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IDEALS OF FIBER TYPE AND POLYMATROIDS

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Abstract

In the first half of this paper, we complement the theory on discrete polymatroids. More precisely, (i) we prove that a discrete polymatroid satisfying the strong exchange property is, up to an affinity, of Veronese type; (ii) we classify all uniform matroids which are level; (iii) we introduce the concept of ideals of fiber type and show that all polymatroidal ideals are of fiber type. On the other hand, in the latter half of this paper, we generalize the result proved by Stefan Blum that the defining ideal of the Rees ring of a base sortable matroid possesses a quadratic Gröbner basis. For this purpose we introduce the concept of "l-exchange property" and show that a Gröbner basis of the defining ideal of the Rees ring of an ideal I can be determined and that I is of fiber type if I satisfies the l-exchange property. Ideals satisfying the l-exchange property include strongly stable ideals, polymatroid ideals of base sortable discrete polymatroids, ideals of Segre-Veronese type and certain ideals related to classical root systems.

Introduction

In the present paper we will continue our study on combinatorics and algebra of discrete polymatroids developed in the previous paper [9]. In the first half of this paper, consisting of Sections 1–3, we complement [9]. In the latter half of this paper, consisting of Sections 4 and 5, we try to generalize as far as possible the fact by Stefan Blum [1] (and [2]) that the defining ideal of the Rees ring of the matroidal ideal of a base sortable matroid possesses a quadratic Gröbner basis.

This paper will be organized as follows. First of all, in Section 1 we show that the set of bases of a discrete polymatroid satisfying the strong exchange property is, up to an affinity, of Veronese type (Theorem 1.1) . This characterization allows us to give another interesting description for this kind of polymatroids. It turns out, see Theorem 1.4, that these polymatroids are 'locally' nothing but uniform matroids. Second, in Section 2 we determine in Theorem 2.1 all uniform matroids which are level. (Recall that a level ring is a graded Cohen-Macaulay K-algebra R such that all of the generators of the canonical module ω_R of R have the same degree.) This classification partially completes DeNegri and Hibi [6], where all Gorenstein algebras of Veronese type are classified.

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A highlight of this paper is Section 3, where we introduce the concept of ideals of fiber type and show that all polymatroidal ideals are of fiber type.

Let K be a field and $I \subset S = K[x_1, \dots, x_n]$ be an equigenerated graded ideal in the polynomial ring S. Then both the symmetric algebra $S(I)$ as well as the Rees ring $R(I)$ are bigraded K-algebras with generators of degree $(1, 0)$ and $(0, 1)$, and the canonical map $\psi\colon S(I) \to R(I)$ is a homomorphism of bigraded K-algebras. The ideal I is called of *linear type* if ψ is an isomorphism. We say I is of *fiber type* if $\operatorname{Ker}(\psi)$ is generated by elements of degree $(0, a)$, $a \in \mathbb{N}$. The terminology is explained as follows: let $\mathfrak{m} = (x_1, \dots, x_n)$ be the graded maximal ideal of S. The ideal I is of fiber type if and only if the relations of the fiber $S(I)/\mathfrak{m}S(I) \to R(I)/\mathfrak{m}R(I)$, called the *fiber relations*, together with the relations of the symmetric algebra generate all the relations of $R(I)$. Therefore, in a sense, an ideal of fiber type is the next 'best' to an ideal of linear type. Note that if $I = (f_1, \dots, f_m)$, then the kernel of the fiber map is isomorphic to the defining ideal J of the K-algebra $K[f_1, \dots, f_m]$.

An important class of ideals of fiber type are polymatroidal ideals, see Theorem 3.3. They are generated by the set of monomials $\{x^u = x_1^{u(1)} \cdots x_n^{u(n)} : u \in B\}$ where B is the set of bases of a discrete polymatroid. Note that for a polymatroidal ideal I the fiber $R(I)/\mathfrak{m}R(I)$ is isomorphic to the base ring $K[B]$ of the discrete polymatroid whose set of bases is B. Thus the relations of the Rees algebra of a polymatroidal ideal I consist of the relations of the symmetric algebra $S(I)$ which arise from the relations of the ideal I (as an S-module) as well as of the toric relations of the base ring.

On other hand, the purpose of Sections 4 and 5 is, as we said, to generalize [1, Theorem 4.3.9] which guarantees that the defining ideal of the Rees ring of a base sortable matroid possesses a quadratic Gröbner basis. Reading the proof of [1, Theorem 4.3.9] as carefully as possible naturally enables us to introduce the concept of "l-exchange property" (Definition 4.1). We show in Section 5 that a Gröbner basis of the ideal of relations of $R(I)$ can be determined and that I is of fiber type if I satisfies the l-exchange property. We present several classes of ideals satisfying the l-exchange property, namely strongly stable ideals, polymatroid ideals of base sortable discrete polymatroids, ideals of Segre-Veronese type or ideals related to classical root systems.

1. Polymatroids with strong exchange property

Fix an integer $n > 0$ and set $[n] = \{1, 2, \dots, n\}$. The canonical basis vectors of $\mathbb{R}^n$ will be denoted by $\varepsilon_1, \dots, \varepsilon_n$. Let $\mathbb{R}_+^n$ denote the set of those vectors $u = (u(1), \dots, u(n)) \in \mathbb{R}^n$ with each $u(i) \geq 0$, and $\mathbb{Z}_+^n = \mathbb{R}_+^n \cap \mathbb{Z}^n$. For a vector $u = (u(1), \dots, u(n)) \in \mathbb{R}_+^n$ and for a subset $A \subset [n]$, we set

$$u(A) = \sum_{i \in A} u(i).$$

Thus in particular $u(\{i\})$ is the ith component $u(i)$ of u. The *modulus* of u is

$$|u| = u([n]) = \sum_{i=1}^{n} u(i).$$

The set of all $u \in \mathbb{Z}_+^n$ of modulus d will be denoted by $V_n^{(d)}$.

A *discrete polymatroid* on the ground set $[n]$ is a non-empty finite subset $P \subset \mathbb{Z}_+^n$ satisfying

(1) if $u \in P$ and $v \in \mathbb{Z}_+^n$ with $v \leq u$, then $v \in P$;
(2) if $u = (u_1, \ldots, u_n) \in P$ and $v = (v_1, \ldots, v_n) \in P$ with $|u| < |v|$, then there is $i \in [n]$ with $u_i < v_i$ such that $u + \varepsilon_i \in P$.

A *base* of P is a vector $u \in P$ such that $u < v$ for no $v \in P$. We denote the set of bases by $B(P)$. Every base of P has the same modulus $\text{rank}(P)$, the *rank* of P.

A set of bases of a polymatroid can be characterized by the following interesting exchange property: a subset $B \subset V_n^{(d)}$ is the set of bases of a discrete polymatroid if and only if for all $u, v \in B$ such that $u(i) > v(i)$ for some i, there exists $j \in [n]$ with $u(j) < v(j)$ such that $u - \varepsilon_i + \varepsilon_j \in B$.

Moreover, the symmetric exchange theorem [9, Theorem 4.1] guarantees that the set of bases $B(P)$ of a discrete polymatroid P possesses the *symmetric exchange property*: for all $u, v \in B(P)$ such that $u(i) > v(i)$ for some i, there exists $j \in [n]$ with $u(j) < v(j)$ such that both $u - \varepsilon_i + \varepsilon_j$ and $v - \varepsilon_j + \varepsilon_i$ belong to $B(P)$.

We say that a discrete polymatroid satisfies the *strong exchange property*, if for all $u, v \in B(P)$ with $u(i) > v(i)$ and $u(j) < v(j)$ for some i and j, one has that $u - \varepsilon_i + \varepsilon_j \in B(P)$.

Examples of discrete polymatroids satisfying the strong exchange property are *polymatroids of Veronese type*, that is, discrete polymatroids whose set of bases $B \subset V_n^{(d)}$ is given as follows: for $i = 1, \ldots, n$ there exist integers $a_i \geq 1$ such that $u \in V_n^{(d)}$ belongs to B if and only if $u(i) \leq a_i$ for $i = 1, \ldots, n$.

One aim of this section is to show discrete polymatroids satisfying the strong exchange property are essentially of Veronese type. To be precise, we say that two sets $A, B \in \mathbb{R}^n$ are *isomorphic*, if there exists an affinity $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(A) = B$.

The main result of this section will be the following

Theorem 1.1. *Let P be a discrete polymatroid with the strong exchange property. Then $B(P)$ is isomorphic to the bases of a polymatroid of Veronese type.*

For the proof of the theorem we need the characterization of discrete polymatroids in terms of rank functions. Let P be a discrete polymatroid. The *ground set rank function* of P is the function $\rho\colon 2^{[n]} \to \mathbb{Z}_+$ defined by setting

$$\rho(A) = \max\{v(A)\colon v \in P\}$$

for all $\emptyset \neq A \subset [n]$ together with $\rho(\emptyset) = 0$.

This function is *nondecreasing*, i.e., if $A \subset B \subset [n]$, then $\rho(A) \leq \rho(B)$, and is *submodular*, i.e.,

$$\rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B)$$

for all $A, B \subset [n]$.

Conversely, given a nondecreasing and submodular function $\rho\colon 2^{[n]} \to \mathbb{Z}_+$, then the set $u \in \mathbb{Z}_+^n$ satisfying

$$u(A) \leq \rho(A) \quad \text{for all } A \in 2^{[n]} \tag{1}$$

is a discrete polymatroid.

It follows from (1) that a discrete polymatroid is the set of integer points of an integral convex polytope. In fact this polytope is the convex hull $\operatorname{conv}(P)$ of P.

