all of $C(X^*)$ is obtained.

We return to the proof of 3.3.

(b)→(a). Since $A \subseteq M$ for every free ideal $M \in \mathfrak{M}_s(C^*(X))$, it follows from Lemma 3.2 that $A \subseteq C_\infty(X)$.

By 3.4, we imbed $C_\infty(X)$ in $C(X^*)$. Now $C(X^*)$ is isomorphic to the subring of $C^*(X)$ consisting of all functions which "approach a limit at ∞". Thus, A may be viewed as an ideal of $C(X^*)$ contained in M^∞. Now for each $p \in X$, there exists an $f \in A$ such that $f(p) \neq 0$. Hence $\Delta(A) = \bigcap_{f \in A} Z(f) = \{\infty\}$. By Lemma 3.1, the ideal of all functions vanishing in a neighborhood of ∞ is contained in A, *i. e.*, $C_s(X) \subseteq A$.

Before stating the next theorem, we point out that it was shown in [3], Theorem 3.3, that every prime ideal of $C(X)$ is contained in a unique maximal ideal.

THEOREM 3.5. *Let X be a compact Hausdorff space; P, a prime ideal of $C(X)$; and M^p, the unique maximal ideal containing P. Then every maximal ideal of P has the form $P \cap M^q$, where $q \neq p$, $M^q \in \mathfrak{M}(C(X))$.*

Proof. Suppose not, and let I be a maximal ideal of P which is not of the form indicated. There is a non-negative function $g \in P - I$. For let $f \in P - I$ be arbitrary. The relations $\max\{f, 0\} \cdot \min\{f, 0\} = 0 \in P$, $\max\{f, 0\} + \min\{f, 0\} = f \in P - I$ imply that both $\max\{f, 0\}$, $\min\{f, 0\}$ are in P but not both are in I; hence, either $\max\{f, 0\}$ or $-\min\{f, 0\}$ is in $P - I$. Now we have also $\sqrt{g} \in P - I$. Thus P/I is not a zero-ring; so I must be a prime ideal of P. It follows that P is isomorphic to an ideal P' of $C^*(X - \{p\})$ satisfying $C_s(X - \{p\}) \subseteq P' \subseteq C_\infty(X - \{p\})$ and having a free primitive ideal. This contradicts Theorem 3.3.

COROLLARY 3.6. *Let X be a locally compact Hausdorff space. Then $C_s(X)$ is an ideal of $C_\infty(X)$ which is contained in no maximal ideal.*

Proof. M^∞ is a prime ideal of $C(X^*)$; by Theorem 3.5, every maximal ideal of M^∞ has the form $M^\infty \cap M^q$, where $q \neq \infty$, $M^q \in \mathfrak{M}(C(X^*))$. But $C_s(X) \not\subset M^q$.

COROLLARY 3.7. *Let X be a compact Hausdorff space; F, a closed subset of X; and A, the subring of $C(X)$ consisting of all functions which vanish on F. Then every maximal ideal of A has the form $A \cap M^q$, where $q \notin F$, $M^q \in \mathfrak{M}(C(X))$.*

Proof. Form a quotient space by reducing F to a point.

EXAMPLE 3.8. The ring $C_\infty(R)$ is a ring in which to every maximal ideal there corresponds an element in its complement having a relative identity (so that every maximal ideal is primitive [K], § 4), but not every element has a relative identity. For, it is clear that any maximal ideal failing to have this property would necessarily contain $C_s(R)$. By 3.6, there are no such maximal ideals. And, the function f in $C_\infty(R)$ defined by $f(x) = 1/(x^2 + 1)$, for all $x \in R$, is an element with no relative identity.

It has been noted that we may view $C_\infty(X)$ as the ideal M^∞ in $C(X^*)$. It is easily seen that, similarly, $C_s(X)$ may be considered as the ideal N^∞ of $C(X^*)$ (see section 2 for the definition of N^p). In fact, $C_s(X)$ and $C_\infty(X)$ are the minimal and maximal ideals associated with the closed set $\{\infty\}$ in X^*, in the sense of Lemma 3.1. If I is a proper ideal of $C_\infty(X)$, it follows from Theorem 3.3 that I is contained in no primitive ideal if and only if $\Delta(I) \cap X$ is empty, or, regarding $\Delta(I)$ as a subset of X^*, if and only if $\Delta(I) = \{\infty\}$. Thus, by Lemma 3.1, $C_s(X)$ is the minimal ideal contained in no primitive ideal.

Now $N^\infty = M^\infty$ if and only if ∞ is a P-point of X^*; equivalently, every countable union of compact subsets of X is contained in a compact set (cf. [3], 4.2). It is easily seen that if X is a σ-compact non-compact space, or a non-countably compact space, then $N^\infty \neq M^\infty$, *i. e.*, $C_s(X) \neq C_\infty(X)$.

According to Lemma 3.2, $C_\infty(X)$ is the intersection of the free maximal ideals of $C^*(X)$. We consider now the intersection D of the free maximal ideals of $C(X)$. The ideal D must be a subring of $C^*(X)$; for if $f \in C(X)$ is unbounded, there is a $p \in \beta X - X$ such that $\hat{f}(p) = \infty$ (cf. section 2), so that $f \notin M^p$. But D is then a subring of $C_\infty(X)$, because $(M^p \cap C^*(X)) \subseteq M^{*p}$ for every p. Since D is an ideal of $C(X)$, it is an ideal of $C^*(X)$.

If X is a locally compact, σ-compact space, then D is contained properly in $C_\infty(X)$. For, $\beta X - X$ is clearly a G_δ-set of βX, and it is closed in βX (cf. [11], p. 163, Exer. G); hence there is an $f \in C^*(X)$ such that $Z(f^\beta) = \beta X - X$ (cf. [11], p. 134, Exer. J). Thus, $f \in C_\infty(X)$, but f is in *no* maximal ideal of $C(X)$, since it is a unit of $C(X)$. It follows that in this case, $C_\infty(X)$ is not an ideal of $C(X)$.

By Lemma 2.1, D coincides with $\{f \in C(X)\colon Z(f)^\beta \supset \beta X - X\}$. Now if $f \in C_s(X)$, then $\beta X = X^\beta = (X - Z(f))^\beta \cup Z(f)^\beta = (\overline{X - Z(f)}) \cup Z(f)^\beta$; so $Z(f)^\beta \supset \beta X - X$, *i. e.*, $f \in D$. Thus $C_s(X) \subseteq D$. We next give two sufficient conditions that $C_s(X) = D$.

THEOREM 3.9. *If either* (a) *X is a P-space (not necessarily locally compact), or* (b) *X is a locally compact Hausdorff space and ∞ is a P-point of X^*, then $C_s(X)$ coincides with the intersection of the free maximal ideals of $C(X)$.*

Proof. (a) Let $f \in D$. Since X is a P-space, $Z(f)$ is open ([3], Theorem 5.3). Hence $X - Z(f)$ and $Z(f)$ are completely separated; so $(X - Z(f))^\beta$ and $Z(f)^\beta$ are disjoint subsets of βX. Now $Z(f)^\beta \supset \beta X - X$ implies that $(X - Z(f))^\beta \subseteq X$. Since $\overline{X - Z(f)} \subseteq (X - Z(f))^\beta$, and $(X - Z(f))^\beta$ is a compact subset of X, it follows that $\overline{X - Z(f)}$ is compact, *i. e.*, that $f \in C_s(X)$. Therefore $D \subseteq C_s(X)$. Combining with the remark preceding the theorem, we have $C_s(X) = D$.

(b) It has been shown that $C_s(X) \subseteq D \subseteq C_\infty(X)$. Thus, if ∞ is a P-point of X^*, we have $C_s(X) = C_\infty(X) = D$.

Note that the two cases considered in 3.9 are mutually exclusive in all spaces with an infinite number of points. For, if X is a locally compact P-space, and ∞ is a P-point of X^*, then X^* is a compact P-space; so X^*, and hence X, is finite ([3], Cor. 5.4).

We designate by $W(\alpha)$ the space of all ordinals less than the ordinal α, with the interval topology. The space $X = W(\omega_1 + 1) \times W(\omega_0 + 1) - \{(\omega_1, \omega_0)\}$ shows that it need not be true that $C_s(X) = D$. Every continuous function of this space is bounded, so D is identical with the intersection of the free maximal ideals of $C^*(X)$, *i. e.*, $D = C_\infty(X)$. But $C_s(X) \neq C_\infty(X)$, in other words, ∞ is not a P-point of X^*, since X is evidently not countably compact.

