Independent University of Moscow
French–Russian Laboratory "J.-V. Poncelet"

International Workshop

IDEMPOTENT AND TROPICAL MATHEMATICS AND PROBLEMS OF MATHEMATICAL PHYSICS

G.L. Litvinov, V.P. Maslov, S.N. Sergeev (Eds.)

Organizing committee: *G.L. Litvinov, V.P. Maslov, S.N. Sergeev, A.N. Sobolevskiĭ*

Web-site: `http://www.mccme.ru/tropical07`

E-mail: `tropical07@gmail.com`

Moscow, August 25–30, 2007

Volume I

Moscow, 2007

Litvinov G.L., Maslov V.P., Sergeev S.N. (Eds.)
Idempotent and tropical mathematics and problems of mathematical physics (Vol. I) – M.: 2007 – 104 pages

This volume contains the proceedings of an International Workshop on Idempotent and Tropical Mathematics and Problems of Mathematical Physics, held at the Independent University of Moscow, Russia, on August 25-30, 2007.

2000 Mathematics Subject Classification: 00B10, 81Q20, 06F07, 35Q99, 49L90, 46S99, 81S99, 52B20, 52A41, 14P99

CONTENTS

Preface 5

Representation of stationary solutions of Hamilton-Jacobi-Bellman equations: a max-plus point of view

Marianne Akian 6

On the assignment problem for a countable state space

M. Akian, S. Gaubert, and V.N. Kolokoltsov 12

Dequantization of coadjoint orbits: the case of exponential Lie groups

Ali Baklouti 21

Quantum Pontryagin principle and quantum Hamilton-Jacobi-Bellman equation: a max-plus point of view

Viacheslav P. Belavkin 22

Tropical Plücker functions

V.I. Danilov, A.V. Karzanov, and G.A. Koshevoy 29

Degree one homogeneous minplus dynamic systems and traffic applications: Part I

N. Farhi, M. Goursat, and J.-P. Quadrat 33

Degree one homogeneous minplus dynamic systems and traffic applications: Part II

N. Farhi, M. Goursat, and J.-P. Quadrat 43

Max-plus cones and semimodules

F. Faye, M. Thiam, L. Truffet, E. Wagneur 50

Duality of cluster varieties

V.V. Fock and A.B. Goncharov 53

From max-plus algebra to non-linear Perron-Frobenius theory: an approach to zero-sum repeated games

Stéphane Gaubert 60

Cyclic projectors and separation theorems in idempotent semimodules
S. Gaubert and S. Sergeev 66

Pseudo-weak convergence of the random sets defined by a pseudo integral based on non-additive measure
T. Grbić and E. Pap 72

The stationary phase method and large deviations
Oleg V. Gulinsky 77

Quantization with a deformed trace
Dmitry Gurevich 81

Transformations preserving matrix invariants over semirings
Alexander E. Guterman 84

Tropical geometry and enumeration of real rational curves
I. Itenberg, V. Kharlamov, and E. Shustin 90

Abstract convexity and cone-vexing abstractions
Semen S. Kutateladze 92

Interval analysis for algorithms of idempotent and tropical mathematics
Grigory L. Litvinov 97

Dequantization procedures related to the Maslov dequantization
G.L. Litvinov and G.B. Shpiz 99

PREFACE

Idempotent mathematics is a new branch of mathematical sciences, rapidly developing and gaining popularity over the last decade. It is closely related to mathematical physics. Tropical mathematics is a very important part of idempotent mathematics. The literature on the subject is vast and includes numerous books and an all but innumerable body of journal papers.

An important stage of development of the subject was presented in the book *Idempotency* edited by J. Gunawardena (Publ. of the Newton Institute, vol **11**, Cambridge University Press, Cambridge, 1998). This book arose out of the well-known international workshop that was held in Bristol, England, in October 1994.

The next stage of development of idempotent and tropical mathematics was presented in the book *Idempotent Mathematics and Mathematical Physics* edited by G.L. Litvinov and V.P. Maslov (Contemporary Mathematics, vol. **377**, American Mathematical Society, Providence, Rhode Island, 2005). The book arose out of the international workshop that was held in Vienna, Austria, in February 2003.

The present volumes contain materials presented for the international workshop *Idempotent and Tropical Mathematics and Problems of Mathematical Physics* (Moscow, Russia, August 25-30, 2007).

It is our pleasure to thank the Independent University of Moscow and the Poncelet Laboratory of this university as well as the Russian Fund for Basic Research and CNRS (France) for their important support. We are grateful to a number of colleagues, especially to L. Kryukova and M. Tsfasman of the Poncelet Laboratory, T. Korobkova and Yu. Torkhov of the Independent University of Moscow, and A. Sobolevskiĭ of the Moscow State University, for their great help. We thank all the authors of the volumes and members of our "idempotent/max-plus/tropical community" for their contributions, help, and useful contacts.

The editors
Moscow, August 2007

Representation of stationary solutions of Hamilton-Jacobi-Bellman equations: a max-plus point of view[1]

Marianne Akian

This talk gathers two recent works: the first one is a joint work with S. Gaubert and C. Walsh first presented in [**5**], the second one is a joint work with B. David and S. Gaubert presented in [**1**].

1. Nonlinear eigenfunctions and stationary solutions of Hamilton-Jacobi-Bellman equations

Let us consider a diffusion control model on a subset X of $\mathbb{R}^n$, that is a stochastic process $\mathbf{x}_t$ with values in X satisfying the stochastic differential equation

$$d\mathbf{x}_t = g(\mathbf{x}_t, \mathbf{u}_t)\ dt + \sigma(\mathbf{x}_t, \mathbf{u}_t)\ d\mathbf{b}_t \tag{1.1}$$

where $(\mathbf{b}_t)$ is a p-dimensional brownian motion, $\mathbf{u} := (\mathbf{u}_t)_{t\geq 0}$ (the control) is a stochastic process with values in a subset U of $\mathbb{R}^p$ and adapted to the filtration of $(\mathbf{b}_t)$, and the drift $g : X \times U \to \mathbb{R}^n$ and the standard deviation $\sigma : X \times U \to \mathcal{M}_{n,p}(\mathbb{R})$ are given.

The stochastic control problem with horizon T consists in maximizing over all the controls $\mathbf{u}$ the quantity

$$\mathbb{E}\left[\int_0^T L(\mathbf{x}_s, \mathbf{u}_s)\ ds + \phi(\mathbf{x}_T)\right], \tag{1.2}$$

where $\mathbf{x}_t$ is the solution of (1.1) with initial condition $\mathbf{x}_0 = x$, L is the Lagrangian and ϕ is a final reward. Let us denote by $v^T(x)$ the value of this optimization problem, and by S^T the map which associates v^T to ϕ. The familly of operators $\{S^t\}_{t\geq 0}$ is the (non linear) evolution semigroup associated to the control problem. Moreover, each operator S^t is additively homogeneous ($S^t(\mu + \phi) = \mu + S^t(\phi)$, where $(\mu + \phi)(x) = \mu + \phi(x)$), and order preserving, thus it is nonexpansive for the sup-norm. If the control problem is purely deterministic then the operators S^t are max-plus linear, that is $S^t(\phi \vee \psi) = S^t(\phi) \vee S^t(\psi)$. In general, the operators S^t are convex, which means that, for all $t \geq 0$ and $x \in X$, the map $\phi \mapsto S^t(\phi)(x)$ is convex on $\mathbb{R}^X$.

We say that λ is an additive eigenvalue of the evolution semigroup if there exists a function $\phi : X \to \mathbb{R}$ such that for all $t \geq 0$, $S^t\phi = \lambda t + \phi$. The

[1]The present work was partially supported by the RFBR/CNRS grant 05-01-02807.

function ϕ is called an additive eigenfunction of the evolution semigroup associated to λ. If X is compact and we restrict ourselves to continuous eigenfunctions, the semigroup has at most one eigenvalue. Moreover, under some regularity assumptions on L, g, σ, the eigenfunctions are exactly the viscosity solutions ϕ of the ergodic Hamilton-Jacobi-Bellman equation

$$\lambda - H(x, D\phi(x), D^2\phi(x)) = 0, \qquad x \in X, \tag{1.3}$$

where the Hamiltonian of the problem is given by

$$H(x,p,A) = \max_{u\in U}\left(\frac{1}{2}\operatorname{tr}(\sigma(x,u)\sigma(x,u)^T A) + \langle p, g(x,u)\rangle + L(x,u)\right). \tag{1.4}$$

In that case, λ is the maximal mean reward by unit of time (the ergodic reward).

Given an eigenvalue λ of $(S^t)_{t\geq 0}$, we are interested in characterizing the set $\mathcal{E}_\lambda$ of associated eigenfunctions, or of solutions ϕ of (1.3).

2. Related results

In the particular deterministic case ($\sigma \equiv 0$), the discrete-time analogue of this problem consists in the characterization of eigenvectors of max-plus linear operators, which has received a considerable amount of attention, see for instance [**7, 8, 16, 4, 3**]. In the finite dimensional setting, eigenvectors are max-plus linear combinations of extremal generators which are themselves in bijection with the "critical classes" (critical classes can be seen as the max-plus analogue of recurrent classes). The deterministic continuous time problem itself has been studied by Maslov, Kolokoltsov, Samborskii and other members of the "idempotent analysis" school [**18, 19**], and by Rouy and Tourin [**20**] in some special cases. More recently, it has been studied as a part of the "weak KAM" theory developped by Fathi [**14, 13, 12**] and Fathi and Siconolfi [**15**]. In this setting, it is shown that when X is a Riemannian manifold and the Lagrangian has smoothness and strict-convexity properties, an eigenfunction is uniquely determined by its restriction to the "projected Aubry set". This set can be thought of as a continuous analogue of the set of "critical states" of the finite dimensional max-plus spectral theory. The case of a non-compact state space X has been studied in different settings by Contreras [**11**], and by Ishii and Mitake [**17**].

In Section 3, we present briefly the results of [**5**], where we showed that general representation results hold in the continuous-time setting, without any regularity assumption on the Lagrangian. These results were inspired

by the discrete-time theory developped in [**3**] and rely on a compactification of the state space X, which is the max-plus analogue of the Martin compactification in potential theory, and is similar to the compactification of metric spaces by horofunctions (generalised Busemann functions).

It is natural to ask whether an analogue of the weak KAM theory can be developped for stochastic control problems. Such an analogue does exist in the simpler finite state space and discrete time case. Indeed, it is shown in [**2**] that the additive eigenvectors are determined by their restriction to a subset of "critical states" obtained by taking exactly one element in each "critical class". Here critical states are defined in terms of subdifferentials, and can be interpreted as follows: a state is critical if there is an optimal stationary randomised strategy for which it is recurrent, and two critical states are in the same critical class if they are in the same recurrence class for an optimal stationary randomised strategy. In the continuous time stochastic case, characterization results were obtained in the uniformly elliptic case and under various settings, by Bensoussan [**10**], Akian, Sulem and Taksar [**6**], and Barles and Da Lio [**9**]: the eigenfunction is then unique up to an additive constant.

In Section 4, we present briefly the results of [**1**] giving a description of the additive eigenspace similar to the one of [**2**], in the simplest degenerate case in which there is only a finite number of "singular points" $x_1, \dots, x_k$ playing the role of critical states and classes. In the deterministic case similar results were obtained in [**20, 18**].

3. Hamilton-Jacobi equations on non-compact spaces

In [**5**], we consider general time continuous semigroups $(S^t)_{t\geq 0}$ of max-plus linear operators with kernel. We assume that $S^t : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^X$ can be written as $S^t g(x) = \sup_{y\in X} (S^t(x,y) + g(y))$. for some function $(x,y) \mapsto S^t(x,y) \in \overline{\mathbb{R}}$. This includes the case of the evolution semigroup associated to the deterministic optimal control problem with dynamics (1.1) with $\sigma \equiv 0$ and criteria (1.2). Without loss of generality, we assume that $\lambda = 0$, in which case an eigenfunction is called a harmonic function. Harmonic functions may take the $-\infty$ value, so that the set $\mathcal{E}_0$ of harmonic functions is a semimodule over the max-plus semiring $\mathbb{R}_{\max}$ (recall that this is the set $\mathbb{R} \cup \{-\infty\}$ endowed with max as addition and $+$ as multiplication).

We need the following assumptions :

(A1) $S^*(x,y) := \sup_{t\geq 0} S^t(x,y)$ is finite for all x and y in X.

(A2) For all $t \geq 0$ and $x, y \in X$, $S^t(x,y) = \sup_\gamma \{I(\gamma)\}$, where the supremum is taken over all paths $\gamma : [0,t] \to X$ from x to y, and

where the reward $I(\gamma)$ is defined as

$$I(\gamma) := \inf_{t_0,\dots,t_n} \sum_{i=0}^{n-1} S^{t_{i+1}-t_i}(\gamma(t_i), \gamma(t_{i+1})),$$

with the infimum taken over all finite increasing sequences (t_i), $i \in \{0, \dots, n\}$, in $[0, t]$ with $t_0 = 0$ and $t_n = t$.

The (max-plus) *Martin kernel* of the semigroup $(S^t)_{t\geq 0}$ with respect to the basepoint $b \in X$ is defined by:

$$K(x, y) = S^*(x, y) - S^*(b, y) \ .$$

The (max-plus) *Martin space* $\mathscr{M}$ of $(S^t)_{t\geq 0}$ is the closure in the topology of pointwise convergence of the set $\mathscr{K} := \{K(\cdot, y) \mid y \in X\} \subset \mathbb{R}^X$. Any element of $\mathscr{M}$ is super-harmonic, which means that it satisfies $S^t\phi \leq \phi$.

For all functions $\xi : X \to \mathbb{R}_{\max}$ and for all $\eta \in \mathscr{M}$, we set:

$$\mu_\xi(\eta) := \limsup_{K(\cdot,x)\to\eta} \ (S^*(b, x) + \xi(x)) \ ,$$

and if $\xi \in \mathscr{M}$ we set: $H(\eta, \xi) := \mu_\xi(\eta)$. The kernel H extends the kernel S^* from $X \times X$ to $\mathscr{M} \times \mathscr{M}$, up to a normalization, since

$$H(K(\cdot, x), K(\cdot, y)) = S^*(b, x) + S^*(x, y) - S^*(b, y) \ .$$

The *minimal boundary* $\mathscr{M}^m$ of $(S^t)_{t\geq 0}$ is the set of elements ξ of $\mathscr{M}$ that are harmonic and satisfy $H(\xi, \xi) = 0$.

THEOREM 3.1 ([**5**, Theorem 3.11]). *Under Assumptions (A1)–(A2), a function $h : X \to \mathbb{R}_{\max}$ is harmonic if and only if it can be written*

$$h = \sup_{w\in\mathscr{M}^m} \ (\nu(w) + w) \ , \tag{3.1}$$

where ν is some upper semicontinuous function from $\mathscr{M}^m$ to $\mathbb{R}_{\max}$. Moreover, μ_h is the greatest ν satisfying this equation.

Here, μ_h plays the role of the spectral measure in the Martin representation theorem. In [**5**], we also show that the elements of the minimal Martin boundary are precisely the extremal generators of $\mathcal{E}_0$ normalized in such a way that $w(b) = 0$, and that they are in correspondance with almost-geodesics.

4. A degenerate Hamilton-Jacobi-Bellman equation on the torus

In [**1**], we study a simple degenerate stochastic control model on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. We consider the stochastic model given by (1.1) with the criteria (1.2) on the set $X = \mathbb{T}^n$. We need the following assumptions:

(A1) L is $\mathcal{C}^2$ on $\mathbb{T}^n \times U$.

(A2) g and σ are Lipschitz continuous on $\mathbb{T}^n \times U$.

(A3) L takes non-positive values.

(A4) There exists k distinct points $x_1, \dots, x_k$ of $\mathbb{T}^n$ such that:
- (a) $\forall i = 1, \dots, k, \exists u_i \in U$ such that $g(x_i, u_i) = 0, L(x_i, u_i) = 0$ and $\sigma(x_i, u_i) = 0$.
- (b) $\forall x \in \mathbb{T}^n \setminus \{x_1, \dots, x_k\}, \forall u \in U$, at least one of the following properties holds: (i) $L(x, u) < 0$ or (ii) $\sigma(x, u)\sigma(x, u)^T$ is a positive definite matrix.

(A5) For all $i = 1, \dots, k$, there exists $u^{(i)} : \mathbb{T}^n \to U$ Lipschitz continuous satisfying $u^{(i)}(x_i) = u_i$, and a continuous function $W^{(i)} : \mathbb{T}^n \to \mathbb{R}$ which vanishes on x_i and is positive elsewhere, and which is a viscosity solution of

$$\begin{aligned} &-\frac{1}{2}\operatorname{tr}(\sigma(x, u^{(i)}(x))\sigma(x, u^{(i)}(x))^T D^2 W^{(i)}(x)) \\ &-\langle g(x, u^{(i)}(x)), \nabla W^{(i)}(x)\rangle + L(x, u^{(i)}(x)) \geq 0, \quad x \in \mathbb{T}^n. \end{aligned} \tag{4.1}$$

Assumptions (A3)–(A5) ensure that the point x_i is stabilized in probability by the control $u^{(i)}$. Since X is compact, the evolution semigroup has a unique eigenvalue, and the previous assumptions imply that this eigenvalue is 0. The following result shows in particular that the set $\{x_1, \dots, x_k\}$ plays a role analogous to the projected Aubry set.

THEOREM 4.1 ([**1**]). *Under Assumptions (A1)–(A5), the map $\mathcal{E}_0 \to \mathbb{R}^k, v \mapsto (v(x_1), ..., v(x_k))$ is a sup-norm isometry, whose image is a non-empty closed convex subset C of $\mathbb{R}^k$, that is invariant by all the translations $(v_1, \dots, v_k) \mapsto (\mu + v_1, \dots, \mu + v_k)$ $(\mu \in \mathbb{R})$.*

A more precise description of the convex set C is given in [**1**].

Bibliography

[1] M. Akian, B. David, and S. Gaubert, *Un théorème de représentation des solutions de viscosité d'une équation d'Hamilton-Jacobi-Bellman ergodique dégénérée sur le tore*, Preprint, 2007.

[2] M. Akian and S. Gaubert, *Spectral theorem for convex monotone homogeneous maps, and ergodic control*, Nonlinear Analysis. Theory, Methods & Applications **52** (2003), no. 2, 637–679.

[3] M. Akian, S. Gaubert, and C. Walsh, *The max-plus Martin boundary*, Déc. 2004, arXiv:math.MG/0412408.

[4] ———, *Discrete max-plus spectral theory*, Idempotent Mathematics and Mathematical Physics (G. L. Litvinov and V. P. Maslov, eds.), Contemporary Mathematics, American Mathematical Society, 2005, Also ESI Preprint 1485, arXiv:math.SP/0405225, pp. 19–51.

[5] ———, *How to find horizon-independent optimal strategies leading off to infinity: a max-plus approach*, Proc. of the 45th IEEE Conference on Decision and Control (CDC'06) (San Diego), 2006.

[6] M. Akian, A. Sulem, and M. Taksar, *Dynamic optimization of long-term growth rate for a portfolio with transaction costs and logarithmic utility*, Math. Finance **11** (2001), no. 2, 153–188.

[7] F. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat, *Synchronization and linearity : an algebra for discrete events systems*, Wiley, New-York, 1992.

[8] R. B. Bapat, *A max version of the Perron-Frobenius theorem*, Linear Algebra Appl. **275/276** (1998), 3–18.

[9] G. Barles and F. Da Lio, *On the boundary ergodic problem for fully nonlinear equations in bounded domains with general nonlinear Neumann boundary conditions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), no. 5, 521–541.

[10] A. Bensoussan, *Perturbation methods in optimal control*, Wiley/Gauthier-Villars Series in Modern Applied Mathematics, John Wiley & Sons Ltd., Chichester, 1988, Translated from the French by C. Tomson.

[11] G. Contreras, *Action potential and weak KAM solutions*, Calc. Var. Partial Differential Equations **13** (2001), no. 4, 427–458.

[12] A. Fathi, *Weak KAM theorem in lagrangian dynamics*, Cambridge University Press, to appear.

[13] ———, *Solutions KAM faibles conjuguées et barrières de Peierls*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 6, 649–652.

[14] ———, *Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 9, 1043–1046.

[15] A. Fathi and A. Siconolfi, *Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation*, Invent. Math. **155** (2004), no. 2, 363–388.

[16] M. Gondran and M. Minoux, *Graphes, dioïdes et semi-anneaux*, TEC & DOC, Paris, 2002.

[17] H. Ishii and H. Mitake, *Representation formulas for solutions of Hamilton-Jacobi equations with convex Hamiltonians*, Indiana Univ. Math. J. (2007), to appear.

[18] V. N. Kolokoltsov and V. P. Maslov, *Idempotent analysis and its applications*, Mathematics and its Applications, vol. 401, Kluwer Academic Publishers Group, Dordrecht, 1997.

[19] V. P. Maslov and S. N. Samborskiĭ, *Idempotent analysis*, Advances In Soviet Mathematics, vol. 13, Amer. Math. Soc., Providence, 1992.

[20] E. Rouy and A. Tourin, *A viscosity solutions approach to shape-from-shading*, SIAM J. Numer. Anal. **29** (1992), no. 3, 867–884.

On the assignment problem for a countable state space[1]

M. Akian, S. Gaubert, and V.N. Kolokoltsov

1. Introduction and formulation of the results.

Our main results are formulated below as Theorems 1.1 - 1.3. The first two theorems are proved in Section 2. A proof of Theorem 1.3 will be given elsewhere. Section 3 contains the algebraic interpretation of our results and methods.

Let X be either the set of natural numbers $\mathbb{N}$ or that of all integer numbers $\mathbb{Z}$. Further we work with infinite matrices $B = (b_{ij})$, $i, j \in X$ that will always have entries in $\mathbb{R} \cup \{-\infty\}$ and satisfy the following condition:

(C) For any i there is a j such that $b_{ij} \neq -\infty$, for any j there is an i such that $b_{ij} \neq -\infty$, and $b_{ij} \to -\infty$ as $|i - j| \to \infty$.

Any such matrix B defines the mapping B from the space of functions bounded below on X to the set $\mathbb{R}^X$ of all real valued functions on X, by the formula

$$(Bf)_i = \sup_j \{b_{ij} - f_j\} \tag{1.1}$$

By B^T we shall denote the transpose matrix of B and the corresponding operator

$$(B^T g)_j = \sup_i \{b_{ij} - g_i\}. \tag{1.2}$$

A crucial fact about B^T is that the pair (B, B^T) defines a Galois connection in $\mathbb{R}^X$, which means in particular (see [**2**]) that B^T is a generalized inverse to B in the sense that if the equation $Bf = g$ with a given $g \in \mathbb{R}^X$ has a solution $f \in \mathbb{R}^X$, then necessarily $\tilde{f} = B^T g$ is also a solution of this equation.

The infinite dimensional theory depends crucially on the class of functions, in which the solutions to the equation $Bf = g$ are sought, and on the corresponding definitions of solutions to the assignment problem. Let l_∞; l_1; and l_0 denote respectively the spaces of functions $s = (s_i)$, $i \in X$ on X such that $\sup_i |s_i| < \infty$; $\sup_n \sum_{|i| \leq n} |s_i| < \infty$; and the (finite) limit $\lim_{|n| \to \infty} s_n$ exists. Let $\mathbf{l} = \mathbf{l}(X)$ be any of these spaces.

Definition 1.1. A matrix B (satisfying (C)) will be called $\mathbf{l}$-*strongly regular* if there exists a function $g \in \mathbf{l}$ such that (i) $f = B^T \in \mathbf{l}$, (ii) f is the unique solution in $\mathbb{R}^X$ of the equation $Bf = g$ and (iii) g is the unique

[1]Partially supported by the joint RFBR/CNRS grant 05-01-02807.

solution in $\mathbb{R}^X$ of the equation $f = B^T g$. In this case g (respectively, f) is said to belong to the $\mathbf{l}$-*simple image* of B (respectively, of B^T).

Of course, it follows from this definition that B is $\mathbf{l}$-strongly regular if and only if B^T is strongly regular.

Remark 1.1. One can show (though this is not obvious) that in the case of finite X our definition coincides with the standard definition of strong regularity given by P. Butkovič, see [**5**]. In fact, we added a crucial additional condition in our definition, which turns out to be automatically fulfilled for finite, but not for infinite X.

Definition 1.2. For any two bijections $F, G : X \mapsto X$, we define the *distance* between them by

$$\rho(F, G) = \sup_n |F(n) - G(n)|.$$

The binary relation $F \sim G$ iff $\rho(F, G) < \infty$ is clearly an equivalence relation on the set of bijections defining the decomposition of this set into non-intersecting classes. We shall say that F is *locally bounded* if it is equivalent to the identity map.

Definition 1.3. A bijection $F : X \mapsto X$ is called a *(global) solution* or a *strong solution* respectively to the assignment problem for a matrix B if

$$\liminf_{n\to\infty} \sum_{|i|\le n} (b_{iF(i)} - b_{iG(i)}) \ge 0 \tag{1.3}$$

for any other bijection $G : X \mapsto X$ or if

$$\liminf_{n\to\infty} \sum_{|i|\le n} (b_{iF(i)} - b_{iG(i)}) > 0, \tag{1.4}$$

respectively. We say that this solution is a *locally bounded* $\mathbf{l}$-*solution*, if F is locally bounded and the "solution sequence" $b_{iF(i)}$ belongs to $\mathbf{l}(X)$. We say that F is a *local solution* if (1.3) (or (1.4) respectively) holds for all G such that the distance between F and G is finite.

If a strong solution exists, then the solution to the assignment problem is obviously unique.

Definition 1.4. A matrix B is called *normal* (respectively, *strongly normal*) if all its non-diagonal entries are non-positive (respectively, negative) and $b_{ii} = 0$ for all i.

This definition is literally the same as the usual finite- dimensional one (see [**4**]). The normal (respectively, strongly normal) matrices present a class of examples, where the identity map is an obvious locally bounded l_1-solution (respectively, unique solution) to the assignment problem. As our first result will show, this class of matrices presents natural "normal forms" for strongly regular matrices.

DEFINITION 1.5. Matrices B and C are called *(locally bounded)* **l**-*similar* if there exist two locally bounded bijections $H : X \mapsto X$, $K : X \mapsto X$ and two vectors g and f from $\mathbf{l}(X)$ such that

$$c_{ij} = b_{H(i)K(j)} - \phi_i - \psi_j. \tag{1.5}$$

This is also a standard definition in the case of finite X (see e.g. [**4**] and Section 4 below for an intuitive interpretation). The importance of this notion is basically due to the following result.

PROPOSITION 1.1. *(i) Conditions (C) and* **l**-*strong regularity for matrices with entries in* $\mathbb{R} \cup \{-\infty\}$ *are all invariant under* **l**-*similarity. (ii) The property to have an* l_1- *solution (in particular locally bounded or strong) to the assignment problem for matrices with entries in* $\mathbb{R} \cup \{-\infty\}$ *is invariant under* l_1-*similarity. (iii) The property to have a locally bounded local* l_0- *solution (in particular strong) to the assignment problem for matrices with entries in* $\mathbb{R} \cup \{-\infty\}$ *is invariant under* l_0-*similarity.*

PROOF. (i) The invariance of condition (C) is obvious. The invariance of **l**-strong regularity follows from the observation that if C and B are related by (1.5) then the equation $g = Cf$ is equivalent to the equation

$$(g + \phi)(H^{-1}) = B((f + \psi)(K^{-1})).$$

(ii) Let a bijection F be a solution to the assignment problem of a matrix C. Notice that

$$\sum_{|i|\le n} (c_{iF(i)} - c_{iG(i)}) = \sum_{|i|\le n} (b_{H(i)KF(i)} - b_{H(i)KG(i)}) + \sum_{|i|\le n} (\psi_{G(i)} - \psi_{F(i)}). \tag{1.6}$$

Clearly, the last sum on the r.h.s. tends to zero as $n \to \infty$ whenever $\psi \in l_1$. Hence, F is an l_1-solution (respectively, a strong l_1-solution) to the assignment problem for the matrix C if and only if the mapping KFH^{-1} is an l_1-solution (respectively, a strong l_1-solution) to the assignment problem for the matrix B.

(iii) By the previous discussion, it suffices to show that the sequence

$$b_n = \sum_{|i|<n} (\psi_{G(i)} - \psi_{F(i)})$$

tends to zero as $n \to \infty$ if and only if both F and G are not infinitely far from the identity map. To this end, observe that due to the last condition, there exists a natural number p such that for every n the sets $\{F(i) : |i| < n\}$ and $\{G(i) : |i| < n\}$ both contain the set $\{i : |i| < n - p\}$. Hence

$$b_n = \psi_{G(i_1)} + \psi_{G(i_2)} + ... + \psi_{G(i_p)} - \psi_{F(j_1)} - ... - \psi_{G(j_p)},$$

where i_k (respectively j_k) are such that $|G(i_k)| > n - p$ (respectively $|F(j_k)| > n - p$). We recall now that the function ψ_n has a finite limit as $n \to \infty$, which immediately implies that b_n tends to zero (the statement that we wanted to prove). This was the crucial application of this assumption, which seems to be the weakest possible to provide a link between the solutions to the assignment problem for similar matrices. □

THEOREM 1.1. *A matrix B (satisfying (C)) is* $\mathbf{1}$*-strongly regular if and only if it is* $\mathbf{1}(X)$ *similar to a strongly normal matrix.*

It is of course interesting to know what can be said about the assignment problem for a regular matrix itself (not just for some of its similar matrices). In the analysis of this question (as well as the inverse one), an important role is played by the following functions describing in some sense the size of the problem. Namely, let F be a (possibly local) solution to the assignment problem of a matrix B. Let "the optimal distance" between the points i, j be defined as

$$\begin{aligned}\tilde{b}_{ij} = \sup[&b_{iF(i_1)} - b_{i_1F(i_1)} + b_{i_1F(i_2)} - b_{i_2F(i_2)} + ... \\ &+ b_{i_{n-1}F(i_n)} - b_{i_nF(i_n)} + b_{i_nF(j)} - b_{jF(j)}],\end{aligned} \tag{1.7}$$

where sup is taken over all $n = 1, 2, ...$ and all collections $i_1, i_2, ..., i_n$ of points from X, and "the potential" and "the inverse potential" as the functions on X are given respectively by

$$\phi_i = \sup_j \tilde{b}_{ij}, \qquad \tilde{\phi}_i = \sup_j \tilde{b}_{ji}. \tag{1.8}$$

The following simple properties of these functions are crucial:

(1) the values of $\tilde{b}$, ϕ and $\tilde{\phi}$ do no change if one takes the sup only over families $i_1, ..., i_n$ with pairwise disjoint points (in fact, any cycle gives a non-positive contribution due the assumption that F is a solution to the assignment problem);
(2) $\tilde{b}_{ii}$, ϕ_i and $\tilde{\phi}_i$ are nonnegative for all i (in fact, take $n = 1$ and $i_1 = i$ in (1.7));

(3) the functions ϕ and $-\tilde{\phi}$ satisfy the equation

$$f_i = \sup_j [b_{iF(j)} - b_{jF(j)} + f_j], \quad \forall i \in X, \tag{1.9}$$

or, equivalently,

$$f = B\psi, \qquad \psi_j = b_{F^{-1}(j)j} - f_{F^{-1}j}. \quad \forall i \in X, \tag{1.10}$$

(4) the functions ϕ and $-\tilde{\phi}$ satisfy the equation

$$f_i = \sup_j [\tilde{b}_{ij} + f_j], \quad \forall i \in X. \tag{1.11}$$

Observe that if B is normal then $\tilde{b}_{ij} \leq 0$ and $\phi_i = \tilde{\phi}_i = 0$ for all i, j.

THEOREM 1.2. *(i) If a matrix B (satisfying (C)) is l_0-strongly regular, then it has a (necessarily unique) locally bounded local strong l_0-solution to its assignment problem such that*

$$\limsup_{i,j\to\infty} \tilde{b}_{ij} \leq 0, \tag{1.12}$$

and that the potentials ϕ and $\tilde{\phi}$ are bounded functions. (ii) If B is l_1-strongly regular, then this solution is also a global l_1-solution.