We say that $\emptyset \neq A \subset [n]$ is ρ-*closed* if any subset $B \subset [n]$ properly containing A satisfies $\rho(A) < \rho(B)$, and that $\emptyset \neq A \subset [n]$ is ρ-*separable* if there exist two nonempty subsets A_1 and A_2 of A with $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$ such that $\rho(A) = \rho(A_1) + \rho(A_2)$. A set $\emptyset \neq A \subset [n]$ is ρ-*inseparable* if A is not ρ-separable.

A theorem of Edmonds [7] says that the supporting hyperplanes of $\operatorname{conv}(P)$ are the hyperplanes $H_A = \{x \in \mathbb{R}^n\colon |x| = \rho(A)\}$, where A ranges over all ρ-closed and ρ-inseparable subsets on $[n]$.

Proof of Theorem 1.1. Let $[n]$ be the ground set of P and d its rank. We denote the rank function of P by ρ.

Suppose that for some A with $|A| = n-1$, say for $A = \{1, \ldots, n-1\}$, we have $c = \rho(A) < \rho([n])$. Then for any $u \in B(P)$ it follows that $u(n) \geq d - c$. Let $\tau\colon \mathbb{R}^n \to \mathbb{R}^n$ be the translation with $\tau(v) = v-(d-c)\varepsilon_n$ for all $v \in \mathbb{R}^n$. Then obviously $B' = \tau(B(P))$ is the set of bases of a discrete polymatroid P' of rank c on the ground set $[n]$ whose rank function ρ' satisfies $\rho'([n]) = \rho(A)$.

Since $B(P')$ is affinely isomorphic to $B(P)$ we may assume from the beginning that

$$\rho(A) = \rho([n]) \quad \text{for all } A \subset [n] \quad \text{with } |A| = n-1. \tag{2}$$

Next we claim that there is no ρ-closed and ρ-inseparable set A with $2 \leq |A| \leq n-2$. This will imply our assertion, because then the only hyperplanes defining the facets of P, besides the hyperplanes $x_i = 0$ and $x([n]) = d$, are hyperplanes of the form $x_i = a_i$ for certain i. Such a discrete polymatroid is of Veronese type.

In order to prove the claim we may assume that $A = [k]$ with $2 \leq k \leq n-2$. For $u \in P$ we set $\pi_1(u) = (u(1), \ldots, u(k))$ and $\pi_2(u) = (u(k+1), \ldots, u(n))$.

We first show

(a) There are elements $u, v \in B(P)$ with $u(A) = v(A)$ but $\pi_1(u) \neq \pi_1(v)$.

In fact by [9, Lemma 3.2], there exists for each $i = 1, \ldots, k$ an element $u_i \in P$ such that

$$u_i(i) = \rho(\{i\}), \quad u_i(A) = \rho(A) \quad \text{and} \quad u_i([n]) = \rho([n]).$$

The last equation says that $u_i \in B(P)$.

Suppose that $\pi_1(u_1) = \pi_1(u_2) = \cdots = \pi_1(u_k)$. Then $\rho(A) = \sum_{i=1}^{k} \rho(\{i\})$, a contradiction since A is ρ-inseparable.

Next we claim

(b) There are elements $u, v \in B(P)$ with $u(A) = v(A)$ but $\pi_2(u) \neq \pi_2(v)$.

Again by [9, Lemma 3.2] there exists for each $i = k+1, \dots, n$ an element $u_i \in P$ with

$$u_i(A) = \rho(A), \quad u_i([n] \setminus \{i\}) = \rho([n] \setminus \{i\}) \quad \text{and} \quad u_i([n]) = \rho([n]).$$

The last equation implies that all $u_i \in B(P)$, while by assumption (2) and the second equation it follows that $u_i(i) = 0$ for $i = k+1, \dots, n$. Suppose that $\pi_2(u_{k+1}) = \cdots = \pi_2(u_n)$. Then $\rho(A) = \rho([n])$, a contradiction since A is ρ-closed.

Now let $H = \{u \in B(P) \colon u(A) = \rho(A)\}$. We consider two cases.

CASE 1. $\pi_1(u) = \pi_1(v) \Longrightarrow \pi_2(u) = \pi_2(v)$ for all $u, v \in H$.

By (b) there exist $u, v \in H$ with $\pi_2(u) \neq \pi_2(v)$. Thus in this case $\pi_1(u) \neq \pi_1(v)$, too.

CASE 2. There exist $u, v \in H$ with $\pi_1(u) = \pi_1(v)$ and $\pi_2(u) \neq \pi_2(v)$.

By (a) there exists $w \in H$ with $\pi_1(w) \neq \pi_1(u)(= \pi_1(v))$. Then either $\pi_2(w) \neq \pi_2(u)$ or $\pi_2(w) \neq \pi_2(v)$.

Hence our discussion shows that in both cases we can find $u, v \in H$ such that

$$\pi_1(u) \neq \pi_1(v) \quad \text{and} \quad \pi_2(u) \neq \pi_2(v).$$

Since $\pi_1(u) \neq \pi_1(v)$ there exists $i \in [k]$ with $u(i) > v(i)$. Since $u, v \in H$ it follows that $u([k+1, n]) = v([k+1, n])$. Hence since $\pi_2(u) \neq \pi_2(v)$ it follows that there exists $j \in [k+1, n]$ such that $u(j) < v(j)$. The strong exchange property implies that $v' = v + \varepsilon_i - \varepsilon_j \in B(P)$. This is a contradiction since $v'(A) = \rho(A) + 1$. $\square$

Let $u, v \in V_n^{(d)}$. Then the set

$$[u, v] = \left\{ w \in V_n^{(d)} \colon \min\{u(i), v(i)\} \leq w(i) \leq \max\{u(i), v(i)\} \quad \text{for all } i \right\}$$

is called the *interval* between u and v. The following characterization of discrete polymatroids satisfying the strong exchange property will be used later.

Lemma 1.2. *Suppose that B is a set of integer vectors u in $\mathbb{R}^n$ with $u \geq 0$ and $u([n]) = d$. Then B is the set of bases of a discrete polymatroid which satisfies the strong exchange property if and only if $B = \bigcup_{u,v \in B} [u, v]$.*

Proof. Suppose that $B = \bigcup_{u,v\in B}[u,v]$, and let $u_1, u_2 \in B$ such that $u_1(i) > u_2(i)$ and $u_1(j) < u_2(j)$. Since $u_1 - \varepsilon_i + \varepsilon_j \in [u_1, u_2]$ it follows that $u_1 - \varepsilon_i + \varepsilon_j \in B$. So B satisfies the strong exchange property.

Conversely, suppose that B is the set of bases of a discrete polymatroid which satisfies the strong exchange property. We recall the definition of the distance between bases u and v:

$$\operatorname{dis}(u,v) = \frac{1}{2}\sum_{i=1}^{n} |u(i) - v(i)|.$$

In order to prove that $[u,v] \subset B$ we use induction on $p = \operatorname{dis}(u,v)$. In the case $p = 1$ we have $2 = \sum_{i=1}^{n} |u(i) - v(i)|$. Because $u([n]) = v([n])$ we must have two different indices i and j such that $u(i) = v(i) + 1$ and $v(j) = u(j) + 1$ and $u(k) = v(k)$ for all the others $k \in [n]$. It follows then $v = u - \varepsilon_i + \varepsilon_j$, $[u,v] = \{u,v\}$ and hence $[u,v] \subseteq B$.

Now let $u, v \in B$ with $\operatorname{dis}(u,v) = p > 1$. Without loss of generality we may assume that there exists integers l and m with $1 \leq l < m \leq n$ such that $u(i) > v(i)$ for $1 \leq i \leq l$ and $u(j) < v(j)$ for $l+1 \leq j \leq m$, and $u(k) = v(k)$ for $k > m$. If we denote by $u_{ij} := u - \varepsilon_i + \varepsilon_j$, then we claim that

$$[u,v] = \bigcup_{\substack{1\leq i\leq l\\ l+1\leq j\leq m}} [u_{ij}, v] \cup \{u\}.$$

This implies our assertion because $\operatorname{dis}(u_{ij}, v) = p - 1$.

It is clear that the union of the sets on the right side is contained in $[u,v]$. Conversely, let $w \in [u,v]$. We may assume that $w \neq u, v$. Then, because $\operatorname{dis}(u,w) \geq 1$, there exist i and j with $1 \leq i \leq l$ and $l+1 \leq j \leq m$ such that $u(i) - w(i) \geq 1$ and $w(j) - u(j) \geq 1$. Since

$$u_{ij}(k) = \begin{cases} u(k), & \text{for } k \neq i, j, \\ u(i) - 1, & \text{for } k = i, \\ u(j) + 1, & \text{for } k = j. \end{cases}$$

we have $w(i) \leq u_{ij}(i)$ and $w(j) \geq u_{ij}(j)$, and hence we see that $w \in [u_{ij}, v]$. This completes the proof. $\square$

REMARK 1.3. An easy consequence of 1.2 is that the smallest set of bases of a discrete polymatroid with strong exchange property which contains a finite set $A_1 := \{u_1, u_2, \ldots, u_k\}$ of integer vectors of the same modulus d can be obtained as follows: Let $A_2 := \bigcup_{1\leq i<j\leq k}[u_i, u_j]$. If $A_2 = A_1$, then the previous lemma implies that A_1 is the set we want. If $A_2 \neq A_1$ then we take $A_3 := \bigcup_{u,v\in A_2}[u,v]$. Assuming that we have defined A_i and $A_i \neq A_{i-1}$, then we consider $A_{i+1} := \bigcup_{u,v\in A_i}[u,v]$. If $A_{i+1} = A_i$ then A_i is the set we want, otherwise we continue this procedure. Because we have

$|A_i| < |A_{i+1}|$ and $A_i \subseteq V_n^{(d)}$ for any i and $|V_n^{(d)}|$ is finite, then surely after a finite number of steps we obtain the desired set of bases.