A familiar question related to these matters is, for what spaces X is it true that $X^* = \beta X$ (cf. *e. g.*, [8], p. 62-63)? It is of course necessary that X be locally compact Hausdorff. And for this class, $X^* = \beta X$ if and only if every function in $C(X)$ has a continuous extension to X^*, which is clearly equivalent to the condition that $C(X)$ coincide with the set of functions which "approach a limit at ∞". Thus, it certainly suffices that every function in $C(X)$ be constant outside a compact set; this is equivalent to the statement that each $f \in C(X)$ has the form $g + r \cdot 1$, $g \in C_\infty(X)$, $r \in R$, and that ∞ is a P-point of X^*. That this is not necessary is shown by the space

$$W(\omega_1 + 1) \times W(\omega_0 + 1) - \{(\omega_1, \omega_0)\}.$$

The final result of this section indicates that an interesting condition on $C(X)$ (see [3], Theorem 5.3) is too restrictive when applied to $C_\infty(X)$.

THEOREM 3.10. *Let X be a locally compact Hausdorff space. Then $C_\infty(X)$ is a regular* (²) *ring if and only if X is finite.*

Proof. If X is finite, then X is discrete; so $C_\infty(X)$ is the direct sum of a finite number of fields (each isomorphic to R), and hence is regular.

Conversely, suppose that $C_\infty(X)$ is regular. Now $C_\infty(X)$ is an ideal of $(C_\infty(X); R)$, which by Lemma 3.4, is isomorphic to $C(X^*)$. It follows from [12], Theorem 1, and the fact that R is regular, that $C(X^*)$ is regular. Thus X^* is a P-space ([3], Theorem 5.3). But X^* is compact; so X^*, and hence X, is finite ([3], Cor. 5.4).

4. Algebraic characterizations. In this section we give ring characterizations of the $\mathfrak{Z}$-ideals of $C^*(X)$ (cf. section 2) containing $C_s(X)$ and contained in $C_\infty(X)$, where X is a locally compact Hausdorff space. Characterizing conditions are given separately for the ideals of greatest interest, $C_s(X)$ and $C_\infty(X)$.

Before giving these theorems, we state in 4.1 an essentially known result, which will be needed in the proofs. We include a proof of 4.1 based on a different viewpoint, which we believe to be interesting in itself; it is developed from the method used by Heider in his characterization of the lattice of all continuous real-valued functions on a compact Hausdorff space [6].

Let A be a subring of a commutative ring B. If the mapping α defined by $\alpha(M) = M \cap A$ is one-to-one from $\mathfrak{M}(B)$ onto $\mathfrak{M}(A)$, we shall say that B is an $\mathfrak{M}$-*extension* of A. Throughout this section, the symbol α will denote a mapping defined as above for the pair of rings under discussion.

There are many examples of $\mathfrak{M}$-extensions which are *proper* extensions. The ring $C([0,1])$ is a proper $\mathfrak{M}$-extension of the ring of continuous rational functions over R defined on $[0,1]$, as well as of the ring of differentiable real-valued functions on $[0,1]$. More generally, $C([0,1])$ is a proper $\mathfrak{M}$-extension of any subring A of $C([0,1])$ having the properties: (1) For each $p \in [0,1]$, $\{r \in R\colon f(p) = r$ for some $f \in A\}$ is a field; (2) if $f \in A$ and $Z(f)$ is empty, then f is a unit of A. For, (1) evidently implies that for each $p \in [0,1]$, $\{f \in A\colon f(p) = 0\}$ is a primitive ideal of A; while it follows from (2), by a familiar argument of Gelfand and Kolmogoroff, that every primitive ideal of A is fixed. Many proper subrings of $C([0,1])$ satisfying (1) and (2) can be constructed merely by restricting the functions at a single point p. For instance, the collection of rings $A_1, A_2, A_3, A_4, \ldots$, consisting of all functions f in $C([0,1])$ such

(²) For the definition and simple properties, see [15], p. 147-149.

that $f(p)$ lies in Ra, Ra$(\sqrt{2})$, Ra$(\sqrt{2},\sqrt{3})$, Ra$(\sqrt{2},\sqrt{3},\sqrt{5})$, ..., respectively, (where Ra denotes the rationals) is a sequence of subrings such that A_{i+1} is a proper $\mathfrak{M}$-extension of A_i $(i=1,2,3,...)$. Also, $C([0,1])$ is a proper $\mathfrak{M}$-extension of each A_i. In every example given above, the mapping α is actually a homeomorphism if the two spaces are given the Stone topology.

Any ring A satisfying $(\mathfrak{R})$ of 4.1, (1) possesses an induced partial order defined as follows: Given $a \in A$, we set $a \geqslant 0$ if and only if the image of a in A/M is non-negative for each $M \in \mathfrak{M}(A)$. The symbol $\leqslant$ used in the statement of condition $(\mathfrak{B})$ of 4.1, (1) and in condition $(\mathfrak{B}_s)$ of 4.6, signifies this partial order.

It would be incorrect to say that Theorem 4.1 is merely a translation of Heider's result on lattices into the terminology of rings. If the "maximality" condition is omitted from [6], Theorem 5.1, the analogue of our 4.1, (1) is not obtained. For example, the rings of rational functions and differentiable functions mentioned above satisfy all the hypotheses of 4.1, (1); but they do not satisfy the remaining conditions of [6], Theorem 5.1, since they are not lattices.

Whenever it is convenient, we shall identify a ring with any ring with which it is known to be isomorphic, without further notice.

THEOREM 4.1 [3]. (1) *Let A be a commutative ring satisfying*

$(\mathfrak{R})$ *A is a semi-simple algebra over R such that for each $M \in \mathfrak{M}(A)$, we have $A/M \cong R$.*

$(\mathfrak{J})$ *A has an identity* (*denoted by* 1).

$(\mathfrak{B})$ *For each $a \in A$, there exists an $r \in R$ such that $a \leqslant r\cdot 1$.*

Then A is isomorphic to a dense subring of $C(\mathfrak{M}_K(A))$, where K is a suitable compact Hausdorff topology.

(2) *If, in addition, A satisfies*

$(\mathfrak{E})$ *Any $\mathfrak{M}$-extension of A satisfying* $(\mathfrak{R})$, $(\mathfrak{J})$ *and* $(\mathfrak{B})$ *coincides with A,*

then A is isomorphic to $C(\mathfrak{M}_K(A))$.

(3) *Conversely, if X is a compact Hausdorff space, then $C(X)$ satisfies* $(\mathfrak{R})$, $(\mathfrak{J})$, $(\mathfrak{B})$ *and* $(\mathfrak{E})$.

[3] The author is indebted to J. E. Kist for pointing out the similarity of 4.1 to the ordered algebra theorem of Stone (see, e. g., [10], Theorem 3.1). (Added in proof: Stone's theorem is more general than our 4.1; but we have since obtained a substantial improvement in 4.1, which is almost the same as Stone's theorem. The rest of the section can be correspondingly improved. In effect, we assume that A is an "almost Archimedean" ordered algebra (rather than Archimedean); in compensation, only an algebra isomorphism can be obtained – the order is not necessarily preserved. The additions and changes in the proofs are too lengthy to be indicated here.)

Proof. (1) By $(\mathfrak{R})$, A is a ring of functions from $\mathfrak{M}(A)$ to R; and by $(\mathfrak{J})$ and $(\mathfrak{B})$, each function is bounded. Thus, A is a subring of $C^*(\mathfrak{M}(A))$ (where $\mathfrak{M}(A)$ has the discrete topology). We form $\beta\mathfrak{M}(A)$, and extend each $a \in A$ to a^β, an element of $C(\beta\mathfrak{M}(A))$. Now for each $x \in \beta\mathfrak{M}(A)-\mathfrak{M}(A)$, there is an $M \in \mathfrak{M}(A)$ such that $a^\beta(x)=a^\beta(M)$ for all $a \in A$, namely, the kernel of the homomorphism $\tau\colon A \to R$ defined by $\tau(a)=a^\beta(x)$. Furthermore, there is not more than one M corresponding to x. For, if $M_1, M_2 \in \mathfrak{M}(A)$, $M_1 \neq M_2$, then there is an $a \in A$ such that $a(M_1) \neq a(M_2)$; so if M_1 corresponds to x, $a^\beta(M_2) \neq a^\beta(x)$.