To prove the converse to Theorem 1.2, we shall use the following additional technical assumption on a solution to the assignment problem:

(B($\mathbf{l}(X)$)) Either the potential ϕ or the inverse potential $\tilde{\phi}$ belong to $\mathbf{l}(X)$.

THEOREM 1.3. *If $\mathbf{l}$ is either l_0 or l_1 and if the assignment problem for a matrix B has a (possibly local) locally bounded strong $\mathbf{l}$-solution satisfying condition (B($\mathbf{l}(X)$)), then B is strongly $\mathbf{l}$-regular.*

We have to indicate an unpleasant small gap between the necessary condition and the sufficient condition: from strong l_0-regularity it follows that the potential ϕ belongs to l_∞, but in Theorem 1.3 we assume that $\phi \in l_0$ (which implies (1.12)). However, this discrepancy vanishes when we consider classes of similar matrices, as the following direct corollary of Theorem 1.1 and 1.3 suggests.

COROLLARY 1.1. *Let $\mathbf{l}(X)$ be either l_0 or l_1. Then a matrix B is strongly $\mathbf{l}(X)$-regular if and only if it is $\mathbf{l}(X)$-similar to a matrix having a strong solution to the assignment problem satisfying condition (B($\mathbf{l}(X)$)).*

2. Coverings and sub-differentials. Proofs of Theorems 1.1 and 1.2.

For the analysis of the equation $Bf = g$ (also in a more general setting of uncountable X), an important role belongs to the notion of (abstract) sub-differentials.

DEFINITION 2.1. For a matrix B and $f, g \in \mathbb{R}^X$, the abstract sub-differentials (or B-sub-differentials) are defined as follows (see [**2**] and references therein)

$$\partial f(j) = \{k \in X : (Bf)_k = \sup_l \{b_{kl} - f_l\} = b_{kj} - f_j\},$$

$$\partial^T g(i) = \{k \in X : (B^T g)_k = \sup_l \{b_{lk} - g_l\} = b_{ik} - g_i\}.$$

For a given f the sub-differential is a mapping from X to the set $P(X)$ of subsets of X. For any such mapping G its inverse mapping $G^{-1} : X \mapsto P(X)$ is naturally defined as $G^{-1}(j) = \{i : j \in G(i)\}$.

We shall start with the following well known basic property of sub-differentials that we prove here for completeness.

PROPOSITION 2.1. *If $g = BB^T g \in \mathbb{R}^X$, then $(\partial^T g)^{-1} = \partial B^T g$.*

PROOF.

$$(\partial^T g)^{-1}(j) = \{i : (B^T g)_j = \sup_l \{b_{lj} - g_l\} = b_{ij} - g_i\},$$

which is the same as $g_i = b_{ij} - (B^T g)_j$, or equivalently $B(B^T g)_i = b_{ij} - (B^T g)_j$, and which means that $i \in \partial(B^T g)(j)$. □

PROPOSITION 2.2. *Suppose that functions g, $B^T g \in \mathbb{R}^X$ are bounded from below. Then $B^T g$ is a solution to the equation $Bf = g$ if and only if $\partial^T g(i) \neq \emptyset$ for all i or, equivalently, if the family of the sets $(\partial^T g)^{-1}(j)$, $j \in X$, is a covering of X.*

PROOF. This is a direct consequence of a more general Theorem 3.5 from [**2**], where one only has to observe that the assumption that $f = B^T g$ is bounded from below ensures that the set $\{j : b_{ij} - f_j \geq \beta\}$ is finite for any $i \in X$ and $\beta \in \mathbb{R}$, which is the crucial condition for the applicability of this theorem. □

DEFINITION 2.2. Let G be a mapping from X to the set of its subsets $P(X)$ and let the family of subsets $\{G(j)\}_{j \in X}$ be a covering of X. An element $j \in X$ is called *essential* (with respect to this covering) if

$$\exists i \in X : i \notin \cup_{k \in X \setminus j} G(k).$$

The covering is called *minimal* if all elements of X are essential.

PROPOSITION 2.3. *Suppose that functions g, $B^T g \in \mathbb{R}^X$ are bounded from below. Then $B^T g$ is the unique solution of equation $Bf = g$ in $\mathbb{R}^X$ if and only if $(\partial^T g)^{-1}(j)$, $j \in X$, is a minimal covering of X.*

PROOF. This is again a consequence of a more general Theorem 4.7 from [**2**]. □

The key point in proving Theorems 1.1 and 1.2 is contained in the following statement.

PROPOSITION 2.4. *Suppose that functions g, $B^T g \in \mathbb{R}^X$ are bounded from below, and such that $f = B^T g$ is the unique solution of equation $Bf = g$, and g is the unique solution to the equation $B^T g = f$. Then there exists a locally bounded bijection $F : X \mapsto X$ such that*

$$j = F(i) \iff \partial f(j) = \{i\} \iff \partial^T g(i) = \{j\} \iff j = F(i). \tag{2.1}$$

In particular,

$$\begin{aligned} &\forall k \neq F(i) \qquad b_{iF(i)} - f_{F(i)} > b_{ik} - f_k, \\ &\forall k \neq i \qquad\quad b_{iF(i)} - g_i > b_{kF(i)} - g_k. \end{aligned} \tag{2.2}$$

REMARK 2.1. As one easily checks, the inverse statement holds as well: if a locally bounded bijection F and bounded below functions f, g satisfy (2.1), then $f = B^T g$ is the unique solution of the equation $Bf = g$, and g is the unique solution of the equation $B^T g = f$.

PROOF OF PROP. 2.4. Applying Proposition 2.3 to the equation $B^T g = f$ one concludes that for all i there exists j such that $j \in (\partial f)^{-1}(i)$, but $j \notin (\partial f)^{-1}(k)$ for any $k \neq i$. In other words $(\partial f)(j) = \{i\}$, which by Proposition 2.1 means that $(\partial^T g)^{-1}(j) = \{i\}$. Hence, defining the mapping $F : X \mapsto P(X)$ by the formula

$$F(i) = \{j : (\partial f)(j) = \{i\}\} = \{j : (\partial^T g)^{-1}(j) = \{i\}\},$$

one deduces that $F(i) \neq \emptyset$ for all i and F is injective in the sense that $F(i) \cap F(k) = \emptyset$ whenever $i \neq k$. Applying now Proposition 2.3 to the equation $Bf = g$, one finds that for all j there exists i such that $(\partial^T g)(i) = \{j\}$. From this, one easily concludes that each set $F(i)$ contains precisely one point and that F is surjective, which finally implies that F is a bijection $X \mapsto X$ such that (2.1) holds.

Let us show that F is bounded. In fact, since B satisfies (C) and f is bounded from below, it follows that for any $C > 0$ there exists N such that $b_{ij} - f_j < -C$ whenever $|i - j| > N$. On the other hand, as g is

bounded from below, $b_{iF(i)} - f_{F(i)} = g_i$ is bounded from below, and hence $|i - F(i)| < N$ for large enough N and all i. □

Proof of Theorem 1.1 . Let F be a bijection constructed in Proposition 2.4. From the equation $b_{iF(i)} - f_{F(i)} = g_i$ it follows that $\{b_{iF(i)}\} \in \mathbf{l}(X)$ whenever $f, g \in \mathbf{l}(X)$. Hence the matrix C with entries

$$c_{ij} = b_{iF(j)} - f_{F(j)} - (b_{iF(i)} - f_{F(i)})$$

is strongly normal and is $\mathbf{l}$-similar to B. □

Proof of Theorem 1.2 . The existence of required solution follows from Proposition 1.1. This solution is actually given by the bijection F constructed in Proposition 2.4. The boundedness of ϕ and $\tilde{\phi}$ follows of course from (1.11). To prove the latter, one observes that according to the second inequality of (2.2)

$$\tilde{b}_{ij} \leq g_i - g_j,$$

and the r.h.s. of this inequality tends to 0 as $i, j \to \infty$, since $g \in l_0$. □

3. Algebraic interpretation

A natural algebraic language for the analysis of discrete event systems and optimal control is supplied by the so called idempotent algebra, in particular the $(\max, +)$-algebra (see e.g. [**6**], [**8**]). The $(\max, +)$-algebra deals with the semiring $\mathbb{R} \cup \{-\infty\}$ equipped with the binary operations $\oplus = \max$ and $\otimes = +$ and with finite-dimensional semimodules over this semiring. The main impetus to the development of this algebra (and further its infinite-dimensional generalizations, see [**8**], [**9**]) was a simple observation that the basic Bellman operator

$$(Bf)_i = \sup_j (b_{ij} + f_j), \quad i \in X \tag{3.1}$$

of the optimal control theory is linear in this structure, i.e.

$$B(\alpha_1 \otimes f_1 \oplus \alpha_2 \otimes f_2 = \alpha_1 \otimes B(f_1) \oplus \alpha_2 \otimes B(f_2)$$

for $\alpha_1, \alpha_2 \in \mathbb{R} \cup \{-\infty\}$ and $f_1, f_2 \in (\mathbb{R} \cup \{-\infty\})^X$. (Note that previously we denoted by Bf the operator which would now be denoted by $B(-f)$; this was more convenient for the study of the inversion of B.) In fact the operators of type (3.1) are the most natural $(\max, +)$-linear operators, though they do not exhaust all of them (see e.g. [**1**], [**7**], [**9**] and references therein for classical and recent results on this "kernel type" representations).

The introduction of main notions and objects studied in this article was motivated by the development of the $(\max, +)$-algebra that supplies a clear

intuitive interpretation for them. For instance, the strong regularity turns to be a linear algebraic problem connected with the inversion of matrices (or more generally linear operators having kernel representation). Our notion of similarity is obtained by rewriting the classical algebraic notion of similarity of matrices in $(\max, +)$. Next, the solution to the assignment problem

$$\sup_F \sum_{i\in X} b_{iF(i)} = \oplus_F \otimes_{i\in X} b_{iF(i)}$$

turns out to be the $(\max, +)$ analogue of the classical algebraic notion of matrix permanent. Namely, solving the assignment problem means finding the $(\max, +)$- permanent of a matrix (in our case infinite dimensional). If this solution is strong, then one says that this matrix has a strong permanent.

An important tool in algebra is given by the so called Kleene star that for a given matrix A is defined by

$$A^\star = \oplus_{k=0}^\infty A^k$$

(the powers A^k are understood in the operations of a given algebra). In $(\max, +)$-algebra the elements of $A^\star$ clearly define the longest path on the graph associated with A (see details e.g. in [**3**] for finite and respectively infinite space X), and its columns are natural candidates for the solution of the eigenvalue - eigenvector equation for A (equation of type (1.9)). Hence the non-surprising appearance of $A^\star$ in our setting (our functions $\tilde{b}_{ij}$ and potentials ϕ_i represent appropriately normalized elements of $A^\star$). As was observed in [**10**], the functions $\tilde{b}_{ij}$ turn out to be useful also in the analysis of the Monge-Kantorovich mass transfer problem, which is a natural analogue of the assignment problem for general measurable (uncountable) state space X.

Bibliography

[1] M. Akian. Densities of idempotent measures and large devoations. Trans. Amer. Math. Soc. **351:11** (1999), 4515-4543.

[2] M. Akian, S. Gaubert, V. Kolokoltsov. Set coverings and invertibility of functional Galois connections. Contemporary Mathematics, v. 377 (2005), 19-51.

[3] M. Akian, S. Gaubert and C. Walsh. Discrete max-plus spectral theory. Contemporary Mathematics, v. 377 (2005), 53-77.

[4] P. Butkovič. Simple image set of (max,+) linear mappings. Discrete Appl. Math. **105** (2000), 73-86.

[5] P. Butkovič, Max-algebra: linear algebra of combinatorics? Linear Algebra Appl. **367** (2003), 313-335.

[6] J. Gunawardena (Ed.). Idempotency. Cambridge University Press, 1998.

[7] V.N. Kolokoltsov. On linear, additive and homogeneous operators in idempotent analysis. In: V.P. Masov, S.N. Samborskii (Eds.) Idempotent Analysis. Adv. Sov. Math. **13**, AMS Providence (1992), 87-101.
[8] V.N. Kolokoltsov and V.P. Maslov. Idempotent analysis and its Applications. Kluwer Academic, 1997.
[9] G.L. Litvinov, V.P. Maslov, G.B. Shpiz. Idempotent functional analysis. An algebraic approach. Mathematical Notes **69: 5** (2001), 696-729, also arXiv:math.FA/0009128 (2000) (http://arXiv.org).
[10] L. Rüschendorf. On c-optimal random variables. Statistics and Probability Letters **27** (1996), 267-270.

Dequantization of coadjoint orbits: the case of exponential Lie groups[1]

Ali Baklouti

It is well known that the unitary dual $\hat{G}$ of an exponential solvable Lie group G is homeomorphic to the space of coadjoint orbits. Given a coadjoint orbit $\mathcal{O}$, the orbit method enables us to construct the associated unitary and irreducible representation. The inverse procedure is called *Dequantization* and it consists in going backwards from an irreducible unitary representation $\pi : G \longrightarrow U(H)$ to the coadjoint orbit $\mathcal{O}$ and its associated geometric objects. Here, $U(H)$ denotes the group of unitary operators on some complex inner product space H. Towards dequantization, we consider the Poisson characteristic variety of some topological unitary modules over a deformed algebra appropriately associated with the representation in question. In the case of nilpotent Lie groups, we showed that the Poisson characteristic variety coincides with the associated coadjoint orbit. For exponential Lie groups, we conjecture that such a variety coincides with the Zariski closure of the orbit. In this work, we prove such a conjecture for some restrictive classes of exponential Lie groups.

[1]Partially supported by the D.G.R.S.T Research Unity:00 UR 1501.

Quantum Pontryagin principle and quantum Hamilton-Jacobi-Bellman equation: a max-plus point of view

Viacheslav P. Belavkin

We exploit the separation of the filtering and control aspects of quantum feedback control to consider the optimal control as a classical stochastic problem on the space of quantum states. We derive the corresponding Hamilton-Jacobi-Bellman equations using the elementary arguments of classical control theory and show that this is equivalent to a Hamilton-Pontryagin setup. We show that, for cost functionals that are linear in the state, the theory yields the traditional Bellman equations treated so far in quantum feedback. A controlled qubit with a feedback is considered as example.

1. Introduction

Quantum measurement, by its very nature, leads always to partial information about a system in the sense that some quantities always remain uncertain, and due to this the measurement typically alters the prior to a posterior state in process. The Belavkin nondemolition principle [**3, 5**] states that this state reduction can be effectively treated within a non-demolition scheme [**5**],[**6**] when measuring the system over time. Hence we may apply a quantum filter for either discrete [**1**] or time-continuous [**3**] non-demolition state estimation, and then consider feedback control based on the results of this filtering. The general theory of continuous-time nondemolition estimation developed in [**6**],[**8**],[**9**] derives for quantum posterior states a stochastic filtering evolution equation not only for diffusive but also for counting measurements, however we will consider here the special case of Belavkin quantum state filtering equation based on a diffusion model described by a single white noise innovation, see e.g. [**7**]. Once the filtered dynamics is known, the optimal feedback control of the system may then be formulated as a distinct problem.

The separation of the classical world from the quantum world is, of course, the most notoriously troublesome task faced in modern physics. At the very heart of this issue is the very different meanings we attach to the word *state*. What we want to exploit is the fact that the separation of the control from the filtering problem gives us just the required separation of classical from quantum features. By the quantum state we mean the von Neumann density matrix which yields all the (stochastic) information

available about the system at the current time - this we also take to be the state in the sense used in control engineering. All the quantum features are contained in this state, and the filtering equation it satisfies may then to be understood as classical stochastic differential equation which just happens to have solutions that are von Neumann density matrix valued stochastic processes. The ensuing problem of determining optimal control may then be viewed as a classical problem, albeit on the unfamiliar state space of von Neumann density matrices rather than the Euclidean spaces to which we are usually accustomed. Once we get accustomed to this setting, the problem of dynamical programming, Bellman's optimality principle, etc., can be formulated in much the same spirit as before.

2. Notations and Facts

The Hilbert space for our fixed quantum system will be a complex, separable Hilbert space $\mathfrak{h}$. We shall use the following spaces of operators:

$\mathcal{A} = \mathfrak{B}(\mathfrak{h})$	- the Banach algebra of bounded operators on $\mathfrak{h}$;
$\mathcal{A}_\star = \mathfrak{T}(\mathfrak{h})$	- the predual space of trace-class operators on $\mathfrak{h}$;
$\mathcal{S} = \mathfrak{S}(\mathfrak{h})$	- the positive, unital-trace operators (states) on $\mathfrak{h}$;
$\mathcal{T}_0 = \mathfrak{T}_0(\mathfrak{h})$	- the tangent space of zero-trace operators on $\mathfrak{h}$;
$\mathcal{T}_0^\star = \mathfrak{T}_0(\mathfrak{h})^\star$	- the cotangent space (see below).

The space $\mathcal{A}_\star$ equipped with the trace norm $\|\varrho\|_1 = \operatorname{tr}|\varrho|$ is a complex Banach space, the dual of which is identified with the algebra $\mathcal{A}$ with usual operator norm. The natural duality between the spaces $\mathcal{A}_\star$ and $\mathcal{A}$ is indicated by

$$\langle \varrho, A\rangle := \operatorname{tr}\{\varrho A\}, \tag{2.1}$$

for each $\varrho \in \mathcal{A}_\star, A \in \mathcal{A}$. The positive elements, in the sense of positive definiteness $\varrho \geq 0$, form a cone $\mathcal{T}_+$ of the real subspace $\mathcal{T} \subset \mathcal{A}_\star$ of all Hermitian elements $\varrho = \varrho^\dagger$, and the unit trace elements $\varrho \in \mathcal{T}_+$ normalized as $\|\varrho\|_1 = 1$ are called normal states. Thus $\mathcal{S} = \mathcal{T}_+ \cap \mathcal{T}_1$, where $\mathcal{T}_1 = \{\tau \in \mathcal{T} : \operatorname{tr}\tau = 1\}$, and the extremal elements $\varrho \in \mathcal{S}$ of the convex set $\mathcal{S} \subset \mathcal{T}_+$ correspond to pure quantum states. Every state ϱ can be parametrized as $\varrho(q) = \varrho_0 - q$ by a *tangent element* $q \in \mathcal{T}_0$ with respect to a given state $\varrho_0 \in \mathcal{S}$. We may use the duality (2.1) to introduce *cotangent elements* $p \in \mathcal{T}_0^\star$. Knowledge of $\langle q, p\rangle$ for each $q \in \mathcal{T}_0$ will only serve to determine $p \in \mathcal{A}$ up to an additive constant (as the q's are trace-free): for this reason we should think of cotangents elements p as equivalence classes

$$p[X] = \{A \in \mathcal{A} : A = X + \lambda I, \text{ for some } \lambda \in \mathbb{R}\}. \tag{2.2}$$

The symmetric tensor power $\mathcal{A}_{sym}^{\otimes 2} = \mathcal{A} \otimes_{sym} \mathcal{A}$ of the algebra $\mathcal{A}$ is the subalgebra of $\mathfrak{B}\left(\mathfrak{h}^{\otimes 2}\right)$ of all bounded operators on the Hilbert product space $\mathfrak{h}^{\otimes 2} = \mathfrak{h} \otimes \mathfrak{h}$, commuting with the unitary involutive operator $S = S^\dagger$ of permutations $\eta_1 \otimes \eta_2 \mapsto \eta_2 \otimes \eta_1$ for any $\eta_i \in \mathfrak{h}$.

A map $\mathfrak{L}(t, \cdot)$ from $\mathcal{A} = \mathfrak{B}(\mathfrak{h})$ to itself is said to be a Lindblad generator if it takes the form

$$\mathfrak{L}(t, X) = i[H(t), X] + \sum_\alpha \mathfrak{L}_{R_\alpha}(X), \tag{2.3}$$

$$\mathfrak{L}_R(X) = R^\dagger X R - \frac{1}{2} R^\dagger R X - \frac{1}{2} X R^\dagger R \tag{2.4}$$

with H self adjoint, the $R_\alpha \in \mathcal{A}$ (and the summations in (3) understood to be ultraweakly convergent [27] for an infinite set $\{R_\alpha\}$). The generator is Hamiltonian if it just takes form $i[H(t), \cdot]$. The pre-adjoint $\mathfrak{L}' = \mathfrak{L}_\star$ of a generator $\mathfrak{L}$ is defined on the pre-adjoint space $\mathcal{A}_\star$ through the relation $\langle \mathfrak{L}'(\varrho), X\rangle = \langle \varrho, \mathfrak{L}(X)\rangle$. We note that Lindblad generators have the property $\mathfrak{L}(I) = 0$ corresponding to conservation of the identity operator $I \in \mathcal{A}$ or, equivalently, $\mathrm{tr}\{\mathfrak{L}'(\varrho)\} = 0$ for all $\varrho \in \mathcal{A}_\star$.

In quantum control theory it is necessary to consider time-dependent generators $\mathfrak{L}(t)$, through an integrable time dependence of the controlled Hamiltonian $H(t)$, and, more generally, due to a square-integrable time dependence of the coupling operators $R_\alpha(t)$. We shall always assume that these integrability conditions, ensuring existence and uniqueness of the solution $\varrho(t)$ to the quantum state Master equation

$$\frac{d}{dt}\varrho(t) = \mathfrak{L}'(t, \varrho(t)) \equiv \vartheta(t, \varrho(t)), \tag{2.5}$$

for all for $t \geq t_0$ with given initial condition $\varrho(t_0) = \varrho_0 \in \mathcal{S}$, are fulfilled.

Let $\mathsf{F} = \mathsf{F}[\cdot]$ be a (nonlinear) functional $\varrho \mapsto \mathsf{F}[\varrho]$ on $\mathcal{A}_\star$ (or on $\mathcal{S} \subset \mathcal{A}_\star$), then we say it admits a (Frechet) derivative if there exists an $\mathcal{A}$-valued function $\nabla_\varrho \mathsf{F}[\cdot]$ on $\mathcal{A}_\star$ ($\mathcal{T}_0^\star$-valued functional on $\mathcal{T}_0$) such that

$$\lim_{h\to 0} \frac{1}{h}\{\mathsf{F}[\cdot + h\tau] - \mathsf{F}[\cdot]\} = \langle \tau, \nabla_\varrho \mathsf{F}[\cdot]\rangle, \tag{2.6}$$

for each $\tau \in \mathcal{A}_\star$ (for each $\tau \in \mathcal{T}_0$). In the same spirit, a Hessian $\nabla_\varrho^{\otimes 2} \equiv \nabla_\varrho \otimes \nabla_\varrho$ can be defined as a mapping from the functionals on $\mathcal{S}$ to the $\mathcal{A}_{sym}^{\otimes 2}$-valued functionals, via

$$\lim_{h,h'\to 0} \frac{1}{hh'}\{\mathsf{F}[\cdot + h\tau + h'\tau'] - \mathsf{F}[\cdot + h\tau] - \mathsf{F}[\cdot + h'\tau'] + \mathsf{F}[\cdot]\}$$
$$= \langle \tau \otimes \tau', \nabla_\varrho \otimes \nabla_\varrho \mathsf{F}[\cdot]\rangle. \tag{2.7}$$

and we say that the functional is twice continuously differentiable whenever $\nabla_\varrho^{\otimes 2}\mathsf{F}[\cdot]$ exists and is continuous in the trace norm topology.

Likewise, a functional $f : X \mapsto f[X]$ on $\mathcal{A}$ is said to admit an $\mathcal{A}_\star$-derivative if there exists an $\mathcal{A}_\star$-valued function $\nabla_X f[\cdot]$ on $\mathcal{A}$ such that

$$\lim_{h\to 0}\frac{1}{h}\{f[\cdot+hA]-f[\cdot]\}=\langle\nabla_X f[\cdot],A\rangle \tag{2.8}$$

for each $A \in \mathfrak{B}(\mathfrak{h})$. The derivative $\nabla_X f[\cdot]$ has zero trace, $\nabla_X f[A] \in \mathcal{T}_0$ for each $A \in \mathcal{A}$, if and only if the functional $f[X-\lambda I]$ does not depend on λ, i.e. is essentially a function $f(p)$ of the class $p[X] \in \mathcal{T}_0^\star$.

With the customary abuses of differential notation, we have for instance

$$\nabla_\varrho f(\langle\varrho,X\rangle)=f'(\langle\varrho,X\rangle)\,X,\quad \nabla_X f(\langle\varrho,X\rangle)=f'(\langle\varrho,X\rangle)\,\varrho,$$

for any differentiable function f of the scalar $x = \langle\varrho, X\rangle$. Typically, we shall use ∇_ϱ more often, and denote it by just ∇.

3. Quantum Optimal Control

From now on we will assume that the Hamiltonian H and therefore ϑ (and υ) are functions of a controlled parameter $u \in \mathcal{U}$ depending on t such that the time dependence of the generator $\mathfrak{L}$ is of the form $\mathfrak{L}(u(t))$. Moreover, we do not require at this stage the linearity of $\vartheta(u,\varrho)$ with respect to ϱ, as well as the quadratic dependence $\sigma(\varrho)$, which means that what follows below is also applicable to more general quantum stochastic kinetic equations

$$d\varrho_\bullet(t)=\vartheta(u(t),\varrho_\bullet(t))\,dt+\sigma(\varrho_\bullet(t))\,dw(t)$$

of Vlassov and Boltzmann type, with only the positivity and trace preservation requirements $\mathrm{tr}\{\vartheta(u,\varrho)\}=0=\mathrm{tr}\{\sigma(\varrho)\}$. A choice of the control function $\{u(r) : r \in [t_0,t]\}$ is required before we can solve the filtering equation (Belavkin equation) at the time t for a given initial state ϱ_0 at time t_0. From what we have said above, this is required to be a $\mathcal{U}$-valued function which we take to be continuous for the moment.

The cost for a control function $\{u(r)\}$ over any time-interval $[t,T]$ is random and taken to have the integral form

$$\mathsf{J}_\omega[\{u(r)\};t,\varrho]=\int_t^T\mathsf{C}(u(r),\varrho_\omega(r))\,dr+\mathsf{G}(\varrho_\omega(T)) \tag{3.1}$$

where $\{\varrho_\bullet(r) : r \in [t,T]\}$ is the solution to the filtering equation with initial condition $\varrho_\bullet(t)=\varrho$. We assume that the *cost density* C and the *terminal cost, or bequest function,* G will be continuously differentiable

in each of its arguments. In fact, due to the statistical interpretation of quantum states, we should consider only the linear dependence

$$\mathsf{C}(u,\varrho)=\langle\varrho,C(u)\rangle\,,\ \mathsf{G}(\varrho)=\langle\varrho,G\rangle \tag{3.2}$$

of C and G on the state ϱ as it was already suggested in [**4**],[**6**],[**10**]. We will explicitly consider this case later, but for the moment we will not use the linearity of C and G. We refer to $C(u)\in\mathcal{A}$ as *cost observable* for $u\in\mathcal{U}$ and $G\in\mathcal{A}$ as the *bequest observable*.

The feedback control $u(t)$ is to be considered a random variable $u_\omega(t)$ adapted with respect to the innovation process $w(t)$, in line with our causality requirement, and so we therefore consider the problem of minimizing its average cost value with respect to $\{u_\bullet(t)\}$. To this end, we define the optimal average cost on the interval $[t,T]$ to be

$$\mathsf{S}(t,\varrho):=\inf_{\{u_\bullet(r)\}}\mathbb{E}\left[\mathsf{J}_\bullet\left[\{u_\bullet(r)\}\,;t,\varrho\right]\right], \tag{3.3}$$

where the minimum is considered over all measurable adapted control strategies $\{u_\bullet(r):r\geq t\}$. The aim of feedback control theory is then to find an optimal control strategy $\{u^*_\bullet(t)\}$ and evaluate $\mathsf{S}(t,\varrho)$ on a fixed time interval $[t_0,T]$. Obviously that the cost $\mathsf{S}(t,\varrho)$ of the optimal feedback control is in general smaller than the minimum of $\mathbb{E}\left[\mathsf{J}_\bullet\left[\{u\}\,;t,\varrho\right]\right]$ over nonstochastic strategies $\{u(r)\}$ only, which gives the solution of the open loop (without feedback) quantum control problem. In the case of the linear costs (3.2) this open-loop problem is equivalent to the following quantum deterministic optimization problem which can be tackled by the classical theory of optimal deterministic control in the corresponding Banach spaces.