Let $1 \le k \le m \le n$. Recall that the *uniform matroid* $U_{k,m}$ on the ground set $[n]$ is the discrete polymatroid on $[n]$ whose set of bases $B(U_{k,m})$ consists of those $(0,1)$-vectors $\varepsilon_{i_1} + \varepsilon_{i_2} + \cdots + \varepsilon_{i_k}$ with $1 \le i_1 < i_2 < \cdots < i_k \le m$.

Theorem 1.4. *Let $u = (u(1), u(2), \dots, u(n))$ be a given point in $\mathbb{R}^n \setminus \mathbb{Z}^n$ with $u([n]) \in \mathbb{N}$, and such that $u \ge 0$, and let $I = \{i \in [n]\colon u(i) \notin \mathbb{Z}\}$. Then, with respect to inclusion, there exists a unique smallest discrete polymatroid P_u of rank $d = u([n])$ with $u \in \operatorname{conv}(B(P_u))$ satisfying the strong exchange property. Moreover the set of bases $B(P_u)$ of P_u is isomorphic to the set of bases of the uniform matroid $U_{k,m}$ where $k = \sum_{i\in I}(u(i) - \lfloor u(i)\rfloor)$ and $m = |I|$.*

Proof. First we fix some notation. Let

$$B' = \{v \in \mathbb{Z}^n \colon v(i) \in \{\lfloor u(i)\rfloor, \lceil u(i)\rceil\} \text{ for each } i \in [n] \text{ and } v([n]) = u([n])\},$$

where $\lfloor x\rfloor$ is the biggest integer $\le x$ and $\lceil x\rceil$ is the smallest integer $\ge x$.

Then B' is the base of a discrete polymatroid P_u which satisfies the strong exchange property and $u \in \operatorname{conv}(B')$. Indeed, if $v_1, v_2 \in B'$ and $i, j \in [n]$ such that $v_1(i) > v_2(i)$, $v_1(j) < v_2(j)$, it follows from the definition of B' that $i, j \in I$ and $v_1(i) = \lceil u(i)\rceil$, $v_1(j) = \lfloor u(j)\rfloor$. So $v_1 - \varepsilon_i + \varepsilon_j \in B'$ and therefore B' satisfies the strong exchange property.

The subset $\mathcal{Q}$ of $\mathbb{R}^n$ defined by

$$\lfloor u(i)\rfloor \le x(i) \le \lceil u(i)\rceil \text{ for each } i \in [n] \text{ and } x([n]) = u([n]),$$

is an integral convex polytope whose set of vertices is B'. It follows that $\mathcal{Q} = \operatorname{conv}(B')$, and since $u \in \mathcal{Q}$ we conclude that $u \in \operatorname{conv}(B')$, as desired.

In order to prove the uniqueness we show that for each discrete polymatroid P of rank d with strong exchange property such that $u \in \operatorname{conv}(B(P))$ we have $B' \subset B(P)$. Let $B(P) := \{u_1, \dots, u_k\}$. Then, $u \in \operatorname{conv}(B(P))$ implies that $u(i) = \sum_{j=1}^k \alpha_j u_j(i)$, for each i with $1 \le i \le n$, and for some non-negative real numbers $\alpha_1, \dots, \alpha_k$ such that $\sum_{j=1}^k \alpha_j = 1$.

We claim that for each $i \in [n]$ there exist some $j, l \in \{1, \dots, k\}$ such that $u_j(i) \le \lfloor u(i)\rfloor$ and $u_l(i) \ge \lceil u(i)\rceil$. Indeed, if for all $i \in [n]$ we would have $u_j(i) > \lfloor u(i)\rfloor$ for any $j \in \{1, \dots, k\}$, then $u(i) \ge \lfloor u(i)\rfloor + 1$, which is a contradiction. A similar argument shows us that there exists $l \in \{1, \dots, k\}$ such that $u_l(i) \ge \lceil u(i)\rceil$.

Consider now the affine translation $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\sigma(v) = v - \sum_{i=1}^n \left(\min_{j=1,\dots,k}\{u_j(i)\}\right)\varepsilon_i.$$

Replacing u by $u' := \sigma(u)$, $B(P)$ by $\sigma(B(P))$ and B' by $\sigma(B')$ we may assume by Theorem 1.1 that $B(P)$ is of Veronese type defined by

$$B(P) = \{v : v(i) \text{ is an integer with } 0 \le v(i) \le a_i \text{ and } |v| = d'\},$$

where

$$a_i = \max_{j=1,\dots,k}\{u_j(i)\}$$

and

$$d' = d - \sum_{i=1}^{n}\left(\min_{j=1,\dots,k}\{u_j(i)\}\right).$$

Now let $w \in B'$. Then $w(i) = \lfloor u(i) \rfloor$ or $w(i) = \lceil u(i) \rceil$. Therefore $0 \le w(i) \le a_i$ for all i, and hence $w \in B(P)$.

Therefore we have proved the existence and uniqueness of the smallest discrete polymatroid P_u satisfying the strong exchange property and containing u, and whose set of bases $B(P_u) = B'$. Consider now the affine translation $\tau : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\tau(v) = v - \sum_{i=1}^{n}(\lfloor u(i) \rfloor)\varepsilon_i.$$

It is easy to see that $\tau(B') = B(U_{k,m})$ with k and m as in the theorem. □

Corollary 1.5. *Let $B(P)$ be the set of bases of a discrete polymatroid P. Then the following conditions are equivalent*:
(a) *P satisfies the strong exchange property,*
(b) *For all $u \in \operatorname{conv}(B(P))$ we have $P_u \subset P$.*

Proof. (a) $\Longrightarrow$ (b) follows from Theorem 1.4. For the converse let $v_1, v_2 \in B(P)$. Without loss of generality we may assume that there exist integers l and m with $1 \le l < m \le n$ such that $v_1(i) > v_2(i)$ for $1 \le i \le l$ and $v_1(j) < v_2(j)$ for $l + 1 \le j \le m$, and $v_1(k) = v_2(k)$ for $k > m$. Now if we consider λ to be a real number such that

$$\max_{\substack{1 \le i \le l\\ l+1 \le j \le m}}\left\{1 - \frac{1}{v_1(i) - v_2(i)}, 1 - \frac{1}{v_2(j) - v_1(j)}\right\} < \lambda < 1,$$

then $u := \lambda v_1 + (1 - \lambda)v_2$ has the following properties: $v_1(i) - 1 < u(i) < v_1(i)$ for $1 \le i \le l$, $v_1(j) < u(j) < v_1(j) + 1$ for $l + 1 \le j \le m$ and $u(k) = v_1(k)$ for $k > m$. But u belongs to the line segment S between v_1 and v_2 and $S \subset \operatorname{conv}(B(P))$, so that $u \in \operatorname{conv}(B(P))$. Therefore by our hypothesis we have $P_u \subset P$. Now it is easy to see that $v_1 - \varepsilon_i + \varepsilon_j \in P_u$ for all i, j such that $1 \le i \le l$ and $l + 1 \le j \le m$. Hence the conclusion. □

Corollary 1.6. *Let $\mathcal{P}$ be a set of discrete polymatroids all of the same rank, satisfying the strong exchange property. Then the following conditions are equivalent*:
(a) $\bigcap_{P\in\mathcal{P}} B(P) \neq \emptyset$,
(b) $\bigcap_{P\in\mathcal{P}} \operatorname{conv}(B(P)) \neq \emptyset$.

Proof. (a) $\Longrightarrow$ (b) is trivial. For the converse, let $u \in \bigcap_{P\in\mathcal{P}} \operatorname{conv}(B(P))$. Then, by Theorem 1.4, $B(P_u) \subset B(P)$ for all $P \in \mathcal{P}$. Hence the conclusion. $\square$

Corollary 1.7. *Let $\mathcal{P}$ be a set of discrete polymatroids all of the same rank, satisfying the strong exchange property and the equivalent conditions of Corollary* 1.6, *then*

$$\bigcap_{P\in\mathcal{P}} B(P) = B\left(\bigcap_{P\in\mathcal{P}} P\right) \quad \textit{and} \quad \operatorname{conv}\left(\bigcap_{P\in\mathcal{P}} B(P)\right) = \bigcap_{P\in\mathcal{P}} \operatorname{conv}(B(P)).$$

Proof. The first equality follows from the fact (Corollary 1.6) that $\bigcap_{P\in\mathcal{P}} B(P) \neq \emptyset$. The inclusion $\operatorname{conv}\left(\bigcap_{P\in\mathcal{P}} B(P)\right) \subset \bigcap_{P\in\mathcal{P}} \operatorname{conv}(B(P))$ is trivial. Conversely, let $u \in \bigcap_{P\in\mathcal{P}} \operatorname{conv}(B(P))$. Then by Theorem 1.4 we have $B(P_u) \subset B(P)$ for all $P \in \mathcal{P}$. Therefore $u \in \operatorname{conv}(B(P_u)) \subset \operatorname{conv}\left(\bigcap_{P\in\mathcal{P}} B(P)\right)$. $\square$

The next examples show that all the hypotheses in Corollary 1.6 are needed.

EXAMPLES 1.8. (a) The intersection of discrete polymatroids is in general not a discrete polymatroid, even if they have the same rank and the intersection of their set of bases is non-empty. Consider the discrete polymatroids P_1 and P_2, whose sets of bases are:

$$B(P_1) = \{(1,0,1,0), (1,1,0,0), (0,1,0,1), (0,0,1,1)\},$$
$$B(P_2) = \{(1,0,1,0), (0,1,1,0), (0,1,0,1), (1,0,0,1)\}.$$

Then $B(P_1) \cap B(P_2) = \{(1,0,1,0), (0,1,0,1)\}$ does not satisfy the exchange property, so it is not the set of bases of a discrete polymatroid.
(b) The condition $\bigcap_{P\in\mathcal{P}} B(P) \neq \emptyset$ is essential, even if all $P \in \mathcal{P}$ satisfy the strong exchange property. Let P_1, P_2, P_3 be the discrete polymatroids, whose sets of bases are:

$$B(P_1) = \{(2,0,2), (3,0,1), (2,1,1)\},$$
$$B(P_2) = \{(2,1,1), (1,1,2), (1,2,1)\},$$
$$B(P_3) = \{(0,2,2), (0,3,1), (1,2,1)\}.$$

Then P_1, P_2 and P_3 satisfy the strong exchange property but

$$\begin{aligned} &P_1 \cap P_2 \cap P_3 \\ &= \{(1,1,1),(1,1,0),(1,0,1),(0,1,1),(0,0,2),(1,0,0),(0,1,0),(0,0,1),(0,0,0)\}, \end{aligned}$$

is not a discrete polymatroid.