We partition $\beta\mathfrak{M}(A)$ by identifying all points which are not distinguished by elements of A, *i. e.*, we stipulate that for any $x, y \in \beta\mathfrak{M}(A)$, $x \equiv y$ if and only if $a^\beta(x)=a^\beta(y)$ for all $a \in A$. It has just been shown that the points of the resulting quotient space are in one-to-one correspondence with the points of $\mathfrak{M}(A)$. It can easily be shown that the quotient topology is compact Hausdorff. From this, a compact Hausdorff topology K may be given to $\mathfrak{M}(A)$ in the natural way. Thus, A is a subalgebra of $C(\mathfrak{M}_K(A))$. Furthermore, A separates points of $\mathfrak{M}(A)$. By the Stone-Weierstrass Theorem, A is dense in $C(\mathfrak{M}_K(A))$.

(2) Let X be a compact Hausdorff space. It is well known that $C(X)$ satisfies $(\mathfrak{R})$; and it is evident that $C(X)$ satisfies $(\mathfrak{J})$ and $(\mathfrak{B})$. Thus, $C(\mathfrak{M}_K(A))$ is an $\mathfrak{M}$-extension of A satisfying $(\mathfrak{R})$, $(\mathfrak{J})$ and $(\mathfrak{B})$. From $(\mathfrak{E})$, $A=C(\mathfrak{M}_K(A))$.

(3) Let X be a compact Hausdorff space. It remains only to show that $C(X)$ satisfies $(\mathfrak{E})$. Suppose B is an $\mathfrak{M}$-extension of $C(X)$ satisfying $(\mathfrak{R})$, $(\mathfrak{J})$ and $(\mathfrak{B})$. It follows from (1) that B is a subring of $C(\mathfrak{M}_{K'}(B))$, where K' is a compact Hausdorff topology. Since the elements of B are continuous, K' is at least as strong as the Stone topology (Lemma 2.3). Thus, α is continuous from $\mathfrak{M}_{K'}(B)$ to $\mathfrak{M}_s(C(X))$. Since α is also one-to-one and onto, it is a homeomorphism.

Now X is homeomorphic to $\mathfrak{M}_s(C(X))$ under the natural mapping $p \to M^p$ $(p \in X)$. Since $\alpha\colon \mathfrak{M}_{K'}(B) \to \mathfrak{M}_s(C(X))$ is defined by $\alpha(M)=M \cap A$, $M \in \mathfrak{M}_{K'}(B)$, it is clear that X and $\mathfrak{M}_{K'}(B)$ are homeomorphic under the natural mapping. Thus, B may be *identified* with a subring of $C(X)$. Since also $B \supseteq C(X)$, we have $B=C(X)$.

Let A be a commutative ring satisfying $(\mathfrak{R})$. As in [K], § 6, we may imbed A in the ring with identity $(A; R)$. From [K], Theorems 6.1 and 6.3, it follows that $(A; R)$ also satisfies $(\mathfrak{R})$. Thus, as above, $(A; R)$ possesses an induced partial order. The symbol $\leqslant$ used in the statement of $(\mathfrak{B}')$, 4.2, signifies this partial order.

THEOREM 4.2. *Let A be a commutative ring satisfying* ($\mathfrak{R}$) *and*

($\mathfrak{B}'$) *For each $a \in A$, there exists an $s \in R$ such that $(a,0) \leqslant s(0,1)$ in $(A;R)$.*

Then A is isomorphic to a dense subring of $C_\infty(\mathfrak{M}_L(A))$, where L is a suitable locally compact Hausdorff topology.

Proof. Imbed A in $(A;R)$. As observed above, $(A;R)$ satisfies ($\mathfrak{R}$); and it is evident that $(A;R)$ satisfies ($\mathfrak{I}$). Finally, ($\mathfrak{B}'$) implies that $(A;R)$ satisfies ($\mathfrak{B}$). For let $(a,t) \in (A;R)$ be given, and let $s \in R$ be such that $(a,0) \leqslant s(0,1)$. Then $(a,t) \leqslant (s+t)(0,1)$.

We conclude that $(A;R)$ is isomorphic to a dense subring of $C(\mathfrak{M}_K(A;R))$, where K is some compact Hausdorff topology. Let L denote the relative topology on $\mathfrak{M}(A_0)$; then $\mathfrak{M}_L(A_0)$ is locally compact Hausdorff, and $\mathfrak{M}_K(A;R)$ is its one-point compactification, with $\infty = A_0$ (notation as in [K], § 6). Given $f \in C(\mathfrak{M}_K(A;R))$, the elements of $(A;R)$ which approximate f may always be chosen so as to coincide with f at ∞. (If $f(\infty)=r$, and $|(a,s)-f|<\varepsilon/2$, then $|(a,r)-f|<\varepsilon$, since $|(a,r)-(a,s)| = |r-s| < \varepsilon/2$). In particular, every element of $C_\infty(\mathfrak{M}_L(A_0))$ may be approximated by elements of $(A;R)$ that vanish at ∞, *i. e.*, by elements of A_0. Thus, A is isomorphic to a dense subring of $C_\infty(\mathfrak{M}_L(A_0))$, or, of $C_\infty(\mathfrak{M}_L(A))$.

We shall now utilize the concept of $\mathfrak{Z}$-ideal which was introduced in section 2. It is clear that every intersection of $\mathfrak{Z}$-ideals is a $\mathfrak{Z}$-ideal.

LEMMA 4.3. *If A is a subring of $C_\infty(X)$, there is a smallest $\mathfrak{Z}$-ideal $\mathfrak{Z}(A,X)$ of $C^*(X)$ such that $A \subseteq \mathfrak{Z}(A,X) \subseteq C_\infty(X)$.*

Proof. $C_\infty(X)$ is a $\mathfrak{Z}$-ideal, since it is the intersection of the free maximal ideals of $C^*(X)$ (Lemma 3.2). Therefore the desired ideal $\mathfrak{Z}(A,X)$ is simply the intersection of all the $\mathfrak{Z}$-ideals containing A.

THEOREM 4.4. (1) *Let A be a commutative ring satisfying* ($\mathfrak{R}$), ($\mathfrak{B}'$) *and*

($\mathfrak{E}'$) *Any $\mathfrak{M}$-extension of A satisfying* ($\mathfrak{R}$) *and* ($\mathfrak{B}'$) *such that $\alpha(\{\mathfrak{M}(a)\colon a \in \mathfrak{Z}(B,\mathfrak{M}(B))\}) = \{\mathfrak{M}(a)\colon a \in \mathfrak{Z}(A,\mathfrak{M}(A))\}$ coincides with A.*

Then A is isomorphic to a $\mathfrak{Z}$-ideal of $C^*(\mathfrak{M}_L(A))$ containing $C_s(\mathfrak{M}_L(A))$ and contained in $C_\infty(\mathfrak{M}_L(A))$, where L is a suitable locally compact Hausdorff topology.

(2) *Conversely, if X is a locally compact Hausdorff space, then every $\mathfrak{Z}$-ideal of $C^*(X)$ containing $C_s(X)$ and contained in $C_\infty(X)$ satisfies* ($\mathfrak{R}$), ($\mathfrak{B}'$) *and* ($\mathfrak{E}'$).

Proof. (1) Let X be a locally compact Hausdorff space, and let H be a $\mathfrak{Z}$-ideal of $C^*(X)$ such that $C_s(X) \subseteq H \subseteq C_\infty(X)$. Then H satisfies ($\mathfrak{R}$) and ($\mathfrak{B}'$). For, by Theorem 3.3, each ideal of $\mathfrak{M}_s(H)$ is the intersection of H with a fixed ideal of $C^*(X)$. Condition ($\mathfrak{R}$) then follows immediately.

Now imbed H in $(H;R)$. Since $\mathfrak{M}_s(H;R)$ is compact, every element of $(H;R)$ is bounded, so H satisfies ($\mathfrak{B}'$).