3.1. Bellman & Hamilton-Pontryagin Optimality. Let us first consider nonstochastic quantum optimal control theory assuming that the state $\varrho(t)\in\mathcal{S}$ obeys the master equation (2.5) where $\vartheta(u,\varrho)$ is the adjoint $\mathfrak{L}'(u)$ of some Lindblad generator for each u with, say, the control being exercised in the Hamiltonian component $i[\cdot,H(u)]$ as before. (More generally, we could equally well consider a nonlinear quantum kinetic equation.) The control strategy $\{u(t)\}$ will be here non-random, as will be any specific cost $\mathsf{J}\left[\{u\}\,;t_0,\varrho_0\right]$. As for $\mathsf{S}(t,\varrho)=\inf\mathsf{J}\left[\{u\}\,;t,\varrho\right]$ at the times $t<t+\varepsilon<T$, one has

$$\mathsf{S}(t,\varrho)=\inf_{\{u\}}\left\{\int_t^{t+\varepsilon}\mathsf{C}(u(r),\varrho(r))\,dr+\int_{t+\varepsilon}^{T}\mathsf{C}(u(r),\varrho(r))\,dr+\mathsf{G}(\varrho(T))\right\}.$$

Suppose that $\{u^*(r) : r \in [t,T]\}$ is an optimal control when starting in state ϱ at time t, and denote by $\{\varrho^*(r) : r \in [t,T]\}$ the corresponding state trajectory starting at state ϱ at time t. Bellman's optimality principle observes that the control $\{u^*(r) : r \in [t+\varepsilon, T]\}$ will then be optimal when starting from $\varrho^*(t+\varepsilon)$ at the later time $t+\varepsilon$. It therefore follows that

$$\mathsf{S}(t,\varrho) = \inf_{\{u(r)\}} \left\{ \int_t^{t+\varepsilon} \mathsf{C}(u(r), \varrho(r))\, dr + \mathsf{S}(t+\varepsilon, \varrho(t+\varepsilon)) \right\}.$$

For ε small we expect that $\varrho(t+\varepsilon) = \varrho + \vartheta(u(t), \varrho)\,\varepsilon + o(\varepsilon)$ and provided that S is sufficiently smooth we may make the Taylor expansion

$$\mathsf{S}(t+\varepsilon, \varrho(t+\varepsilon)) = \left[1 + \varepsilon \frac{\partial}{\partial t} + \varepsilon \langle \vartheta(u(t), \varrho), \nabla \rangle\right] \mathsf{S}(t,\varrho) + o(\varepsilon). \tag{3.4}$$

In addition, we approximate

$$\int_t^{t+\varepsilon} \mathsf{C}(u(r), \varrho(r))\, dr = \varepsilon \mathsf{C}(u(t), \varrho) + o(\varepsilon)$$

and conclude that (note the convective derivative!)

$$\mathsf{S}(t,\varrho) = \inf_{u \in U} \left\{ \left[1 + \varepsilon \left(\mathsf{C}(u,\varrho) + \frac{\partial}{\partial t} + \langle \vartheta(u,\varrho), \nabla \cdot \rangle \right)\right] \mathsf{S}(t,\varrho) \right\} + o(\varepsilon)$$

where now the infimum is taken over the point-value of $u(t) = u \in U$. In the limit $\varepsilon \to 0$, one obtains the equation

$$-\frac{\partial}{\partial t}\mathsf{S}(t,\varrho) = \inf_{u \in \mathcal{U}} \{\mathsf{C}(u,\varrho) + \langle \vartheta(u,\varrho), \nabla \mathsf{S}(t,\varrho) \rangle\}, \tag{3.5}$$

where $\nabla = \nabla_\varrho$. The equation is then to be solved subject to the terminal condition

$$\mathsf{S}(T,\varrho) = \mathsf{G}(\varrho). \tag{3.6}$$

We may introduce the Pontryagin Hamiltonian function on $\mathcal{T}_0 \times \mathcal{T}_0^\star$ defined by the Legendre-Fenchel transform

$$\mathcal{H}_\vartheta(q, p[X]) := \sup_{u \in \mathcal{U}} \{\langle \vartheta(u, \varrho(q)), \lambda I - X \rangle - \mathsf{C}(u, \varrho(q))\}. \tag{3.7}$$

Here we use a parametrization $\varrho(q) = \varrho_0 - q$, $q \in \mathcal{T}_0$ and the fact that the supremum does not depend on $\lambda \in \mathbb{R}$ since $\langle \vartheta(u,\varrho), I \rangle = 0$. Therefore $\mathcal{H}$ depends on X only through the equivalence class $p[X] \in \mathcal{T}_0^\star$ which is referred to as the *co-state*. It should be emphasized that these Hamiltonians are purely classical devices which may be called super-Hamiltonians

to be distinguished from H. We may then rewrite (3.5) as the (backward) *Hamilton-Jacobi* equation

$$-\frac{\partial}{\partial t}\mathsf{S}\left(t,\varrho\left(q\right)\right)+\mathcal{H}_{\vartheta}\left(q,p\left[\nabla\mathsf{S}\left(t,\varrho\right)\right]\left(q\right)\right)=0. \tag{3.8}$$

Applying the derivative $\nabla_q = -\nabla$ to this equation to $q = \varrho_0 - \varrho$ in the tangent space $\mathcal{T}_0$ we obtain the dynamical equation $\dot{p} = -\nabla_q \mathcal{H}_\vartheta (q,p)$ for the co-state $p_t = Q_t (T,s)$ of the operator–valued function $X(t,\varrho) = \nabla\mathsf{S}(t,\varrho)$, where $Q_t(T,s) = p[X(t,\varrho)]$ is the solution of this equation satisfying the terminal condition $Q_T(T,s) = s := p[G]$ with $p[G(\varrho)](q) = p[G(\varrho(q))]$ for $G = \nabla\mathsf{G}$. We remark that, if $u^*(q,p(X))$ is an optimal control maximizing

$$\mathcal{K}_\vartheta\left(u,q,p\left(X\right)\right)=\left\langle\vartheta\left(u,\varrho\left(q\right)\right),\lambda I-X\right\rangle-\mathsf{C}\left(u,\varrho\left(q\right)\right),$$

then the corresponding state dynamical equation $\frac{d}{dt}\varrho = \vartheta(u^*(\varrho,X),\varrho)$ in terms of its optimal solution $q_t \equiv \varrho_0 - \varrho_t(t_0,\varrho_0)$ corresponding to $\varrho_{t_0} \equiv \varrho_0$ can be written as $\dot{q} = \nabla_p \mathcal{H}_\vartheta(q,p)$, noting that

$$\nabla_p\mathcal{H}_\vartheta\left(q,p\right)=\nabla_p\mathcal{K}_\vartheta\left(u^*\left(q,p\right),q,p\right)=-\vartheta\left(u^*\left(q,p\right),\varrho\left(q\right)\right) \tag{3.9}$$

due to the stationarity condition $\frac{\partial}{\partial u}\mathcal{K}_\vartheta(u,q,p) = 0$ at $u = u^*$. This forward equation with $q_0 = 0$ for $\varrho^*(t_0) = \varrho_0$ together with the co-state backward equation with $p_T = p[G] \equiv s$ is the canonical Hamiltonian system. Thus we may equivalently consider the Hamiltonian boundary value problem

$$\left\{\begin{array}{l}\dot{q}_t-\nabla_p\mathcal{H}_\vartheta\left(q_t,p_t\right)=0,\ q_0=0\\ \dot{p}_t+\nabla_q\mathcal{H}_\vartheta\left(q_t,p_t\right)=0,\ p_T=s\end{array}\right. \tag{3.10}$$

which we refer to as the *Hamilton-Pontryagin problem*, in direct analogy with the classical case. The solution to this problem defines the minimal cost as the path integral

$$\mathsf{S}\left(t_0,\varrho_0\right)=\int_{t_0}^{T}\left[\left\langle\dot{q}_r|p_r\right\rangle-\mathcal{H}\left(q_r,p_r\right)\right]dr+\mathsf{G}\left(\varrho\left(q_T\right)\right).$$

Thus the *Pontryagin maximum principle* for the quantum dynamical system is the observation that the optimal quantum control problem is equivalent to the Hamiltonian problem for state and co-state $\{q\}$ and $\{p\}$ respectively, leading to optimality $\mathcal{K}_\vartheta(u,q,p) \le \mathcal{H}_\vartheta(q,p)$ with equality for $u = u^*(q,p)$ maximizing $\mathcal{K}_\vartheta(u,q,p)$.

Bibliography

[1] V.P. Belavkin. Optimal Quantum Filtration of Markovian Signals. Problems Control Inform. Theory, **7**: no. 5, 345–360 (1978)

[2] V.P. Belavkin, *Optimal Measurement and Control in Quantum Dynamical Systems.* Preprint No. 411, Inst. of Phys., Nicolaus Copernicus University, Torun', February 1979

[3] V.P. Belavkin, *Quantum Filtering of Markov Signals with Wight Quantum Noise.* Radiotechnika and Electronika, **25**: 1445–1453 (1980). English translation in: *Quantum Communications and Measurement.* V. P. Belavkin et al, eds., 381–392 (Plenum Press, 1994).

[4] V.P. Belavkin, Theory of the control of observable quantum systems. Autom. Remote Control, **44**: 178-188, (1983)

[5] V.P. Belavkin, Nondemolition measurement and control in quantum dynamical systems. Information complexity and control in quantum physics (Udine, 1985), 311–329, CISM Courses and Lectures, 294, Springer, Vienna, 1987.

[6] V.P. Belavkin, Nondemolition measurements, nonlinear filtering and dynamical programming of quantum stochastic processes. In: Modelling and Control of Systems (Lecture Notes in Control and Information Sciences), ed A Blaquiere, **121**: 381–92 (Berlin: Springer, 1988)

[7] V.P. Belavkin, A new wave equation for continuous nondemolition measurement. Phys. Lett. A, **140**: 355–8 (1989).

[8] V.P. Belavkin, Stochastic posterior equations for quantum nonlinear filtering. Probability Theory and Mathematical Statistics, ed B Grigelionis, **1**: 91–109 (Vilnius: VSP/Mokslas, 1990).

[9] V.P. Belavkin, Quantum stochastic calculus and quantum nonlinear filtering. Journal of Multivariate Analysis, **42**: 171-201, (1992)

[10] V. P. Belavkin, Measurement, filtering and control in quantum open dynamical systems. Rep. on Math. Phys.**43**: no. 3, 405-425 (1999).

Tropical Plücker functions

V.I. Danilov, A.V. Karzanov and G.A. Koshevoy

1. Introduction

Totally positive matrices play an important role in different areas of mathematics, from differential equations to combinatorics. Studying parametrizations of canonical bases, Berenstein, Fomin and Zelevinsky [**1**] established the so-called Chamber ansatz for flag minors of matrices. This result relies on Plücker relations between flag minors. In this talk, we study functions which satisfy tropical Plücker relations. We consider two approaches to tropicalization of Plücker relations. One approach is based on

tropicalization of the so-called special Plücker relation (3-term relation) after writing it as a subtraction-free expression. On this way we get the class of TP-functions (on Boolean cube 2^N) which can be seen as the tropicalization of flag minors. We show that such functions are determined by their restrictions to the interval family of subsets of N, and that the class of submodular TP-functions coincides with the class of submodular functions on the interval family. This might be seen as the tropicalization of the Chamber ansatz. Our proof is based on a construction of DMTP-functions via normal flows in weighted digraphs. A DMTP-function is a function that satisfies tropical Plücker relations, and at this point we consider the tropicalization in the sense of the second approach, meaning that we tropicalize an algebraic formula with subtractions in the right hand side as an inequality.

2. Flag minors and Plücker relations

For X an $n \times n$ matrix and I a subset of $N = \{1, \dots, n\}$, denote by $\Delta_I(X)$ the determinant of the submatrix located in the intersection of the first $|I|$ rows and the columns indexed by I. These determinants are called *flag minors* of X. It is known (see Fulton and Harris [**3**], p.235), that flag minors of a matrix X satisfy the Plücker relations

$$\Delta_{A\cup I}\Delta_{A\cup J} = \sum_{i\in I}(-1)^{d(I,J,i,j)}\Delta_{A\cup(I\setminus i)\cup j}\Delta_{A\cup(J\setminus j)\cup i},$$

for any pairwise disjoint $A, I, J \subset N$, $|I| \geq |J|$, any fixed $j \in J$, and an appropriate function $d(I, J, i, j)$.

The Plücker relations with $2 \geq |I| \geq |J| \geq 1$ are of special interest. They are given by

$$\Delta_{A\cup ik}\Delta_{A\cup j} = \Delta_{A\cup ij}\Delta_{A\cup k} + \Delta_{A\cup jk}\Delta_{A\cup i}$$

with any A and $i < j < k$, and

$$\Delta_{A\cup ik}\Delta_{A\cup jl} = \Delta_{A\cup ij}\Delta_{A\cup kl} + \Delta_{A\cup jk}\Delta_{A\cup il}$$

with any A and $i < j < k < l$.

Definition 2.1. A function $F : 2^N \to \mathbb{R}$ is said to be *P-function* if F satisfies the above Plücker relations for all A and $i < j < k$, that is

$$F(A\cup ik)F(A\cup j) = F(A\cup ij)F(A\cup k) + F(A\cup jk)F(A\cup i). \tag{2.1}$$

Denote by $\mathcal{PF}$ the set of P-functions.

We collected properties of P-functions in the following

THEOREM 2.1. a) *Any P-function satisfies all Plücker relations.*
b) *Let $F : 2^N \to \mathbb{R}_{\neq 0}$ be a P-function. Then there exists a unique upper-triangular $N \times N$ matrix X such that $F(I) = \Delta_I(X)$.*
c) *Let $\mathcal{I} \subset 2^N$ denote the interval family constituted from intervals $\{i, i+1, \dots, j\}$, $i \le j$, and let $\mathrm{res}_I : \mathbb{R}^{2^N} \to \mathbb{R}^{\mathcal{I}}$ denote the restriction map from 2^N to $\mathcal{I}$. Then the mapping res_I is a bijection between the subspace PF ($\subset \mathbb{R}^{2^N}$) and $\mathbb{R}^{\mathcal{I}}$.*

3. Tropical Plücker functions

We consider two ways of tropicalization of P-functions. Firstly, we can tropicalize (2.1) as follows:

$$f(A\cup ik)+f(A\cup j) = \max(f(A\cup ij)+f(A\cup k), f(A\cup jk)+f(A\cup i)), \tag{3.1}$$

with any A and $i < j < k$.

DEFINITION 3.1. A function $f : 2^N \to \mathbb{R}$ is said to be a *TP-function* if F satisfies the tropicalization (3.1) of Plücker relations for all A and $i < j < k$.

Secondly, we can think of tropicalization in the form of inequality:

$$f(A \cup I) + f(A \cup J) \le \max_{i\in I}(f(A \cup (I \setminus i) \cup j) + f(A \cup (J \setminus j) \cup i)), \tag{3.2}$$

for any pairwise disjoint A, I, J, $|I| \ge |J|$, and any fixed $j \in J$.

Specializing this to tropicalization of the 3-term Plücker relation given by (2.1), we have that, for all disjoint $A \subset N$ and $\{i, j, k\} \subset N$,

the maximum is attained at least twice among the three values

$$a = f(A\cup ik)+f(A\cup j),\ b = f(A\cup ij)+f(A\cup k),\ c = f(A\cup jk)+f(A\cup i). \tag{3.3}$$

Similarly, for a 4-term Plücker relation, that is, for all disjoint $A \subset N$ and $\{i, j, k, l\} \subset N$, we have that

the maximum is attained at least twice among the three values

$$x = f(A\cup ik)+f(A\cup jl),\ y = f(A\cup ij)+f(A\cup kl),\ z = f(A\cup jk)+f(A\cup il). \tag{3.4}$$

DEFINITION 3.2. A function $f : 2^N \to \mathbb{R}$ is called a *DMTP-function* if f satisfies (3.3) and (3.4).

We will show in the following section that the class of TP-functions is a subclass of DMTP-functions.

We have the following property of DMTP-functions, which is closely related to valuated matroids, see Dress and Wenzel [**2**].

THEOREM 3.1. *A DMTP-function f satisfies all tropical Plücker relations (3.2).*

4. Flows in digraphs and DMTP-functions

Here we propose a construction of DMTP-functions.

We deal with a digraph $G = (V, E)$, a function $c : E \to \mathbb{R}$ of weights on the edges, and disjoint subsets $S, T \subset V$. We assume that $|S| = |T|$ and that T is ordered: $T = (t_1, t_2, \ldots, t_{|T|})$, and denote $\{t_1, \ldots, t_i\}$ by T_i.

For $F \subseteq E$ and $X \subseteq V$, denote:

the sets of edges in F leaving X and entering X by $\delta^+_F(X)$ and by $\delta^-_F(X)$, respectively;

the set $\delta^+_F(X) \cup \delta^-_F(X)$ by $\delta_F(X)$;

the number $|\delta^+_F(X)| - |\delta^-_F(X)|$ by $\mathrm{div}_F(X)$.

DEFINITION 4.1. Let us say that $F \subset E$ is a *normal flow* from $S' \subseteq S$ if

$$\mathrm{div}_F(v) = \begin{cases} 1 & \forall\ v \in S', \\ -1 & \forall\ v \in T_{|S'|}, \\ 0 & \text{for the other } v \in V, \end{cases}$$

where $\mathrm{div}(v)$ stands for $\mathrm{div}(\{v\})$.

This gives rise to the following important function $f = f_c$ on 2^S:

$$f(S') := \max\{c(F),\ F \text{ is a normal flow from } S'\}, \quad S' \subseteq S. \tag{4.1}$$

THEOREM 4.1. *f defined in (4.1) is a DMTP-function.*

The claim that TP-functions constitute a subslass of DMTP-functions can be obtained as a consequence of the following

THEOREM 4.2. *Let $f : 2^N \to \mathbb{R}$ be a TP-function. Then there exists a planar digraph $G = (V, E)$ and a weight function $c : E \to \mathbb{R}$, such that f is defined by (4.1).*

REMARK 4.1. For a TP-function on the Boolean 2^N, we can take the unique planar of the following from: $V := \{(i, j)\, 1 \le i, j \le n\}$, and two edges emanate from the vertex (i, j), which terminate either in $(i - 1, j)$ or in $(i, j + 1)$, respectively, if both terminate points are vertices.

The following property is important, see Kamnitzer [**4**], for applications to crystal bases construction via MV-polytopes:

THEOREM 4.3. *A TP-function $f : 2^N \to \mathbb{R}$ is submodular (that is $f(A) + f(B) \ge f(A \cup B) + f(A \cap B)$, $A, B \subset N$) if and only if the function $\mathrm{res}_{\mathcal{I}}(f)$ is submodular on $\mathcal{I}$.*

Bibliography

[1] Berenstein A., Fomin S. and Zelevinsky A., Parametrizations of canonical bases and totally positive matrices, *Adv. in Math.* **122** (1996), 49–149.
[2] Dress A.W.M. and W.Wenzel, Valuated matroids: A new look at the greedy algorithm, *Appl. Math. Lett.*, 4 (1991), 33-35
[3] Fulton W. and J.Harris, *Rrepresentation Theory*, Graduate Texts in Mathematics, 129, Springer, 1991
[4] Kamnitzer J., The crystal structure on the set of Mirkovic-Vilonen polytopes, ArXiv, math:QA/0505398

Degree one homogeneous minplus dynamic systems and traffic applications: Part I[1]

N. Farhi, M. Goursat, and J.-P. Quadrat

We show that car traffic on a town can be modeled using a Petri net extension where arcs have negative weights. The corresponding minplus dynamics is not linear but homogeneous of degree one. Possibly depending on the initial condition, homogeneous of degree 1 minplus systems may be periodic or have a chaotic behavior (to which corresponds a constant throughput) or may explode exponentially. In traffic systems, when this constant throughput exists, it has the interpretation of the average car speed. In this first part we recall the derivation of the 1-homogeneous dynamics of traffic system and show the existence of such systems with chaotic behavior and a constant throughput.

1. Introduction

At macroscopic level, the traffic on a road can be seen from various points of view:

- The Lighthill-Whitham-Richards Model [**6**] is the more standard one
$$\begin{cases} \partial_t \rho + \partial_x q = 0 \ , \\ q = f(\rho), \end{cases}$$
where $q(x,t) =$ denotes the flow at time t and position x on the road, $\rho(x,t) =$ denotes density, f is a given function, called the fundamental traffic law. For the traffic, it plays the role analogous to that of the perfect gas law in the fluid dynamics.

[1]Partially supported by the joint RFBR/CNRS grant No. 05-01-02807.

- The kinetic model (Prigogine-Herman [**7**]) gives the evolution of the particle density $\rho(t,x,v)$ as a function of t, x and v, where v is the speed of particle

$$\partial_t \rho + v\partial_x \rho = C(\rho,\rho) \ ,$$

with $C(\rho,\rho)$ an interacting term in general quadratic in ρ.

The second model is more costly in terms of computation time and therefore not used in practice. The first one assumes the knowledge of the function f, which is usually obtained experimentally, or theoretically using a simple microscopic model. Here, we will recall a way to derive a good approximation of this law f from a simple minplus linear system based on a Petri net.

The main purpose of this paper is to generalize this fundamental law to the 2D cases where roads have crossings. The original minplus linear model on a unique road cannot be generalized easily in term of Petri nets. We have proposed in a previous paper a way to solve the difficulty by using Petri net with negative weights. The dynamics of this Petri net can be written easily. It is not linear in minplus algebra any more, but it is homogeneous of degree 1. We recall here the derivation of this 1-homogeneous dynamics.

In the first part of this paper we show that we can compute the eigenvalues for this 1-homogeneous system but that chaotic dynamics may appear. In the second part we discuss the phases appearing in the fundamental diagram, obtained numerically, and describe new situations where we can prove that the system is periodic.

2. Traffic on a circular road

Let us recall the simplest model which allows to derive the fundamental traffic law on a single road. The simplest way is to study the stationary regime on a circular road with a given number of vehicles and then to consider that this stationary regime is reached locally when the density is given on a standard road. We present two ways to obtain this law : – by logical deduction from an exclusion process, – by computing the eigenvalue of a minplus system derived from a simple Petri net describing the road with the vehicles.

2.1. Exclusion process modeling. Following [**3**] we can consider the dynamic system defined by the rule $10 \to 01$ applied to a binary word describing the car positions on a road cut in sections (each bit representing

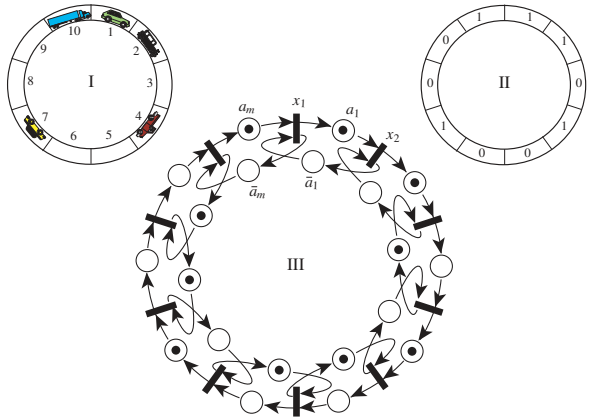

FIGURE 1. A circular road.

a section, 1 meaning occupied and 0 meaning free, see II in Figure 1). Let us take an example :

$$m_1 = 1101001001, \quad m_2 = 1010100101, \quad m_3 = 0101010011,$$
$$m_4 = 1010101010, \quad m_5 = 0101010101,$$

Let us define : – the *density* ρ to be the number of vehicles n divided by the number of places m : $\rho = n/m$, – the flow $q(t)$ at time t to be the number of vehicles going one step forward at time t divided by the number of places. Then the *fundamental traffic law* gives the relation between $q(t)$ and d.

If $\rho \leq 1/2$, then all the vehicle groups split off after a transient period, and all the vehicles can move forward without other vehicles in the way, and we have :

$$q(t) = q = n/m = d \, .$$

If $\rho \geq 1/2$, then the free place groups split off after a finite time and move backward without other free place in the way. Then $m - n$ vehicles move forward and we have

$$q(t) = q = (m - n)/m = 1 - d \, .$$

Therefore :

$$\exists T : \ \forall t \geq T \quad q(t) = q = \begin{cases} \rho & \text{if } \ \rho \leq 1/2 \, , \\ 1 - \rho & \text{if } \ \rho \geq 1/2 \, . \end{cases}$$

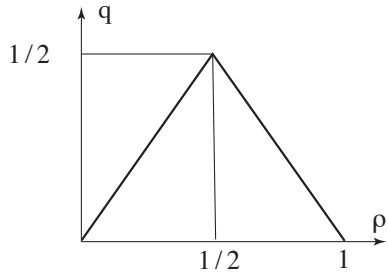

FIGURE 2. The fundamental traffic law.

2.2. Event Graph modeling. Consider the Petri net given in III of Figure 1 which describes in a different way the same dynamics. In fact this Petri net is an event graph and therefore its dynamics is linear in minplus algebra. The number of vehicles entered in the place i before time k is denoted x_i^k. The initial vehicle position is given by the boolean number a_i, which takes the value 1 when the cell contains a vehicle and is 0 otherwise.

We use the notation $\bar{a} = 1 - a$, then the dynamics is given by :

$$x_i^{k+1} = \min\{a_{i-1} + x_{i-1}^k, \bar{a}_i + x_{i+1}^k\} \ ,$$

which can be written linearly in minplus algebra :

$$x_i^{k+1} = a_{i-1} x_{i-1}^k \oplus \bar{a}_i x_{i+1}^k \ .$$

This event graph has three kinds of elementary circuits : – the outside circuit with average mean n/m, – the inside circuit with average mean $(m-n)/m$, – the circuits corresponding to make some step forward and coming back, with average mean $1/2$, Therefore its eigenvalue is

$$q = \min(n/m, (m-n)/m, 1/2) = \min(\rho, 1-\rho) \ ,$$

which gives the average speed as a function of the car density.

3. 2D traffic

Let us generalize the second approach to derive the fundamental diagram to a regular town described in Figure 3. The complete town can be modeled as a set of subsystems corresponding to a unique crossing and

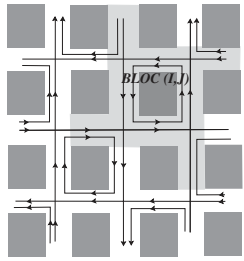

FIGURE 3. A town.

two adjacent roads. To write the dynamics of the town we have first to give the Petri net describing a crossing.

A first trial is to consider the Petri net given in Figure 4. This Petri net is not an event graph any more. However, following L. Libeaut[**5**], it is possible to write the nonlinear implicit minplus equation describing a general Petri net :

$$\min_{p\in x^{in}}\left[a_p+\sum_{x'\in x^{in}}x'(k-1)-\sum_{x''\in x^{out}}x''(k)\right]=0,\ \forall x\ ,\forall k. \tag{3.1}$$

where $x(k)$ denotes the firing number of transition x and p is a place of the Petri Net. But this equation does not determine the dynamics completely,

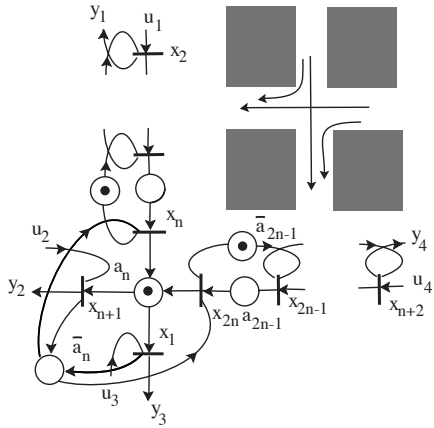

FIGURE 4. A simplified crossing.

since solution to the Cauchy problem may be non-unique. Indeed : – at place a_n we may have a *routing policy* giving the proportion of cars going towards y_2 and the proportion going towards y_3 (which is not described by the Petri net 4) – at place $\bar{a}_n$ we may follow *the first arrived the first served* rule with the right priority if two cars arrive simultaneously at the crossing (which is also not described by the Petri net 4).

Precising the dynamics of Petri net in such a way that the trajectories are uniquely defined corresponds to giving another Petri net having only one arc leaving each place. Let us discuss more precisely these points on a simple system given in the first picture of Figure 5.

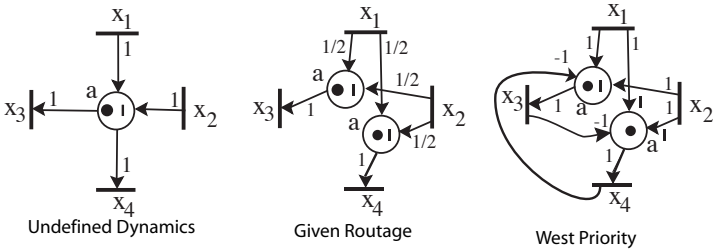

FIGURE 5. Dynamic Completion.

The incomplete dynamics of this system can be written in minplus algebra $x_4^n x_3^n = a x_1^{n-1} x_2^{n-1}$. Clearly x_4 and x_3 are not defined uniquely. We can complete the dynamics, for example, in the two following ways useful for the traffic application : – by precising the routing policy

$$x_4^n = x_3^n = \sqrt{a x_1^{n-1} x_2^{n-1}}$$

– by choosing a priority rule

$$\begin{cases} x_3^n = a x_1^{n-1} x_2^{n-1} / x_4^{n-1} \\ x_4^n = a x_1^{n-1} x_2^{n-1} / x_3^n. \end{cases}$$

In these two cases we obtain a *degree one homogeneous minplus* system.

This method can be applied to the crossing and we obtain a Petri net with negative weights which has only one arc leaving each place (that we call deterministic Petri net), see Figure 6.

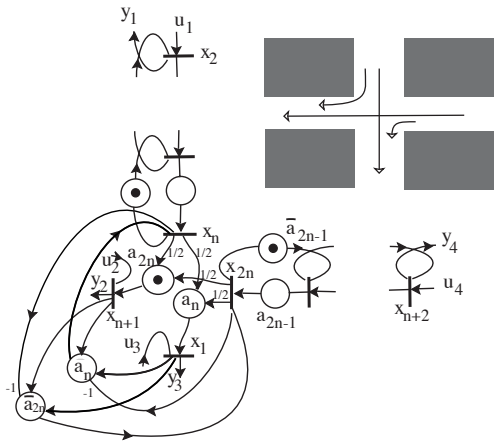

FIGURE 6. A Complete Crossing.

If we neglect the roundings, then the system can be written with minplus notations :

$$\begin{cases} x_i/\delta = a_{i-1}x_{i-1} \oplus \bar{a}_i x_{i+1}, \\ x_n/\delta = \bar{a}_n x_1 x_{n+1}/x_{2n} \oplus a_{n-1}x_{n-1} \ , \\ x_{2n}/\delta = \bar{a}_{2n} x_1 x_{n+1}/(x_n/\delta) \oplus a_{2n-1}x_{2n-1} \ , \\ x_1/\delta = a_n\sqrt{x_n x_{2n}} \oplus \bar{a}_1 x_2 \ , \\ x_{n+1}/\delta = a_{2n}\sqrt{x_n x_{2n}} \oplus \bar{a}_{n+1} x_{n+2} \ , \end{cases}$$

where δ denotes the forward shifting operator acting on sequences. It is a general degree 1 homogeneous minplus system.

Simulation of this system starting from 0 shows that

$$\lim_k x_i^k/k = \lambda, \ \ \forall i \ .$$

The constant λ has the interpretation of the average speed. The fundamental diagram gives the relation between the average speed and the vehicle density of the system. In Figure 7 we give this law in the cases of two circular roads with one crossing for different relative size of the two roads. We see that three phases appear on each fundamental diagram. These phases will be discussed in the second part of this paper. The experimental existence of this λ motivates the study of the eigenvalue of 1-homogeneous minplus system.

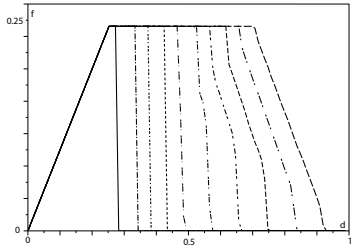

FIGURE 7. 2D-traffic fundamental diagrams.

4. Eigenvalues of 1-homogeneous minplus systems

The eigenvalue problem for 1-homogeneous system $f : \mathbb{R}^n_{\min} \mapsto \mathbb{R}^n_{\min}$ can be formulated as finding $x \in \mathbb{R}^n_{\min}$ non zero, and $\lambda \in \mathbb{R}_{\min}$ such that :

$$\lambda x = f(x) \ .$$

Since f is 1-homogeneous, assuming without loss of generality that if x exists then $x_1 \neq \epsilon$, the eigenvalue problem becomes :

$$\begin{cases} \lambda & = f_1(x/x_1) \ , \\ x_2/x_1 & = (f_2/f_1)(x) \ , \\ \cdots & = \cdots \\ x_n/x_1 & = (f_n/f_1)(x) \ , \end{cases}$$

Denoting $y = (x_2/x_1, \cdots, x_n/x_1)$, the problem is reduced to the computation of the fixed point problem $y = g(y)$ (with $g_{i-1}(y) = (f_i/f_1)(0, y)$). It is a problem of computing a normalized eigenvector, from which the eigenvalue is deduced by : $\lambda = f_1(0, y)$. But now g is a general minplus function.

The fixed point problem does not always have a solution. There are cases where we are able to solve the problem – f is affine in standard algebra, – f is minplus linear, – f is a positive power function. In the first case there is a unique eigenvalue as soon as $\dim(\ker(f - I)) = 1$.

In the two last cases, the problem can be reduced to the minimization of the average cost by time unit using dynamic programming methods. The corresponding fixed points are unique and stable.

Moreover, since $\max(x, y) = xy/(x \oplus y)$, games problem are also 1-homogeneous minplus systems and the solution to the corresponding eigenvalue problem is known.

In general, we may have unstable fixed points that, nevertheless, we can compute by Newton method (which is exactly the policy iteration) but which don't give the information about the asymptotic behavior of the system any more. In this case the asymptotics is obtained by an averaging based on invariant measure which may be difficult to compute. Let us give an example of chaotic system which has a 1-homogeneous minplus dynamics.

5. A Chaotic system example

Let us consider the 1-homogeneous minplus dynamic system

$$\begin{cases} x_1^{k+1} = (x_1^k)^2/x_2^k \oplus 2(x_2^k)^3/(x_1^k)^2 \ , \\ x_2^{k+1} = x_2^k \ . \end{cases}$$

The corresponding eigenvalue problem is

$$\begin{cases} \lambda x_1 = x_1^2/x_2 \oplus 2x_2^3/x_1^2, \\ \lambda x_2 = x_2 \ . \end{cases}$$

The solutions are $\lambda = 0$ and $y = x_1/x_2$. They satisfy the equation

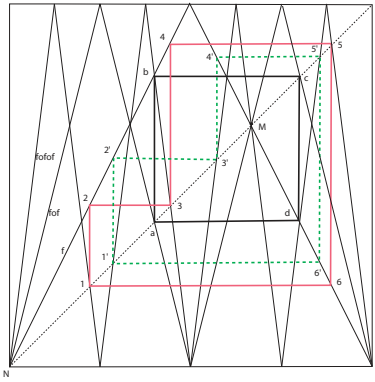

FIGURE 8. Cycles of tent transformation.

$$y = y^2 \oplus 2/y^2,$$

which has solutions $y = 0$ and $y = 2/3$. These two solutions are unstable fixed points of the transformation $f(y) = y^2 \oplus 2/y^2$. But the system $y_{n+1} = f(y_n)$ is a chaotic system since f is the tent transform (see e.g. [**2**]

for a comprehensive discussion of this dynamics). In Figure 8 we show graphs of the functions f, $f \circ f$, $f \circ f \circ f$, their fixed points and the corresponding periodic trajectories.