2. Level rings

Let K be a field. Recall that a standard graded Cohen-Macaulay K-algebra R is called a *level ring*, if all generators of the canonical module ω_R have the same degree. For example all Veronese subrings of the polynomial ring are level. More generally if R is level, then any Veronese subring $R^{(d)}$ is level, too. This follows easily from the fact that for a standard graded Cohen-Macaulay K-algebra R one has $\omega_R^{(d)} \cong \bigoplus_i (\omega_R)_{id}$, see [3, Exercise 3.6.21 (c)]. Actually this formula even shows that $R^{(d)}$ is level for all d greater than or equal to the highest degree of a generator of ω_R.

The question arises which rings of Veronese type are level. At present we can give a complete answer only in the case of squarefree Veronese rings, that is, the base rings of uniform matroids. For the proof we use the characterization of the canonical module of a normal semigroup ring, given by Danilov and Stanley (cf. [3, Theorem 6.3.5]):

Let C be a normal semigroup. Then the ideal I generated by the monomials x^c with $c \in \operatorname{relint}(C)$ is the canonical module of $K[C]$.

Now let $K[B]$ be the base ring of the discrete polymatroid P of rank d on the ground set $[n]$ with rank function ρ. In [9] it is shown that $K[B]$ is a normal semigroup ring. Let C be the semigroup generated by B, and let $\Omega = \operatorname{relint}(C)$. Then a vector c belongs to Ω if and only if $c(i) > 0$ for $i \in [n]$, and for some integer r we have $|c| = rd$ and $c(A) < r\rho(A)$ for all $A \subset [n]$. For each integer r we set $\Omega_r = \Omega \cap V_n^{(rd)}$, and call the elements of Ω_r the inner points of rB.

Let k be the smallest integer such that $\Omega_k \neq \emptyset$. It follows from the theorem of Danilov and Stanley that $K[B]$ is level, if and only if for all $r > k$ and all $v \in \Omega_r$ there exists $u \in \Omega_k$ such that $v - u \in (r-k)B$.

In the particular case, that the rank function ρ depends only on the cardinality of A, we call the polymatroid P *uniform*. Let $c = (c(1), \ldots, c(n))$ be a vector, and π a permutation of the elements in $[n]$. Then we set $\pi(c) = (c(\pi(1)), \ldots, c(\pi(n)))$. It is clear that if P is uniform, then $c \in rB$ if and only if $\pi(c) \in rB$, and $c \in \Omega_r$ if and only if $\pi(c) \in \Omega_r$. In the orbit of each c under the action of the symmetric group is a unique *descending* vector, that is, a vector v with $v(1) \geq v(2) \geq \cdots \geq v(n)$. Thus for a uniform polymatroid it suffices to check the level condition only for descending vectors.

Theorem 2.1. *Let K be a field and $R = K[B(U_{d,n})]$ the base ring of the uniform matroid $U_{d,n}$. Then R is level if and only if $d = 1$, $d = n-1$, or $d \geq 2$ and $n = 2d-1$,*

$n = 2d$ *or* $n = 2d + 1$.

Proof. If $d = 1$ or $d = n - 1$, then R is a polynomial ring. Also note that $K[B(U_{d,n})] \cong K[B(U_{n-d,n})]$. Thus we may assume from now on that $2 \leq d \leq \lfloor n/2 \rfloor$.

We set $B = B(U_{d,n})$ and denote by k the smallest positive integer such that kB has inner points. Note that $u = (u(1), \ldots, u(n))$ is an inner point of rB if and only if $|u| = rd$ and $0 < u(i) < r$ for $i \in [n]$. We consider two cases.

CASE 1. $d|n$.

In this case we have $k = n/d$ and kB has only one inner point, namely $(1, \ldots, 1)$. If $d < k$, then $(d + 1, 1, \ldots, 1)$ is an inner point of $(k + 1)B$ and $(d + 1, 1, \ldots, 1) - (1, \ldots, 1) = (d, 0, \ldots, 0) \notin B$, because $d \geq 2$. Therefore, for $d < k$, R is not level.

If $d \geq k$, then $(k + 1)B$ has an inner point u with $u(1) = k$. Indeed, $k + (n - 1) < d(k + 1)$ and $kn > d(k + 1)$.

If $k \geq 3$, we have $(k, u(2), \ldots, u(n)) - (1, \ldots, 1) = (k-1, u(2)-1, \ldots, u(n)-1) \notin B$ and hence R is not a level ring.

If $k = 2$, then $n = 2d$. We will show that R is level in this case. For $r \geq 3$ let $u \in rB$ be an inner point which is a descending vector. Then $u - (1, \ldots, 1) \in (r-2)B$, and therefore R is level.

CASE 2. d does not divide n.

Now we have $k = \lceil n/d \rceil$ and $n = kd - s$ with $0 < s < d$. Note also that $k \geq 3$.

If $d < k$, then kB has the inner point $u_0 = (s+1, 1, \ldots, 1)$. Note that for any other inner point $u \in kB$ we have $u(1) \leq s + 1$. The vector $v = (d + 1, s + 1, 1, \ldots, 1)$ is an inner point of $(k + 1)B$ and $v - u_0 = (d - s, s, 0, \ldots, 0)$.

If $d - s \geq 2$ or $s \geq 2$, then R is not level.

Suppose now that $s = 1$ and $d = 2$. Then $n = 2k - 1$ and $(2, 1, \ldots, 1)$ is the only descending vector which is an inner point of kB. If $k \geq 4$, then $(k + 1)B$ contains the inner point $(4, 1, \ldots, 1)$ and hence R is not level. If $k = 3$, then $n = 5$ and we have a level ring as we shall see below.

If $d = k$, then $n = d^2 - s$ with $0 < s < d$.

If $s + 1 < d$, then $(s + 1, 1, \ldots, 1) \in dB$ is an inner point with the largest first coordinate, $(d + 1)B$ has the inner point $(d, s + 2, 1, \ldots, 1)$, and $(d, s + 2, 1, \ldots, 1) - (s + 1, 1, \ldots, 1) = (d - s - 1, s + 1, 0, \ldots, 0) \notin B$. Therefore R is not level.

If $s+1 = d$ and $d = 2$, we have $n = 3$ and then R is the polynomial ring. If $d > 2$, then dB contains $(d-1, 2, 1, \ldots, 1)$, $(d+1)B$ has the inner point $(d, d, 2, 1, \ldots, 1)$ and $(d, d, 2, 1, \ldots, 1) - (d - 1, 2, 1, \ldots, 1) = (1, d - 2, 1, 0, \ldots, 0)$. Therefore if $d \geq 4$, then R is not level. If $d = 3$, then $n = 7$ and R is level as we shall see below.

Assume now that $d > k$. Let t be the largest integer such that $t(k-1)+(n-t) \leq dk$. Then $t = \lfloor s/(k - 2) \rfloor$ and $t < n$.

If $t \geq 1$, then kB has an inner point u such that $u(1) = \cdots = u(t) = k - 1$, $k - 1 > u(t + 1) \geq \cdots \geq u(n) \geq 1$ and $t(k - 1) + u(t + 1) + \cdots + u(n) = dk$. Since $tk + (u(t + 1) + 2) + u(t + 2) + \cdots + u(n) = dk + t + 2$, it follows that for the case $t + 2 \leq d$

the set $(k+1)B$ has an inner point v such that $v(1) = \cdots = v(t) = k$, $v(t+1) = u(t+1)+2$ and $v(j) \geq u(j)$ for $j \geq t+2$. Therefore R is not level.

If $k \geq 4$, then $t \leq s/2$ and $t+2 \leq d$, and R is not level.

So it remains to study the case $k = 3$. Then $t = s$, and R is not level if $s+2 \leq d$. If n is even, then d and s have the same parity and therefore $s+2 \leq d$. If n is odd, then except in the case $s = d-1$ we have $s+2 \leq d$. If $s = d-1$, then $n = 2d+1$ and we will show that R is level. We may assume that $d \geq 2$: This will include the cases $d = 2$, $n = 5$ and $d = 3$, $n = 7$. The descending inner point u of $3B$ satisfies $u(1) = \cdots = u(d-1) = 2$ and $u(d) = \cdots = u(n) = 1$. Let v be an inner point in rB with $r \geq 4$ and $r-1 \geq v(1) \geq \cdots \geq v(n) \geq 1$. We want to show that $v-u \in (r-3)B$. First we shall see that $v \geq u$. Indeed, if we suppose that $v(d-1) = 1$, then $|v| \leq (d-2)(r-1)+d+3 < dr$, a contradiction. So $v(d-1) \geq 2$, and hence we are done. Now we want to prove that $v(j) - u(j) \leq r-3$ for all $j \in [n]$. This is obvious for $j \leq d-1$. Assume now that $v(d)-1 = r-2$. This implies that $v(1) = \cdots = v(d) = r-1$, and hence $|v| \geq d(r-1)+d+1 > dr$, which is a contradiction. Therefore R is a level ring.