Suppose A satisfies ($\mathfrak{R}$), ($\mathfrak{B}'$) and ($\mathfrak{E}'$). By Theorem 4.2, A is isomorphic to a dense subring of $C_\infty(\mathfrak{M}_L(A))$, where L is a suitable locally compact Hausdorff topology. In view of Theorem 3.3, it follows that $\mathfrak{Z}(A,\mathfrak{M}_L(A))$ is an $\mathfrak{M}$-extension of A as in ($\mathfrak{E}'$). From ($\mathfrak{E}'$), we have $A = \mathfrak{Z}(A,\mathfrak{M}_L(A))$.

(2) Let X be a locally compact Hausdorff space, and let H be such that $C_s(X) \subseteq H \subseteq C_\infty(X)$. It remains only to show that H satisfies ($\mathfrak{E}'$). Suppose B is an $\mathfrak{M}$-extension of H as in ($\mathfrak{E}'$). It follows from 4.2 that B is isomorphic to a dense subring of $C_\infty(\mathfrak{M}_{L'}(B))$, where L' is a locally compact Hausdorff topology. Since the elements of B are continuous, L' is at least as strong as the Stone topology (Lemma 2.3). Thus, α is continuous from $\mathfrak{M}_{L'}(B)$ to $\mathfrak{M}_s(H)$.

Next, let $\mathfrak{K} \subseteq \mathfrak{M}_s(H)$ be compact. By Theorem 3.3, the Stone topology on $\mathfrak{M}_s(H)$ is locally compact Hausdorff. Thus, using the same method as in 3.1, we find an $a \in C_s(\mathfrak{M}_s(H))$ whose support contains $\mathfrak{K}$, and repeating the argument, a non-negative $b \in C_s(\mathfrak{M}_s(H))$ exceeding 1 everywhere on the support of a. Let $e = \min\{b,1\}$. Then $e \in C_s(\mathfrak{M}_s(H))$, and e is a relative identity for a. By hypothesis, $H \supseteq C_s(\mathfrak{M}_s(H))$; so $a, e \in H$. We now view a and e as functions on $\mathfrak{M}_{L'}(B)$. Since $ae = a$ in B also, it is clear that $e(M)=1$ for each $M \in \mathfrak{M}_{L'}(B)$ in the support of a. Thus, a has a support which is a closed subset of $\mathfrak{D} = \{M \in \mathfrak{M}_{L'}(B)\colon e(M)=1\}$. The set $\{x \in \beta\mathfrak{M}(B;R)\colon (e,0)^\beta(x)=1\}$ is compact, so its continuous image in the quotient space which yields the one-point compactification of $\mathfrak{M}_{L'}(B)$ is also compact. But this latter set, since it does not contain the point at infinity, coincides with $\mathfrak{D}$. It follows that $a \in C_s(\mathfrak{M}_{L'}(B))$. Since α is continuous and $\mathfrak{K}$ is closed, $\alpha^{-1}(\mathfrak{K})$ is compact. Thus, we have shown that α^{-1} takes compact sets into compact sets.

Finally, we extend α to a mapping $\alpha^*\colon \mathfrak{M}_*(B) \to \mathfrak{M}_*(H)$, where $\mathfrak{M}_*(B) = \mathfrak{M}_{L'}(B) \cup \{\infty_B\}$, $\mathfrak{M}_*(H) = \mathfrak{M}_s(H) \cup \{\infty_H\}$ are the one-point compactifications of $\mathfrak{M}_{L'}(B)$, $\mathfrak{M}_s(H)$, respectively, by stipulating that $\alpha^*(\infty_B) = \infty_H$. If $\mathfrak{V} \subseteq \mathfrak{M}_*(H)$ is any open set containing ∞_H, then $\mathfrak{M}_*(H) - \mathfrak{V}$ is compact; so $\alpha^{*-1}(\mathfrak{M}_*(H)-\mathfrak{V}) = \alpha^{-1}(\mathfrak{M}_*(H)-\mathfrak{V})$ is compact, and hence is the complement of an open set containing ∞_B. It follows that α^* is continuous, and hence a homeomorphism. Thus, α is a homeomorphism from $\mathfrak{M}_{L'}(B)$ to $\mathfrak{M}_s(H)$.

By Theorem 3.3, X is homeomorphic to $\mathfrak{M}_s(H)$ under the mapping $p \to M^{*p} \cap H$, $(p \in X)$. Hence X and $\mathfrak{M}_{L'}(B)$ are homeomorphic under the natural mapping. Thus, $\mathfrak{Z}(B,\mathfrak{M}_{L'}(B))$ may be identified with $\mathfrak{Z}(H,X)$, *i. e.*, with H. Since $B \subseteq \mathfrak{Z}(B,\mathfrak{M}_{L'}(B))$, and $B \supseteq H$, we have $B = H$.

COROLLARY 4.5 (4). (1) *Let A be a commutative ring satisfying* ($\mathfrak{R}$), ($\mathfrak{B}'$) *and*

($\mathfrak{E}_\infty$) *Any $\mathfrak{M}$-extension of A satisfying* ($\mathfrak{R}$) *and* ($\mathfrak{B}'$) *coincides with A.*

Then A is isomorphic to $C_\infty(\mathfrak{M}_L(A))$, where L is a suitable locally compact Hausdorff topology.

(2) *Conversely, if X is a locally compact Hausdorff space, then $C_\infty(X)$ satisfies* ($\mathfrak{R}$), ($\mathfrak{B}'$) *and* ($\mathfrak{E}_\infty$).

Proof. The only statement which might require a remark is that if X is a locally compact Hausdorff space, then $C_\infty(X)$ satisfies ($\mathfrak{E}_\infty$). A proof of this can be given by using the proof of 4.4, (2) in a manner similar to 4.6, (2) below.

The final theorem is a direct generalization of Theorem 4.1, (2) and (3).

THEOREM 4.6 (5). (1) *Let A be a commutative ring satisfying* ($\mathfrak{R}$) *and*

($\mathfrak{I}_s$) *Every element of A has a relative identity.*

($\mathfrak{B}_s$) *For each $a \in A$, there exists an $r \in R$ such that $a \leqslant re$, where e is a suitable relative identity for a.*

($\mathfrak{E}_s$) *Any $\mathfrak{M}$-extension of A satisfying* ($\mathfrak{R}$), ($\mathfrak{I}_s$) *and* ($\mathfrak{B}_s$) *coincides with A.*

Then A is isomorphic to $C_s(\mathfrak{M}_L(A))$, where L is a suitable locally compact Hausdorff topology.

(2) *Conversely, if X is a locally compact Hausdorff space, then $C_s(X)$ satisfies* ($\mathfrak{R}$), ($\mathfrak{I}_s$), ($\mathfrak{B}_s$) *and* ($\mathfrak{E}_s$).

Proof. (1) We modify the proof of Theorem 4.2 as follows: By ($\mathfrak{I}_s$) and ($\mathfrak{B}_s$), each element of A_0 in $(A; R)$ is a bounded function. Now [K], Theorem 6.3, shows that each element of $(A; R)$ is the sum of a function in A_0 and a constant function, and hence is bounded. Since condition ($\mathfrak{B}$) of Theorem 4.1, (1) is used only to obtain boundedness, we may again apply 4.1, (1) to $(A; R)$. Thus, from 4.2, we conclude that A is isomorphic to a dense subring of $C_\infty(\mathfrak{M}_L(A))$, where L is a locally compact Hausdorff topology. We now show that, in fact, $A \subseteq C_s(\mathfrak{M}_L(A))$. Let $a \in A$, and let $e \in A$ be a relative identity for a. Since $e \in C_\infty(\mathfrak{M}_L(A))$, $\{M \in \mathfrak{M}_L(A): |e(M)| \geqslant 1\}$ is compact. The support of a is a closed subset of $\{M \in \mathfrak{M}_L(A): |e(M)| \geqslant 1\}$. Hence $a \in C_s(\mathfrak{M}_L(A))$.

Now if X is a locally compact Hausdorff space, then $C_s(X)$ satisfies ($\mathfrak{R}$), ($\mathfrak{I}_s$) and ($\mathfrak{B}_s$). For, ($\mathfrak{R}$) was shown to hold in 4.4; ($\mathfrak{I}_s$) follows from a construction like that used in 4.4, (2); and ($\mathfrak{B}_s$) is evident. Thus, in view of Theorem 3.3, it follows that $C_s(\mathfrak{M}_L(A))$ is an $\mathfrak{M}$-extension of A satisfying ($\mathfrak{R}$), ($\mathfrak{I}_s$) and ($\mathfrak{B}_s$). From ($\mathfrak{E}_s$), we have $A = C_s(\mathfrak{M}_L(A))$.