In Figure 9 we show a trajectory for an initial condition chosen randomly with the uniform law on the set $\{(i-1)/10^5, i = 1, \cdots, 10^5\}$. The diagonal line in the picture is a decreasing sort applied to the trajectory. It shows that the invariant empirical density is uniform. We can prove that

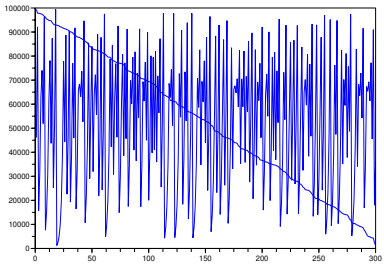

FIGURE 9. A tent iteration trajectory.

the tent iteration has a unique invariant measure absolutely continuous with respect to the Lebesgue measure : the uniform law on $[0, 1]$.

More generally, a chaotic 1-homogeneous minplus system will grow linearly with a value λ given by :

$$\lambda = \int f_1(y) d\mu(y) \ ,$$

where μ is the invariant probability measure of y depending on the initial value y^0. For example, according to the initial value y^0, the tent iterations y^k stay in circuits or follow trajectories without circuit (possibly dense in $[0, 1]$).

Bibliography

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat : *Synchronization and Linearity*, Wiley (1992).

[2] N. Berglund : *Geometrical Theory of Dynamical Systems* ArXiv:math (2001).

[3] M. Blank : *Variational principles in the analysis of traffic flows*, Markov Processes and Related Fields, pp.287-305, vol.7, N.3 (2000).

[4] N. Farhi, M. Goursat, J.-P. Quadrat : *Derivation of the fundamental traffic diagram for two circular roads and a crossing using minplus algebra and Petri net modeling*, in Proceedings IEEE-CDC, 2005, Seville (2005).
[5] L. Libeaut : *Sur l'utilisation des dioïdes pour la commande des systèmes à événements discrets*, Thèse, Laboratoire d'Automatique de Nantes (1996).
[6] J. Lighthill, J. B. Whitham : *On kinetic waves: II) A theory of traffic Flow on long crowded roads*, Proc. Royal Society A229 p. 281-345 (1955).
[7] I. Prigogine, R. Herman : *Kinetic Theory of Vehicular Traffic*, Elsevier (1971).

Degree one homogeneous minplus dynamic systems and traffic applications : Part II[1]

N. Farhi, M. Goursat, and J.-P. Quadrat

In this second part we discuss the phases appearing in the fundamental diagrams of traffic systems modeled by 1-homogeneous minplus dynamics. We show the improvement obtained by traffic light control. We give a new subclass of 1-homogeneous dynamics having periodic trajectories. It generalizes the standard cases which need a monotony property. Finally we show that the periodicity of this new, but still restrictive class, has applications to justify the existence of the fundamental diagram for regular town traffic with crossings but without turning possibilities.

1. The traffic fundamental diagram phases.

The fundamental diagrams of quite different systems are similar to the one given in part I. We have studied the cases of two circular roads with one crossing and two crossings and the cases of regular towns with various number of roads on a torus. In all these cases we suppose the existence of "right priority".

The fundamental diagrams have always three phases corresponding respectively to low, middle and high densities. We see on the fundamental diagram of Part I that : – for low densities the flow increases linearly with the density, – for middle densities the flow is constant, – for high density there are deadlocks and the flows are null.

On Figures 1, 2 and 3 we show the typic asymptotic distribution of vehicles in the three phases (see [**3, 4**]).
We see that :

[1]Partially supported by RFBR/CNRS grant 05-01-02807.

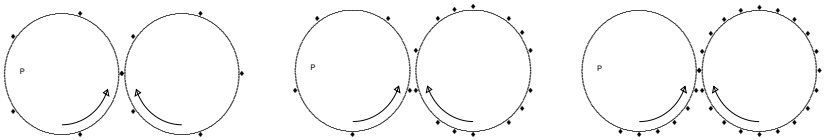

FIGURE 1. Two circular roads with one crossing case. Car distributions in the low, middle and high density phases.

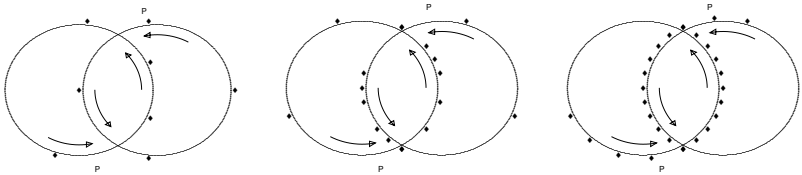

FIGURE 2. Four roads with two crossings. Car distributions in the low, middle and high density phases.

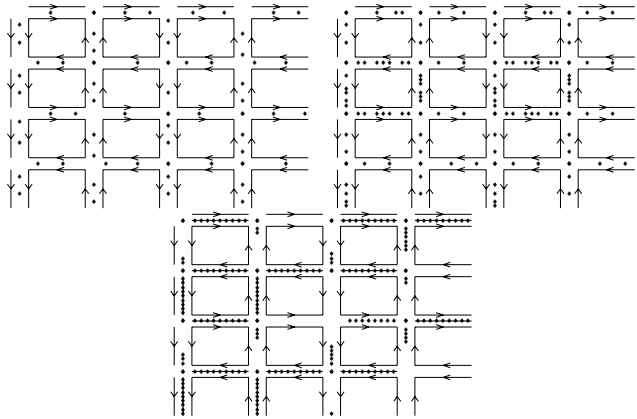

FIGURE 3. A regular town. Car distributions in the low, middle and high density phases.

- *Low density phase.* There are so few vehicles in the network that after a transient regime, they move without obstructing each other on the roads and in the crossings. Thus, the "priority to the right" is not used, the vehicles moves as on a unique circular

road and the average flow is equal to the vehicle density in the network. This phase corresponds to densities less than 1/4.

- *Middle density phase.* When the density is between 1/4 and 1/2 (in the symmetric road cases), the vehicles can neither move freely on the roads, nor avoid each other on the crossings. Therefore "priority to the right" happens. The cars on the priority road move freely and the waiting cars are all in the non priority road. The flow reaches the maximum value 1/4 corresponding to the full use of the crossings.
- *High density phase.* When the density exceeds a quantity equal to 1/2 (in the symmetric case), at asymptotic regime, a closed circuit of vehicles on some nonpriority roads appear which creates a complete deadlock of the system.

2. Traffic light control.

To avoid the deadlock due to the right priority we can use traffic light controls. A Petri net describing the junction with the traffic light control is shown on Figure 4. The negative weight extension of Petri net is necessary to model the light phases in a time invariant way. The part of the Petri net modeling the traffic light control corresponds to the places $a_g, a_c, \bar{a}_g, \bar{a}_c$. As long as a_c contains a token ($a_c = 1$) the green light is for the North street, when $\bar{a}_c = 1$ the green light is for the East street. As long as $a_c = 1$ we have $a_g = 1$ and q_v is authorized to fire (since thanks to the loop q_v, a_g, q_v as soon as a token is consumed another one is generated in the place a_g). The main point is that when the token in a_c goes in $\bar{a}_c$ (phase change) the tokens in a_g must be removed (this is done by the input arc with weight -1 of the place a_g). More generally without negative weight we cannot model tokens staying less than a prescribed time.

In Figure 5, we compare the fundamental diagrams of three crossing policies for a system composed of two circular roads of same size with two junctions. The three policies are : – right priority, – standard given phase duration, – feedback controlled duration (based on the road congestion) computed by LQG method.

The control improves the middle and the high density phases, without spoiling the low density one. The improvement given by the feedback control achieves the throughput obtained on a unique circular road without crossing but doubling the time spent in a place representing the crossing place.

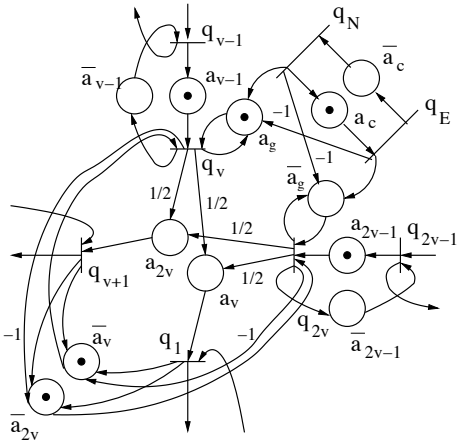

FIGURE 4. Traffic lights modeling.

The high density phase for the openloop standard traffic light control is unstable but for the time being we have not been able to find an explanation in terms of chaotic dynamics. It will be the source of future refexions.

Furthermore, the feedback control dissolves more efficiently the jams (that can appear locally in transient regimes) than the other policies would do.

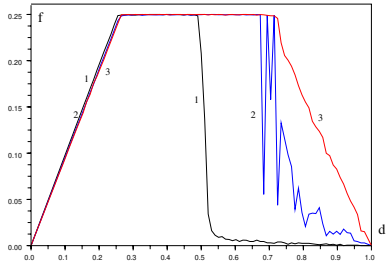

FIGURE 5. Comparison of three crossing policies : – right priority (1), – open loop light control (2), –feedback light control (3).

3. A subclass of triangular homogeneous dynamics

In this section we study a subclass of 1-homogeneous minplus linear systems for which we can prove the periodicity. Their dynamics belongs to a subclass of 1-homogeneous triangular systems :

$$\left\{ \begin{array}{l} u_{k+1} = C \otimes u_k, \\ x_{k+1} = A(u_k) \otimes x_k \oplus B(u_k) \otimes u_k. \end{array} \right. \tag{3.1}$$

where $\{u_k\}_{k\in\mathbb{N}}$ and $\{x_k\}_{k\in\mathbb{N}}$ are minplus column vectors, C is a minplus square matrix, $A(u_k)$ and $B(u_k)$ are two minplus 0-homegeneous matrices depending of u_k. We call this type of systems Triangular 1-Homogeneous (T1H).

We call linear periodic dynamic (LP) a dynamic given by :

$$x_{k+1} = E_k \otimes x_k, \quad x_0 \text{ given},$$

where E_k are minplus matrices periodic in k.

We can prove the following theorems (see the proofs in [**5**]).

THEOREM 3.1. *Every T1H dynamics behaves asymptotically as a LP dynamics.*

THEOREM 3.2. *A T1H system with $A(u)$ irreducible for every $u \in \mathbb{R}_{min}$ satisfies :*

$$\max_{u_0\in\mathbb{R}_{\min}} \mu_x(u_0) = \max_{\bar{u}\in\mathcal{V}} \mu_x(\bar{u}),$$

where (u_0, x_0) denotes the initial condition of (3.1), $-\mu_x(u_0)$ is equal to $\lim_{k\to\infty} x_k/k$, and $\mathcal{V}$ is the set of the minplus eigenvectors of the matrix C.

THEOREM 3.3. *Every LP dynamic $y_{k+1} = E_k \otimes y_k$, such that the matrices E_k have the same support, is realizable by a T1H dynamics.*

4. Application to traffic

We show that the traffic of regular towns with traffic light, buffered junction but without turning possibilities can be modeled with a T1H dynamics.

On the Petri net of Figure 6 the traffic light is modeled by the subsystem corresponding to the transitions u_1, u_2, u_3, u_4, which has no input coming from the rest of the system. The dynamic of this subsystem is minplus linear. If the initial condition $u_0 = (0, 0, 0, 0)$ the number of tokens in the places a_0 and b_0 is boolean and periodic. To a cycle corresponds the four phases given in the Table 1. The junction has a buffer place in each direction (a_1, b_1) to avoid blocking. The phases 2 and 4 gives the time,

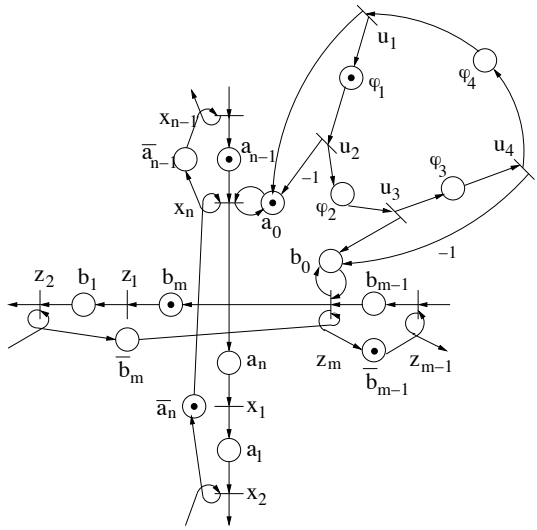

FIGURE 6. Traffic light intersection without possibility of turning.

Phase	a_0	b_0	Vertical light color	Horizontal light color
1	1	0	green	red
2	0	0	red	red
3	0	1	red	green
4	0	0	red	red

TABLE 1. The phases of the traffic light.

for car entering in the junction, to go in the buffer and then to free the crossing. Indeed, a vehicle entering in the crossing (represented by the two places a_n and b_m) leaves it surely in one unit of time.

The green duration of phase 1 and 3 is the sojourn time of tokens in the place φ_i. The phases 2 and 4 have a duration of one unit.

PROPOSITION 4.1. *The Petri net of Figure 6 has the T1H following dynamics :*

$$u^{k+1} = \begin{bmatrix} \cdot & \cdot & \cdot & \varphi_4 \\ \varphi_1 & \cdot & \cdot & \cdot \\ \cdot & \varphi_2 & \cdot & \cdot \\ \cdot & \cdot & \varphi_3 & \cdot \end{bmatrix} \otimes u^k, \qquad \begin{bmatrix} x^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} A_1(u^k) & \cdot \\ \cdot & A_2(u^k) \end{bmatrix} \otimes \begin{bmatrix} x^k \\ z^k \end{bmatrix},$$

where $\cdot$ denotes ∞, with

$$A_1(u)_{i,j} = \begin{cases} a_0 u_1/u_2, & \text{if } (i,j) = (n,n), \\ \text{independent of } u \text{ elsewhere.} \end{cases}$$

and

$$A_2(u)_{i,j} = \begin{cases} b_0 u_3/u_4, & \text{if } (i,j) = (m,m), \\ \text{independent of } u \text{ elsewhere.} \end{cases}$$

We are able to explicit the asymptotic flows which are different according the direction followed by the vehicles. Lets us give the result in the particular case where all the phases have a duration of one time unit $\varphi_i = 1$ for all i.

THEOREM 4.1. *The average flow on the horizontal (resp. vertical) road is given by $\lambda/4$ where λ is the unique eigenvalue of the irreducible matrix $\bigotimes_{k=0}^{3} A_1(u^k)$ [resp. $\bigotimes_{k=0}^{3} A_2(u^k)$].*

Bibliography

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat : *Synchronization and Linearity*, Wiley, 1992.

[2] M. Gondran, M. Minoux : *Graphs and Algorithms*, J. Wiley & Sons, 1986.

[3] N. Farhi, M. Goursat, and J.P. Quadrat : *Derivation of the fundamental traffic diagram for two circular roads and a crossing using minplus algebra and Petri net modeling*, In Proceedings of the 44th IEEE - CDC-ECC Seville December 2005.

[4] N. Farhi, M. Goursat, and J.P. Quadrat: *Fundamental Traffic Diagrams of Elementary Road Networks*, to appear in Proceedings of ECC Kos July 2007.

[5] N. Farhi : *A class of periodic degree one homogeneous minplus systems*, to appear.

[6] G. Cohen, S. Gaubert and J.P. Quadrat : *Asymptotic Throughput of Continuous Timed Petri Nets* Proceedings of the 34th Conference on Decision and Control, New Orleans, Dec, 1995.

[7] S. Gaubert : *Théorie des systémes linéaires dans les dioides*, Thesis dissertation, École des Mines de Paris, 1992.

Max-plus cones and semimodules[1]

F. Faye, M. Thiam, L. Truffet, and E. Wagneur

The concept of moduloïd over a dioïd has been introduced in [**7**]. These algebraic structures have been considered in the context of production systems [**5**], computer systems [**9**], network systems [**4**], or more generally for the modeling and analysis of discrete event systems [**1**], [**8**]. For G. Gondran and M. Minoux ([**7**]), a moduloïd over a dioïd is the algebraic structure left invariant under the action of a matrix A with entries in a dioïd (the "space" of proper "vectors" of A). This structure is also very similar to that of band-space over a belt of R.A. Cunninghame-Green [**6**].

The problem of solving linear equations of the type $Ax = Bx$ in the max-algebra has been considered by many authors (cf [**4**], and [**2**], where additional references may be found). In [**2**] the authors show how to compute all solutions to a system of linear equations over a totally ordered idempotent semifield.

In [**3**] the authors consider subsets of the positive cone $\mathbb{R}_+^n$ endowed with the max operator as the first composition law, together with "$+$" as the second composition law. They show how to relate subsets of $\mathbb{R}_+^n$ to the concept of generating vectors and bases defined in [**11**].

Since the early years, the terminology evolved, and the concept of idempotent semimodule over an idempotent semiring (or semifield) has emerged as the counterpart of that of vector space or, more generally, of module over a ring.

The most general definition of a (finite dimensional) idempotent semimodule M is the following one: take two matrices A, B of finite size with coefficients in a semifield S, consider the inequalities $Ax \leq Bx$, and then define M as the set of solutions to this set of inequalities. Another way to define (or to represent) a (finite dimensional) semimodule over S is to give its basis, for example as the (independent) columns of a matrix with coefficients in S, It is then natural to ask how to get from one representation to the other.

The aim of this paper is to study n-dimensional semimodules over a completely ordered and complete idempotent semifield defined by a pair (A, B).

Recall that an idempotent semigroup $(S, \vee)$ is ordered by the relation $s, t \in S\,,\ s \leq t \iff s \vee t = t$. If $(S, \vee, \mathbf{0})$ is an idempotent semigroup with the neutral element $\mathbf{0}$, then $\mathbf{0}$ is the least element of S, since for

[1]Research supported by NRC grant RGPIN-143068-05.

every $s \in S, s \vee \mathbf{0} = s$. We will assume here that the set of scalars S is an idempotent completely ordered semifield, which is complete, i.e. S is totally ordered and complete, with least element $\mathbf{0}$, endowed with two composition laws: $\vee$, and $\cdot$ such that :
i) $(S, \vee)$ is an idempotent commutative monoïd, with neutral element $\mathbf{0}$.
ii) $(S \setminus \{\mathbf{0}\}, \cdot)$ is an abelian group — hence $(S \setminus \{\mathbf{0}\}, \cdot, \leq)$ is an ℓ-group –, with neutral element written 1.
iii) $\cdot$ is distributive over $\vee$,
iv) $\mathbf{0}$ is absorbent (for every $s \in S, \mathbf{0} \cdot s = \mathbf{0}$).

An idempotent semifield is also called a dioïd [**1**].

Next we give a brief summary of our talk. In Part 1, we state some general results on semimodules defined by a pair (A, B). In particular, we give an explicit formula for the semimodule of solutions to a single equation $a_i x \leq b_i x$ in terms of the coefficients a_{ij}, and b_{ij} of a_i and b_i (here and in the sequel, scalar product is denoted by concatenation). Take a permutation σ_i from the symmetrical group S_n such that, for $j = 1, \ldots, k$, $a_{i,\sigma_i(j)} \leq b_{i,\sigma_i(j)}$, while $b_{i,\sigma_i(j)} < a_{i,\sigma_i(j)}$, $j = k+1, \ldots n$. Let $J_k(i) = \{j\, (1 \leq j \leq n) |\ a_{i,\sigma_i(j)} \leq b_{i,\sigma_i(j)}\}$. Clearly for every $j \in J_k(i)$, $e_{\sigma_i(j)}$ is a solution to $a_i x \leq b_i x$, where the e_j's are the elements of the canonical basis of S^n.
Also, for every $j \in J_k(i)$, $\ell \notin J_k(i)$, we have:

$$a_{i,\sigma_i(j)} \vee b_{i,\sigma_i(j)} = a_{i,\sigma_i(j)} \vee a_{i,\sigma_i(\ell)}(a^{-1}_{i,\sigma_i(\ell)} b_{i,\sigma_i(j)}) =$$
$$= b_{i,\sigma_i(j)} = b_{i,\sigma_i(j)}(\mathbf{1} \vee b_{i,\sigma_i(\ell)} a^{-1}_{i,\sigma_i(\ell)}) = b_{i,\sigma_i(j)} \vee b_{i,\sigma_i(\ell)}(a^{-1}_{i,\sigma_i(\ell)} b_{i,\sigma_i(j)}).$$

Hence $e_{\sigma_i(j)} \vee (a^{-1}_{i,\sigma_i(\ell)} b_{i,\sigma_i(j)}) e_{\sigma_i(\ell)}$ also satisfies $a_i x \leq b_i x$.

We have proved the following statement.

Proposition 1. *If $b_i \not< a_i$, then the set of solutions to $a_i x \leq b_i x$ is a semimodule M_i generated by the $k\,(n+1-k)$ vectors given, for every $j \in J_k(j)$, by:*
$e_{\sigma_i(j)}$, and $e_{\sigma_i(j)} \vee (a^{-1}_{i,\sigma_i(\ell)} b_{i,\sigma_i(j)}) e_{\sigma_i(\ell)}$, $\ell \notin J_k(j)$.

The semimodule M_i is generated by the columns of V_i given by the concatenation over $J_k(i)$ of the matrices

$$V_j(i) = [\ e_{\sigma_i(j)} | e_{\sigma_i(j)} \vee (a^{-1}_{i,\sigma_i(\ell_1)} b_{i,\sigma_i(j)}) e_{\sigma_i(\ell_1)} |$$
$$e_{\sigma_i(j)} \vee (a^{-1}_{i,\sigma_i(\ell_2)} b_{i,\sigma_i(j)}) e_{\sigma_i(\ell_2)} | \cdots | e_{\sigma_i(j)} \vee (a^{-1}_{i,\sigma_i(\ell_{n-k})} b_{i,\sigma_i(j)}) e_{\sigma_i(\ell_{n-k})}\ ],$$

where $\{\ell_1, \ell_2, \ldots, \ell_{n-k}\} = \{\ell |\, \ell \notin J_k(i)\}$.

In Part 2, we give a geometric interpretation of the results of Part 1. Let M_p stand for the semimodule generated by the solutions to $a_p x \leq b_p x$,

with $\sigma_p \in S_n$, and $J_q(p) = \{r\ (1 \leq r \leq n) | a_{\sigma_p(r)} \leq b_{\sigma_p(r)}\}$. We have the following statement.

THEOREM 1. *For $1 \leq k \leq n-1$, we have $M_i \cap M_p \neq \{0\}$ iff one of the following conditions holds*
i) $J_k(i) \bigcap J_q(p) \neq \emptyset$
ii) $\bigvee_{\ell \in J_q(p)} a_{i\sigma_p(\ell)} b^{-1}_{p,\sigma_p(\ell)} \geq \bigwedge_{j \in J_k(i)} a^{-1}_{p,\sigma_i(j)} b_{i,\sigma_i(j)}$.

In Part 3, we solve the system of inequalities $Ax \leq Bx$ using combinatorial decomposition of the inequalities $a_i x \leq b_i x$.
In particular, we show that $a_i x \leq b_i x$ is equivalent to a series of inequalities :
$b^{-1}_{i1}\big(a_{i1}x_{i1} \vee (a_{i2} \vee b_{i2})x_2 \vee \ldots \vee (a_{in} \vee b_{in})x_n\big) \leq x_1$ or
$b^{-1}_{i2}\big((a_{i1} \vee b_{i1})x_1 \vee a_{i2}x_2 \vee (a_{i3} \vee b_{i3})x_3 \vee \ldots \vee (a_{in} \vee b_{in})x_n\big) \leq x_2$ or
$\ldots\ldots$ or
$b^{-1}_{in}\big((a_{i1} \vee b_{i1})x_1 \vee \ldots \vee a_{in}x_n\big) \leq x_n$.

The first inequality may be written in matrix form (add the trivial inequalities $x_2 \leq x_2\,,\ \ldots, x_n \leq x_n$) as: $P_{i1}x \leq x$, with

$$P_{i1} = \begin{bmatrix} b^{-1}_{i1}a_{i1} & b^{-1}_{i1}(a_{i2} \vee b_{i2}) & \cdot & \cdot & b^{-1}_{i1}(a_{in} \vee b_{in}) \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdot & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \cdot & \mathbf{0} \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \mathbf{0} & \cdot & \cdot & \cdot & \mathbf{1} \end{bmatrix}$$

It is well-known that $P_{i1}x \leq x \iff P^*_{i1}x = x$. This equation has a nontrivial solution iff $a_{i1} \leq b_{i1}$, and in this case the solutions are given by the columns of P^*_{i1}.

Proceeding similarly for each line, we get $P^*_2, \ldots, P^*_n$, and all solutions lie in the concatenation $[P_{i1}|P_{i2}|\ldots|P_{in}]$.

Then we look at all the intersections of the type $P_{ik} \cap P_{j\ell}$, etc. in a combinatorial way, and devise an algorithm for the solution to these systems of inequalities.

Finally, we give a complete description, both algebraic and geometric, for cases $n = 2, 3$.

Bibliography

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J-P. Quadrat. *Synchronization and Linearity*. John Wiley and Sons, 1992.

[2] P. Butkovič, and G. Hegedüs An Elimination Method for Finding All solutions of the System of Linear Equations over an Extremal Algebra, *Economicko-Matematický Obzor*, 20, 1984, 203-215.

[3] P. Butkovič, H. Schneider, and S. Sergeev, Generators, extremals and bases of max-cones, *Linear Algebra and its Applications*, 421, 2007, 394-406.
[4] B. A. Carré An algebra for Networks Routing problems *J. Inst. Math Appl.*7, 1971, 273-294.
[5] G.Cohen, D. Dubois, J.P. Quadrat, and M. Viot, A Linear System Theoretic View of Discrete-Event Processes and its use for Performance Evaluation in Manufacturing, *IEEE Trans. on Automatic Control, AC–30, 1985, 210–220.*
[6] R.A. Cunninghame-Green, *Minimax Algebra* , Lecture Notes in Economics and Mathematical Systems, 166, Springer Verlag, 1979.
[7] M. Gondran, and M. Minoux Valeurs propres et vecteurs propres dans les dioïdes et leur interprétation en théorie des graphes. *E.D.F. Bulletin de la Direction des Études et Recherches. Série C-Math. Info.*, #2 ,1977, pp. 25–41.
[8] V. Kolokoltsov, and V. Maslov, *Idempotent Analysis and its Applications*, Mathematics and its Applications, Kluwer, 1997.
[9] C.E.Leiserson and J.B. Saxe, Optimizing synchronous systems, *Journal of VLSI and compiuter systems*, 1, 1983, 41-67.
[10] P. Moller, *Théorie algébrique des systèmes à événements discrets.* Thèse, École des Mines de Paris, Paris, 1988.
[11] E. Wagneur, Moduloids and Pseudomodules. 1. Dimension theory *Discrete Mathematics*, 98, 1991, 57-73.
[12] E. Wagneur, Dequantisation: Direct and Semi-direct Sum Decomposition of Idempotent Semimodules. – In: G.L. Litvinov and V.P. Maslov (eds.), *Idempotent Mathematics and Mathematical Physics*, Contemporary Mathematics, 377, 2005, pages 339-352.

Duality of cluster varieties

V.V. Fock and A.B. Goncharov

Cluster variety is an algebraic variety (strictly speaking, a scheme) defined by combinatorial data by explicit set of coordinate charts and transition functions. More precisely, for any collection of combinatorial data, called *seed* one associates three varieties $\mathcal{A}_{|\mathbb{I}|}$, $\mathcal{X}_{|\mathbb{I}|}$, and $\mathcal{D}_{|\mathbb{I}|}$. These varieties possess canonical pre-symplectic, Poisson and symplectic structures, respectively. One defines also a discrete group $\mathfrak{D}_{|\mathbb{I}|}$ acting on all the three types of varieties and preserving the respective structures. The manifolds $\mathcal{X}_{|\mathbb{I}|}$ and $\mathcal{D}_{|\mathbb{I}|}$ admit a quantisation (noncommutative deformation of the algebra of functions) which is also $\mathfrak{D}_{|\mathbb{I}|}$-invariant.

Varieties admitting cluster descriptions are simple Lie groups, moduli spaces of Stokes parameters, moduli of flat connections on Riemann surfaces, configuration spaces of flags, Teichmüller spaces and their generalisations, the spaces of measured laminations and some others. One of the important features of cluster varieties is that they are defined not only

over a field but also over semifields (semigroups w.r.t. addition and groups w.r.t. the multiplication). For example, one can consider Teichmüller space, space of measured laminations and the space of flat $PSL(2,\mathbb{F})$-connections over a surface Σ as the same cluster manifold but defined over the semifield $\mathbb{R}_{>0}$ of positive real numbers, tropical semifield $\mathbb{R}^t$ (which is ordinary R as a set with maximum for the addition operation and ordinary addition for the multiplication), and a field $\mathbb{F}$, respectively.

Let us give the precise definitions:

A *cluster seed*, or just *seed*, $\mathbf{I}$ is a quadruple (I, I_0, ε, d), where

i) I is a finite set;

ii) $I_0 \subset I$ is its subset;

iii) ε is a matrix ε_{ij}, where $i, j \in I$, such that $\varepsilon_{ij} \in \mathbb{Z}$ unless $i, j \in I_0$.

iv) $d = \{d_i\}$, where $i \in I$, is a set of positive integers, such that the matrix $\widehat{\varepsilon}_{ij} = \varepsilon_{ij} d_j$ is skew-symmetric.

The elements of the set I are called *vertices*, the elements of I_0 are called *frozen vertices*. The matrix ε is called *exchange matrix*, the numbers $\{d_i\}$ are called *multipliers*, and the function d on I whose value at i is d_i is called *multiplier function*. We omit $\{d_i\}$ if all of them are equal to one, and therefore the matrix ε is skew-symmetric, and we omit the set I_0 if it is empty.

An isomorphism σ between two seeds is a map $\mathbf{I} = (I, I_0, \varepsilon, d)$ and $\mathbf{I}' = (I', I_0', \varepsilon', d')$ is an isomorphism of finite sets $\sigma : I \to I'$ such that $\sigma(I_0) = I_0'$, $d_{\sigma(i)} = d_i$ and $\varepsilon_{\sigma_i, \sigma_j} = \varepsilon_{ij}$. Observe that the automorphism group of a seed may be nontrivial.

For a seed $\mathbf{I}$ we associate a torus $\mathcal{X}_{\mathbf{I}} = (\mathbb{F}^\times)^I$, called $\mathcal{X}$-*torus*, another torus $\mathcal{A}_{\mathbf{I}} = (\mathbb{F}^\times)^I$, called $\mathcal{X}$-*torus* and the third one $\mathcal{D}_{\mathbf{I}} = (\mathbb{F}^\times)^{I \times I}$ called $\mathcal{D}$-*torus* or a *double torus*. We denote the standard coordinates on these tori by $\{x_i | i \in I\}$, $\{a_i | i \in I\}$ and $\{y_i, b_i | i \in I\}$, respectively.

The $\mathcal{X}$-torus is equipped with the Poisson structure

$$\{x_i, x_j\} = \widehat{\varepsilon}_{ij} x_i x_j \tag{1}$$

The $\mathcal{A}$-torus is equipped with the pre-symplectic structure (closed 2-form ω possibly degenerate)

$$\omega = \frac{1}{2} \sum_{i,j} \widehat{\varepsilon}_{ij} \frac{da_i \wedge da_j}{a_i a_j} \tag{2}$$

The $\mathcal{D}$-torus is equipped with the symplectic form

$$\omega_{\mathcal{D}} = \frac{1}{2} \sum_{i,j} \widehat{\varepsilon}_{ij} \frac{db_i \wedge db_j}{b_i b_j} + \sum_i d_i^{-1} \frac{db_i \wedge dy_i}{b_i y_i} \tag{3}$$

The inverse of this form is a nondegenerate Poisson structure which can be written as

$$\{y_i, y_j\} = \widehat{\varepsilon}_{ij} y_i y_j, \quad \{y_i, b_j\} = \delta^i_j d^i y_i b_j, \quad \{b_i, b_j\} = 0 \tag{4}$$

Observe that these sructures are constant in logarithmic coordinates.