If $t = 0$, then we have $s < k-2$. Let q be the maximal integer such that there exists a descending inner point u in kB with $u(1) = q$. Then let l be the maximal number for which there exists a descending inner point u in kB with $u(1) = u(2) = \cdots = u(l) = q$. Hence we have $l = \lfloor s/(q-1) \rfloor$ and $l \leq s$. Also $t = 0$ implies that $q \leq k-2$. Then in the same manner as in the case $t \geq 1$ we show that there exists an inner point v in $(k+1)B$ such that $v(1) = \cdots = v(l) = q+1$, $v(l+1) = u(l+1)+2$ and $v(j) \geq u(j)$ for all $j \geq l+2$. Because $l+2 \leq s+2 < k < d$, it follows as in the case $t \geq 1$ that R is not level. $\square$

3. Ideals of fiber type

Let K be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring, $I \subset S$ an equigenerated graded ideal, that is, a graded ideal whose generators $f_1, \ldots, f_m$ are all of same degree. Then the Rees ring

$$R(I) = \bigoplus_{j \geq 0} I^j t^j = S[f_1 t, \ldots, f_m t] \subset S[t]$$

is naturally bigraded with $\deg(x_i) = (1,0)$ for $i = 1, \ldots, n$ and $\deg(f_i t) = (0,1)$ for $i = 1, \ldots, m$.

Let $T = S[y_1, \ldots, y_m]$ be the polynomial ring over S in the variables $y_1, \ldots, y_m$. We define a bigrading on T by setting $\deg(x_i) = (1,0)$ for $i = 1, \ldots, n$, and $\deg(y_j) = (0,1)$ for $j = 1, \ldots, m$. Then there is a natural surjective homomorphism of bigraded K-algebras $\varphi\colon T \to R(I)$ with $\varphi(x_i) = x_i$ for $i = 1, \ldots, n$ and $\varphi(y_j) = f_j t$ for $j = 1, \ldots, m$.

If h is a bihomogeneous element of bidegree (a,b). Then we call a the x-degree, and b the y-degree of h.

Let $\alpha = (a_{ij})_{\substack{i=1,\dots,r\\ j=1,\dots,m}}$ be the relation matrix of I. Then for $i = 1, \dots, r$, the bihomogeneous polynomials $g_i = \sum_{j=1}^m a_{ij} y_j$ belong to $\operatorname{Ker}(\varphi)$, and

$$T/L \quad \text{with} \quad L = (g_1, \dots, g_r)$$

is isomorphic to the symmetric algebra $S(I)$ of I. The generators g_i of L are all linear in the variables y_j.

Let $\mathfrak{m} = (x_1, \dots, x_n)$ be the graded maximal ideal of S. The K-algebra $R(I)/\mathfrak{m}R(I)$ is called the *fiber ring of* I.

Note that the standard graded subalgebra $R(I)_{(0,*)} = \bigoplus_{j\geq 0} R(I)_{(0,j)}$ of $R(I)$ is isomorphic to $K[f_1, \dots, f_m] \subset S$, and that the composition of the natural K-algebra homomorphisms $R(I)_{(0,*)} \to R(I) \to R(I)/\mathfrak{m}R(I)$ is an isomorphism. Therefore the fiber ring of I is isomorphic to $K[f_1, \dots, f_m]$.

The homomorphism $\varphi\colon T \to R(I)$ induces a surjective K-algebra homomorphism

$$\varphi'\colon K[y_1, \dots, y_m] = T/\mathfrak{m}T \to R(I)/\mathfrak{m}R(I) = K[f_1, \dots, f_m].$$

The elements in $\operatorname{Ker}(\varphi')$ are called the *fiber relations*. We note that

$$\varphi' = \varphi_{(0,*)}\colon T_{(0,*)} = K[y_1, \dots, y_m] \to R(I)_{(0,*)} = K[f_1, \dots, f_m].$$

Therefore $\operatorname{Ker}(\varphi') \subset \operatorname{Ker}(\varphi)$. We set $R = K[y_1, \dots, y_m]$ and $J = \operatorname{Ker}(\varphi')$. Then $K[f_1, \dots, f_m] = R/J$.

The natural map $\psi\colon S(I) \to R(I)$ is a surjective homomorphism of bigraded K-algebras. Recall that I is called of *linear type*, if ψ is an isomorphism, that is, if $\operatorname{Ker}(\varphi) = L$. The next best situation is given by

DEFINITION 3.1. The ideal I is called of *fiber type*, if $\operatorname{Ker}(\varphi) = (L, JT)$.

Note that I is of fiber type if and only if $\operatorname{Ker}(\psi)$ is generated by elements of x-degree 0.

We begin with an example which is due to Villareal ([17, Theorem 8.2.1]). Let $f, g \in S$ be monomials. We denote by $[f, g]$ the least common multiple of f and g. Let $f_1, \dots, f_m \in S$. If $\alpha = (i_1, \dots, i_s)$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq m$ we set $f_\alpha = f_{i_1} f_{i_2} \cdots f_{i_s}$.

Theorem 3.2 (Villareal). *Suppose $I = (f_1, \dots, f_m)$ is an equigenerated monomial ideal satisfying*:
$(*)$ *for all non-decreasing sequences $\alpha = (i_1, \dots, i_s)$ and $\beta = (j_1, \dots, j_s)$ with $i_k, j_k \in [m]$ for $k = 1, \dots, s$ for which $f_\alpha \neq f_\beta$, there exist integers r and t such that $f_{i_r}(f_\alpha/f_{i_t})$ divides $[f_\alpha, f_\beta]$.*
Then I is of fiber type.

Moreover, condition $(*)$ *is satisfied if* I *is generated by quadratic monomials.*

Here is another class of monomial ideals of fiber type.

Theorem 3.3. *Let* $I \subset S$ *be a polymatroidal ideal. Then* I *is of fiber type.*

Proof. There exists a discrete polymatroid P with set of bases B such that I is generated by the monomials x^v with $v \in B$. We denote the variable in T which is mapped onto $x^v t \in R(I)$ by y_v.

Since I is a monomial ideal, the Rees algebra $R(I)$ is a toric ring, and hence $\mathrm{Ker}(\varphi)$ is generated by bihomogeneous binomials

$$f = x^a y_{u_1} y_{u_2} \cdots y_{u_r} - x^b y_{v_1} y_{v_2} \cdots y_{v_r}, \tag{3}$$

such that the first and second term in the binomial have no factor in common.

We show that f can be reduced modulo (L, JT) to a binomial which is bihomogeneous of degree $(0, *)$, that is, contains no factors x_i.

We have $\deg(f) \geq 2$. If $\deg(f) = 2$, then there is nothing to prove. Suppose now that $\deg(f) > 2$. We may assume $\max\{a_i, b_i\} \neq 0$ for some i, because otherwise there is nothing to show. Let $k = \max\{i : \max\{a_i, b_i\} \neq 0\}$. Then $k \geq 2$, since $|a| = |b|$. We may assume that $a_k > 0$ and $b_k = 0$.

Let $u = u_1 + u_2 + \cdots + u_r$ and $v = v_1 + v_2 + \cdots + v_r$. Then $u(k) < v(k)$ and $u(i) = v(i)$ for $i > k$. Since u and v belong to the set of bases of the discrete polymatroid rP, there exists $j < k$ such that $u - \varepsilon_j + \varepsilon_k$ is again a base of rP. That is, we find $u'_1, \ldots, u'_r \in B$ such that

$$x_j y_{u'_1} \cdots y_{u'_r} - x_k y_{u_1} \cdots y_{u_r} \in \mathrm{Ker}(\varphi). \tag{4}$$

Modulo this relation f can be rewritten as

$$x^{a'} y_{u'_1} \cdots y_{u'_r} - x^b y_{v_1} y_{v_2} \cdots y_{v_r},$$

with $x^{a'} = (x_j / x_k) x^a$.

We will show that f can be reduced to zero modulo relations of type (4). Note that $a'_k = a_k - 1$. If $a'_k > 0$, we use a similar reduction, so that after a finite number of steps we may assume that f satisfies $\max\{i : \max\{a_i, b_i\} \neq 0\} < k$.

If $\min\{a_i, b_i\} = 0$ for all $i < k$, then by induction on k we may assume that f can be reduced to zero modulo relations of type (4).

Suppose for some $i < k$ we have $a_i > 0$ and $b_i > 0$ (which after these reductions may of course happen). Then $f = x_i g$ with $g = (x^a / x_i) y_{u_1} y_{u_2} \cdots y_{u_r} - (x^b / x_i) y_{v_1} y_{v_2} \cdots y_{v_r}$. Since $R(I)$ is a domain, and since $x_i \notin \mathrm{Ker}(\varphi)$ we conclude that $g \in \mathrm{Ker}(\varphi)$. Therefore by induction on the degree we see that f can be reduced to zero modulo relations of type (4).

Thus it remains to show that the relations of type (4) can be reduced to zero modulo (L, JT). In other words we may assume that

$$f = x_i y_{u_1} y_{u_2} \cdots y_{u_r} - x_j y_{v_1} y_{v_2} \cdots y_{v_r},$$

and that the two terms in f have no common factor. We denote by $\operatorname{dis}(f) = \sum_{i=1}^{r} \operatorname{dis}(u_i, v_i)$ and show by induction on $\deg(f)$ and $\operatorname{dis}(f)$ that f can be reduced to zero modulo (L, JT).