(2) Let X be a locally compact Hausdorff space. It remains only to show that $C_s(X)$ satisfies ($\mathfrak{E}_s$). Suppose B is an $\mathfrak{M}$-extension of $C_s(X)$ satisfying ($\mathfrak{R}$), ($\mathfrak{I}_s$) and ($\mathfrak{B}_s$). It follows from the preceding part of the proof that B is a subring of $C_s(\mathfrak{M}_{L'}(B))$, where L' is a locally compact Hausdorff topology. As in the proof of 4.4, (2), X is homeomorphic to $\mathfrak{M}_{L'}(B)$. Thus $B = C_s(X)$.

(4) For a characterization of $C_\infty(X)$ as a ring and lattice, see [1], Theorem 5.

(5) A similar characterization of $C_s(X)$, using its vector lattice properties, has been obtained by J. E. Kist.

5. βF-points (6). The set U is called an X-*neighborhood* of $p \in \beta X$ if U has the form $X \cap \Omega$, where Ω is a neighborhood of p in βX. Thus when $p \in X$, the set of X-neighborhoods of p coincides with the set of neighborhoods of p in X. For convenience, when U is an X-neighborhood of p, we shall refer to the set $U - \{p\}$ as a *deleted X-neighborhood of p* — even when $p \in \beta X - X$, so that $U - \{p\} = U$.

Let $f \in C(X)$, and Y be a subset of X. If a statement about $f(x)$ is true for each $x \in Y$, we shall say the statement is true for f *on* Y.

We recall that for any $f \in C(X)$, $\hat{f}$ denotes the extension to βX of f (as a function into the one-point compactification of R). For any maximal ideal M^p of $C(X)$, $f \in M^p$ implies $\hat{f}(p) = 0$. But the converse is false in general, as can be seen from Lemma 2.1.

In the following definition, we generalize several concepts discussed in [4].

DEFINITION 5.1. Let $p \in \beta X$. We define p to be a:

(1) βF-*point* (with respect to X), if for each $f \in C(X)$ such that $\hat{f}(p) = 0$, there is an X-neighborhood of p on which one of the relations $f \geqslant 0$, $f \leqslant 0$ holds;

(2) $\beta P'$-*point* (with respect to X), if for each $f \in C(X)$ such that $\hat{f}(p) = 0$, there is a deleted X-neighborhood of p on which one of the relations $f > 0$, $f < 0$, $f = 0$ holds;

(3) βP-*point* (with respect to X), if for each $f \in C(X)$ such that $\hat{f}(p) = 0$, there is an X-neighborhood of p on which $f = 0$.

We observe that if $\hat{f}(p) = 0$, then $\hat{f}$ is continuous and finite-valued in some neighborhood of p in βX. Thus, since X is dense in βX, each of the conclusions in 5.1, (1) and (3), is equivalent to that obtained by replacing "X-neighborhood" with "neighborhood in βX", and f with $\hat{f}$. However, this is not the case for (2).

(6) It is interesting to compare this section with the results obtained in [7] for the ring of entire functions. (Added in proof: We hawe since shown that the conclusions of 5.6 hold for any prime ideal of $C(X)$, and that all conclusions of 5.8, 5.11 and 5.13 hold when p is *any* βF-point. The proofs will be given elsewhere.)

A point $p \in X$ is a βP-point (resp. $\beta P'$-point) if and only if it is a P-point (resp. P'-point) as defined in [3], 4.1 (resp. [4], 8.1).

When $p \in vX - X$, the concept of βP-point coincides with that of "P-point with respect to X" given in [3], § 4; but when $p \notin vX$, the concept of βP-point is more restrictive. For, whenever $p \notin vX$, there is an $f \notin M^p$ such that $\hat{f}(p) = 0$ (cf. [8], Theorem 45). Now there is a restriction placed on f if p is a βP-point, but not if p is a P-point with respect to X. As an example, let X be the discrete countable space $\{e_1, e_2, \ldots, e_n, \ldots\}$; then $vX = X$. Let p be any point in $\beta X - X$. Since X is a P-space, p is a P-point with respect to X (cf. [3], Theorem 5.3, (4)). But p cannot be a βP-point, since the function f defined by $f(e_n) = 1/n$ $(n = 1, 2, \ldots)$, vanishes on no X-neighborhood of p, although $\hat{f}(p) = 0$.

If $p \notin X$ is a $\beta P'$-point, then p is a P-point with respect to X. For, if p is a $\beta P'$-point, every $f \in C(X)$ satisfying $\hat{f}(p) = 0$ and such that neither $f > 0$, $f < 0$ holds on any X-neighborhood of p, must vanish on some X-neighborhood of p. In particular, each $f \in M^p$ satisfies these conditions (cf. 2.1).

Now suppose $p \notin X$ is a βF-point, and a P-point with respect to X. Then every $f \in C(X)$ satisfying $\hat{f}(p) = 0$ either (1) belongs to M^p, so that, by the second condition, it vanishes on an X-neighborhood of p; or (2) is non-zero on some X-neighborhood of p, so that, by the first condition, it is positive or negative on some X-neighborhood of p. Thus, p is a $\beta P'$-point.

THEOREM 5.2. *Let $p \in \beta X$. Then p is a βF-point if and only if the ideal N^p is prime.*

The proof is almost identical with the last part of the proof of [4], Theorem 2.5.

Since N^p coincides with the intersection of the prime ideals contained in M^p ([4], Theorem 1.4), 5.2 justifies the formulation of the hypothesis that p is a βF-point which was given in the introduction. Theorem 5.2 and [4], Theorem 2.5 together show that every point of βX is a βF-point if and only if X is an F-space ([4], Definition 2.1).

We next relate the concept of βF-point to zero-sets of functions in $C(X)$.

THEOREM 5.3. *Let $p \in \beta X$. Then p is a βF-point if and only if for each pair of functions $f, g \in C(X)$ satisfying $\hat{f}(p) = \hat{g}(p) = 0$, there is an X-neighborhood U of p such that at least one of the relations $(Z(f) \cap U) \supseteq (Z(g) \cap U)$, $(Z(f) \cap U) \subseteq (Z(g) \cap U)$ is valid.*

Proof. Let p be a βF-point, and let $f, g \in C(X)$ satisfy $\hat{f}(p) = \hat{g}(p) = 0$. Set $k = |f| - |g|$. Then $\hat{k}(p) = 0$, so there is an X-neighborhood U of p such that either $k \geq 0$ or $k \leq 0$ on U. If $k \geq 0$ on U, then for any $x \in U$, $f(x) = 0$ implies $g(x) = 0$, *i. e.*, $(Z(f) \cap U) \subseteq (Z(g) \cap U)$; and similarly if $k \leq 0$ on U.

Conversely, if p is not a βF-point, let $f \in C(X)$ satisfy $\hat{f}(p) = 0$ and change sign in every X-neighborhood of p. Define $g = \max\{f, 0\}$, $h = \min\{f, 0\}$. For every X-neighborhood U of p, there exist $x, y \in U$ such that $g(x) \neq 0$, $h(y) \neq 0$, whence $g(y) = 0$, $h(x) = 0$. Thus, neither $(Z(g) \cap U) \supseteq (Z(h) \cap U)$ nor $(Z(g) \cap U) \subseteq (Z(h) \cap U)$ is valid.

Let $p \in \beta X$. If p is a non-isolated point of X, we define M'^p to be the set of all $f \in C(X)$ such that $Z(f)$ meets every deleted neighborhood of p. Otherwise, we set $M'^p = M^p$. Evidently, $M^p \supseteq M'^p \supseteq N^p$.

THEOREM 5.4. *If p is a βF-point, then M'^p is a prime ideal.*

Proof. Since M^p is a prime ideal, it suffices to consider the case where p is a non-isolated point of X. Let $f, g \in M'^p$. By 5.3, there is an X-neighborhood U of p such that (say) $(Z(f) \cap U) \supseteq (Z(g) \cap U)$. Let V' be an arbitrary deleted X-neighborhood of p. Since $U \cap V'$ is a deleted X-neighborhood of p, there is a point x in $Z(g) \cap U \cap V'$. Now we have also $x \in Z(f) \cap U \cap V'$. But $Z(f-g) \supseteq Z(f) \cap Z(g)$; so $x \in Z(f-g) \cap U \cap V'$. Hence $Z(f-g)$ meets V'. It follows that $f - g \in M'^p$.