Isomorphism between two $\mathcal{X}$-tori $\mathcal{X}_{\mathbb{I}}$ and $\mathcal{X}_{\mathbb{I}'}$ is a map given in coordinates by $x_{\sigma(i)} = x_i$, where σ is an isomorphism of the seeds. Observe that there are much less isomorphisms of $\mathcal{X}$-tori then just isomorphisms of the corresponding Poisson manifolds. Isomorphisms of $\mathcal{A}$- and $\mathcal{D}$-tori are defined analogously.

There exist the following maps between the tori:

$$\mathcal{A}_{\mathbf{I}} \to \mathcal{X}_{\mathbf{I}}, \quad x_i = \prod_j a_j^{\varepsilon_{ij}}; \tag{5}$$

$$\mathcal{A}_{\mathbf{I}} \times \mathcal{A}_{\mathbf{I}} \to \mathcal{D}_{\mathbf{I}}, \quad y_i = \prod_j a_j^{\varepsilon_{ij}}, \quad b_i = a_i/\tilde{a}_i, \tag{6}$$

Here $\tilde{a}_i$ are coordinates on the second $\mathcal{A}_{\mathbf{I}}$-factor.

$$\mathcal{D}_{\mathbf{I}} \to \mathcal{X}_{\mathbf{I}}, \quad x_i = y_i, \tag{7}$$

and

$$\mathcal{D}_{\mathbf{I}} \to \mathcal{X}_{\mathbf{I}}, \quad x_i = y_i \prod_j b_j^{\varepsilon_{ij}}. \tag{8}$$

All the maps are compatible with the respective symplectic, pre-symplectic and Poisson structures. Namely the map (5) is a composition of the quotient by the kernel of the pre-symplectic form and a symplectic map to a symplectic leaf. The map (6) maps the symplectic form to the pre-symplectic one. The map (7) is Poisson, the map (8) is anti-Poisson (Poisson with the opposite Poisson structure on the $\mathcal{X}$-torus). The maps (7) and (8) are dual to each other in the sense on Poisson pairs.

Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ and $\mathbf{I}' = (I', I'_0, \varepsilon', d')$ be two seeds, and $k \in I - I_0$. A *mutation in the vertex* k is an isomorphism $\mu_k : I \to I'$ satisfying the following conditions:

- $\mu_k(I_0) = I'_0$,
- $d'_{\mu_k(i)} = d_i$,
- $\varepsilon'_{\mu_k(i)\mu_k(j)} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k \text{ otherwise} \\ \varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{kj} < 0 \\ \varepsilon_{ij} + \varepsilon_{ik}|\varepsilon_{kj}| & \text{if } \varepsilon_{ik}\varepsilon_{kj} \geq 0 \end{cases}$

Two seeds related by a sequence of mutations are called equivalent.

Mutations induce rational maps between the corresponding seed tori, which are denoted by the same symbol μ_k and are given by the formulae

$$x_{\mu_k(i)} = \begin{cases} x_k^{-1} & \text{if} \quad i = k \\ x_i(1+x_k)^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \geq 0 \\ x_i(1+(x_k)^{-1})^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \leq 0 \end{cases} .$$

for the $\mathcal{X}$-torus,

$$a_{\mu_k(i)} = \begin{cases} \dfrac{\prod\limits_{j|\varepsilon_{jk}>0} a_j^{\varepsilon_{jk}} + \prod\limits_{j|\varepsilon_{jk}<0} a_j^{-\varepsilon_{jk}}}{a_k} & \text{if } i = k \\ a_i & \text{if } i \neq k \end{cases}$$

for the $\mathcal{A}$-torus and

$$b_{\mu_k(i)} = \begin{cases} \dfrac{(1+x_k)^{-1} \prod\limits_{j|\varepsilon_{jk}>0} b_j^{\varepsilon_{jk}} + (1+(x_k)^{-1})^{-1} \prod\limits_{j|\varepsilon_{jk}<0} b_j^{-\varepsilon_{jk}}}{b_k} & \text{if } i = k \\ b_i & \text{if } i \neq k \end{cases}$$

$$y_{\mu_k(i)} = \begin{cases} y_k^{-1} & \text{if} \quad i = k \\ y_i(1+y_k)^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \geq 0 \\ y_i(1+(y_k)^{-1})^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \leq 0 \end{cases} .$$

for the $\mathcal{D}$-torus.

Since in the sequel we shall extensively use compositions of mutations called also *cluster transformations* we would like to introduce a shorthand notation for them. Namely, we denote an expression $\mu_{\mu_i(j)}\mu_i$ by $\mu_j\mu_k$, $\mu_{\mu_{\mu_i(j)}\mu_i(k)}\mu_{\mu_i(j)}\mu_i$ by $\mu_k\mu_j\mu_i$, and so on.

Mutations have the following properties (valid for mutation of seeds as well as for mutations of respective tori):

- Every seed $\mathbf{I} = (I, I_0, \varepsilon, d)$ seed is related to other seeds by exactly $\sharp(I - I_0)$ mutations.

A_1: $\mu_i\mu_i = id$

$A_1 \times A_1$ If $\varepsilon_{ij} = \varepsilon_{ji} = 0$ then $\mu_i\mu_j\mu_j\mu_i = id$.

A_2: If $\varepsilon_{ij} = -\varepsilon_{ji} = -1$ then $\mu_i\mu_j\mu_i\mu_j\mu_i = id$. (This is called the *pentagon relation.*)

B_2: If $\varepsilon_{ij} = -2\varepsilon_{ji} = -2$ then $\mu_i\mu_j\mu_i\mu_j\mu_i\mu_j = id$.

G_2: If $\varepsilon_{ij} = -3\varepsilon_{ji} = -3$ then $\mu_i\mu_j\mu_i\mu_j\mu_i\mu_j\mu_i = id$.

By *id* we mean here an isomorphism of the seeds or tori. Conjecturally all relations between mutation follow from these ones.

Given a seed one can produce a $\sharp(I - I_0)$ seeds by mutations. Continuing this procedure one obtains a $\sharp(I - I_0)$-valent tree whose vertices are seeds (or seed tori) and edges are pairs of mutually inverse mutations. Obviously if we start from any other seed from the tree we obtain the same tree. Every two tori of the tree are related by exactly one composition of mutations. Call two points of two different tori equivalent if they are related by the composition of mutations. The cluster manifold (denoted by $\mathcal{X}_{|\mathbf{I}|}$, $\mathcal{A}_{|\mathbf{I}|}$ or $\mathcal{D}_{|\mathbf{I}|}$ depending on which kind of tori are used) is the affine closure of disjoint union of the tori quotiented by the equivalence relation.

Each particular seed tori can be considered as a coordinate chart of the corresponding cluster manifolds and compositions of mutations can be considered as transition functions between the charts.

Mutations respect the Poisson structure when acting on $\mathcal{X}$ tori, presymplectic structure when acting on $\mathcal{A}$-tori and symplectic when acting on $\mathcal{D}$-tori. Thus the cluster manifolds $\mathcal{X}_{|\mathbf{I}|}$, $\mathcal{A}_{|\mathbf{I}|}$ and $\mathcal{D}_{|\mathbf{I}|}$ acquire the respective structures. (In fact the formula for mutation of the matrix ε can be considered as a corollary of this property and the mutation formulae for, say, $\mathcal{X}$-tori).

Mutations commute with the maps (5),(6),(7) and (8) thus these maps are defined between the respective cluster varieties compatible with presymplectic, symplectic and Poisson structures thereof.

Mutations are rational maps with positive integral coefficients and thus the cluster manifold can be defined not only over a field but over any semifield as well. For semifields without -1 (like the semifields of positive real numbers or the tropical semifields) the mutations are isomorphisms and thus the whole manifold is isomorphic to every coordinate torus.

The symmetry group $\mathfrak{D}_{|\mathbf{I}|}$ of a cluster manifold permuting the seed tori is called the (generalised) *mapping class group* of the cluster manifold. The name comes from the case of Teichmüller space, when this group is the actual mapping class group. The group depends on the equivalence class of a seed only and is common for cluster manifolds of types $\mathcal{X}$, $\mathcal{A}$ and $\mathcal{D}$. Every sequence of mutations together with an isomorphism of the initial and the final seed gives an element of the mapping class group. Conversely, given a seed, every mapping class group element can be presented by a sequence of mutations starting from the given seed together with the isomorphism between the final seed and the initial one. Two sequences of mutations different by the relations A_1–G_2 correspond to the same mapping class group elements.

Consider the ring of algebraic functions on a cluster manifold in more details. The ring of algebraic functions $\mathcal{O}(\mathcal{X}_{\mathbf{I}})$ (resp. $\mathcal{O}(\mathcal{D}_{\mathbf{I}})$, $\mathcal{O}(\mathcal{A}_{\mathbf{I}})$) on every torus is the ring of Laurent polynomials of cluster variables. This ring contains a subring of Laurent polynomials with integral coefficients $\mathcal{O}^{\mathbb{Z}}$ and a semiring of Laurent polynomials with positive integral coefficients $\mathcal{O}^{\mathbb{Z}}_{>0}$ also depending of course of the seed and of the type of the torus. A cluster transformation in general does not presereve the ring $\mathcal{O}$ since it is birational. The ring of algebraic functions on the whole cluster manifold is the intersection of inverse images of the rings $\mathcal{O}$ under all possible cluster transformations of a seed tori. In other words the ring $\mathcal{O}$ consists of Laurent polynomials which stay Laurent under all possible cluster transformations. The celebrated result of Fomin and Zelevinsky called *Laurent phenomenon* claims that for any cluster variety $\mathcal{A}_{|\mathbf{I}|}$ of type $\mathcal{A}$ the coordinate functions belong to the ring $\mathcal{O}^{\mathbb{Z}}$. The ring $\mathcal{O}$ contains a subring $\mathcal{O}^{\mathbb{Z}}$ and a subsemiring $\mathcal{O}^{\mathbb{Z}}_{>0}$. The latter is additively generated by Laurent polynomials from $\mathcal{O}$ with positive integral coefficients indecomposable into a sum of two such polynomials. Such elements of the semiring $\mathcal{O}^{\mathbb{Z}}_{>0}$ are called *irreducibles*. The main conjecture, proven for a sufficiently wide class of cluster manifolds describes the structure of the set of irreducible Laurent polynomials:

- The set of irreducible Laurent polynomials is a basis in the ring $\mathcal{O}$.
- The set of irreducible Laurent polynomials for the cluster variety $\mathcal{X}_{|\mathbf{J}|}$ (resp. $\mathcal{A}_{|\mathbf{I}|}$, $\mathcal{D}_{|\mathbf{I}|}$) is canonically isomorphic to the set of points of the cluster variety $\mathcal{A}_{|\mathbf{I}|}(\mathbb{Z}^t)$ (resp. $\mathcal{X}_{|\mathbf{I}|}(\mathbb{Z}^t)$, $\mathcal{D}_{|\mathbf{I}|}(\mathbb{Z}^t)$).

One can consider this property as a duality between cluster varieties of type $\mathcal{X}$ (resp. $\mathcal{A}$, $\mathcal{D}$) and the tropical cluster varieties of type $\mathcal{A}$, $\mathcal{X}$ and $\mathcal{D}$, respectively.

The correspondence between irreducible Laurent polynomials and points of the dual tropical variety is especially simple for the variety of type $\mathcal{X}$. In this case the coordinates of the corresponding point of the tropical variety are given by multidegree of the highest term of the corresponding Laurent polynomial.

Example

Let us consider the simplest nontrivial example: the seed $\mathbf{I} = \{I, \varepsilon\}$ with $I = \{1, 2\}$ and $\varepsilon_{12} = 1$. There are exactly 5 isomorphism classes of seed tori equivalent to a given one, however all the five seeds are isomorphic, thus the mapping class group is $\mathbb{Z}/5\mathbb{Z}$.

The simplest geometric meaning has the space $\mathcal{X}$. It is the space of 5-tuples of points $(p_1, \dots, p_5)$ on the projective line P^1 such that $p_i \neq p_{i+1 \pmod 5}$ and modulo the automorphisms of P^1. The 5-tuple of coordinate systems on this space is numerated by triangulations of the pentagon with vertices $1, \dots, 5$. For every internal diagonal one associates the cross-ratio of the four points of the quadrilateral which this diagonal cuts into halves. Mutations correspond to removing a diagonal and replacing it by another one of the quadrilateral. The same variety over $\mathbb{R}_{>0}$ is the configuration space of 5-tuples of points on $\mathbb{R}P^1$ with prescribed cyclic order.

The $\mathcal{A}$-space is the space of collections of 10 nonvanishing vectors $v_1, \dots, v_{10}$ in $\mathbb{F}^2$ equipped with a nonzero bivector Vol. The collections are considered up to the action of the group $SL(2, \mathbb{F})$ of linear transformations preserving Vol and subject to the relations $v_i = -v_{i+5 \pmod{10}}$ and $v_i \wedge v_{i+1 \pmod{10}} = Vol$. The map $\mathcal{X}_{|\mathbf{I}|} \to \mathcal{X}_{|\mathbf{I}|}$ is given by the obvious projection of $\mathbb{F}^2 - \{0\} \to P^1$. For the internal diagonal of the pentagon with ends i and j one associates the coordinate $v_i \wedge v_j / Vol$. The $\mathcal{D}$ variety is the space of flat $SL(2, \mathbb{F})$ connections on a sphere with 5 different points on the equator removed with parabolic monodromy around these points. Consider the associated vector bundle and choose a monodromy invariant section about each singular points. Then trivialise the bundle over the northern hemisphere. The five chosen sections give five vectors $v_1, \dots, v_5$ in $\mathbb{F}^2$. The same procedure over the southern hemisphere gives five vectors $w_1, \dots, w_5$ in another copy of $\mathbb{F}^2$. Given a triangulation of the pentagon we associate to every internal diagonal two coordinates x and b. The coordinate x is just the cross ratio of four points in P^1 defined by the vectors v_i standing at the corners of the quadrilateral cut by the diagonal (just like for the $\mathcal{X}$-space). The coordinate b is given by $b = (v_i \wedge v_j)/(w_i \wedge w_j)$, where i and j are the ends of our diagonal. The two projections to the $\mathcal{X}$ variety are obviously given by projectivising the collections of vectors $\{v_i\}$ and $\{w_i\}$, respectively. The same manifold over $\mathbb{R}_{>0}$ can be identified with the space of complex structures on a sphere with five punctures on the equator.

Given a triangulation of the pentagon one can describe the basis of the ring $\mathcal{O}^{\mathbb{Z}}$ of the corresponding $\mathcal{X}$-variety explicitly as a set of Laurent polynomials $P_{\mathsf{a},\mathsf{b}}(x, y)$ of two variables x, y parameterised by two integers a, b as follows:

$$P_{\mathsf{a},\mathsf{b}}(x,y) = \begin{cases} x^{\mathsf{a}}y^{\mathsf{b}} & \text{if} \quad \mathsf{a} \leq 0, \mathsf{b} \leq 0 \\ x^{\mathsf{a}}y^{\mathsf{b}}(1+x^{-1})^{-\mathsf{b}} & \text{if} \quad \mathsf{a} \leq 0, \mathsf{b} \geq 0 \\ x^{\mathsf{a}}y^{\mathsf{b}}(1+x^{-1})^{-\mathsf{b}}(1+y^{-1}+x^{-1}y^{-1})^{\mathsf{a}} & \text{if} \quad \mathsf{a} \geq 0, \mathsf{b} \leq 0 \\ x^{\mathsf{a}}y^{\mathsf{b}}(1+y^{-1})(1+y^{-1}+x^{-1}y^{-1})^{\mathsf{a}-\mathsf{b}} & \text{if} \quad \mathsf{a} \geq \mathsf{b} \geq 0 \\ x^{\mathsf{a}}y^{\mathsf{b}}(1+y^{-1})^{\mathsf{a}} & \text{if} \quad \mathsf{b} \geq \mathsf{a} \geq 0 \end{cases}$$

One can easily check that this set of Laurent polynomials is invariant under simultaneous mutation of the variables x, y and of the variables a, b.

Bibliography

[1] V.V. Fock, A.B. Goncharov *Cluster X-varieties, amalgamation and Poisson-Lie groups*, n *Algebraic Geometry Theory and Number Theory*, Birkhäuser, Progress in math. Vol. **253**, 2006, `arXiv:math.RT/0508408`

[2] V.V. Fock, A.B. Goncharov *The quantum dilogarithm and unitary representations of the cluster mapping class groups*, `arXiv:math/0702397`

From max-plus algebra to non-linear Perron-Frobenius theory: an approach to zero-sum repeated games[1]

Stéphane Gaubert

This talk is based essentially on two joint works, with Akian and Nussbaum [**AGN07**], on the one hand, and with Akian and Lemmens [**AGL07**], on the other hand.

1. Introduction

The analysis of zero-sum repeated games by the dynamic programming method classically leads to studying discrete time dynamical systems of the form

$$v(k,\cdot) = f(v(k-1,\cdot)) \tag{1.1}$$

where the map f is order preserving. Here, $v(k,\cdot)$ is the *value function*, which associates to any initial state the value of the corresponding game in horizon k. The map f is the "one day" dynamic programming operator. The case of a finite state space is already interesting. Then, denoting by n the number of states, we may identify the value function to a vector in $\mathbb{R}^n$, and the map f to a self-map of $\mathbb{R}^n$.

[1]This work was partially supported by the joint RFBR/CNRS grant 05-01-02807

The explicit form of f depends on the details of the game. However, the map f may be written abstractly as:

$$f(v) = \inf_{\sigma} \sup_{\pi} r^{\sigma\pi} + P^{\sigma\pi} v \ , \tag{1.2}$$

where the infimum is taken over the strategies σ of the first player and the supremum is taken over the strategies π of the second player, $r^{\sigma\pi} \in \mathbb{R}^n$ is a vector of payments, and $P^{\sigma\pi}$ is a $n \times n$ nonnegative matrix. In the case of games with undiscounted payoff, the matrices $P^{\sigma\pi}$ are stochastic. When there is a positive discount rate, or when the game may halt with a positive probability, the matrices $P^{\sigma\pi}$ are substochastic. These (sub)-stochasticity properties imply that f is nonexpansive in the sup-norm, meaning that

$$\|f(v) - f(w)\|_\infty \leq \|v - w\|_\infty \ .$$

The relevance of the order and nonexpansiveness properties to control and game problems has been brought to light by several authors, see in particular [**CT80, Kol92, RS01, Ney03**].

The dynamic programming operators (1.2) may be thought of as generalizations of linear positive maps in several different ways. First, linear maps of the form $x \mapsto Px$, where P is a (sub)stochastic matrix, correspond to the zero-player case, in which every player has only one possible strategy, if we assume in addition that the payments are zero. Another special situation concerns the deterministic one player case, in which one of the two players has only one possible strategy, and the entries of the matrices $P^{\sigma\pi}$ are only 0 or 1. Then, f becomes an affine map over the min-plus or max-plus semiring. Further connections with Perron-Frobenius theory become apparent when using a familiar tropical instrument, the "logarithmic/exponential" glasses or "dequantization", as in [**LMS01, Vir01**]. This leads us to consider the conjugate map:

$$g = \exp \circ f \circ \log$$

where log denotes the map from the interior of the standard positive cone $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x \geq 0\}$ to $\mathbb{R}^n$ which does log entrywise, and $\exp := \log^{-1}$. Then, the map g is an order preserving self-map of the interior of $\mathbb{R}^n_+$, and it is positively homogeneous or subhomogeneous of degree one, meaning that $g(tx) = tg(x)$ or $g(tx) \leq tg(x)$ for all scalars $t \geq 1$ and for all $x \in \operatorname{int} \mathbb{R}^n_+$. Such maps belong to non-linear Perron-Frobenius theory, which deals with the nonlinear extensions of the spectral theory of positive linear maps. We refer the reader to [**Nus88**] for a general account of this topic and for references.

I will present some results concerning zero-sum games, which have been obtained by exploiting methods from non-linear Perron-Frobenius

theory with a max-plus or tropical point of view. The main results are taken from the two joint works [**AGN07, AGL07**].

2. Nonlinear spectral radius of dynamic programming operators

The classical notion of spectral radius has been extended to nonlinear maps in several ways [**MPN02**]. We assume here that g is a continuous positively homogeneous of degree one map leaving invariant a (closed, convex, pointed) cone C in a Banach space X, and that g preserves the order induced by C, which is such that $x \leq y$ if $y - x \in C$. *Bonsall's cone spectral radius* of g is defined by:

$$\tilde{r}_C(g) = \lim_k \|g^k\|_C^{1/k}$$

where, for all continuous, positively homogeneous of degree one self-maps h of C,

$$\|h\|_C := \sup_{x \in C \setminus \{0\}} \frac{\|h(x)\|}{\|x\|} \ .$$

Another natural definition of the spectral radius arises when considering the nonlinear eigenproblem:

$$g(u) = \lambda u$$

where the *nonlinear eigenvector* u belongs to $C \setminus \{0\}$, and the *nonlinear eigenvalue* λ is a nonnegative number. The *cone eigenvalue spectral radius*, $\hat{r}_C(g)$, is by definition the maximal nonlinear eigenvalue λ. Under some assumptions involving measures of non-compactness, it has been shown in [**MPN02**] that $\tilde{r}_C(g) = \hat{r}_C(g)$. Other useful notions of spectral radius, which coincide with the previous ones under reasonable assumptions, are studied in [**MPN02**].

We shall discuss here the related notion of *Collatz-Wielandt number*, which is obtained by considering super-eigenvectors in the interior of the cone instead of eigenvectors in the closed cone:

$$\bar{r}_C(g) := \inf\{\lambda > 0 \mid \exists u \in \operatorname{int} C,\ g(u) \leq \lambda u\} \ .$$

The term "Collatz-Wielandt number" arises from Wielandt's proof of the finite dimensional Perron-Frobenius theorem, in which the same formula is seen to characterize the Perron root of an irreducible nonnegative matrix.

The main result of [**AGN07**] shows that $\tilde{r}_C(g) = \bar{r}_C(g)$, when the cone C is normal, and when g satisfies some compactness assumptions.

We apply these tools to dynamic programming maps of the form (1.2), when the payments $r^{\sigma\pi}$ are 0, so that f is positively homogeneous of degree one. Under some standard assumptions (compactness of the action

spaces, continuous dependence of the reward and transition probabilities in the actions), which imply that the infimum and supremum are attained in (1.2) for all v, it is shown in [**AGN07**] that

$$\tilde{r}_C(f) = \hat{r}_C(f) = \bar{r}_C(f) = \inf_\sigma \sup_\pi r(P^{\sigma\pi}) \tag{2.1}$$

where $C = \mathbb{R}^n_+$, and $r(P^{\sigma\pi})$ denotes the Perron root of $P^{\sigma\pi}$.

We derive from the previous result an explicit formula for the geometrical convergence rate of the iterates of the dynamic programming map f, this time with nonzero payments $r^{\sigma\pi}$. To this end, we use the notion of *subdifferential*. Maps of the form (1.2) may not be differentiable, in particular, if the action spaces are finite, f is piecewise affine. However, f may often be assumed to be semidifferentiable, meaning that for all v and h, we can write $f(v+h) = f(v) + f'_v(h) + o(\|h\|)$, where f'_v, the *semidifferential* of f at point v, is a continuous positively homogeneous of degree one map, which is defined uniquely by the latter property.

When f is of the form (1.2), it can be shown that under fairly general assumptions, the semidifferential f'_v at point v exists, and is given by:

$$f'_v(h) = \inf_{\sigma\in\Sigma^*(v)} \sup_{\pi\in\Pi^*(v,\sigma)} P^{\sigma\pi}h \ ,$$

where $\Sigma^*(v)$ denote the set of policies σ which attain the infimum in (1.2), and for all σ, $\Pi^*(v,\sigma)$ denotes the set of strategies π which attain the supremum in the internal term in (1.2). (We need not assume that the inf and sup commute.)

We show that if f has a fixed point $v \in \mathbb{R}^n$, and if

$$\rho := \max(r_C(f'_v), r_{-C}(f'_v)) < 1 \ ,$$

then any orbit of f converges to v at a geometric rate which is bounded from above by ρ (this bound is tight). We eventually get the following explicit convergence rate:

$$\rho = \max(\inf_{\sigma\in\Sigma^*(v)} \sup_{\pi\in\Pi^*(v,\sigma)} r(P^{\sigma\pi}), \sup_{\sigma\in\Sigma^*(v)} \inf_{\pi\in\Pi^*(v,\sigma)} r(P^{\sigma\pi})) \ .$$

3. Order preserving convex functions

The techniques of the previous section are mostly appropriate when f has a unique fixed point, perhaps up to an additive or multiplicative constant.

Therefore, a basic problem is to give a complete description of the fixed point set of the map (1.2). As a partial answer, a precise description of the set of stable fixed points is given in [**AGL07**], when the map f is

convex (this corresponds to the one player case). This extends our earlier results [**AG03**] which concerned the undiscounted case. Here, we do not require any more the matrices $P^{\sigma\pi}$ in (1.2) to be (sub)stochastic. In other words, we allow the possibility of a negative discount rate. Despite its apparently unphysical nature, negative discount is of practical interest: for instance, the study of static analysis problems by abstract interpretation [**GGTZ07**] leads to fixed point problems involving maps which are always order preserving but not necessarily nonexpansive in some norm. Another motivation may come for fixed point problems for polynomials with positive coefficients, leading to maps like:

$$f_i(v) = \log(\sum_{j\in\mathbb{N}^n} a_{ij} \exp(j \cdot v)),$$

where for all $1 \leq i \leq n$, $(a_{ij})_{j\in\mathbb{N}^n}$ is an almost zero family of real nonnegative numbers.

A convenient notion of stability, in the present setting, is the following one: we say that a fixed point v is $\star$-*stable* if every orbit of the semidifferential f'_v is bounded from above. It can be checked that a Lyapunov stable fixed point is $\star$-stable.

Recall that a (communication) *class* of a nonnegative matrix P is by definition a strongly connected component of the digraph of P. We say that a class is *critical* if the corresponding principal submatrix of P has Perron root 1. The *critical graph* of P is the union of the subgraphs of the graph of P induced by the critical classes. If v is a $\star$-stable fixed point of f, we define the *critical graph* of f, $G^c(f)$ to be the union of the critical graphs of the matrices in the subdifferential

$$\partial f(v) := \{P \mid f(w) - f(v) \geq P(w - v), \forall w\} \ .$$

Of course, $\partial f(v)$ depends on v, but $G^c(f)$ is independent of the choice of the $\star$-stable fixed point v. The critical nodes of f are defined to be the nodes of $G^c(f)$.

We show in [**AGL07**] that a $\star$-stable fixed point is uniquely determined by its restriction to the set of critical nodes. Moreover, the restriction to the critical nodes allows us to identify the set of $\star$-stable fixed points of f to a convex inf-subsemilattice of $\mathbb{R}^p$, where p is bounded by the number of critical nodes. Some dynamical information, including a characterization of the possible lengths of "$\star$-stable" periodic orbits of f, is also derived in [**AGL07**].

The results of [**AG03, AGL07**] concern the one player case but have applications to the two player case. Indeed, the representation of the fixed point set has been used in [**CTG06**] to design a policy iteration

algorithm for zero-sum two player stochastic games. It allows one to handle "degenerate" iterations, in which the policies which are selected yield dynamic programming maps with several fixed points. Some other applications of these ideas, to static analysis of programs, are presented in [**CGG$^+$05, GGTZ07**].

Bibliography

[AG03] M. Akian and S. Gaubert. Spectral theorem for convex monotone homogeneous maps, and ergodic control. *Nonlinear Anal.*, 52(2):637–679, 2003.

[AGL07] M. Akian, S. Gaubert, and B. Lemmens. Stable periodic points of discrete convex monotone dynamical systems. 2007. preprint.

[AGN07] M. Akian, S. Gaubert, and R. Nussbaum. The Collatz-Wielandt theorem for order-preserving homogeneous maps on cones. 2007. preprint.

[CGG$^+$05] A. Costan, S. Gaubert, E. Goubault, M. Martel, and S. Putot. A policy iteration algorithm for computing fixed points in static analysis of programs. In *Proceedings of the 17th International Conference on Computer Aided Verification (CAV'05)*, LNCS, pages 462–475, Edinburgh, July 2005. Springer. doi:10.1007/11513988_46.

[CT80] M. G. Crandall and L. Tartar. Some relations between non expansive and order preserving maps. *Proceedings of the AMS*, 78(3):385–390, 1980.

[CTG06] J. Cochet-Terrasson and S. Gaubert. A policy iteration algorithm for zero-sum stochastic games with mean payoff. *C. R. Math. Acad. Sci. Paris*, 343(5):377–382, 2006.

[GGTZ07] S. Gaubert, E. Goubault, A. Taly, and S. Zennou. Static analysis by policy iteration in relational domains. In *Proc. of the 16th European Symposium on Programming (ESOP'07)*. Springer, October 2007. to appear in the LCNS series.

[Kol92] V. N. Kolokoltsov. On linear, additive and homogeneous operators in idempotent analysis. In V. P. Maslov and S. N. Samborskiĭ, editors, *Idempotent analysis*, volume 13 of *Advances In Soviet Mathematics*. Amer. Math. Soc., Providence, 1992.

[LMS01] G. L. Litvinov, V. P. Maslov, and G. B. Shpiz. Idempotent functional analysis: an algebraical approach. *Math. Notes*, 69(5):696–729, 2001. Also eprint arXiv:math.FA/0009128.

[MPN02] J. Mallet-Paret and Roger Nussbaum. Eigenvalues for a class of homogeneous cone maps arising from max-plus operators. *Discrete and Continuous Dynamical Systems*, 8(3):519–562, July 2002.

[Ney03] A. Neyman. Stochastic games and nonexpansive maps. In *Stochastic games and applications (Stony Brook, NY, 1999)*, volume 570 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 397–415. Kluwer Acad. Publ., Dordrecht, 2003.

[Nus88] R. D. Nussbaum. Hilbert's projective metric and iterated nonlinear maps. *Memoirs of the AMS*, 75(391), 1988.

[RS01] D. Rosenberg and S. Sorin. An operator approach to zero-sum repeated games. *Israel J. Math.*, 121:221–246, 2001.

[Vir01] O. Viro. Dequantization of real algebraic geometry on logarithmic paper. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.*, pages 135–146. Birkhäuser, Basel, 2001.

Cyclic projectors and separation theorems in idempotent semimodules[1]

Stéphane Gaubert and Sergeĭ Sergeev

1. Introduction

In an idempotent semiring, there is a canonical order relation, for which every element is "nonnegative". Therefore, idempotent semimodules have much in common with the semimodules over the semiring of nonnegative numbers, that is, with *convex cones* [**6**]. One of the first results based on this idea is the separation theorem for convex sets over "extremal algebras" proved by K. Zimmermann in [**8**]. Generalizations of this result were obtained in a work by S.N. Samborskiĭ and G.B. Shpiz [**7**] and in works by G. Cohen, J.-P. Quadrat, I. Singer, and the first author [**1**], [**2**].

The main result of this paper, Theorem 4.3, shows that in the setting of finite-dimensional semimodules over max-plus semiring, *several* closed subsemimodules which do not have common nonzero points can be separated from each other. This means that for each of these subsemimodules, we can select an idempotent halfspace containing it, in such a way that these halfspaces also do not have common nonzero points.

Even in the case of two semimodules, this statement has not been proved in the idempotent literature. Indeed, the earlier separation theorems deal with the separation of a point from an (idempotent) convex set or semimodule, rather than with the separation of two convex sets or semimodules.

In order to prove the main result, Theorem 4.3, we investigate the spectral properties of idempotent cyclic projectors. By idempotent cyclic projectors we mean finite compositions of certain nonlinear projectors on idempotent semimodules. The continuity and homogeneity of these nonlinear projectors enables us to apply to their compositions, i.e. to the cyclic projectors, a result of R.D. Nussbaum[**5**] (non-linear Perron-Frobenius

[1]Supported by the RFBR grant 05-01-00824 and the joint RFBR/CNRS grant 05-01-02807.

theory). We also show that the orbit of an eigenvector of a cyclic projector maximizes a certain objective function. We call this maximum the Hilbert value of semimodules, as it is a natural generalization of Hilbert's projective metric, and characterize the spectrum of cyclic projectors in terms of these Hilbert values (Theorem 4.2).