Since $\varepsilon_i + u_1 + u_2 + \cdots + u_r = \varepsilon_j + v_1 + v_2 + \cdots + v_r$, there exists s such that $u_s(i) < v_s(i)$, and hence by the exchange property there exists an index k such that $u_s(k) > v_s(k)$ and such that $u'_s = u_s + \varepsilon_i - \varepsilon_k \in B$. Thus modulo $x_k y_{u'_s} - x_i y_{u_s} \in L$ the relation f can be rewritten as

$$f = x_k y_{u_1} \cdots y_{u'_s} \cdots y_{u_r} - x_j y_{v_1} y_{v_2} \cdots y_{v_r}.$$

If $k = j$, then $f = x_k g$ with $g \in JT$, and we are done. If $u'_s = v_i$ for some i, then $f = y_{u'_s} g$ with g a relation of type (4) and $\deg(g) < \deg(f)$, and we are done by induction on $\deg(f)$. Otherwise the new f has no common factor. However since $\operatorname{dis}(u'_s, v_s) < \operatorname{dis}(u_s, v_s)$ it follows that $\operatorname{dis}(f)$ has dropped, and again induction concludes the proof. □

In the next proposition we want to describe a condition which implies that an ideal is of fiber type.

Proposition 3.4. *Let $I \subset S$ be an equigenerated ideal. Assume that for some $h \in \mathfrak{m}$*
(1) *IS_h is of linear type;*
(2) $(0 :_{S(I)} h) \cap \mathfrak{m}S(I) = 0$.
Then I is of fiber type, and $\operatorname{Ker}(\psi) = (0 :_{S(I)} h) = (0 :_{S(I)} \mathfrak{m}S(I))$. *In other words, $J = \{f \in R : \mathfrak{m}f \in L\}$.*

Proof. We first show that $\operatorname{Ker}(\psi) = (0 :_{S(I)} h)$. Let $f \in (0 :_{S(I)} h)$, then $fh = 0$, and hence $h\psi(f) = \psi(hf) = 0$. Since $R(I)$ is a domain, it follows that $\psi(f) = 0$, and hence $f \in \operatorname{Ker}(\psi)$. Conversely, suppose that $f \in \operatorname{Ker}(\psi)$. Since IS_h is of linear type it follows that $\operatorname{Ker}(\psi)_h = 0$. Therefore $h^n f = 0$ for some integer n. If $n = 1$, then $f \in (0 :_{S(I)} h)$, as desired. Otherwise, $n > 1$ and $h(h^{n-1} f) = 0$, so that $h^{n-1} f \in (0 :_{S(I)} h) \cap \mathfrak{m}S(I)$. Since this intersection is 0, by assumption, it follows that $h^{n-1} f = 0$. Backwards induction yields that $f \in (0 :_{S(I)} h)$.

Finally, since $\operatorname{Ker}(\psi) = (0 :_{S(I)} h)$ and since $(0 :_{S(I)} h) \cap \mathfrak{m}S(I) = 0$, we see that $\operatorname{Ker}(\psi)$ contains no elements of positive x-degree. Therefore I is of fiber type. □

Let M be the finitely generated $K[y_1, \ldots, y_m]$-module $S(I)_{(1,*)}$. Then M is generated by $x_1, \ldots, x_n$, and $(0 :_{S(I)} \mathfrak{m}S(I))_{(0,*)} = \operatorname{Ann}_R(M)$. Under the assumptions of

Proposition 3.4 we have

$$\operatorname{Ker}(\psi) = (0 :_{S(I)} \mathfrak{m}S(I)).$$

Since $\operatorname{Ker}(\psi)_{(0,*)} = J$, we conclude that

$$J = \operatorname{Ann}_R(M). \tag{5}$$

Let A be the matrix of linear relations of $I = (f_1, \dots, f_m)$. If I has r linear relations, then A is an $r \times m$-matrix whose coefficients are linear forms in $S = K[x_1, \dots, x_n]$. There is a unique $r \times n$-matrix B whose coefficients are linear forms in $R = K[y_1, \dots, y_m]$ such that

$$Ay = Bx.$$

Here x is the transpose of $(x_1, \dots, x_n)$ and y the transpose of $(y_1, \dots, y_m)$. This matrix B is the relation matrix of the R-module M.

Note that the annihilator of M equals the radical of the ideal $I_n(B)$ of n-minors of B, where $I_n(B) = 0$ if $r < n$. Therefore, since J is a prime ideal, it follows from (5).

Corollary 3.5. *Let I be an equigenerated ideal with* $\operatorname{Ker}(\psi) = (0 :_{S(I)} \mathfrak{m}S(I))$. *Then*

$$J = \sqrt{I_n(B)}.$$

In particular, the degree of the fiber relations is bounded by n.

Quite generally one has $(0 :_{S(I)} x_i) \subset \operatorname{Ker}(\psi)$. Hence if $\operatorname{Ker}(\psi) = (0 :_{S(I)} \mathfrak{m}S(I))$, then $(0 :_{S(I)} x_i) = (0 :_{S(I)} \mathfrak{m}S(I)) = \operatorname{Ker}(\psi)$ for all i. This case seems to be rather rare. On the other hand, since $\sum_{i=1}^n (0 :_{S(I)} x_i) \subset \operatorname{Ker}(\psi)$, it is more likely that in some cases $\sum_{i=1}^n (0 :_{S(I)} x_i) = \operatorname{Ker}(\psi)$.

Conjecture 3.6. *Let $I \subset S$ be a polymatroidal ideal. Then*

$$\operatorname{Ker}(\psi) = \sum_{i=1}^{n} (0 :_{S(I)} x_i).$$

Equivalently, J is the ideal generated by $\{f \in R : f x_i \in L$ for some $i\}$.

REMARK 3.7. The conjecture is true if the fiber ideal J is generated by symmetric exchange relations. Indeed, let $f = y_{u-\varepsilon_i+\varepsilon_j} y_{v-\varepsilon_j+\varepsilon_i} - y_u y_v$ be such a relation. Then

$$x_i f = (x_i y_{u-\varepsilon_i+\varepsilon_j} - x_j y_u) y_{v-\varepsilon_j+\varepsilon_i} + y_u (x_j y_{v-\varepsilon_j+\varepsilon_i} - x_i y_v)$$

is an element of L.

4. *l*-exchange property

In this section we introduce the l-exchange property for sets of integer vectors of modulus d, and give classes of such sets having this property. We will then see in the next section that the monomial ideals corresponding to sets with the l-exchange property are of fiber type.

Let $V_n^{(d)} \subset \mathbb{Z}_+^n$ denote the set of vectors $u \in \mathbb{Z}_+^n$ of modulus d. Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over K. Given a nonempty subset $B \subset V_n^{(d)}$, we write $K[B]$ for the K-subalgebra of S generated by those monomials $\mathbf{x}^u = \prod_{i=1}^n x_i^{u(i)}$ with $u = (u(1), \dots, u(n)) \in B$.

Let $R = K[\{y_u\}_{u \in B}]$ denote the polynomial ring over K with the standard grading, i.e., each $\deg y_u = 1$. The *toric ideal* of $K[B]$ is the ideal $I_{K[B]} \subset R$ which is the kernel of the surjective homomorphism $\pi \colon R \to K[B]$ defined by setting $\pi(y_u) = \mathbf{x}^u$ for all $u \in B$. If $\mathbf{a} = (a_u)_{u \in B}$ is a vector with each $0 \leq a_u \in \mathbb{Z}$, then we write $\mathbf{y}^{\mathbf{a}}$ for the monomial $\prod_{u \in B} y_u^{a_u}$ belonging to R.

Let $<$ be a monomial order on R and $\mathrm{in}_<(I_{K[B]})$ the initial ideal of $I_{K[B]}$ with respect to $<$. Recall that a *standard monomial* of $I_{K[B]}$ with respect to $<$ is a monomial $\mathbf{y}^{\mathbf{a}} \in R$ with $\mathbf{y}^{\mathbf{a}} \notin \mathrm{in}_<(I_{K[B]})$.

DEFINITION 4.1. We say that a nonempty subset $B \subset V_n^{(d)}$ satisfies the *l-exchange property* with respect to a monomial order $<$ on R if B possesses the following property: If $\prod_{\mu=1}^N y_{u_\mu}$ and $\prod_{\mu=1}^N y_{v_\mu}$ are standard monomials of $I_{K[B]}$ of degree N with respect to $<$ such that
(i) $\sum_{\mu=1}^N u_\mu(i) = \sum_{\mu=1}^N v_\mu(i)$ for $i = 1, 2, \dots, q-1$ (with $q \leq n-1$);
(ii) $\sum_{\mu=1}^N u_\mu(q) < \sum_{\mu=1}^N v_\mu(q)$,
then there exist $1 \leq \delta \leq N$ and $q < j \leq n$ with $u_\delta + \varepsilon_q - \varepsilon_j \in B$.

One of the most fundamental examples is

EXAMPLE 4.2. A nonempty subset $B \subset V_n^{(d)}$ is called *strongly stable* if $u \in B$ and $u(j) > 0$, then $u - \varepsilon_j + \varepsilon_i \in B$ for all $i < j$. A strongly stable subset $B \subset V_n^{(d)}$ satisfies the l-exchange property with respect to any monomial order $<$ on R. In fact, if $\sum_{\mu=1}^N u_\mu(i) = \sum_{\mu=1}^N v_\mu(i)$ for $i = 1, 2, \dots, q-1$ and $\sum_{\mu=1}^N u_\mu(q) < \sum_{\mu=1}^N v_\mu(q)$, then there is $1 \leq \delta \leq N$ and $q < j \leq n$ such that $u_\delta(j) > 0$. Hence $u_\delta - \varepsilon_j + \varepsilon_q \in B$.

A class of finite sets $B \subset V_n^{(d)}$ satisfying the l-exchange property naturally arises from the theory of discrete polymatroids.