It is clear that M'^p is closed under multiplication by arbitrary elements of $C(X)$, and that the complement of M'^p is a multiplicative system. Hence M'^p is a prime ideal.

We note that if p is a $\beta P'$-point, then $M'^p = N^p$; and if p is a βF-point such that $M'^p = N^p$, then p is a $\beta P'$-point.

When p is not a βF-point, M'^p need not even be an ideal. For example, let $X = R$, $p = 0$, and let $f, g \in C(X)$ be defined by $f(x) = x \sin^2 1/x$, $x \neq 0$, $f(0) = 0$, and $g(x) = x \cos^2 1/x$, $x \neq 0$, $g(0) = 0$. Then $f, g \in M'^p$, but $f + g \notin M'^p$. On the other hand, M'^p can be an ideal at a non-βF-point. For example, let X be a linearly ordered space, $p \in X$ a point with character c_{10} (see, *e. g.*, [3], § 6). Then it is easily seen that p is not a βF-point. But $M'^p = M^p$, since every continuous function which is zero at p is zero on a whole interval to the left of p; so M'^p is an ideal.

We point out next that when p is a βF-point but not a $\beta P'$-point (whence $N^p \neq M'^p$), then either $M'^p = M^p$ or $M'^p \neq M^p$ can occur. In both our examples, we shall make use of the space $E = N \cup \{e\}$ defined as follows ([4], 8.5): $N = \{e_1, e_2, \ldots\}$ is the denumerable discrete space, and $e \in \beta N - N$. Thus every e_n is an isolated point, while deleted neighborhoods of e are the members of some free ultra-filter on N (*i. e.*, maximal filter on N with total intersection void). It follows that e is a $\beta P'$-point of E ([4], 8.6).

For the first case, let X be the space described in [4], 8.10: $X=E\cup W(\omega_1)$, where all points of $N\cup W(\omega_1)$ are isloated, and a neighborhood of e in X is the union of a neighborhood of e in E with the complement in $W(\omega_1)$ of a countable subset of $W(\omega_1)$. Then e is a βF-point but not a $\beta P'$-point, and $M'^e=M^e$.

The second case is illustrated by the following example (due to L. Gillman).

Example 5.5. For $n=1,2,\ldots$, let $E_n=\{e_{n1},e_{n2},\ldots,e_n\}$ be a copy of E (with e_n corresponding in E_n to e in E), and $X=\left(\bigcup_n E_n\right)\cup E=\left(\bigcup_n E_n\right)\cup\{e\}$, with the following topology: each e_{nm} is isolated; the neighborhoods of e_n are its neighborhoods in E_n; while each neighborhood of e is the union of a neighborhood U of e in E with a neighborhood U_n of e_n in E_n for each $e_n\in U$.

Then e is a βF-point. For consider any $f\in C(X)$ satisfying $f(e)=0$. If there is a deleted neighborhood V of e such that $f>0$ or $f<0$ on $V\cap E$, then the defining property obviously holds for f. If no such neighborhood exists, then, since e is a $\beta P'$-point of E, there is a neighborhood W of e such that $f=0$ on $W\cap E$. Set

$W_1=\{e_n\in W\colon f>0$ on some deleted neighborhood of $e_n\}$,

$W_2=\{e_n\in W\colon f<0$ on some deleted neighborhood of $e_n\}$,

$W_3=\{e_n\in W\colon f=0$ on some deleted neighborhood of $e_n\}$.

Precisely one of W_1, W_2, W_3 is a deleted neighborhood of e in E. The union of this set with suitable neighborhoods of its elements in the associated E_n's is a neighborhood of e in X on which $f\geqslant 0$ or $f\leqslant 0$.

The function $g\in C(X)$ defined by $g(e_n)=g(e)=0$, $g(e_{nm})=1/m$ $(m,n=1,2,\ldots)$ shows that e is not a $\beta P'$-point.

Finally, $M'^e\neq M^e$, as shown by the function $h\in C(X)$ defined as follows: $h(e_n)=h(e_{nm})=1/n$ $(m,n=1,2,\ldots)$, $h(e)=0$.

We now investigate the properties of the quotient-rings $C(X)/N^p$ and $C(X)/M'^p$.

Theorem 5.6. *Let p be a βF-point.* (a) *The ring $C(X)/N^p$ is an ordered integral domain containing R, in which the image of M^p forms the unique maximal ideal; and $C(X)/N^p$ has infinitely large elements if and only if $p\notin vX$.*

(b) *The corresponding statement for $C(X)/M'^p$ also holds.*

Proof. If $f\in C(X)$, $g\in N^p$, and $0\leqslant f\leqslant g$, then, trivially, $f\in N^p$. For any $h\in C(X)$, $h^2-|h|^2=0\in N^p$. By Theorem 5.2, N^p is prime; so at least one of the congruences $h\equiv|h|\ (\mathrm{mod}\ N^p)$, $h\equiv-|h|\ (\mathrm{mod}\ N^p)$ is valid. Thus, by [2], Theorem 4.4, $C(X)/N^p$ is an ordered integral domain. Explicitly, the image of $f\in C(X)$ in $C(X)/N^p$ is positive if $f\geqslant 0$ on some X-neighborhood of p (*i. e.*, $f\equiv|f|\ (\mathrm{mod}\ N^p)$), but $f=0$ on no X-neighborhood of p (*i. e.*, $f\not\equiv 0\ (\mathrm{mod}\ N^p)$). Finally, it is clear that the functions which are constant in some X-neighborhood of p map into a subset of $C(X)/N^p$ which is isomorphic to R.

Since M^p is the only maximal ideal of $C(X)$ containing N^p, it is evident that $C(X)/N^p$ has a unique maximal ideal, namely, the image of M^p.

If $p\in vX$, then for each $f\in C(X)$, there is an $r\in R$ such that $\hat{f}(p)=r$. Thus, the image of f differs from the element corresponding to r by at most an infinitely small element. Conversely, if $p\notin vX$, there is a $g\in C(X)$, $g\geqslant 0$, such that $\hat{g}(p)=\infty$. Hence g is unbounded on every X-neighborhood of p; so the image of g is an infinitely large element.

The proof for $C(X)/M'^p$ is similar.

A commutative ring A with identity is a *valuation ring* if for any $a,b\in A$, either a divides b or b divides a.

Theorem 5.7. *If p is a βF-point, then $C(X)/M'^p$ is a valuation ring. If p is a $\beta P'$-point, or if X is an F-space, then $C(X)/N^p$ is a valuation ring.*

Proof. It is easily seen that every element of $C(X)/N^p$ which is not infinitely small has an inverse. The same statement then follows for its homomorphic image $C(X)/M'^p$. Thus, to show that either $C(X)/N^p$ or $C(X)/M'^p$ is a valuation ring, it suffices to consider an arbitrary pair of distinct infinitely small elements.

Let p be a βF-point, and let $\gamma,\delta\in C(X)/M'^p$ be infinitely small positive elements such that $\gamma<\delta$. It will be shown that δ divides γ.

When p is isolated or $p\notin X$, then $M'^p=M^p$, so the desired conclusion is obvious. We therefore suppose that p is a non-isolated point of X.

Let $f,g\in C(X)$ map into γ, δ, respectively. We show first that it may be assumed that $Z(f)=Z(g)=\{p\}$, and that $0\leqslant f\leqslant g\leqslant 1$ on X.

Since $f\geqslant 0$ on some X-neighborhood of p, we may suppose that $0\leqslant f\leqslant 1$ on X, replacing f by $\min\{|f|,1\}$ if necessary. Now $f>0$ on some deleted X-neighborhood of p, so $Z(f)-\{p\}$ is a closed subset of X. By complete regularity, there is an $f_1\in C^*(X)$ such that $0\leqslant f_1\leqslant 2$, $f_1(x)=0$ for $x\in Z(f)-\{p\}$, and $f_1(p)=2$. Set $h=f+f_1$. Then $0\leqslant h\leqslant 3$, $Z(h)=Z(f)-\{p\}$, and $h(p)=2$. Let $k=1-\min\{h,1\}$. Then $k(x)=1$ when $x\in Z(h)$, and $k(x)=0$ when $h(x)\geqslant 1$; so $Z(k)$ is a closed X-neighborhood of p. Thus, $Z(f+k)=\{p\}$, and $0\leqslant f+k\leqslant 1$; moreover, $f+k$ is congruent to f modulo N^p, and hence modulo M'^p.