Our main results apply to the finite-dimensional semimodules over max-plus semiring. Some of our results still hold in a more general setting, see Sect. 3. However, the separation of several semimodules in such a generality remains an open question.

The results of this paper are presented as follows. Sect. 2 describes the main assumptions, and some preliminary notions and facts that will be used in the paper. Sect. 3 is devoted to the results obtained in the most general setting, with respect to the assumptions of Sect. 2. The main results are obtained in Sect. 4. They include separation of several semimodules and characterization of the spectrum of cyclic projectors.

The proofs of our results are contained in [**3**], which is an extended version of this text.

2. Preliminaries

We recall that a semiring (essentially, a ring without subtraction) is called *idempotent*, if its addition $\oplus$ is idempotent:$a \oplus a = a$. The order relation mentioned above is given by $a \oplus b = b \Leftrightarrow a \leq b$. An example of idempotent semiring that will be important to us is $\mathbb{R}_{\max,\mathrm{m}}$, it is the set of nonnegative numbers $\mathbb{R}^+$ equipped with operations $a \oplus b := \max(a, b)$ and $a \odot b = a \times b$. It is isomorphic to the max-plus semiring (the set $\mathbb{R} \cup -\infty$ equipped with $a \oplus b := \max(a, b)$ and $a \odot b = a + b$). The "spaces" over semirings are called semimodules.

An idempotent semiring or an idempotent semimodule will be called *b-complete*, following [**4**], if it is closed under the sum (i.e. the supremum) of any subset bounded from above, and if the multiplication distributes over such sums. We shall consider semirings $\mathcal{K}$ and semimodules $\mathcal{V}$ over $\mathcal{K}$ that satisfy the following assumptions:

($A0$): the semiring $\mathcal{K}$ is a b-complete idempotent semifield, and the semimodule $\mathcal{V}$ is a b-complete semimodule over $\mathcal{K}$;

($A1$): for all elements x and $y \neq \mathbf{0}$ from $\mathcal{V}$, the set $\{\lambda \in \mathcal{K} \mid \lambda y \leq x\}$ is bounded from above.

Note that both assumptions are true for the semimodules $\mathcal{K}^I$ of $\mathcal{K}$-valued functions on a set I, where $\mathcal{K}$ is a b-complete semifield.

Assumptions $(A0, A1)$ imply that the operation

$$x/y = \max\{\lambda \in \mathcal{K} \mid \lambda y \leq x\}. \tag{2.1}$$

is defined for all elements x and $y \neq \mathbf{0}$ from $\mathcal{V}$.

Definition 2.1. A subsemimodule V of $\mathcal{V}$ is a *b-(sub)semimodule*, if V is closed under the sum of any of its subsets bounded from above in $\mathcal{V}$.

Let V be a b-subsemimodule of the semimodule $\mathcal{V}$. Consider the operator P_V defined by

$$P_V(x) = \max\{u \in V \mid u \leq x\}, \tag{2.2}$$

for every element $x \in \mathcal{V}$. Here we use "max" to indicate that the least upper bound belongs to the set. The operator P_V is a *projector* onto the subsemimodule V, as $P_V(x) \in V$ for any $x \in \mathcal{V}$ and $P_V(v) \in V$ for any $v \in V$.

In idempotent geometry, the role of halfspace is played by the following object.

Definition 2.2. A set H given by

$$H = \{x \mid u/x \geq v/x\} \cup \{\mathbf{0}\} \tag{2.3}$$

with $u, v \in \mathbb{R}^n_{\max,\mathrm{m}}$, $u \leq v$, will be called *(idempotent) halfspace*.

Any halfspace is a semimodule. If $\mathcal{V} = \mathcal{K}^n$, an n-dimensional semimodule over $\mathcal{K}$, and all coordinates of u and v are nonzero, then we have that

$$H = \{x \mid \bigoplus_{\{1,\dots,n\}} x_i u_i^{-1} \leq \bigoplus_{\{1,\dots,n\}} x_i v_i^{-1}\}. \tag{2.4}$$

The following theorem is a version of idempotent separation theorems [**1, 2**], see also [**4**].

Theorem 2.1. *Let V be a b-complete subsemimodule of $\mathcal{V}$ and let $u \in \mathcal{V}$ be not in V. Then the set*

$$H = \{x \mid P_V(u)/x \geq u/x\} \cup \{\boldsymbol{0}\}$$

contains V but not u.

For any subsemimodule V and $y \in \mathcal{V}$, we denote

$$V^y = \{x \in V \mid y/x > \mathbf{0}\}. \tag{2.5}$$

It is a subsemimodule of V.

DEFINITION 2.3. A vector x is called *archimedean*, if $x/y > \mathbf{0}$ for all $y \in \mathcal{V}$. A subsemimodule of $\mathcal{V}$ is called *archimedean*, if it contains archimedean vectors. A halfspace H defined by (2.3) will be called *archimedean* if both u and v are archimedean.

Obviously, Def. 2.3 makes sense only under

($A2$): The semimodule $\mathcal{V}$ has an archimedean vector.

This assumption is true in particular for the semimodules of type $\mathcal{K}^n$. In these semimodules we have that $y/x > \mathbf{0}$ if and only if the support of x, i.e. the set $\mathrm{supp}(x) = \{i \mid x_i \neq \mathbf{0}\}$, is a subset of $\mathrm{supp}(y)$ (the support of y). In this case V^y has the form

$$V^M = \{x \in V \mid \mathrm{supp}(x) \subseteq M\}, \tag{2.6}$$

for some index set M. A vector in $\mathcal{K}^n$ is archimedean if and only if it is positive. Regular halfspaces in this case are given by (2.4).

3. General results

We shall study cyclic projectors, that is, compositions of projectors

$$P_{V_k} \cdots P_{V_1},$$

where $V_1, \ldots, V_k$ are b-subsemimodules of $\mathcal{V}$. We assume ($A0, A1$), which means in particular that $\mathcal{K}$ is an idempotent semifield. For the notational convenience, we will write P_t instead of P_{V_t}. We will also adopt a convention of cyclic numbering of indices of projectors and semimodules, so that $P_{l+k} = P_l$ and $V_{l+k} = V_l$ for all l.

DEFINITION 3.1. Let $x^1, \ldots, x^k$ be nonzero elements of $\mathcal{V}$. The value

$$d_{\mathrm{H}}(x^1, \ldots, x^k) = (x^1/x^2)\ (x^2/x^3) \ldots (x^k/x^1).$$

will be called *the Hilbert value* of $x^1, \ldots, x^k$.

The Hilbert value of two vectors x^1, x^2 was studied in [**1**]. For two comparable vectors in $\mathbb{R}^n_{\max,\mathrm{m}}$, that is, for two vectors with common support M it is given by

$$d_{\mathrm{H}}(x^1, x^2) = \min_{i,j \in M} (x^1_i (x^2_i)^{-1} x^2_j (x^1_j)^{-1}),$$

so that $-\log(d_{\mathrm{H}}(x^1, x^2))$ coincides with Hilbert's projective metric

$$\delta_{\mathrm{H}}(x^1, x^2) = \log(\max_{i,j \in M} (x^1_i (x^2_i)^{-1} x^2_j (x^1_j)^{-1})) = -\log(d_{\mathrm{H}}(x^1, x^2)).$$

DEFINITION 3.2. The *Hilbert value* of k subsemimodules $V_1, \ldots, V_k$ of $\mathcal{V}$ is defined by

$$d_{\mathrm{H}}(V_1, \ldots, V_k) = \sup_{x^1 \in V_1, \ldots, x^k \in V_k} d_{\mathrm{H}}(x^1, \ldots, x^k)$$

We establish two results on the spectrum of cyclic projectors and on their iterations.

THEOREM 3.1. *Suppose that the operator $P_k \circ \ldots \circ P_1$ has an eigenvector y with eigenvalue λ, and define $\bar{x}^i = P_i \circ \ldots \circ P_1 y$. Then*

$$\lambda = d_H(V_1^y, \ldots, V_k^y) = d_H(\bar{x}^1, \ldots, \bar{x}^k).$$

THEOREM 3.2. *For any sequence of nonzero vectors $\{x^i,\ i = 1, \ldots\}$ such that $x^1 \in V_1$ and $x^i = P_i x^{i-1}$ for $i = 2, \ldots$, the Hilbert value $d_H(x^{l+1}, \ldots, x^{l+k})$ is nondecreasing with l.*

The following is an extension of Theorem 2.1, under assumptions ($A0-A2$).

THEOREM 3.3. *Suppose that $V_1, \ldots, V_k$ are b-closed semimodules and that $P_k \circ \ldots \circ P_1$ has an archimedean eigenvector y with nonzero eigenvalue λ. The following are equivalent:*

(1) *there exists an archimedean vector x and a scalar $\mu < \mathbf{1}$ such that*
$$P_k \circ \ldots \circ P_1 x \leq \mu x;$$
(2) *for all $i = 1, \ldots, k$ there exist regular halfspaces H_i such that $V_i \subseteq H_i$ and $\cap_i H_i = \{\mathbf{0}\}$;*
(3) *$\cap_i V_i = \{\mathbf{0}\}$;*
(4) *$\lambda < \mathbf{1}$.*

4. Projectors and separation in max algebra

In $\mathbb{R}^n_{\max,\mathrm{m}}$, it is natural to consider semimodules that are closed in the Euclidean topology. One can easily show that such semimodules are b-semimodules. Theorem 3.11 of [**2**] implies that projectors onto closed subsemimodules of $\mathbb{R}^n_{\max,\mathrm{m}}$ are continuous.

In order to relax the assumption concerning archimedean vectors in Theorem 3.3, we use some results from nonlinear spectral theory, that we next recall. By Brouwer's fixed point theorem, a continuous homogeneous operator $x \mapsto Fx$ that maps $\mathbb{R}^n_+$ to itself has a nonzero eigenvector. This allows us to define the nonlinear spectral radius of F,

$$\rho(F) = \max\{\lambda \in \mathbb{R}_+ \mid \exists x \in (\mathbb{R}^n_+) \setminus 0,\ Fx = \lambda x\}\ . \tag{4.1}$$

Suppose in addition that F is isotone, then the maximum in (4.1) is attained and we can use the following nonlinear generalization of the Collatz-Wielandt formula for the spectral radius of a nonnegative matrix.

THEOREM 4.1. (R.D. Nussbaum, Theorem 3.1 of [**5**]) *For any isotone, homogeneous, and continuous map F from $\mathbb{R}^n_+$ to itself, we have:*

$$\rho(F) = \inf_{x \in (\mathbb{R}_+ \setminus \{0\})^n} \max_{1 \leq i \leq n} [F(x)]_i x_i^{-1}.$$

This result implies that the spectral radius of such operators is isotone: if $F(x) \leq G(x)$ for any $x \in \mathbb{R}^n_+$, then $\rho(F) \leq \rho(G)$.

As the projectors on subsemimodules of $\mathbb{R}^n_{\max,\mathrm{m}}$ are isotone, homogeneous and continuous, so are their compositions, i.e. cyclic projectors. Consequently, we can apply Theorem 4.1 to them. This allows us to refine the general results from the previous section. The following result refines Theorem 3.1 (the spectrum of cyclic projections).

THEOREM 4.2. *Let $V_1, \ldots, V_k$ be closed semimodules in $\mathbb{R}^n_{\max,m}$. Then the Hilbert value of $V_1, \ldots, V_k$ is the spectral radius of $P_k \circ \ldots \circ P_1$. Every eigenvalue of $P_k \circ \ldots \circ P_1$ is equal to $d_H(V_1^M, \ldots, V_k^M)$ for some M. Conversely, every such Hilbert value is an eigenvalue of $P_k \circ \ldots \circ P_1$.*

The following result refines Theorem 3.3 (separation).

THEOREM 4.3. *Suppose that $V_i,\ i = 1, \ldots, k$ are closed semimodules of $\mathbb{R}^n_{\max,m}$, and that $\cap_i V_i = \{\boldsymbol{0}\}$. Then there exist archimedean halfspaces $H_i,\ i = 1, \ldots, k$ such that $V_i \subseteq H_i$, for $i = 1, \ldots, k$, and $\cap_i H_i = \{\boldsymbol{0}\}$.*

A particular case of Theorem 4.3 is the following separation theorem for two semimodules.

THEOREM 4.4. *Suppose that U and V are two closed max cones, and that $U \cap V = \boldsymbol{0}$. Then there exists an archimedean halfspace H_U, which contains U and does not intersect with V, and there exists an archimedean halfspace H_V, which contains V and does not intersect with U.*

Bibliography

[1] G. Cohen, S. Gaubert, and J.P. Quadrat, *Duality and separation theorems in idempotent semimodules.* Linear Algebra Appl., 379:395–422, 2004. E-print arXiv:math.FA/0212294.

[2] G. Cohen, S. Gaubert, J.P. Quadrat, and I. Singer, *Max-plus convex sets and functions.* In G. Litvinov and V. Maslov, editors, *Idempotent Mathematics and Mathematical Physics*, volume 377 of *Contemporary Mathematics* , pages 105–129. AMS, Providence, 2005. E-print arXiv:math.FA/0308166.

[3] S. Gaubert and S. Sergeev, *Cyclic projectors and separation theorems in idempotent semimodules.* E-print arXiv:0706.3347.
[4] G.L. Litvinov, V.P. Maslov, and G.B. Shpiz, *Idempotent functional analysis. An algebraical approach.* Math. Notes, 69(5):696–729, 2001. E-print arXiv:math.FA/0009128.
[5] R.D. Nussbaum, *Convexity and log convexity for the spectral radius.* Linear Algebra Appl., 73:59–122, 1986.
[6] R.T. Rockafellar, *Convex analysis.* Princeton Univ. Press, 1970.
[7] S.N. Samborskiĭ and G.B. Shpiz, *Convex sets in the semimodule of bounded functions.* In V.P. Maslov and S.N. Samborskiĭ, editors, *Idempotent analysis*, volume 13 of *Advances in Soviet Math.*, pages 135–137. American Mathematical Society, Providence, 1992.
[8] K. Zimmermann, *A general separation theorem in extremal algebras.* Ekonomicko-matematický obzor, 13(2):179–201, 1977.

Pseudo-weak convergence of the random sets defined by a pseudo integral based on non-additive measure[1]

T. Grbić and E. Pap

1. Introduction

The weak convergence of sequence of probability measures is the main subject for a large class of limit theorems in the probability theory. In the classical probability theory, it works with σ-additive measures and the Lebesgue integral ([**B**]). Several conditions equivalent to the weak convergence are provided by the theorem of Portmanteau ([**B**]). The main aim of this paper is to prove a Portmanteau-type theorem, with capacity functionals instead of probability measures, and with the general pseudo integral instead of the Lebesgue integral.

Since the convergence in distribution of sequence of random closed sets on $\mathbb{R}$ can be tricky, it is often more appropriate to study the convergence of the corresponding sequence of capacity functionals. In this paper we study the convergence of sequences of random closed sets on $\mathbb{R}$ by looking at the convergence of the corresponding sequence of capacity functionals. Theoretical foundations of the theory of random sets, as generalization of random variables, were layed down by Kendall ([**G**]) and Matheron ([**J**]).

[1]Partially supported by the Project MNZŽSS 144012, grant of MTA HTMT, French-Serbian project "Pavle Savić", and by the project "Mathematical Models for Decision Making under Uncertain Conditions and Their Applications" supported by Vojvodina Provincial Secretariat for Science and Technological Development.

Recall that random closed sets are random elements on the space of closed subsets of $\mathbb{R}$.

Our paper is organized as follows. Sect. 2 contains some preliminary notions, such as pseudo-operations and general pseudo integral [**A, O, S**]. In Sect. 3, we recall some basic notions and definitions from the theory of random sets ([**C, G, J, K, L**]). The main results of this paper, also contained in Sect. 3, are concerned with the weak convergence of sequence of random closed sets, i.e., of the corresponding sequence of capacity functionals with respect to the general pseudo integral.

2. Preliminary notions

Following [**A, O, P**], we recall the notions of pseudo operations and general pseudo integral. Let $\leq$ be the total order on $[0, \infty]$.

DEFINITION 2.1. A binary operation $\oplus : [0,\infty]^2 \to [0,\infty]$ is called *pseudo-addition* if the following properties are satisfied:
(A1) $a \oplus b = b \oplus a$ *(commutativity)*
(A2) $a \leq a' \wedge b \leq b' \Rightarrow a \oplus b \leq a' \oplus b'$ *(monotonicity)*
(A3) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ *(associativity)*
(A4) $a \oplus 0 = a$ *(neutral element)*
(A5) $a_n \to a \wedge b_n \to b \Rightarrow (a_n \oplus b_n) \to a \oplus b$ *(continuity)*

EXAMPLE 2.1. The following operations are pseudo-additions ([**A, O, P**]): (i) $x \oplus y = g^{-1}(g(x)) + g(y)$, where $g : [0,\infty]^2 \to [0,\infty]$ is an increasing bijection;
(ii) $x \oplus y = \max(x, y)$ (note that this operation is idempotent).

DEFINITION 2.2. For a given pseudo-addition $\oplus$ *pseudo-difference* is the binary operation $\ominus : [0,\infty]^2 \to [0,\infty]$ given by

$$a \ominus b = \inf\{x \in [0,\infty] : b \oplus x \geq a\}.$$

EXAMPLE 2.2. Obviously, $a \ominus b = 0$ for $a \leq b$ and $a \ominus b > 0$ for $a > b$, see [**A, I**]. For pseudo-additions from Example 2.1 and $a > b$ corresponding pseudo-differences are
(i) $a \ominus b = g^{-1}(g(a) - g(b))$; (ii) $a \ominus b = a$.

DEFINITION 2.3. For a given pseudo-addition $\oplus$ the *pseudo-multiplication* is a binary operation $\odot : [0,\infty]^2 \to [0,\infty]$ such that the following conditions are satisfied
(M1) $a \odot 0 = 0 \odot b = 0$ *(zero element)*
(M2) $a \leq a' \wedge b \leq b' \Rightarrow a \odot b \leq a' \odot b'$ *(monotonicity)*
(M3) $(a \oplus b) \odot c = (a \odot c) \oplus (a \odot c)$ *(right distributivity)*
(M4) $a \odot \mathbf{1} = \mathbf{1} \odot a = a$ *(unit element)*

(M5) $a \odot (b \odot c) = (a \odot b) \odot c$ *(associativity)*
(M6) $a_n \to a \wedge b_n \to b \Rightarrow (a_n \odot b_n) \to a \odot b$ *(continuity)*

EXAMPLE 2.3. (i) For the pseudo-addition from Example 2.1 (i), define pseudo-multiplication by $a \odot b = g^{-1}(g(a)g(b))$.

(ii) For the pseudo-addition from Example 2.1 (ii), one of the possible pseudo-multiplications is $a \odot b = a + b$, see [**E, F**].

The algebraic structure $([0,\infty], \oplus, \odot)$ is a *semiring*.

Let Ω be an abstract space, $\mathcal{A}$ a σ-algebra of subsets of Ω and $m : \mathcal{A} \to \mathbb{R}$ a non-decreasing set function with $m(\emptyset) = 0$. We consider the space $(\Omega, \mathcal{A}, m)$ and a family of $\mathcal{A}$-measurable functions $f : \Omega \to [0,\infty]$, denoted by $\mathcal{F}$. A *simple function* is a measurable function $s : \Omega \to [0,\infty]$ whose range is finite. Let $Rang(s) = \{a_1, a_2, \ldots, a_k\}$ such that $0 < a_1 < a_2 < \ldots < a_k$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. The *standard $\oplus$-step representation* of a simple function s is given by $s = \bigoplus_{i=1}^{k} b(c_i^*, C_i^*)$, where $c_1^* = a_1,\ c_2^* = a_2 \ominus a_1,\ \ldots, c_m^* = a_m \ominus a_{m-1}$, $C_i^* = \bigcup_{j=i}^{m} A_i$ and $b : \Omega \to [0,\infty]$ is a *basic function* of the form $b(c_i^*, C_i^*)(\omega) = \begin{cases} c_i^*, & \omega \in C_i^*, \\ 0, & \omega \notin C_i^*. \end{cases}$

DEFINITION 2.4. (i) *The general pseudo integral of a simple function* s with the standard $\oplus$-step representation is given by

$$\int^{\oplus} s \odot\ dm = \bigoplus_{i=1}^{m} c_i^* \odot m(C_i^*).$$

(ii) The *general pseudo integral* of a measurable function $f \in \mathcal{F}$ is given by

$$\int^{\oplus} f \odot\ dm = \sup\{\int^{\oplus} s \odot\ dm : s \in \mathcal{S}_f\},$$

where $\mathcal{S}_f$ is the family of all simple function s such that $s \leq f$.

The general pseudo integral has the following properties:

(i) $\int^{\oplus} b(c, C) \odot\ dm = c \odot m(C)$.

(ii) $f \leq g \Rightarrow \int^{\oplus} f \odot\ dm \leq \int^{\oplus} g \odot\ dm$.

(iii) For the pseudo characteristic function $\chi_A : \Omega \to [0,\infty]$ of a set $A \subset \Omega$, defined by $\chi_A(x) = \begin{cases} \mathbf{1}, & x \in A, \\ 0, & x \notin A, \end{cases}$ we have

$$\int^{\oplus} \chi_A \odot\ dm = m(A).$$

3. Weak convergence of the sequence of capacity functionals

3.1. Random closed sets and capacity functionals. We start with a short overview of the theory of random closed sets ([**C, G, J, K, L, M, N**]). Denote collections of closed, open and compact subsets of $\mathbb{R}$ by $\mathcal{F}$, $\mathcal{O}$ and $\mathcal{K}$, respectively. A very important role in the theory of random closed sets is played by collections of closed sets $\mathcal{F}$, and its sub-collections $\mathcal{F}_G = \{F \in \mathcal{F} : F \cap G \neq \emptyset\},\ G \in \mathcal{O},$ and $\mathcal{F}^K = \{F \in \mathcal{F} : F \cap K = \emptyset\},\ K \in \mathcal{K}$. Collections $\{\mathcal{F}_G : G \in \mathcal{O}\}$ and $\{\mathcal{F}^K : K \in \mathcal{K}\}$ generate a topology $\tau(\mathcal{F})$ on $\mathcal{F}$. This topology is known as *hit-or-miss-topology*. The collection $\mathcal{F}$ endowed with the hit-or-miss topology is a compact, separable and Hausdorff space ([**J**]). Taking countable unions and intersections of open sets of the topological space $(\mathcal{F}, \tau(\mathcal{F}))$, we obtain a σ-field $\Sigma(\mathcal{F})$.

Definition 3.1. A *random closed set* S is a measurable mapping from the probability space $(\Omega, \mathcal{A}, \mathrm{P})$ into the measurable space $(\mathcal{F}, \Sigma(\mathcal{F}))$.

A random closed set S generates a probability distribution $\mathbf{P}_\mathrm{S}$ in the following way

$$\mathbf{P}_\mathrm{S}(A) = \mathrm{P}(\{\omega \in \Omega : \mathrm{S}(\omega) \in A\}) = \mathbf{P}_\mathrm{S}(\mathrm{S} \in A), \text{ for all } A \in \Sigma(\mathcal{F}).$$

Definition 3.2. For a random closed set S its *capacity functional* $\mathrm{T}_\mathrm{S} : \mathcal{K} \to [0,1]$ for $K \in \mathcal{K}$ is defined by

$$\mathrm{T}_\mathrm{S}(K) = \mathbf{P}_\mathrm{S}(\mathrm{S} \in \mathcal{F}_K) = \mathbf{P}_\mathrm{S}(\mathrm{S} \cap K \neq \emptyset).$$

The capacity functional T_S is defined on $\mathcal{K}$, and it can be extended onto the family $\mathcal{P}$ of all subsets of $\mathbb{R}$. A subset $M \subset \mathbb{R}$ is called *capacitable* if the following equality $\mathrm{T}_\mathrm{S}(M) = \sup\{\mathrm{T}_\mathrm{S}(K) : K \in \mathcal{K}, K \subset M\}$ is true. All Borel sets B are capacitable ([**L**]). For a given random closed set S, and a sequence of random closed sets $\{\mathrm{S}_n\}$ corresponding capacity functionals will be denoted by T and $\{\mathrm{T}_n\}$, respectively.

3.2. $(\oplus, \odot)$-weak convergence.

Definition 3.3. A sequence of capacity functionals $\{\mathrm{T}_n\}$ $(\oplus, \odot)$-*weak converges* to a capacity functional T (shortly, *pseudo-weak converges*) if and only if for each continuous, bounded function $f : \mathbb{R} \to [0, \infty]$ we have that $\lim\limits_{n\to\infty} \int^\oplus f \odot d\mathrm{T}_n = \int^\oplus f \odot d\mathrm{T}$.

We have proved in [**D**]the following three theorems.

Theorem 3.1. *If a sequence of capacity functionals* $\{\mathrm{T}_n\}$ *pseudo-weak converges to capacity functional* T*, then* $\limsup\limits_n \mathrm{T}_n(F) \leq \mathrm{T}(F)$ *for all closed sets* $F \subseteq \mathbb{R}$.

THEOREM 3.2. *If a sequence of capacity functionals* $\{\mathrm{T}_n\}$ *pseudo-weak converges to capacity functional* T*, then* $\liminf_n \mathrm{T}_n(G) \geq \mathrm{T}(G)$ *for all open sets* $G \subset \mathbb{R}$.

THEOREM 3.3. *If for a sequence of capacity functionals* $\{\mathrm{T}_n\}$ *and for all closed sets* F *holds* (A) $\limsup_n \mathrm{T_n(F)} \leq \mathrm{T(F)}$ *and for all open sets* G *holds* (B) $\liminf_n \mathrm{T_n(G)} \geq \mathrm{T(G)}$, *then* $\{\mathrm{T}_n\}$ *pseudo-weak converges to capacity functional* T.

COROLLARY 3.1. *For a random closed set* S *and a sequence of random closed sets* $\{\mathrm{S}_n\}$, *which are defined in the following way:* $\mathrm{S} = \{\mathrm{X}\}$ *and* $\mathrm{S}_n = \{\mathrm{X}_n\}$, *where* X *is a random variable and* $\{\mathrm{X}_n\}$ *is a sequence of random variables, the* $(\oplus, \odot)$*-weak convergence is equivalent to the weak convergence (with respect to continuous, bounded function* $f : \mathbb{R} \to [0, \infty]$*).*

PROOF. For $\mathrm{S} = \{\mathrm{X}\}$ and $\mathrm{S}_n = \{\mathrm{X}_n\}$, we have that $\mathrm{T}(K) = \mathrm{P}(\mathrm{X} \in K)$ and $\mathrm{T}_n(K) = \mathrm{P}(\mathrm{X}_n \in K)$ ([**C**]), where T and T_n are capacity functionals of random sets S and S_n, respectively. Since for each Borel set B we have that $\mathrm{T}(B) = \sup\{\mathrm{T}(K) : K \in \mathcal{K}, K \subset B\}$, it follows that $\mathrm{T}(B) = \mathrm{P}(\mathrm{X} \in B)$. For all $n \in \mathbb{N}$ we have that $\mathrm{T}_n(B) = \mathrm{P}(\mathrm{X}_n \in B)$. Suppose that $\{\mathrm{S}_n\}$ $(\oplus, \odot)$-weak converges to S. Then by Theorem 3.1, $\limsup_n \mathrm{P}(\mathrm{X}_n \in F) \leq \mathrm{P}(\mathrm{X} \in F)$ for all closed sets F. From the classical theorem of Portmanteau ([**B**]) we obtain that $\int f d\mathrm{P}_n \to \int f d\mathrm{P}$, i.e., that the sequence of random closed sets $\{\mathrm{S}_n\}$ weak converges to S.
The weak convergence of the sequence of probability measures for any open set G implies that $\liminf_n \mathrm{P}(\mathrm{X}_n \in G) \geq \mathrm{P}(\mathrm{X} \in G)$, and for any closed set F it implies that $\liminf_n \mathrm{P}(\mathrm{X}_n \in F) \leq \mathrm{P}(\mathrm{X} \in F)$. Then, by Theorem 3.3, the sequence of random closed sets $\{\mathrm{S}_n\}$ $(\oplus, \odot)$-weak converges to S. □

REMARK 3.1. (i) For the special case described by corollary 3.1, the capacity functional reduces to the probability measure and then Theorem 3.3 can be proved by taking into the account only one of the assumptions, (A) or (B).

(ii) Weak convergence of the sequence of capacity functionals with respect to Choquet integral is investigated in [**M**]. Some equivalent conditions for the weak convergence of the sequence of probability measures, induced by sequence of random closed sets, are obtained in [**N, Q**].

(iii) The results obtained in this paper will serve for the investigation of further convergence properties of a sequence of capacity functionals of the sequence of random closed sets, based on the idempotent sup-measure and related integrals, see ([**D, R**].

Bibliography

[A] Benvenuti P., Mesiar R., Vivona D., *Monotone Set Functions-Based Integrals*, in Handbook of Measure Theory (Ed. E. Pap), Volume II, Elsevier, North-Holland, (2002), 205–232.

[B] Billingslay, P., *Probability Measures*, John Wiley & Sons, Inc., New York, 1968.

[C] Goutsias J., *Modeling Random Shapes: An Introduction to the Random Closed Set Theory*, Technical Report JHU/ECE 90-12, 1990.

[D] Grbić, T., Pap, E., *Generalization of Portmanteau theorem with respect to pseudo-weak convergence* (under preparation).

[E] Litvinov, G.L., *The Maslov Dequantization, Idempotent and Tropical Mathematics: a very Brief Introduction*, Cont. Mathematics 377, AMS, (2005), 1-17.

[F] Maslov, V.P., Samoborskiĭ, S.N., (Eds.), *Idempotent analysis*, Adv, in Sov. Math., Vol. 13, AMS, RI, 1992.

[G] Kendall D.G., *Fondutations of a theory of random sets*, In Stohastic Geometry, E.F. Harding and D.G. Kendall, eds., London, (1974), 322–376.

[H] Klein, E., Thompson,A., *Theory of Correspondences*, John Wiley, 1984.

[I] Klement, E. P., Mesiar, R., Pap, E., Triangular Norms. Dordrecht: Kluwer Academic Publishers, 2000.

[J] Matheron, G., *Random Sets and Integral Geometry*, John Wiley, 1975.

[K] Molchanov, I., *Limit Theorems for Unions of Random Closed Sets*, vol. 1561 of Lect. Notes Math., Springer, Berlin, 1993.

[L] Molchanov, I., *Theory of Random Sets*, Springer-Verlag, 2005.

[M] Nguyen, H.T., *Choquet Weak Convergence of Capacity Functionals of Random Sets*, in Soft Methodology and Random Information Systems, Springer, (2004), 19–31.

[N] Nguyen, H.T., Bouchon-Meunier B., *Random sets and large deviations principle as a foundation for possibility measures*, Soft Computing 8, (2004), 61–70.

[O] Pap E., *Null-Additive Set Functions*, Kluwer Academic Publishers, 1995.

[P] Pap E., *Pseudo-Additive Measures and Their Applications*, in Handbook of Measure Theory (Ed. E. Pap), Volume II, Elsevier, North-Holland, (2002), 1403–1468.

[Q] Pap, E., Grbić, T., Nedović, Lj., Ralević, N.M., *Weak Convergence of Random Sets*, 3rd Serbian-Hungarian Joint Symposium on Intelligent Systems, Subotica, (2005), 73–80.

[R] Puhalskii, A., *Large deviations and idempotent probability*, Chapman & Hall/CRC, 2001.

[S] Wang Z., Klir G.J.,*Fuzzy measure theory*, Plenum Press, New York, 1992.