Let $B \subset V_n^{(d)}$. Let $u, v \in B$, and write $\mathbf{x}^u \mathbf{x}^v = x_{i_1} x_{i_2} \cdots x_{i_{2d}}$ with $i_1 \leq i_2 \leq \cdots \leq i_{2d}$. Set $\mathbf{x}^{u'} = \prod_{j=1}^d x_{2j-1}$ and $\mathbf{x}^{v'} = \prod_{j=1}^d x_{2j}$, and we define the map

$$\mathrm{sort} \colon B \times B \to V_n^{(d)} \times V_n^{(d)}, \quad (u, v) \to (u', v').$$

We call $B \subset V_n^{(d)}$ *sortable* if $\mathrm{Im}(\mathrm{sort}) \subset B \times B$. It is known ([16, Theorem 14.2], [5]) that if $B \subset V_n^{(d)}$ is sortable, then the toric ideal $I_{K[B]}$ possesses a Gröbner basis consisting of the sorting relations $y_u y_v - y_{u'} y_{v'}$ with $u, v \in B$ and $(u', v') = \mathrm{sort}(u, v)$ with respect to the "sorting order" $<_{\mathrm{sort}}$. It is shown in [9] that the set of bases of a discrete polymatroid with the strong exchange property is sortable.

Theorem 4.3. *If B is sortable and is the set of bases of a discrete polymatroid, then B satisfies the l-exchange property with respect to $<_{\mathrm{sort}}$.*

Proof. Let $\prod_{\mu=1}^{N} y_{u_\mu}$ and $\prod_{\mu=1}^{N} y_{v_\mu}$ are standard monomials of $I_{K[B]}$ of degree N with respect to $<_{\mathrm{sort}}$ and suppose that $\mathrm{sort}(u_\mu, u_{\mu'}) = (u_\mu, u_{\mu'})$ and $\mathrm{sort}(v_\mu, v_{\mu'}) = (v_\mu, v_{\mu'})$ for all $1 \le \mu < \mu' \le n$. If $\sum_{\mu=1}^{N} u_\mu(i) = \sum_{\mu=1}^{N} v_\mu(i)$ for $i = 1, 2, \ldots, q-1$, then $u_\mu(i) = v_\mu(i)$ for all $1 \le \mu \le N$ and for all $1 \le i < q$. If, in addition, $\sum_{\mu=1}^{N} u_\mu(q) < \sum_{\mu=1}^{N} v_\mu(q)$, then there is $1 \le \delta \le N$ such that $u_\delta(q) < v_\delta(q)$. Since $u_\delta(i) = v_\delta(i)$ for all $1 \le i < q$, the symmetric exchange property of B guarantees that there is $q < j \le n$ with $u_\delta + \varepsilon_q - \varepsilon_j \in B$, as desired. $\square$

Another natural class of finite sets $B \subset V_n^{(d)}$ satisfying the l-exchange property comes from algebras of Segre-Veronese type ([12]).

Fix nonnegative integers $1 \le r \le n$ and $n_0, n_1, \ldots, n_r$ with $0 = n_0 < n_1 < \cdots < n_{r-1} < n_r = n$. Let $\{\alpha_1, \ldots, \alpha_r\}, \{\beta_1, \ldots, \beta_r\}$ and $\{c_1, \ldots, c_n\}$ be sets of nonnegative integers. Write

$$B = B\left(\{\alpha_j\}_{j=1}^{r}, \{\beta_j\}_{j=1}^{r}, \{c_i\}_{i=1}^{n}\right)$$

for the subset of $V_n^{(d)}$ consisting of all vectors $u = (u(1), \ldots, u(n)) \in V_n^{(d)}$ such that
(i) $u(i) \le c_i$ for each $1 \le i \le n$;
(ii) $\beta_j \le \sum_{k=n_{j-1}+1}^{n_j} u(k) \le \alpha_j$ for each $1 \le j \le r$.

Theorem 4.4. *If $B = B(\{\alpha_j\}_{j=1}^{r}, \{\beta_j\}_{j=1}^{r}, \{c_i\}_{i=1}^{n})$ is nonempty, then $B \subset V_n^{(d)}$ is sortable and satisfies the l-exchange property with respect to $<_{\mathrm{sort}}$.*

Proof. In [12] it is noticed that B is sortable. Thus, except for the last sentence with using the symmetric exchange property, the proof of Theorem 4.3 is valid in the present situation. What we must prove is that, for u and v belonging to B with $u(i) = v(i)$ for all $1 \le i < q$ and with $u(q) < v(q)$, there is $q < j \le n$ such that $u + \varepsilon_q - \varepsilon_j \in B$.

Let $n_{j-1} < q \le n_j$. Since $u(q) < c_q$, if $q < n_j$ and if there is $q < q' \le n_j$ with $u(q') > 0$, then $u + \varepsilon_q - \varepsilon_{q'} \in B$.

Let either (i) $q = n_j$ or (ii) $n_{j-1} < q < n_j$ with $u(q+1) = \cdots = u(n_j) = 0$. Since

$$\beta_j \leq \sum_{k=n_{j-1}+1}^{n_j} u(k) = \sum_{k=n_{j-1}+1}^{q} u(k) < \sum_{k=n_{j-1}+1}^{q} v(k) \leq \sum_{k=n_{j-1}+1}^{n_j} v(k) \leq \alpha_j,$$

there is $j < j' \leq r$ with

$$\beta_{j'} \leq \sum_{k=n_{j'-1}+1}^{n_{j'}} v(k) < \sum_{k=n_{j'-1}+1}^{n_{j'}} u(k) \leq \alpha_{j'}.$$

If $n_{j'-1} < i' \leq n_{j'}$ with $u(i') > 0$, then $u + \varepsilon_q - \varepsilon_{i'} \in B$, as required. □

We now turn to the discussion of a class of finite sets $B \subset V_n^{(d)}$ satisfying the l-exchange property related with classical root systems.

Let $n \geq 3$. Let $\mathbf{A}_{n-1}^{(+)}$ denote the set of positive roots of the root system $\mathbf{A}_{n-1}$, i.e., $\mathbf{A}_{n-1}^{(+)} = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}$. Set $\varepsilon = \varepsilon_1 + \cdots + \varepsilon_n$, $f_{i,j} = \varepsilon + (\varepsilon_i - \varepsilon_j)$ with $1 \leq i < j \leq n$, and

$$B(\mathbf{A}_{n-1}) = \{f_{i,j} : 1 \leq i < j \leq n\} \cup \{\varepsilon\} \subset V_n^{(n)}.$$

Using the notation $y_{i,j}$ instead of $y_{f_{i,j}}$ and y instead of y_ε, we write $<_{\mathrm{rev}}$ for the reverse lexicographic order $<_{\mathrm{rev}}$ on the polynomial ring $R = K[y, \{y_{i,j}\}_{1 \leq i < j \leq n}]$ in $\binom{n}{2}+1$ variables over K induced by the ordering of the variables

$$\begin{aligned} y < y_{1,n} < y_{1,n-1} < \cdots < y_{1,3} < y_{1,2} < y_{2,n} < \cdots < y_{2,3} \\ < \cdots < y_{n-2,n} < y_{n-2,n-1} < y_{n-1,n}. \end{aligned}$$

It is known ([8], [13]) that the reduced Gröbner basis of $I_{K[B]}$ with respect to $<_{\mathrm{rev}}$ consists of the binomials $y_{i,j}y_{j,k} - yy_{i,k}$ with $1 \leq i < j < k \leq n$ and of $y_{i',j}y_{i,j'} - y_{i',j'}y_{i,j}$ with $1 \leq i' < i < j < j' \leq n$.

Since $\mathrm{sort}((2,0,1,1),(1,1,2,0)) = ((2,0,2,0),(1,1,1,1))$ and $(2,0,2,0) \notin B(\mathbf{A}_3)$, it follows that $B(\mathbf{A}_{n-1})$ is *not* sortable for $n \geq 4$.

It follows from [14, Theorem 2.1] that

Lemma 4.5. *Let B be a subset of $B(\mathbf{A}_{n-1})$ with $\varepsilon \in B$ and with the property*

($\sharp$) *if $1 \leq i' \leq i < j \leq j' \leq n$ and if $f_{i,j} \in B$, then $f_{i',j'} \in B$,*

and fix a monomial order $<$ on $R = K[y, \{y_{i,j}\}_{f_{i,j} \in B}]$ satisfying the conditions

(α) *if $1 \leq i' < i < j < j' \leq n$ and if $f_{i,j} \in B$, then $y_{i',j'}y_{i,j} < y_{i',j}y_{i,j'}$;*

(β) *if $1 \leq i < j < k \leq n$ and if $f_{i,j}, f_{j,k} \in B$, then $yy_{i,k} < y_{i,j}y_{j,k}$.*

Then the reduced Gröbner basis of the toric ideal $I_{K[B]}$ with respect to $<$ consists of all binomials $y_{i,j}y_{j,k} - yy_{i,k} \in I_{K[B]}$ with $1 \leq i < j < k \leq n$ and of all binomials $y_{i',j}y_{i,j'} - y_{i',j'}y_{i,j} \in I_{K[B]}$ with $1 \leq i' < i < j < j' \leq n$.