Similarly, we may suppose that $0\leqslant g\leqslant 1$, and that $Z(g)=\{p\}$. Finally, since $\gamma<\delta$ in $C(X)/M'^p$, we have $f\leqslant g$ on some X-neighborhood of p. Thus, $\min\{f,g\}\equiv f\ (\mathrm{mod}\ M'^p)$, so replacing f by $\min\{f,g\}$ if necessary, we may assume that $f\leqslant g$ everywhere.

The remainder of this part of the proof is a simple modification of the argument used in [4], Theorem 2.3, III. On $X-\{p\}$, we define:

(1) $d=f/g$.

For every real r, define a function $\mu_r \in C^*(X)$ by

(2) $\mu_r(x)=f(x)-rg(x)$.

Obviously, if $r>s$, then $\mu_r \leqslant \mu_s$ (since $g \geqslant 0$). Furthermore, $\mu_r(p)=0$ for every real r.

We have $\mu_0=f\geqslant 0$. Now there is a deleted neighborhood of p not meeting $Z(f-g)$. Thus, since $f-g\leqslant 0$, for each neighborhood U of p, there is a $y \in U$ such that $\mu_1(y)=f(y)-g(y)<0$. We may put

(3) $d(p)=\sup\{r\colon \mu_r\geqslant 0$ on some neighborhood of $p\}$.

It must be shown that d is continuous at p. By (3), for every $r>d(p)$, and for every neighborhood U of p, there is an $x \in U$ such that $\mu_r(x)<0$. Since $\mu_r(p)=0$, $\mu_r\leqslant 0$ on some neighborhood of p. From this point we may follow the proof of [4], 2.3, exactly, changing only the notation. We then have: For every $\varepsilon>0$, there is a neighborhood U of p such that $|d(x)-d(p)|\leqslant\varepsilon$, $x \in U-\{p\}$. Thus, $d \in C(X)$. Clearly $f=dg$, so $f\equiv dg$ (mod M'^p). This concludes the proof that $C(X)/M'^p$ is a valuation ring.

If p is a $\beta P'$-point, then $M'^p=N^p$, so $C(X)/N^p$ is a valuation ring by the result just established.

Now suppose X is an F-space. Given a pair of distinct infinitely small positive elements of $C(X)/N^p$, let a, b be functions in $C(X)$ which map into these elements. It is easily seen that we may assume that $0\leqslant a\leqslant b\leqslant 1$. Clearly $Z(a)\supseteq Z(b)$. Set $c=a/b$ on $X-Z(b)$. Then $c \in C^*(X-Z(b))$. By [4], Theorem 2.6, c has a continuous extension $c' \in C^*(X)$. Clearly $a=c'b$, so $a\equiv c'b$ (mod N^p). Thus, $C(X)/N^p$ is a valuation ring.

Corollary 5.8. *If p is a βF-point, then the prime ideals containing M'^p form a chain. If p is a $\beta P'$-point, or if X is an F-space, then the set of all prime ideals contained in M^p form a chain.*

Proof. It is easily seen that the set of all ideals in a valuation ring form a chain. The sets in question are the sets of inverse images, under the natural mapping, of the prime ideals in $C(X)/M'^p$, $C(X)/N^p$, respectively (cf. also [2], Theorem 3.10).

We shall show next that the conclusions of the second parts of 5.7 and 5.8 never hold when p fails to be a βF-point.

Theorem 5.9. *If p is not a βF-point, then the prime ideals contained in M^p do not form a chain.*

Proof. By Theorem 5.2, if p is not a βF-point, then N^p is not prime. From [4], Theorem 1.4, we have that the intersection of the prime ideals contained in M^p is not a prime ideal. But then the prime ideals contained in M^p do not form a chain (cf. [2], Theorem 3.9).

Corollary 5.10. *If p is not a βF-point, then $C(X)/N^p$ is not a valuation ring.*

Proof. By 5.9 there are incomparable prime ideals contained in M^p (and containing N^p); these map into incomparable prime ideals in $C(X)/N^p$. As already noted, the ideals of a valuation ring form a chain.

The following alternative proof of 5.9 seems interesting. By 5.3, if p is not a βF-point, there are functions $g,h \in C(X)$ satisfying $\hat{g}(p)=\hat{h}(p)=0$, and such that $Z(g)$, $Z(h)$ are incomparable in every X-neighborhood of p. We show that $\{g,g^2,\ldots,g^n,\ldots\}\cap(N^p,h)$ is empty. Suppose not; then there exist $d \in N^p$, $k \in C(X)$ and a positive integer m such that $g^m=d+kh$. Let V be an X-neighborhood of p on which $d=0$. Then $Z(g)\cap V=Z(g^m)\cap V=([Z(k)\cup Z(h)]\cap V)\supseteq(Z(h)\cap V)$, a contradiction. Similarly, $\{h,h^2,\ldots,h^n,\ldots\}\cap(N^p,g)$ is empty.

By [15], Lemma 2, p. 105, there are prime ideals P,Q containing (N^p,g), (N^p,h), respectively, and disjoint from the multiplicative systems $\{h,h^2,\ldots,h^n,\ldots\}$, $\{g,g^2,\ldots,g^n,\ldots\}$, respectively. Since P and Q contain N^p, they are contained in M^p; and they are clearly incomparable.

This method can be used to obtain a prime ideal contained properly in M^p whenever p is not a P-point with respect to X (cf. [3], Theorem 3.5). That is, we choose a function $f \in M^p-N^p$, and let P be an ideal which is maximal in the class of ideals containing N^p and disjoint from the multiplicative system $\{f,f^1,\ldots,f^n,\ldots\}$. Now P is never the entire complement of $\{f,f^2,\ldots,f^n,\ldots\}$ in M^p. For, let $g=\max\{f,0\}$, $h=\min\{f,0\}$; then the relation $g+h=f\notin P$ implies that not both g and h are in P, while $gh=0 \in P$ implies that at least one of the elements g,h is in P. Thus, exactly one of g, h is in P. From this, we conclude that $|f|=g-h\notin P$. Thus $|f|^{1/k}\notin P$, so that $|f|^{1/k}\in M^p-(P\cup\{f,f^2,\ldots,f^n,\ldots\})$, $(k=2,3,\ldots)$.

Theorem 5.11. *If p is a βF-point, and I is a proper $\mathfrak{Z}$-ideal of $C(X)$ containing M'^p, then I is a prime ideal. If p is a $\beta P'$-point or if X is an F-space, and I is a $\mathfrak{Z}$-ideal of $C(X)$ containing N^p, then I is a prime ideal.*

Proof. By Lemma 2.2, I is an intersection of proper prime ideals. From 5.8, the prime ideals containing I form a chain. Thus, any intersection of prime ideals containing I is a prime ideal.

Finally, we consider a type ofi deal which may be viewed as a generalization of N^p.

Lemma 5.12. *Let h be a fixed element of a maximal ideal M^p of $C(X)$, and let I be the set of all f in M^p such that $(Z(f)\cap U)\supseteq(Z(h)\cap U)$, where U is an X-neighborhood of p (depending on f). Then I is an ideal of $C(X)$.*

Proof. Given $f, g \in I$, choose X-neighborhoods U, V of p satisfying $\overline{(Z(f) \cap U)} \supseteq \overline{(Z(h) \cap U)}$, $\overline{(Z(g) \cap V)} \supseteq \overline{(Z(h) \cap V)}$. Then $\overline{(Z(f-g) \cap U \cap V)} \supseteq \overline{(Z(f) \cap Z(g) \cap U \cap V)} \supseteq \overline{(Z(h) \cap U \cap V)}$; so $f-g \in I$. Since it is clear that I is closed under multiplication by arbitrary elements of $C(X)$, I is an ideal.

THEOREM 5.13. *If p is a $\beta P'$-point or if X is an F-space, then the ideal I defined in 5.12 is prime.*

Proof. Since I is clearly a $\mathfrak{Z}$-ideal containing N^p, this follows from 5.11.