The stationary phase method and large deviations

Oleg V. Gulinsky

Let $\{P_\lambda\}$ be a family of probability measures on a measurable space $(X, \mathcal{F})$ and let I be a nonnegative function on X with compact level sets. $\{P_\lambda\}$ obeys the large deviation principle with a rate function I if and only

if

$$\lim_{\lambda\to\infty}\Big[\int_X (g(x))^\lambda P_\lambda(dx)\Big]^{1/\lambda} = \sup_{x\in X} g(x)e^{-I},$$

for all bounded continuous nonnegative functions g on X [**1**], [**4**].

We say that in this sense $\{P_\lambda\}$ converges to an idempotent measure $\exp\{-I\}$. The r.h.s. of the last display is called a **sup - integral** or **idempotent integral** with respect to the idempotent measure and defines rough logarithmic asymptotics of the Laplace method.

In this report we discuss logarithmic asymptotics of the integral

$$J(\lambda) = \int_X \exp\{\imath\lambda u(x)\}(g(x))^\lambda P_\lambda(dx),$$

where $X = R^d$ (in what follows R^1 for the simplicity) and u,g ($g \geq 0$) are smooth enough real-valued functions.

We consider this problem as a natural generalization of the stationary phase method which imbeds the classical one in the context of large deviations. The interest in the problem is motivated by the slicing approximation approach to infinite dimension oscillatory integrals as well (see, for example [**2**]).

Our approach is based on the technique of an almost analytic extension and follows the ideas of [**3**] where the classical method of stationary phase was extended to the case of complex-valued phase function. The new difficulty in our problem is the following. The function which plays the role corresponding to the imaginary part of the phase function in [**3**], is just the rate function I defined asymptotically by the large deviation principle.

Nevertheless, we consider $f(x) = u(x) + \imath I(x)$ as complex-valued "phase function" and assume that $f(x)$ is C^∞ function in a neighborhood of the origin, which in turn is a non-degenerate stationary point of f with $I(0) = 0$.

We introduce an almost analytic extension of f as follows:

$$f(z) = [u(x)\chi(y) - I^{'}(x)y\chi(t_1 y) - \frac{1}{2!}u''(x)y^2\chi(t_2 y) + ...]$$
$$+\imath[I(x)\chi(y) + u'(x)y\chi(t_1 y) - \frac{1}{2!}I''(x)y^2\chi(t_2 y) + ...] =$$
$$u(x,y) + \imath v(x,y),$$

where $\chi(y) \in C_0^\infty$ is equal to one in a neighborhood of the origin and vanishes for $\mid y \mid \geq 1$. The numbers $t_k \geq 1$ are chosen sufficiently large so that the series converges.

To examine the asymptotic behavior of the integral $J(\lambda)$, we replace the integration along R by the integration along a suitable chain in the complex domain passing through the critical point of $f(z)$. We show that on this chain the problem is reduced to the standard variational principle of large deviations.

To fulfil the program following [**3**], we first find new coordinates $\tilde{z}$ in C for which $f(z)-f(0)$ is a quadratic form in $\tilde{z}$. To this end, using Taylor's formula we write

$$f(z) - f(0) =< z, R(z)z > /2,$$

where by definition $R(z)$ is an almost analytic function of z. Since all non-degenerate quadratic forms on C are equivalent, there is a linear transformation A such that

$$A^T R(0) A = \imath \mathbf{I}.$$

In turn, the equation

$$\begin{aligned} \imath Q^T(z) Q^T(z) &= R(z), \\ Q(0) &= A^{-1}, \end{aligned}$$

has a C^∞ solution $Q(z)$ defined near the origin, since the map $Q \to \imath Q^T Q$ is analytic with surjective differential at $Q = A^{-1}$. Moreover, Q is an analytic function of $R(z)$ and therefore an almost analytic function of z.

The map $z \to \tilde{z} = \tilde{z}(z) = Q(z)z$ defines new coordinates in the neighborhood of the origin and in this coordinates we have

$$f(z) = f(0) + \imath < \tilde{z}, \tilde{z} > /2,$$

where $\tilde{z} = \tilde{x} + \imath\tilde{y}$ and $< \tilde{z}, \tilde{z} >= \tilde{x}^2 - \tilde{y}^2 + 2\imath < \tilde{x}, \tilde{y} >$.

Since $I(x) \geq 0$ with $I(0) = 0$, it follows that $\tilde{x}^2 - \tilde{y}^2 \geq 0$ on the tangent space at the origin and so there is a C^∞ function φ, defined in a neighborhood of 0, such that in the new coordinates R is given by the equation $\tilde{y} = \varphi(\tilde{x})$.

We are now in a position to define a family of chains Γ_s in C and examine the behavior of f on them. Let $\tilde{z} \to z = z(\tilde{z})$ be the inverse of the map $z \to \tilde{z}(z)$. For $0 \leq s \leq 1$, putting

$$\Gamma_s = \{z : z = z(\tilde{z}_s), \tilde{z}_s = \tilde{z}_s(\tilde{x}) = \tilde{x} + \imath\varphi(\tilde{x})s, \tilde{x} \in R\}$$

one gets the estimate

$$\mathtt{Im} f(z(\tilde{z}_s)) \geq C \mid \mathtt{Im} z(\tilde{z}_s) \mid^2 .$$

To replace the integration, we first consider the chain Γ_1 and note that in a small enough neighborhood of $z = 0$ the integrals $J(\lambda)$ and

$$\int_{\Gamma_1} \exp\{\imath\lambda f(z)\}(g(z))^\lambda \exp\{\lambda I(x)\chi(y)\}P_\lambda(dx)dy,$$

where $g(z)$ is almost analytic extension of g, are equivalent(we may assume that the support of g w.r.t. z belongs to a small fixed neighborhood of the origin).

Finally we have to show that $\int_{\Gamma_1}$ differs from $\int_{\Gamma_0}$ with a very small error. We are able to do that with the help of Stokes's formula by the following arguments: $(1) f(z)$ and $g(z)$ are almost analytical functions, $(2)\mathtt{Im} f(z(\tilde{z}_s)) \geq C \mid \mathtt{Im} z(\tilde{z}_s) \mid^2$.

Thus, it suffices compute the logarithmic asymptotics of the integral

$$\int_{\Gamma_0} \exp\{\imath\lambda f(z)\}(g(z))^\lambda \exp\{\lambda I(x)\chi(y)\}P_\lambda(dx)dy = \exp\{\imath\lambda u(0)\} \times$$
$$\int \exp\{-\lambda[I(0)+ \mid \tilde{x} \mid^2 /2]\}(g(z(\tilde{x}))^\lambda \exp\{\lambda I(x(\tilde{x}))\}G(\tilde{x})P_\lambda(d\tilde{x}),$$

where $G(\tilde{x}) = \mathtt{det}(\frac{dz}{d\tilde{x}})$. One can easily recognize that the asymptotics of the last integral coincides with the asymptotics of

$$\exp\{\imath\lambda u(0)\} \times \int (g(z(\tilde{x}))^\lambda G(\tilde{x})P_\lambda(d\tilde{x}).$$

Thus we reduced the initial problem to the problem of large deviations.

Bibliography

[1] W. Bryc. Large deviations by the asymptotic value method. in: M. Pynsky (ed.). Diffusion processes and related problems in analysis. *Birkhäuser*, 447-472, 1990.

[2] N.Kumano-go. Feynman path integrals as analysis on path space by time slicing approximation. Bull. Sci. math. **128**, 197-251, 2004.

[3] A.Melin and J.Sjöstrand. Fourier integral operators with complex-valued phase functions. Springer Lecture Notes in Math. **459**, 120-223, 1974.

[4] A. Puhalskii. Large deviations of semimartingales via convergence of the predictable characteristics. Stochastics. **49**, 27-85, 1994.

Quantization with a deformed trace

Dmitry Gurevich

The standard quantization scheme of a Poisson structure on a variety M consists in the following. First, one looks for an associative $\star$-product satisfying the so-called correspondence principle. Existence of such a product is shown by Kontsevich. Second, one represents the constructed associative algebra $\mathcal{A}_\hbar$ in a linear (hopefully, Hilbert) space. However, if the initial Poisson structure is not symplectic, such a representation is usually associated to each symplectic leaf of the bracket.

In the 80's the author considered some Poisson pencils whose quantization leads to "braided" algebras (cf. [**G1, G2**] and the references therein). This means that in a sense they are related to a braiding, i.e. a solution to the Quantum Yang-Baxter equation (YBE)

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}, \text{ where } R_{12} = R \otimes I, \ R_{23} = I \otimes R,$$

V is a vector space over the ground field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), and $R : V^{\otimes 2} \to V^{\otimes 2}$ is a linear operator. Such a braiding plays the role of the usual flip in all related constructions and operations. In particular, generalized Lie algebras and their enveloping algebras were defined in this way. However, braidings entering their definitions were assumed to be involutive ($R^2 = I$).

Semiclassical counterpart of such braiding is a classical r-matrix. Given a Lie algebra $\mathbf{g}$. By a classical r-matrix we mean an element $r \in \bigwedge^2 \mathbf{g}$ satisfying the classical analog of the YBE

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \text{ where } r_{12} = r \otimes 1, r_{23} = 1 \otimes r.$$

Let $\rho : \mathbf{g} \to \mathrm{Vect}\,(M)$ be a representation of the Lie algebra $\mathbf{g}$ into the vector fields space on a variety M. It is clear that the operator

$$f \otimes g \in \mathbb{K}[M] \to \{f, g\}_r = \circ\, \rho^{\otimes 2}(r)(f \otimes g)$$

where $\circ$ stands for the usual (commutative) product in the coordinate ring $\mathbb{K}[M]$ of the variety M defines a Poisson bracket on it.

A typical example is $M = \mathbf{g}^*$, $\mathbb{K}[\mathbf{g}^*] = \mathrm{Sym}\,(\mathbf{g})$. Given a classical r-matrix $r \in \bigwedge^2(\mathbf{g})$ then the bracket $\{\ ,\ \}_r$ is compatible with the linear Poisson-Lie bracket $\{\ ,\ \}_{PL}$, i.e. these brackets generate a Poisson pencil

$$\{\ ,\ \}_{a,b} = a\{\ ,\ \}_{PL} + b\{\ ,\ \}_r.$$

Moreover, each of them (and consequently, the whole Poisson pencil) can be restricted to any G-orbit $\mathcal{O} \subset \mathbf{g}^*$. (The restriction of the PL bracket to the orbit $\mathcal{O}$ is called Kirillov-Kostant-Souriau bracket.)

A quantization of the Poisson-Lie bracket can be realized in different ways. We consider the enveloping algebra $U(\mathbf{g}_\hbar)$ to be quantum counterpart of the bracket $\{\,,\,\}_{PL}$. Hereafter by $\mathbf{g}_\hbar$ we mean the Lie algebra with the bracket $\hbar[\,,\,]$ where $[\,,\,]$ is the bracket of the Lie algebra $\mathbf{g}$ and $\hbar$ is a deformation (quantization) parameter. As for the the KKS bracket its quantization can be realized as an appropriate quotient of the algebra $U(\mathbf{g}_\hbar)$ and represented in a vector space in the spirit of the Kirillov orbit method.

In order to quantize the whole pencil $\{\,,\,\}_{a,b}$ or its restriction to an orbit $\mathcal{O}$ we apply the following result of Drinfeld. There exists an element $F_\nu \in U(\mathbf{g})\hat{\otimes}U(\mathbf{g})$ such that

$$F_\nu = 1 + \nu\, r + \dots,$$
$$F_\nu(X+Y,Z)\, F_\nu(X,Y) = F_\nu(X,Y+Z)\, F_\nu(Y,Z),$$

and

$$(\epsilon \otimes 1)F_\nu = (1 \otimes \epsilon)F_\nu = 1,$$

where ϵ is the counit in $U(\mathbf{g})$.

By using this element ("quantor" according to Lychagin's terminology) it is possible to quantize the above Poisson pencil and all other operators. Say, by equipping the algebra $\mathcal{A}_\hbar = U(\mathbf{g}_\hbar)$ with a new product

$$f \star_{\hbar,\nu} g = \star_\hbar \rho^{\otimes 2}(F_\nu)(f \otimes g)$$

where the representation $\rho = \mathrm{ad}$ is naturally extended to the algebra $U(\mathbf{g}_\hbar)$ and $\star_\hbar$ stands for the product in this algebra we get a new associative algebra denoted $\mathcal{A}_{\hbar,\nu}$.

The aforementioned braiding can be introduced via the element F_ν. Namely, we put $R = F_\nu^{-1}\, P\, F_\nu$ where P is the usual flip. It is clear that R is involutive. Moreover, it is subject to the quantum YBE, i.e. it is a braiding.

By means of the quantor F_ν the category of finite dimensional modules of the algebra $\mathcal{A}_\hbar$ can be converted into that of $\mathcal{A}_{\hbar,\nu}$-ones. This category is monoidal tensor rigid. Let us consider an object V of this category and the corresponding object $\mathrm{End}\,(V) \cong V \otimes V^*$ of internal endomorphisms. There exists a map

$$\mathrm{Tr}_R : \mathrm{End}\,(V) \to \mathbb{K}$$

which is a deformation of the usual trace and is morphism in this category. Moreover, it is R-symmetric, i.e.

$$\mathrm{Tr}_R\,(X \circ Y) = \mathrm{Tr}_R \circ R(X \otimes Y)$$

where $\circ$ stands for the usual product in the algebra $\mathrm{End}\,(V)$. In a sense it looks like a super-trace for which the role of R is played by a super-flip.

So, by quantizing the above Poisson pencil and by considering representations of the quantum algebra we are forced to replace the usual trace by its braided version. According to [**G1**] the linear term of the deformation of of the trace can be treated as a cocycle on the Lie algebra $\mathbf{g}$. So, the deformation procedure itself can be regarded as a quantization of this cocycle. (Note that the involution operator $A \to A^*$ must be also deformed.)

Recently it was understood what is an analog of the above algebra $\mathcal{A}_{\hbar,\nu}$ corresponding to a non-involutive braiding (of Hecke type) and what is its semiclassical counterpart.

Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a Hecke symmetry, i.e. a braiding which meets the Hecke relation

$$(q\,I - R)(q^{-1}I + R) = 0,\ q \in \mathbb{K} \text{ is generic.}$$

The algebra generated by the unit and elements $l_i^j,\ 1 \le i, j \le n = \dim V$ subject to the equation

$$RL_1RL_1 - L_1RL_1R = \hbar(RL_1 - L_1R),$$

where $L = (l_i^j)$ is the matrix with entries l_i^j and $L_1 = L \otimes 1$ is call modified Reflection Equation Algebra (mREA).

If the Hecke symmetry R comes from the quantum group $U_q(sl(n))$ it is a one parameter deformation of the usual flip. In this case the mREA is two parameter deformation of the commutative algebra $\mathrm{Sym}\,(gl(n))$ (which is a specialization of the $\mathcal{A}_{\hbar,q}$ at $\hbar = 0,\ q = 1$). Its semiclassical counterpart is a Poisson pencil similar to that above but with the bracket $\{\ ,\ \}_r$ defined in another way. Namely, it is an extension to the ambient vector space of the so-called Semenov-Tian-Shansky bracket defined on the group $SL(n)$. Similarly to the pencil above the latter one can be also restricted to any $GL(n)$-orbit in $gl(n)^*$.

The algebra $\mathcal{A}_{\hbar,q}$ possesses a braided bi-algebra structure and has the same category of finite dimensional representations as the quantum group $U_q(sl(n))$ has. However, in contrast with the above monoidal tensor category this one is quasitensor one. Nevertheless, an intrinsic trace Tr_R which is a categorical morphism and a deformation of the usual trace is well defined on any object $\mathrm{End}\,(V)$ of internal endomorphisms. (For simple objects V it is unique up to a factor.)

For instance, if V is the basic space then $\mathrm{Tr}_R\, l_i^j = \delta_i^j$. The defining relations of the algebra $\mathcal{A}_{\hbar,q}$ can be rewritten as follows

$$X \otimes Y - Q(X \otimes Y) = [X, Y],\ X, Y \in \mathbf{L} = \mathrm{span}\,(l_i^j)$$

where $Q : \mathbf{L}^{\otimes 2} \to \mathbf{L}^{\otimes 2}$ is a braiding and $[\,,\,] : \mathbf{L}^{\otimes 2} \to \mathbf{L}$ is a "braided Lie bracket". We would like to emphasize that the latter form of the mREA makes it more similar to an enveloping algebra.

It is easy to check that

$$\mathrm{Tr}_R(X \circ Y) = \mathrm{Tr}_R \circ Q(X \otimes Y)$$

where $\circ$ stands for the usual product in the algebra $\mathrm{End}\,(V)$ (note that the space $\mathbf{L}$ can be naturally identified with $\mathrm{End}\,(V)$). So, we can see that such a trace is Q-symmetric.

A more detailed presentation of the topic can be found in the paper [**GPS**].

In my talk I shall exhibit the role of a deformed (quantum) trace in "braided geometry".

Bibliography

[G1] Gourevitch D. *Equation de Yang-Baxter et quantification des cocycles*, C.R.Acad.Sci. Paris, **310** (1990) 845–848.

[G2] Gurevich D. *Algebraic aspects of the Yang-Baxter equation*, English translation: Leningrad Math. **2** (1991) 801 – 828.

[GPS] Gurevich D., Pyatov P., Saponov P. *Representation theory of (modified) Reflection Equation Algebra of $GL(m|n)$ type*, math/0612815.

Transformations preserving matrix invariants over semirings[1]

Alexander E. Guterman

The investigations of matrix transformations which leave fixed different matrix properties and invariants is an actively developing part of matrix theory. This research was started in the works by Frobenius, see [**6, 9**, Theorem 1.1], and Dieudonné, see [**5, 9**, Theorem 1.2], where bijective linear transformations on matrices over fields which preserve the determinant and the set of singular matrices, correspondingly, were characterized.

[1]Partially supported by the RFBR grant 05-01-01048 and the grant MK-2718.2007.1.

During the last three decades many authors investigated linear transformations on more general algebraic structures, such as matrices over rings and semirings. In this talk we are going to discuss the corresponding problems on max-algebras and related classes of semirings.

DEFINITION 4. A *semiring* is a set $\mathcal{S}$ with two binary operations, addition and multiplication, such that:

- $\mathcal{S}$ is an abelian monoid under addition (identity denoted by 0);
- $\mathcal{S}$ is a semigroup under multiplication (identity, if any, denoted by 1);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

In this paper we will always assume that there is a multiplicative identity 1 in $\mathcal{S}$ which is different from 0.

DEFINITION 5. A semiring $\mathcal{S}$ is called *commutative* if the multiplication in $\mathcal{S}$ is commutative.

DEFINITION 6. A semiring $\mathcal{S}$ is called *antinegative* (or *zero-sum-free*) if $a + b = 0$ implies that $a = b = 0$.

This means that the zero element is the only element with an additive inverse

DEFINITION 7. We say that a semiring $\mathcal{S}$ *has no zero divisors* if from $ab = 0$ in $\mathcal{S}$ it follows that either $a = 0$ or $b = 0$.

DEFINITION 8. A semiring is called a *max-algebra* if the set $\mathcal{S}$ is an ordered group with the multiplication $*$ and the order relation $\leq$, and operations in $\mathcal{S}$ are defined as follows: $a + b = \max\{a, b\}$, $a \cdot b = a * b$ for any $a, b \in \mathcal{S}$.

It is straightforward to see that max-algebra is antinegative. Also it does not contain zero divisors, moreover any non-zero element of a max-algebra has a multiplicative inverse.

Let $\mathcal{M}_{m,n}(\mathcal{S})$ denote the set of $m \times n$ matrices with entries from the semiring $\mathcal{S}$, $\mathcal{M}_n(\mathcal{S}) = \mathcal{M}_{n,n}(\mathcal{S})$. Under natural definitions of matrix addition and multiplication $\mathcal{M}_n(\mathcal{S})$ is obviously a semiring. Matrix theory over semirings has been an object of intensive study during the last decades, see for example the monograph [**7**] and references therein. The development of linear algebra over semirings certainly requires such an important matrix invariant as the determinant function. However it turns out that even over commutative semirings without zero divisors the classical determinant can

not be defined as over fields and commutative rings. The main problem lies in the fact that in semirings which are not rings not all elements possess an additive inverse. A natural replacement of the determinant function for matrices over commutative semirings is the bideterminant known for many years, see [**7**].

DEFINITION 9. A *bideterminant* of a matrix $A = [a_{i,j}] \in \mathcal{M}_n(\mathcal{S})$ is the pair $(\|A\|^+, \|A\|^-)$, where

$$\|A\|^+ = \sum_{\sigma \in A_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}, \ \|A\|^- = \sum_{\sigma \in S_n \setminus A_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

here S_n denotes the symmetric group on the set $\{1, \ldots, n\}$, A_n denotes its subgroup of even permutations.

It is known that the bideterminant function possesses some natural properties. Namely it is invariant under transposition, and for any scalar $\alpha \in \mathcal{S}$, $(\|\alpha A\|^+, \|\alpha A\|^-) = (\alpha^n \|A\|^+, \alpha^n \|A\|^-)$. However, some basic properties of the determinant are no longer true for the bideterminant. For example, if A is invertible then $\|A\|^+ \neq \|A\|^-$ but the converse is not always true.

EXAMPLE 1. Let us consider $A = E_{1,1} + 2E_{1,2} + 3E_{2,1} + 4E_{2,2} \in M_2(\mathcal{S})$, where $\mathcal{S} = (\mathbb{Q}_+, \max, \cdot)$, namely the set of non-negative rationals with the standard multiplication and the addition defined by $a + b = \max\{a, b\}$. Then $(\|A\|^+, \|A\|^-) = (4, 6)$ but A is not invertible.

Note that the bideterminant is not multiplicative in general. However, some weaker versions of this property are true, in particular,

$$\|AB\|^+ + \|A\|^+ \|B\|^- + \|A\|^- \|B\|^+ = \|AB\|^- + \|A\|^+ \|B\|^+ + \|A\|^- \|B\|^-.$$

We prove the following theorem which is a semiring analog of famous Frobenius theorem, see [**6**], on linear transformations preserving the determinant of complex matrices.

THEOREM 1. [**2**] *Let $\mathcal{S}$ be a commutative antinegative semiring without zero divisors and $T : \mathcal{M}_n(\mathcal{S}) \to \mathcal{M}_n(\mathcal{S})$ be a surjective linear transformation. Then $(\|T(X)\|^+, \|T(X)\|^-) = (\|X\|^+, \|X\|^-)$ for all $X \in \mathcal{M}_n(\mathcal{S})$ if and only if there exists permutation matrices P, Q and invertible diagonal matrices D, E, satisfying $(\|PQ\|^+, \|PQ\|^-) = (\|DE\|^+, \|DE\|^-) = (1, 0)$, such that either $T(X) = PDXEQ$ for all $X \in \mathcal{M}_n(\mathcal{S})$ or $T(X) = PDX^tEQ$ for all $X \in \mathcal{M}_n(\mathcal{S})$. Here the matrices P, Q are defined uniquely and the matrices D, E are defined uniquely up to an invertible scalar factor.*

DEFINITION 10. We say that a transformation $T : \mathcal{M}_n(\mathcal{S}) \to \mathcal{M}_n(\mathcal{S})$ is *standard* if it is defined by $T(X) = PDXEQ$ for all $X \in \mathcal{M}_n(\mathcal{S})$ or $T(X) = PDX^tEQ$ for all $X \in \mathcal{M}_n(\mathcal{S})$ for certain permutational matrices P, Q and diagonal matrices D, E.

The following similar function is widely considered in combinatorial matrix theory:

DEFINITION 11. A *permanent* of a matrix $A = [a_{i,j}] \in \mathcal{M}_n(\mathcal{S})$ is

$$\text{per (A)} = \sum_{\sigma \in \mathrm{S_n}} \mathrm{a_{1,\sigma(1)}} \cdots \mathrm{a_{n,\sigma(n)}}.$$

Also the following polynomial is related to this function:

DEFINITION 12. The *rook polynomial* of a matrix $A \in M_{m,n}(\mathcal{S})$ is $R_A(x) = \sum_{j \geq 0} p_j x^j$, where $p_0 = 1$, p_j is the sum of the permanents of all $j \times j$ submatrices of A.

Linear transformations preserving the rook polynomial and permanent itself were characterized by Beasley and Pullman.

Here we provide more general result, namely, we prove that in order to characterize a transformation, it is enough to know that it preserves any single coefficient of a rook polynomial, namely we prove the following:

THEOREM 2. [**8**] *Let $T : M_n(\mathcal{S}) \to M_n(\mathcal{S})$ be a surjective linear transformation and j, $2 \leq j \leq n$, be fixed. Then T preserves the j-th coefficient of the rook polynomial iff T is standard with $p_j(DE) = 1$.*

The following notions of singularity are in use while dealing with matrices over semirings, as usual, we separate left and right singularity.

DEFINITION 13. A matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$ is said to be $\mathcal{S}$-*right singular* if $A\mathbf{x} = \mathbf{0}$ for some nonzero $\mathbf{x} \in \mathcal{S}^n$. $A \in \mathcal{M}_{m,n}(\mathcal{S})$ is $\mathcal{S}$-*left singular* if $\mathbf{x}^tA = \mathbf{0}^t$ for some nonzero $\mathbf{x} \in \mathcal{S}^m$. A matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$ is $\mathcal{S}$-*singular* if A is either $\mathcal{S}$-left singular or $\mathcal{S}$-right singular.

The next example shows that even over antinegative commutative semirings without zero divisors there exist matrices that are $\mathcal{S}$-left singular and are not $\mathcal{S}$-right singular or vice versa.

EXAMPLE 2. Let $(\mathbb{R}, +, \max)$ be a max-algebra,

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbb{R}, +, \max).$$

We have that $A\mathbf{x} = \mathbf{0}$ forces $\mathbf{x} = \mathbf{0}$ since $(\mathbb{R}, +, \max)$ is antinegative, but $[1, 0]A = [0, 0]$. Similar $\mathbf{x}^tB = \mathbf{0}$ forces $\mathbf{x} = \mathbf{0}$ while $B[0, 1]^t = [0, 0]^t$.

Definition 14. A matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$ is $\mathcal{S}$-*nonsingular* if A is not $\mathcal{S}$-singular.

Note that if $\mathcal{S}$ is commutative and A is an $\mathcal{S}$-singular square matrix then $(\|A\|^+, \|A\|^-) = (0,0)$. However the following example shows that there are $\mathcal{S}$-nonsingular matrices with the bideterminant equal to $(0,0)$.

Example 3. Over any commutative antinegative semiring,

$$\left\| \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\|^+ = 0 = \left\| \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\|^- .$$

We obtain the following analog of Dieudonné theorem on singularity preservers, see [**5**], for matrices over semirings.

Theorem 3. [**2**] *Let $\mathcal{S}$ be an antinegative semiring without zero divisors and $T : \mathcal{M}_{m,n}(\mathcal{S}) \to \mathcal{M}_{m,n}(\mathcal{S})$ be a surjective linear operator. Then the following statements are equivalent*

(1) *T preserves the set of $\mathcal{S}$-singular matrices;*
(2) *T preserves the set of $\mathcal{S}$-nonsingular matrices;*
(3) *There are permutational matrices $P, Q \in \mathcal{M}_n(\mathcal{S})$ and a matrix B with all invertible entries such that either $T(X) = P(X \circ B)Q$ for all $X \in \mathcal{M}_n(\mathcal{S})$ or $T(X) = P(X \circ B)^t Q$ for all $X \in \mathcal{M}_n(\mathcal{S})$, here $X \circ B$ is an Hadamard product, i.e., $(X \circ B)_{i,j} = x_{i,j} b_{i,j}$.*

If the semiring $\mathcal{S}$ is also a subsemiring of an associative ring $\mathcal{R}$ without zero divisors we can consider the following notion of singularity as well.

Definition 15. We say that a matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$ is $\mathcal{R}$-*right singular* if $A\mathbf{x} = \mathbf{0}$ for some nonzero $\mathbf{x} \in \mathcal{R}^n$. $A \in \mathcal{M}_{m,n}(\mathcal{S})$ is $\mathcal{R}$-*left singular* if $\mathbf{x}^t A = \mathbf{0}$ for some nonzero $\mathbf{x} \in \mathcal{R}^m$. A is $\mathcal{R}$-*singular* if A is either $\mathcal{R}$-left singular or $\mathcal{R}$-right singular, and $\mathcal{R}$-*nonsingular* if it is not $\mathcal{R}$-singular.

It is straightforward to see that if a semiring $\mathcal{S}$ is a subsemiring of a certain ring $\mathcal{R}$ then $\mathcal{R}$-right (left) singularity follows from $\mathcal{S}$-right (left) singularity. However the following example shows that there are $\mathcal{S}$-nonsingular matrices which are $\mathcal{R}$-singular.

Example 4. For any n the matrix $J_n = \sum_{i,j=1}^{n} E_{i,j} \in \mathcal{M}_n(\mathbb{Z}_+)$ is $\mathbb{Z}$-left and $\mathbb{Z}$-right singular but $\mathbb{Z}_+$-nonsingular.

Note that similarly to the situation over fields all non-square matrices are $\mathcal{R}$-singular, however as the above example shows they may not be $\mathcal{S}$-singular.

An analog of Theorem 3 holds for transformations preserving $\mathcal{R}$-singularity. Corresponding transformations appear to be standard.

DEFINITION 16. Let $\mathcal{S}$ be a max-algebra (operations are denoted by max and $+$). A matrix $A = [a_{ij}] \in M_n(\mathcal{S})$ is said to be *tropically singular* if the maximum in the expression for the permanent

$$\mathrm{per\ (A)} = \max_{\sigma \in \mathrm{S_n}} \{\mathrm{a}_{1\sigma(1)} + \ldots + \mathrm{a}_{\mathrm{n}\sigma(\mathrm{n})}\}$$

is achieved at least twice.

It can be generalized to the case of an arbitrary antinegative semiring $\mathcal{S}$ in the following way:

DEFINITION 17. A matrix $A = [a_{ij}] \in M_n(\mathcal{S})$ is said to be *tropically singular* if there exists a subset $\mathcal{T} \in S_n$ such that

$$\sum_{\sigma \in \mathcal{T}} a_{1\sigma(1)} \cdots a_{n\sigma(n)} = \sum_{\sigma \in S_n \setminus \mathcal{T}} a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

Our further results include the characterization of linear transformations preserving these and related notions of singularity. Also we obtain several analogs of Markus and Moyls result on linear transformations preserving rank, see [**9**, Theorems 3.1, 3.2], for several well-known semiring rank functions.

Bibliography

[1] L. B. Beasley, N. J. Pullman, Term rank, permanent and rook polynomial preservers, *Linear Algebra Appl.* **90** (1987) 33-46.

[2] L. B. Beasley, A. E. Guterman, S.-G. Lee, S.-Z. Song, Frobenius and Dieudonné theorems over semirings, *Linear and Multilinear Algebra*, **55**, no. 1, (2007) 19-34.

[3] P. Butkovič, Max-algebra: the linear algebra of combinatorics? *Linear Algebra Appl.* **367** (2003) 315-335.

[4] M. Develin, F. Santos, B. Sturmfels, On the rank of a tropical matrix, In *Discrete and Computational Geometry* (E. Goodman, J. Pach and E. Welzl, eds.), MSRI Publications, Cambridge Univ. Press, 2005.

[5] J. Dieudonné, Sur une généralisation du groupe orthogonal à quatre variables, *Arch. Math.* **1** (1949) 282-287.

[6] G. Frobenius, Ü*ber die Darstellung der endlichen Gruppen durch lineare Substitutionen*, Sitzungsber., Preuss. Akad. Wiss (Berlin), Berlin (1897) 994-1015.

[7] K. Glazek, *A Guide to the Literature on Semirings and their Applications in Mathematics and Information Sciences*, Kluwer Academic Publishers, 2002.

[8] A. E. Guterman, Transformations of non-negative integer matrices preserving the determinant, *Uspehi Mat. Nauk* **58**, no. 6, (2003) 147-148.

[9] C.-K. Li, S. Pierce, Linear preserver problems, *Amer. Math. Monthly* **108**, no. 7, (2001) 591-605.