Theorem 4.6. *Let B be a subset of $B(\mathbf{A}_{n-1})$ with $\varepsilon \in B$ and with the property $(\sharp)$ and fix a monomial order $<$ on $R = K[y, \{y_{i,j}\}_{f_{i,j} \in B}]$ satisfying the conditions (α) and (β). Then $B \subset V_n^{(n)}$ satisfies the l-exchange property with respect to $<$.*

Proof. Let $y^a y_{i_1,j_1} \cdots y_{i_N,j_N}$ and $y^{a'} y_{i'_1,j'_1} \cdots y_{i'_{N'},j'_{N'}}$ be standard monomials of the toric ideal $I_{K[B]}$ of degree $a+N$ $(= a'+N')$ with respect to $<$. Let $u = a \cdot \varepsilon + \sum_{k=1}^{N}(\varepsilon + (\varepsilon_{i_k} - \varepsilon_{j_k}))$ and $v = a' \cdot \varepsilon + \sum_{k=1}^{N'}(\varepsilon + (\varepsilon_{i'_k} - \varepsilon_{j'_k}))$, and suppose that $u(i) = v(i)$ for $i = 1, 2, \ldots, q-1$ (with $q \leq n-1$) and $u(q) < v(q)$.
(i) If $j_k < q$ for some $1 \leq k \leq N$ and if $j_{k_1} \geq j_k$ for all $1 \leq k_1 \leq N$, then there is $1 \leq k' \leq N'$ with $j_k = j'_{k'}$. If $i_k \neq i'_{k'}$ and, say, $i_k < i'_{k'}$, then there is $1 \leq k_2 \leq N$ with $i_{k_2} = i'_{k'}$. Since $f_{i'_{k'},j'_{k'}} \in B$, in case of $j_{k_2} > j_k$, the monomial $y^a y_{i_1,j_1} \cdots y_{i_N,j_N}$ cannot be a standard monomial. Hence $j_{k_2} = j_k$, and the variable $y_{i'_{k'},j'_{k'}}$ appears in both $y^a y_{i_1,j_1} \cdots y_{i_N,j_N}$ and $y^{a'} y_{i'_1,j'_1} \cdots y_{i'_{N'},j'_{N'}}$. Thus by using induction on $a+N$ the desired result arises. Let $j_k \geq q$ for all $1 \leq k \leq N$.
(ii) If $i_k > q$, then $(\varepsilon + (\varepsilon_{i_k} - \varepsilon_{j_k})) + \varepsilon_q - \varepsilon_{i_k} \in B$. Let $i_k \leq q$ for all $1 \leq k \leq N$.
(iii) If $j_k = q$, then $(\varepsilon + (\varepsilon_{i_k} - \varepsilon_q)) + \varepsilon_q - \varepsilon_{q+1} \in B$. Let $j_k \neq q$ for all $1 \leq k \leq N$. Thus by (i) $j_k > q$ for all $1 \leq k \leq N$.
(iv) Now, since $u(q) < v(q)$ and since $j_k > q$ for all $1 \leq k \leq N$, there is $1 \leq k' \leq N'$ with $i'_{k'} = q$. If $a > 0$, then $\varepsilon + \varepsilon_q - \varepsilon_{j'_{k'}} = f_{i'_{k'},j'_{k'}} \in B$. Let $a = 0$.

Now, by (ii), (iii) and (iv), one has $u(1) + \cdots + u(q) = (q+1)N = (q+1)(a'+N')$. However, $v(1)+\cdots+v(q) \leq q(a'+N')+N' \leq u(1)+\cdots+u(q)$. This contradicts $u(i) = v(i)$ for $i = 1, 2, \ldots, q-1$ and $u(q) < v(q)$. $\square$

Corollary 4.7. *With keeping the notation as above, $B(\mathbf{A}_{n-1}) \subset V_n^{(n)}$ satisfies the l-exchange property with respect to $<_{\mathrm{rev}}$.*

5. Rees algebras

Let, as before, B be a nonempty subset of $V_n^{(d)}$ and $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables over K. Write I_B for the monomial ideal of S generated by those monomials $\mathbf{x}^u = \prod_{i=1}^{n} x_i^{u(i)}$ with $u = (u(1), \ldots, u(n)) \in B$.

Let $T = K[\{x_1, \ldots, x_n\} \cup \{y_u\}_{u \in B}]$ be the polynomial ring over K with the standard grading, i.e., each $\deg x_i = 1$ and each $\deg y_u = 1$. If $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_u)_{u \in B}$ are vectors with each $0 \leq a_i, b_u, \in \mathbb{Z}$, then we write $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}$ for the monomial $(x_1^{a_1} \cdots x_n^{a_n}) \prod_{u \in B} y_u^{b_u}$ belonging to T.

The toric ideal of the Rees algebra $R(I_B)$ is the ideal $P_{R(I_B)} \subset T$ which is the kernel of the surjective homomorphism $\varphi \colon T \to R(I_B)$ defined by setting $\varphi(x_i) = x_i$ for all $1 \leq i \leq n$ and $\varphi(y_u) = x^u t$ for all $u \in B$.

Let $<^{(\sharp)}$ be an arbitrary monomial order on $R = K[\mathbf{y}]$, and $<_{\text{lex}}$ the lexicographic order on S induced by $x_1 > \cdots > x_n$. We introduce the new monomial order $<_{\text{lex}}^{(\sharp)}$ on T as follows: For monomials $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}$ and $\mathbf{x}^{\mathbf{a}'}\mathbf{y}^{\mathbf{b}'}$ belonging to T, one has $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} <_{\text{lex}}^{(\sharp)} \mathbf{x}^{\mathbf{a}'}\mathbf{y}^{\mathbf{b}'}$ if and only if either (i) $\mathbf{x}^{\mathbf{a}} <_{\text{lex}} \mathbf{x}^{\mathbf{a}'}$ or (ii) $\mathbf{x}^{\mathbf{a}} = \mathbf{x}^{\mathbf{a}'}$ and $\mathbf{y}^{\mathbf{b}} <^{(\sharp)} \mathbf{y}^{\mathbf{b}'}$.

We are now in the position to state the main result of this section.

Theorem 5.1. *Let B be a nonempty subset of $V_n^{(d)}$ and $<^{(\sharp)}$ a monomial order on $R = K[\mathbf{y}]$. Let $\mathcal{G}_{<^{(\sharp)}}(J_{K[B]})$ denote the reduced Gröbner basis of the toric ideal $J_{K[B]}$ of $K[B]$ with respect to $<^{(\sharp)}$. Suppose that B satisfies the l-exchange property with respect to $<^{(\sharp)}$. Let $<_{\text{lex}}$ denote the lexicographic order on S induced by $x_1 > \cdots > x_n$. Then the reduced Gröbner basis of the toric ideal $P_{R(I_B)}$ with respect to $<_{\text{lex}}^{(\sharp)}$ consists of all binomials belonging to $\mathcal{G}_{<^{(\sharp)}}(J_{K[B]})$ together with*

$$x_i y_u - x_j y_v, \tag{6}$$

where $i < j$ with $u+\varepsilon_i = v+\varepsilon_j$ and where j is the largest integer for which $u+\varepsilon_i-\varepsilon_j \in B$. In particular, I_B is of fiber type.

Proof. Let $\mathcal{G}$ denote the finite set which consists of all binomials belonging to $\mathcal{G}_{<^{(\sharp)}}(J_{K[B]})$ and all binomials of type (6). Our goal is to show that $\mathcal{G}$ is a Gröbner basis of $P_{R(I_B)}$ with respect to $<_{\text{lex}}^{(\sharp)}$. (Once we know that $\mathcal{G}$ is a Gröbner basis of $P_{R(I_B)}$ with respect to $<_{\text{lex}}^{(\sharp)}$, an easy computation says that $\mathcal{G}$ is the reduced Gröbner basis of $P_{R(I_B)}$ with respect to $<_{\text{lex}}^{(\sharp)}$.)

Let $f \in T$ be an irreducible binomial belonging to $P_{R(I_B)}$. If $\text{in}_{<_{\text{lex}}^{(\sharp)}}(f) \in R$, then $f \in P_{R(I_B)} \cap R = J_{K[B]}$ and $\text{in}_{<_{\text{lex}}^{(\sharp)}}(f)$ is divided by the initial monomial of a binomial belonging to $\mathcal{G}_{<^{(\sharp)}}(J_{K[B]})$. Let $\text{in}_{<_{\text{lex}}^{(\sharp)}}(f) \notin R$ and write $f = x_i\mathbf{x}^{\mathbf{a}}\prod_{\mu=1}^{N} y_{u_\mu} - x_j\mathbf{x}^{\mathbf{a}'}\prod_{\mu=1}^{N} y_{v_\mu}$, where x_i is the biggest variable appearing in f and where $i < j$. We assume that $\prod_{\mu=1}^{N} y_{u_\mu}$ is a standard monomial of $J_{K[B]}$ with respect to $<^{(\sharp)}$. Our work is to show that $x_i \prod_{\mu=1}^{q} y_{u_\mu}$ is divided by the initial monomial of a binomial of type (6).

Now, replacing $\prod_{\mu=1}^{N} y_{v_\mu}$ with its standard monomial of $J_{K[B]}$ with respect to $<^{(\sharp)}$ enables us to assume that both $\prod_{\mu=1}^{N} y_{u_\mu}$ and $\prod_{\mu=1}^{N} y_{v_\mu}$ are standard monomials of $J_{K[B]}$ with respect to $<^{(\sharp)}$.

Since none of the variables x_k with $k < i$ appears in f, it follows that, for each $1 \leq k < i$, the power of the variable x_k appearing in the monomial $\pi\left(\prod_{\mu=1}^{N} y_{u_\mu}\right)$ is equal to the power of the variable x_k appearing in $\pi\left(\prod_{\mu=1}^{N} y_{v_\mu}\right)$. In other words, $\sum_{\mu=1}^{N} u_\mu(k) = \sum_{\mu=1}^{N} v_\mu(k)$ for $k = 1, 2, \ldots, i-1$. Since the variable x_i cannot appear in $\mathbf{x}^{\mathbf{a}'}$, one has $\sum_{\mu=1}^{N} u_\mu(i) < \sum_{\mu=1}^{N} v_\mu(i)$.

The l-exchange property of B with respect to $<^{(\sharp)}$ guarantees the existence of $1 \leq \delta \leq N$ and $i < l \leq n$ such that $u' = u_\delta + \varepsilon_i - \varepsilon_l \in B$. Thus $x_i y_{u_\delta} - x_l y_{u'} \in P_{R(I_B)}$ and its initial monomial divides $x_i \prod_{\mu=1}^{N} y_{u_\mu}$. Consequently, $x_i \prod_{\mu=1}^{N} y_{u_\mu}$ is divided by the initial monomial of a binomial of type (6), as desired. □

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