6. P-spaces, and prime ideals of M^p. We take up first some questions related to [K], § 5. An example was given there of a ring possessing an ideal of an ideal which fails to be an ideal of the whole ring. We give now an example of a ring of continuous functions having the same property. Let $X=R$, and let i be the identity function: $i(x)=x$, for all $x \in X$. Let I be the ideal $\{gi: g \in M^0\}$ of $C(X)$; and let J be the ideal $\{gi^2+ni^2: g \in M^0,\ n \text{ an integer}\}$ of I. Then $i^2 \in J$, but $i^2/2 \notin J$; so J is not an ideal of $C(X)$.

The theorem which follows shows that for rings of continuous functions, this is the usual situation.

THEOREM 6.1. *Let I be a proper ideal of $C(X)$, and let J be a proper ideal of I. Then J is invariably an ideal of $C(X)$ if and only if X is a P-space. In particular, if $p \in X$ is not a P-point, then there is an ideal of an ideal of $C(X)$, contained in M^p, which fails to be an ideal of $C(X)$.*

Proof. If X is a P-space, then $C(X)$ is a regular ring ([3], Theorem 5.3). Let $j \in J$ be given, and let $a \in C(X)$ be arbitrary. There is a $b \in C(X)$ such that $j^2b=j$. Then $jba \in I$; so $ja=j(jba) \in J$. Thus, J is an ideal of $C(X)$.

Conversely, suppose X is not a P-space. Let p be a point of X which is not a P-point, and let $f \in M^p-N^p$. Let I be the ideal $\{gf: g \in M^p\}$ of $C(X)$; and let J be the ideal $\{gf^2+nf^2: g \in M^p,\ n \text{ an integer}\}$ of I. Then $f^2 \in J$; but $f^2/2 \notin J$, since a continuous function which vanishes at p cannot assume the value 1/2 in every neighborhood of p. Thus, J is not an ideal of $C(X)$.

It was also noted in [K], § 5 that a prime ideal of an ideal need not be prime in the whole ring. Now if P and Q are prime ideals of a ring A, $P \cap Q$ is a prime ideal of P; and if P and Q are incomparable, $P \cap Q$ is not prime in A. Thus, an example in function rings may be obtained from any $C(X)$, where X has more than one point; if $q \neq p$, the ideal $M^p \cap M^q$ is a prime ideal of M^p which is not prime in $C(X)$.

By [3], Lemma 3.2, if a prime ideal P of M^p is to be prime in $C(X)$, it is necessary that P contain N^p. In the next theorem, we see that this condition is also sufficient.

THEOREM 6.2. *Let p be any point in βX. Then the prime ideals of $C(X)$ containing N^p coincide with the prime ideals of M^p containing N^p.*

Proof. If P is a prime ideal of $C(X)$ containing N^p, it is contained in M^p; and it is easily seen to be a prime ideal of M^p.

Now let P be a prime ideal of M^p containing N^p. We assume that P is proper, the case $P=M^p$ being trivial. By [K], Theorem 5.1, the set $Q=\{f \in C(X): fM^p \subseteq P\}$ is a prime ideal of $C(X)$ such that $P=Q \cap M^p$. Since $Q \supseteq N^p$, we have $Q \subseteq M^p$ ([3], Theorem 3.3). Hence $P=Q$; so P is a prime ideal of $C(X)$.

We close with a result about subfamilies of $\mathcal{Z}(C(X))$, the family of zero-sets of $C(X)$. By a *$\mathcal{Z}$-filter on X*, we mean a non-empty subfamily $\mathcal{F}$ of $\mathcal{Z}(C(X))$ having the finite intersection property, and such that $A \in \mathcal{F}$, $B \in \mathcal{Z}(C(X))$, $A \subseteq B$ imply $B \in \mathcal{F}$. It is well known that there is a natural correspondence between the proper ideals I of $C(X)$ and the $\mathcal{Z}$-filters $\mathcal{F}$ on X, namely, $I \to \mathcal{F}=\mathcal{Z}(I)$ ([8], Theorem 36). We give first an example to show that in general this correspondence is not one-to-one. Let $X=R$, let i be the identity function: $i(x)=x$, for all $x \in X$, and let $I=(i)$, $J=(i^2)$. Clearly $\mathcal{Z}(I)=\mathcal{Z}(J)=\mathcal{Z}(M^0)$. But $i \notin J$, so $I \neq J$.

Again, our theorem on this question shows that the example illustrates the normal situation.

THEOREM 6.3. *The proper ideals of $C(X)$ and the $\mathcal{Z}$-filters on X are in one-to-one correspondence if and only if X is a P-space. In particular, if $p \in X$ is not a P-point, there are distinct ideals contained in M^p having the same $\mathcal{Z}$-filter on X.*

Proof. Let X be a P-space, and let I, J be proper ideals of $C(X)$ such that $\mathcal{Z}(I)=\mathcal{Z}(J)$. Then $\Delta(I)=\Delta(J)$ (see section 2). Now in a P-space, every ideal is the intersection of all the maximal ideals containing it ([3], Theorem 5.3). Hence $I=\bigcap_{p\in\Delta(I)} M^p=\bigcap_{p\in\Delta(J)} M^p=J$. It follows that the correspondence between ideals of $C(X)$ and $\mathcal{Z}$-filters on X is one-to-one.

Conversely, suppose X is not a P-space; let p be a point of X which is not a P-point, let f be any function in M^p-N^p, and let $I=(f)$, $J=(f^2)$. Since $Z(f)=Z(f^2)$, and in view of the relation $Z(gh)=Z(g) \cup Z(h)$, we have $\mathcal{Z}(I)=\mathcal{Z}(J)$. But $f \notin J$, since a continuous function g cannot satisfy $g(x)f(x)=1$ for values of x arbitrarily near p; so $I \neq J$. It follows that the correspondence between ideals of $C(X)$ and $\mathcal{Z}$-filters on X is not one-to-one.

When X is a P-space, it is clear that the correspondence discussed in Theorem 6.3 is a lattice isomorphism.

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Sur l'unicohérence, les homéomorphies locales et les continus irréductibles

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§ 1. Introduction. Soient X et Y deux espaces métriques compacts et f une fonction dont la variable x parcourt X et dont Y est l'ensemble des valeurs. Appelons f *homéomorphie locale au sens large* lorsqu'il existe, pour tout point $x \in X$, un entourage (ensemble ouvert contenant ce point) U_x tel que la fonction partielle $f|U_x$ est une homéomorphie. Lorsqu'il en existe, pour tout $x \in X$, dont les images $f(U_x)$ sont en outre ouverts (donc des entourages ouverts de $f(x)$ dans Y), la fonction f est dite (voir [2], p. 35) *homéomorphie locale* tout court. Appelons enfin la fonction f *recouvrement* de Y par X (voir le livre [11] de Pontriagin, p. 352, définition 45 ([1])), lorsqu'il existe, pour tout point $y \in Y$, un entourage V_y tel que l'ensemble $f^{-1}(V_y)$, c'est-à-dire celui des x pour lesquels $f(x) \in V_y$, est somme d'une famille d'ensembles ouverts disjoints,

$$f^{-1}(V_y)=\sum_i U_i,$$

sur lesquels les fonctions partielles $f|U$ sont des homéomorphies et $f(U_i)=V_y$.

Les fonctions de ces trois classes sont donc continues par définition.

Toute homéomorphie locale en est trivialement une au sens large, mais pas réciproquement, même lorsque X et Y sont compacts. Par exemple, la fonction $f(x)=e^{ix}$ transforme le segment $0 \leqslant x \leqslant 2\pi$ en circonférence par l'homéomorphie locale au sens large, sans qu'elle soit une homéomorphie locale; en effet, aucun ensemble ouvert dans une circonférence n'est l'image homéomorphe d'un ensemble qui est ouvert dans le segment et en contient le bout $x=0$.

Le même exemple montre qu'une homéomorphie locale au sens large peut augmenter l'*ordre* d'un point, à savoir transformer le bout (donc point d'ordre 1) d'un segment en un point de circonférence (donc point d'ordre 2); mais il sera démontré qu'elle ne peut le diminuer (voir § 3, théorème 2).

([1]) Dans [9] et [11], c'est l'espace X qui est dit *recouvrement de l'espace* Y.