Tropical geometry and enumeration of real rational curves[1]

I. Itenberg, V. Kharlamov, and E. Shustin

The talk is devoted to applications of tropical geometry in enumerative (complex and real) algebraic geometry. We concentrate ourselves at enumeration of real rational curves interpolating fixed collections of real points in a real algebraic surface Σ, and more precisely, at the following question: *given a real divisor D and a generic collection $\boldsymbol{w}$ of $c_1(\Sigma)\cdot D-1$ real points in Σ, how many of the complex rational curves belonging to the linear system $|D|$ and passing through the points of $\boldsymbol{w}$ are real ?* By rational curves we mean irreducible genus zero curves and their degenerations, so that they form in $|D|$ a projective subvariety $S(\Sigma, D)$; this subvariety is called the *Severi variety*. A curve on a real surface Σ is called real, if the curve is invariant under the involution $c : \Sigma \to \Sigma$ defining the real structure of Σ.

While, under mild conditions on Σ and D, the number of complex curves in question is the same for all generic collections $\boldsymbol{w}$ (it equals to the degree of $S(\Sigma, D)$), it is no more the case for real curves (except few very particular situations).

J.-Y. Welschinger [**5, 6**] discovered a way to attribute weights ± 1 to the real solutions in question so that the number of real solutions counted with weights becomes independent of the choice of a generic collection of real points. As an immediate consequence, the absolute value of the Welschinger invariant $W_{\Sigma,D}$ provides a lower bound on the number $R_{\Sigma,D}(\boldsymbol{w})$ of real solutions: $R_{\Sigma,D}(\boldsymbol{w}) \geq |W_{\Sigma,D}|$.

In some cases (for example, in the case of toric Del Pezzo surfaces; recall that there are five toric Del Pezzo surfaces: the projective plane $\mathbb{P}^2$, the product $\mathbb{P}^1 \times \mathbb{P}^1$ of projective lines, and $\mathbb{P}^2$ blown up at k points in general position, where $k = 1, 2$ or 3) Welschinger invariants can be calculated using Mikhalkin's approach [**3, 4**] which deals with a corresponding count of tropical curves. In tropical geometry, complicated non-linear algebro-geometric objects are replaced by simpler piecewise-linear ones. For example, tropical plane curves are piecewise-linear graphs whose edges have rational slopes. Tropical curves can be seen as algebraic curves over the tropical semi-field $(\max, +)$.

Using the tropical approach, we proved (see [**1**]) the logarithmic equivalence for the Welschinger and Gromov-Witten invariants of any toric Del

[1]Partially supported by the joint RFBR/CNRS grant 05-01-02807.

Pezzo surface equipped with its tautological real structure, *i.e.*, the real structure which is provided by the toric structure.

THEOREM 1. (see [**1**]) *Let Σ be a toric Del Pezzo surface equipped with its tautological real structure, and D an ample divisor on Σ. The sequences $\log W_{\Sigma,nD}$ and $\log GW_{\Sigma,nD}$, $n \in \mathbb{N}$, of the Welschinger invariants and the corresponding Gromov-Witten invariants are asymptotically equivalent. More precisely, $\log W_{\Sigma,nD} = \log GW_{\Sigma,nD} + O(n)$ and $\log GW_{\Sigma,nD} = (c_1(\Sigma) \cdot D) \cdot n \log n + O(n)$.*

We also defined (see [**2**]) a series of relative tropical Welschinger-type invariants of real toric surfaces. In the Del Pezzo case, these invariants can be seen as real tropical analogs of relative Gromov-Witten invariants, and are subject to recursive formulas of Caporaso-Harris type.

In the present talk, we consider generic collections of real points on the projective plane blown up at 4 real points in general position and prove that the logarithmic equivalence of the Welschinger and Gromov-Witten invariants holds in this situation as well.

THEOREM 2. *Let Σ be the projective plane $\mathbb{P}^2$ blown up at 4 real points in general position, and D an ample divisor on Σ. The sequences $\log W_{\Sigma,nD}$ and $\log GW_{\Sigma,nD}$, $n \in \mathbb{N}$, of the Welschinger invariants and the corresponding Gromov-Witten invariants are asymptotically equivalent.*

The proof is based on a new version of the correspondence theorem, whose proof in turn uses an appropriate tropical Caporaso-Harris type formulas. In particular, we get recursive formulas that allow one to calculate Welschinger invariants of $\mathbb{P}^2$ blown up at 4 real points in general position.

Bibliography

[1] I. Itenberg, V. Kharlamov, and E. Shustin, *Logarithmic equivalence of Welschinger and Gromov-Witten invariants*, Russian Math. Surveys **59** (2004), no. 6, 1093–1116.

[2] I. Itenberg, V. Kharlamov, and E. Shustin, *A Caporaso-Harris type formula for Welschinger invariants of real toric Del Pezzo surfaces*. Preprint math.AG/0608549, 2006, 1 - 39 (to appear in Commentarii Math. Helvetici).

[3] G. Mikhalkin, *Counting curves via the lattice paths in polygons*, Comptes Rend. Acad. Sci. Paris, Sér. I, **336** (2003), no. 8, 629–634.

[4] G. Mikhalkin, *Enumerative tropical algebraic geometry in $\mathbb{R}^2$*, J. Amer. Math. Soc. **18** (2005), 313–377.

[5] J.-Y. Welschinger, *Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry*, C. R. Acad. Sci. Paris, Sér. I, **336** (2003), 341–344.

[6] J.-Y. Welschinger, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math. **162** (2005), no. 1, 195–234.

Abstract convexity and cone-vexing abstractions

Semen S. Kutateladze

This talk is devoted to some origins of abstract convexity and a few vexing limitations on the range of abstraction in convexity. Convexity is a relatively recent subject. Although the noble objects of Euclidean geometry are mostly convex, the abstract notion of a convex set appears only after the Cantor paradise was founded. The idea of convexity feeds generation, separation, calculus, and approximation. Generation appears as duality; separation, as optimality; calculus, as representation; and approximation, as stability.

1. Generation

Let $\overline{E}$ be a complete lattice E with the adjoint top $\top := +\infty$ and bottom $\bot := -\infty$. Unless otherwise stated, Y is usually a *Kantorovich space* which is a Dedekind complete vector lattice in another terminology. Assume further that H is some subset of E which is by implication a (convex) cone in E, and so the bottom of E lies beyond H. A subset U of H is *convex relative to* H or *H-convex*, in symbols $U \in \mathfrak{V}(H, \overline{E})$, provided that U is the *H-support set* $U_p^H := \{h \in H : h \le p\}$ of some element p of $\overline{E}$.

Alongside the H-convex sets we consider the so-called H-convex elements. An element $p \in \overline{E}$ is *H-convex* provided that $p = \sup U_p^H$; i.e., p represents the supremum of the H-support set of p. The H-convex elements comprise the cone which is denoted by $\mathscr{C}(H, \overline{E})$. We may omit the references to H when H is clear from the context. It is worth noting that convex elements and sets are "glued together" by the *Minkowski diality* $\varphi : p \mapsto U_p^H$. This duality enables us to study convex elements and sets simultaneously.

Since the classical results by Fenchel [**1**] and Hörmander [**2, 3**] it has been well known that the most convenient and conventional classes of convex functions and sets are $\mathscr{C}(A(X), \overline{\mathbb{R}^X})$ and $\mathfrak{V}(X', \overline{\mathbb{R}^X})$. Here X is a locally convex space, X' is the dual of X, and $A(X)$ is the space of affine functions on X (isomorphic with $X' \times \mathbb{R}$).

In the first case the Minkowski duality is the mapping $f \mapsto \operatorname{epi}(f^*)$ where

$$f^*(y) := \sup_{x \in X} (\langle y, x\rangle - f(x))$$

is the *Young–Fenchel transform* of f or the *conjugate function* of f . In the second case we prefer to write down the inverse of the Minkowski duality

which sends U in $\mathfrak{V}(X', \overline{\mathbb{R}}^X)$ to the standard *support function*

$$\varphi^{-1}(U) : x \mapsto \sup_{y \in U} \langle y, x \rangle.$$

As usual, $\langle \cdot, \cdot \rangle$ stands for the canonical pairing of X' and X.

This idea of abstract convexity lies behind many current objects of analysis and geometry. Among them we list the "economical" sets with boundary points meeting the Pareto criterion: capacities, monotone seminorms, various classes of functions convex in some generalized sense, for instance, the Bauer convexity in Choquet theory, etc. It is curious that there are ordered vector spaces consisting of the convex elements with respect to narrow cones with finite generators. Abstract convexity is traced and reflected, for instance, in [**4**]–[**9**].

2. Separation

Consider cones K_1 and K_2 in a topological vector space X and put $\varkappa := (K_1, K_2)$. Given a pair $\varkappa$ define the correspondence $\Phi_\varkappa$ from X^2 into X by the formula

$$\Phi_\varkappa := \{(k_1, k_2, x) \in X^3 : x = k_1 - k_2 \in K_\imath \ (\imath := 1, 2)\}.$$

Clearly, $\Phi_\varkappa$ is a cone or, in other words, a conic correspondence.

The pair $\varkappa$ is *nonoblate* whenever $\Phi_\varkappa$ is open at the zero. Since $\Phi_\varkappa(V) = V \cap K_1 - V \cap K_2$ for every $V \subset X$, the nonoblateness of $\varkappa$ means that

$$\varkappa V := (V \cap K_1 - V \cap K_2) \cap (V \cap K_2 - V \cap K_1)$$

is a zero neighborhood for every zero neighborhood $V \subset X$. Since $\varkappa V \subset V - V$, the nonoblateness of $\varkappa$ is equivalent to the fact that the system of sets $\{\varkappa V\}$ serves as a filterbase of zero neighborhoods while V ranges over some base of the same filter.

Let $\Delta_n : x \mapsto (x, \dots, x)$ be the embedding of X into the diagonal $\Delta_n(X)$ of X^n. A pair of cones $\varkappa := (K_1, K_2)$ is nonoblate if and only if $\lambda := (K_1 \times K_2, \Delta_2(X))$ is nonoblate in X^2.

Cones K_1 and K_2 constitute a nonoblate pair if and only if the conic correspondence $\Phi \subset X \times X^2$ defined as

$$\Phi := \{(h, x_1, x_2) \in X \times X^2 : x_\imath + h \in K_\imath \ (\imath := 1, 2)\}$$

is open at the zero. Recall that a convex correspondence Φ from X into Y is open at the zero if and only if the Hörmander transform of $X \times \Phi$ and the cone $\Delta_2(X) \times \{0\} \times \mathbb{R}^+$ constitute a nonoblate pair in $X^2 \times Y \times \mathbb{R}$.

Cones K_1 and K_2 in a topological vector space X are *in general position* provided that

(1) the algebraic span of K_1 and K_2 is some subspace $X_0 \subset X$; i.e., $X_0 = K_1 - K_2 = K_2 - K_1$;

(2) the subspace X_0 is complemented; i.e., there exists a continuous projection $P : X \to X$ such that $P(X) = X_0$;

(3) K_1 and K_2 constitute a nonoblate pair in X_0.

Let σ_n stand for the rearrangement of coordinates

$$\sigma_n : ((x_1, y_1), \dots, (x_n, y_n)) \mapsto ((x_1, \dots, x_n), (y_1, \dots, y_n))$$

which establishes an isomorphism between $(X \times Y)^n$ and $X^n \times Y^n$.

Sublinear operators $P_1, \dots, P_n : X \to E \cup \{+\infty\}$ are *in general position* if so are the cones $\Delta_n(X) \times E^n$ and $\sigma_n(\operatorname{epi}(P_1) \times \cdots \times \operatorname{epi}(P_n))$. A similar terminology applies to convex operators.

Given a cone $K \subset X$, put

$$\pi_E(K) := \{T \in \mathscr{L}(X, E) : Tk \le 0 \ (k \in K)\}.$$

We readily see that $\pi_E(K)$ is a cone in $\mathscr{L}(X, E)$.

THEOREM. *Let $K_1, \dots, K_n$ be cones in a topological vector space X and let E be a topological Kantorovich space. If $K_1, \dots, K_n$ are in general position then*

$$\pi_E(K_1 \cap \cdots \cap K_n) = \pi_E(K_1) + \cdots + \pi_E(K_n).$$

This formula opens a way to various separation results.

SANDWICH THEOREM. *Let $P, Q : X \to E \cup \{+\infty\}$ be sublinear operators in general position. If $P(x) + Q(x) \ge 0$ for all $x \in X$ then there exists a continuous linear operator $T : X \to E$ such that*

$$-Q(x) \le Tx \le P(x) \quad (x \in X).$$

Many efforts were made to abstract these results to a more general algebraic setting and, primarily, to semigroups. The relevant separation results are collected in [**10**].

3. Calculus

Consider a Kantorovich space E and an arbitrary nonempty set $\mathfrak{A}$. Denote by $l_\infty(\mathfrak{A}, E)$ the set of all order bounded mappings from $\mathfrak{A}$ into E; i.e., $f \in l_\infty(\mathfrak{A}, E)$ if and only if $f : \mathfrak{A} \to E$ and the set $\{f(\alpha) : \alpha \in \mathfrak{A}\}$ is order bounded in E. It is easy to verify that $l_\infty(\mathfrak{A}, E)$ becomes a Kantorovich space if endowed with the coordinatewise algebraic operations and order.

The operator $\varepsilon_{\mathfrak{A},E}$ acting from $l_\infty(\mathfrak{A},E)$ into E by the rule

$$\varepsilon_{\mathfrak{A},E} : f \mapsto \sup\{f(\alpha) : \alpha \in \mathfrak{A}\} \quad (f \in l_\infty(\mathfrak{A},E))$$

is called the *canonical sublinear operator* given $\mathfrak{A}$ and E. We often write $\varepsilon_{\mathfrak{A}}$ instead of $\varepsilon_{\mathfrak{A},E}$ when it is clear from the context what Kantorovich space is meant. The notation ε_n is used when the cardinality of $\mathfrak{A}$ equals n and we call the operator ε_n *finitely-generated*.

Let X and E be ordered vector spaces. An operator $p : X \to E$ is called *increasing* or *isotonic* if for all $x_1, x_2 \in X$ from $x_1 \le x_2$ it follows that $p(x_1) \le p(x_2)$. An increasing linear operator is also called *positive*. As usual, the collection of all positive linear operators in the space $L(X,E)$ of all linear operators is denoted by $L^+(X,E)$. Obviously, the positivity of a linear operator T amounts to the inclusion $T(X^+) \subset E^+$, where $X^+ := \{x \in X : x \ge 0\}$ and $E^+ := \{e \in E : e \ge 0\}$ are the *positive cones* in X and E respectively. Observe that every canonical operator is increasing and sublinear, while every finitely-generated canonical operator is order continuous.

Recall that $\partial p := \partial p(0) = \{T \in L(X,E) : (\forall x \in X)\ Tx \le p(x)\}$ is the *subdifferential* at the zero or *support set* of a sublinear operator p.

Consider a set $\mathfrak{A}$ of linear operators acting from a vector space X into a Kantorovich space E. The set $\mathfrak{A}$ is *weakly order bounded* if the set $\{\alpha x : \alpha \in \mathfrak{A}\}$ is order bounded for every $x \in X$. We denote by $\langle\mathfrak{A}\rangle x$ the mapping that assigns the element $\alpha x \in E$ to each $\alpha \in \mathfrak{A}$, i.e. $\langle\mathfrak{A}\rangle x : \alpha \mapsto \alpha x$. If $\mathfrak{A}$ is weakly order bounded then $\langle\mathfrak{A}\rangle x \in l_\infty(\mathfrak{A},E)$ for every fixed $x \in X$. Consequently, we obtain the linear operator $\langle\mathfrak{A}\rangle : X \to l_\infty(\mathfrak{A},E)$ that acts as $\langle\mathfrak{A}\rangle : x \mapsto \langle\mathfrak{A}\rangle x$. Associate with $\mathfrak{A}$ one more operator

$$p_{\mathfrak{A}} : x \mapsto \sup\{\alpha x : \alpha \in \mathfrak{A}\} \quad (x \in X).$$

The operator $p_{\mathfrak{A}}$ is sublinear. The support set $\partial p_{\mathfrak{A}}$ is denoted by $\operatorname{cop}(\mathfrak{A})$ and referred to as the *support hull* of $\mathfrak{A}$. These definitions entail the following

THEOREM. *If p is a sublinear operator with $\partial p = \operatorname{cop}(\mathfrak{A})$ then $P = \varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle$. Assume further that $p_1 : X \to E$ is a sublinear operator and $p_2 : E \to F$ is an increasing sublinear operator. Then*

$$\partial(p_2 \circ p_1) = \left\{T \circ \langle\partial p_1\rangle : T \in L^+(l_\infty(\partial p_1, E), F) \ \wedge\ T \circ \Delta_{\partial p_1} \in \partial p_2\right\}.$$

Furthermore, if $\partial p_1 = \operatorname{cop}(\mathfrak{A}_1)$ and $\partial p_2 = \operatorname{cop}(\mathfrak{A}_2)$ then

$$\partial(p_2 \circ p_1) = \left\{T \circ \langle\mathfrak{A}_1\rangle : T \in L^+(l_\infty(\mathfrak{A}_1, E), F) \right.$$
$$\left. \wedge\ \left(\exists \alpha \in \partial\varepsilon_{\mathfrak{A}_2}\right)\ T \circ \Delta_{\mathfrak{A}_1} = \alpha \circ \langle\mathfrak{A}_2\rangle\right\}.$$

More details on subdifferential calculus and applications to optimality are collected in [**11**].

4. Approximation

Study of stability in abstract convexity is accomplished sometimes by introducing various epsilons in appropriate places. One of the earliest attempts in this direction is connected with the classical Hyers–Ulam stability theorem for ε-convex functions. The most recent results are collected in [**12**]. Exact calculations with epsilons and sharp estimates are sometimes bulky and slightly mysterious. Some alternatives are suggested by actual infinities, which is illustrated with the conception of *infinitesimal optimality*.

Assume given a convex operator $f : X \to E \cup +\infty$ and a point $\overline{x}$ in the effective domain $\operatorname{dom}(f) := \{x \in X : f(x) < +\infty\}$ of f. Given $\varepsilon \geq 0$ in the positive cone E_+ of E, by the *ε-subdifferential* of f at $\overline{x}$ we mean the set

$$\partial^{\varepsilon} f(\overline{x}) := \big\{T \in L(X,E) : (\forall x \in X)(Tx - Fx \leq T\overline{x} - f\overline{x} + \varepsilon)\big\},$$

with $L(X,E)$ standing as usual for the space of linear operators from X to E.

Distinguish some downward-filtered subset $\mathscr{E}$ of E that is composed of positive elements. Assuming E and $\mathscr{E}$ standard, define the *monad* $\mu(\mathscr{E})$ of $\mathscr{E}$ as $\mu(\mathscr{E}) := \bigcap\{[0,\varepsilon] : \varepsilon \in {}^{\circ}\mathscr{E}\}$. The members of $\mu(\mathscr{E})$ are *positive infinitesimals* with respect to $\mathscr{E}$. As usual, ${}^{\circ}\mathscr{E}$ denotes the external set of all standard members of E, the *standard part* of $\mathscr{E}$.

We will agree that the monad $\mu(\mathscr{E})$ is an external cone over ${}^{\circ}\mathbb{R}$ and, moreover, $\mu(\mathscr{E}) \cap {}^{\circ}E = 0$. In application, $\mathscr{E}$ is usually the filter of order-units of E. The relation of *infinite proximity* or *infinite closeness* between the members of E is introduced as follows:

$$e_1 \approx e_2 \leftrightarrow e_1 - e_2 \in \mu(\mathscr{E}) \wedge e_2 - e_1 \in \mu(\mathscr{E}).$$

Since

$$\bigcap_{\varepsilon \in {}^{\circ}\mathscr{E}} \partial_{\varepsilon} f(\overline{x}) = \bigcup_{\varepsilon \in \mu(\mathscr{E})} \partial_{\varepsilon} f(\overline{x});$$

therefore, the external set on both sides is the so-called *infinitesimal subdifferential* of f at $\overline{x}$. We denote this set by $Df(\overline{x})$. The elements of $Df(\overline{x})$ are *infinitesimal subgradients* of f at $\overline{x}$. If the zero oiperator is an infinitesimal subgradient of f at $\overline{x}$ then $\overline{x}$ is called an *infinitesimal minimum point* of f. We abstain from indicating $\mathscr{E}$ explicitly since this leads to no confusion.

THEOREM. *Let* $f_1 : X \times Y \to E \cup +\infty$ *and* $f_2 : Y \times Z \to E \cup +\infty$ *be convex operators. Suppose that the convolution* $f_2 \triangle f_1$ *is infinitesimally exact at some point* (x, y, z)*; i.e.,* $(f_2 \triangle f_1)(x, y) \approx f_1(x, y) + f_2(y, z)$*. If, moreover, the convex sets* $\mathrm{epi}(f_1, Z)$ *and* $\mathrm{epi}(X, f_2)$ *are in general position then*

$$D(f_2 \triangle f_1)(x, y) = Df_2(y, z) \circ Df_1(x, y).$$

Bibliography

[1] Fenchel W. (1953) *Convex Cones, Sets, and Functions.* Princeton: Princeton Univ. Press.

[2] Hörmander L. (1955) *Sur la fonction d'appui des ensembles convexes dans une espace lokalement convexe.* Ark. Mat., **3**:2, 180–186 [in French].

[3] Hörmander L. (1994) *Notions of Convexity.* Boston: Birkhäuser.

[4] Kutateladze S. S. and Rubinov A. M. (1972) *Minkowski duality and its applications.* Russian Math. Surveys, **27**:3, 137–191.

[5] Kutateladze S. S. and Rubinov A. M. (1976) *Minkowski Duality and Its Applications.* Novosibirsk: Nauka Publishers [in Russian].

[6] Singer I. (1997) *Abstract Convex Analysis.* New York: John Wiley & Sons.

[7] Pallaschke D. and Rolewicz S. (1998) *Foundations of Mathematical Optimization, Convex Analysis Without Linearity.* Dordrecht: Kluwer Academic Publishers.

[8] Rubinov A. M. (2000) *Abstract Convexity and Global Optimization.* Dordrecht: Kluwer Academic Publishers.

[9] Ioffe A. D. and Rubinov A. M. (2002) *Abstract convexity and nonsmooth analysis. Global aspects.* Adv. Math. Econom., **4**, 1–23.

[10] Fuchssteiner B. and Lusky W. (1981) *Convex Cones.* Amsterdam: North-Holland.

[11] Kusraev A. G. and Kutateladze S. S. (2007) *Subdifferential Calculus: Theory and Applications.* Moscow: Nauka Publishers [in Russian].

[12] Dilworth S. J., Howard R., and Roberts J. W. (2006) *A general theory of almost convex functions.* Trans. Amer. Math. Soc., **358**:8, 3413–3445.

Interval analysis for algorithms of idempotent and tropical mathematics[1]

Grigory L. Litvinov

The idempotent interval analysis appears to be best suited for treating problems with order-preserving transformations of input data [1, 2]. It gives exact interval solutions to optimization problems with interval uncertainties in input data without any conditions of smallness on uncertainty intervals. Our aim to generalize results presented in [1, 2] for a very

[1]The work has been supported by the joint RFBR/CNRS grant 05-01-02807 and by the RFBR grant 05-01-00824.

general case of arbitrary algoritms of idempotent mathematics (in particular, tropical mathematics) and algorithms over positive semirings (the semifield of all nonnegative real numbers with usual operations is a typical positive semiring). Algorithms of this type are generated by a collection of basic semiring/semifield operations, the well known star-operations $x \mapsto x^* = 1 \oplus x \oplus x^2 \oplus x^3 \oplus \dots$, and trivial operations.

THEOREM. *Every algorithm of idempotent mathematics (and every algorithm over positive semirings) has an interval version. The complexity of this interval version coincides with the complexity of the initial algorithm. The interval version of the algorithm gives exact interval estimates for the corresponding output data.*

See [1, 2] for some examples.

Note that for the traditional interval analysis the situation is opposite. For example, basic algorithms of the traditional linear algebra are plynomial but the corresponding interval versions are NP-hard and interval estimates are not exact.

Bibliography

[1] G.L. Litvinov and A.N. Sobolevskiĭ, *Exact interval solutions of the discrete Bellman equation and polynomial complexity of problems in interval idempotent linear algebra,* Doklady Mathematics, v. **62**, no. 2, 2000, p.199–201. E-print arXiv: math.LA/0101041.

[2] G.L. Litvinov and A.N. Sobolevskiĭ, *Idempotent interval analysis and optimization problems,* Reliable Computing, v. **7**, no. 5, 2001, p.353–377. E-print arXiv: math.SC/0101080.

Dequantization procedures related to the Maslov dequantization[1]

G.L. Litvinov and G.B. Shpiz

1. The Maslov dequantization

Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers. The well-known max-plus algebra $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ is defined by the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$.

The max-plus algebra can be treated as a result of the *Maslov dequantization* of the semifield $\mathbb{R}_+$ of all nonnegative numbers, see, e.g., [1,2]. The change of variables

$$x \mapsto u = h \log x, \tag{1.1}$$

where $h > 0$, defines a map $\Phi_h \colon \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$. Let the addition and multiplication operations be mapped from $\mathbb{R}_+$ to $\mathbb{R} \cup \{-\infty\}$ by Φ_h, i.e. let

$$u \oplus_h v = h \log(\exp(u/h) + \exp(v/h)), \quad u \odot v = u + v,$$
$$\mathbf{0} = -\infty = \Phi_h(0), \quad \mathbf{1} = 0 = \Phi_h(1).$$

It can easily be checked that $u \oplus_h v \to \max\{u, v\}$ as $h \to 0$. Thus we get the semifield $\mathbb{R}_{\max}$ (i.e. the max-plus algebra) with zero $\mathbf{0} = -\infty$ and unit $\mathbf{1} = 0$ as a result of this deformation of the algebraic structure in $\mathbb{R}_+$.

The semifield $\mathbb{R}_{\max}$ is a typical example of an *idempotent semiring*; this is a semiring with idempotent addition, i.e., $x \oplus x = x$ for arbitrary element x of this semiring, see, e.g., [3-5].

The analogy with quantization is obvious; the parameter h plays the role of the Planck constant [2]. The map $x \mapsto |x|$ and the Maslov dequantization for $\mathbb{R}_+$ give us a natural passage from the field $\mathbb{C}$ (or $\mathbb{R}$) to the max-plus algebra $\mathbb{R}_{\max}$. Following [4], *we will also call this passage the Maslov dequantization.* In fact the Maslov dequantization is the usual Schrödinger dequantization but for imaginary values of the Planck constant (see, e.g., [4]). The passage from numerical fields to the max-plus algebra $\mathbb{R}_{\max}$ (or similar semifields) in mathematical constructions and results generates the so called *tropical mathematics.* The so-called *idempotent dequantization* is a generalization of the Maslov dequantization; idempotent dequantization generates the so-called idempotent mathematics, see, e.g. [4] for details.

[1]This work has been supported by the RFBR grant 05-01-00824 and the joint RFBR/CNRS grant 05-01-02807.

2. The dequantization transform

This transform is defined in [6].

Let X be a topological space. For functions $f(x)$ defined on X we shall say that a certain property is valid *almost everywhere* (a.e.) if it is valid for all elements x of an open dense subset of X. Suppose X is $\mathbb{C}^n$ or $\mathbb{R}^n$; denote by $\mathbb{R}^n_+$ the set $x = \{(x_1, \dots, x_n) \in X \mid x_i \geq 0$ for $i = 1, 2, \dots, n$. For $x = (x_1, \dots, x_n) \in X$ we set $\exp(x) = (\exp(x_1), \dots, \exp(x_n))$; so if $x \in \mathbb{R}^n$, then $\exp(x) \in \mathbb{R}^n_+$.

Denote by $\mathcal{F}(\mathbb{C}^n)$ the set of all functions defined and continuous on an open dense subset $U \subset \mathbb{C}^n$ such that $U \supset \mathbb{R}^n_+$. It is clear that $\mathcal{F}(\mathbb{C}^n)$ is a ring (and an algebra over $\mathbb{C}$) with respect to the usual addition and multiplications of functions.

For $f \in \mathcal{F}(\mathbb{C}^n)$ let us define the function $\hat{f}_h$ by the following formula:

$$\hat{f}_h(x) = h \log |f(\exp(x/h))|, \tag{2.1}$$

where h is a (small) real positive parameter and $x \in \mathbb{R}^n$. Set

$$\hat{f}(x) = \lim_{h \to +0} \hat{f}_h(x), \tag{2.2}$$

if the right-hand part of (2.2) exists almost everywhere. We shall say that the function $\hat{f}(x)$ is a *dequantization* of the function $f(x)$ and the map $f(x) \mapsto \hat{f}(x)$ is a *dequantization transform*. By construction, $\hat{f}_h(x)$ and $\hat{f}(x)$ can be treated as functions taking their values in $\mathbb{R}_{\max}$. Note that in fact $\hat{f}_h(x)$ and $\hat{f}(x)$ depend on the restriction of f to $\mathbb{R}^n_+$ only; so in fact the dequantization transform is constructed for functions defined on $\mathbb{R}^n_+$ only. It is clear that the dequantization transform is generated by the Maslov dequantization and the map $x \mapsto |x|$. Of course, similar definitions can be given for functions defined on $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

Denote by $\partial \hat{f}$ the subdifferential of the function $\hat{f}$ at the origin.

It is well known that all the convex compact subsets in $\mathbb{R}^n$ form an idempotent semiring $\mathcal{S}$ with respect to the Minkowski operations: for $A, B \in \mathcal{S}$ the sum $A \oplus B$ is the convex hull of the union $A \cup B$; the product $A \odot B$ is defined in the following way: $A \odot B = \{ x \mid x = a + b$, where $a \in A, b \in B$. In fact $\mathcal{S}$ is an idempotent linear space over $\mathbb{R}_{\max}$ (see, e.g., [4]). Of course, the Newton polytopes in V form a subsemiring $\mathcal{N}$ in $\mathcal{S}$. If f, g are polynomials, then $\partial(\widehat{fg}) = \partial \hat{f} \odot \partial \widehat{g}$; moreover, if f and g are "in general position", then $\partial(\widehat{f+g}) = \partial \hat{f} \oplus \partial \widehat{g}$. For the semiring of all polynomials with nonnegative coefficients the dequantization transform is a homomorphism of this "traditional" semiring to the idempotent semiring $\mathcal{N}$.

THEOREM 2.1. *If f is a polynomial, then the subdifferential $\partial\hat{f}$ of $\hat{f}$ at the origin coincides with the Newton polytope of f. For the semiring of polynomials with nonnegative coefficients, the transform $f \mapsto \partial\hat{f}$ is a homomorphism of this semiring to the semiring of convex polytopes with respect to the well-known Minkowski operations.*

Using the dequantization transform it is possible to generalize this result to a wide class of functions and convex sets, see [6]. Another approach based on complex analysis is due to A. Rashkovskii, see, e.g., [7,8].

3. Dequantization of linear operators and semigroups of linear operators

The dequantization transform can be rewritten in the following form:

$$f \mapsto \hat{f}(x) = \lim_{h\to+0} h\log(|\, f(\exp(x/h)\,|) = \lim_{s\to+\infty} (1/s)\cdot\log(|\, f(\exp(sx)\,|),$$

where $x \in \mathbb{R}^n$, and h, $s = 1/h$ are real positive parameters.

Our aim is to apply the dequantization transform to matrix elements of operator semigroups generated by linear operators.

Suppose that S is a semigroup and $s \mapsto \pi_s$ is a linear representation of S in a complete (or quasicomplete) barreled locally convex space (by continuous operators). Denote by V' the dual space to V and by $\langle v', v\rangle$ the value of a functional $v \in V'$ on an element $v \in V$. If $s \mapsto \pi_{v',v}(s) = \langle v', \pi_s v\rangle$ is a matrix element of π, then its *dequantization* $\widehat{\pi}_{v',v}$ is defined by the formula:

$$\widehat{\pi}_{v',v} = \overline{\lim}_{s\to\infty}(1/s)\cdot\log(|\, \langle v', \pi_s v\rangle\,|).$$

We discuss the cases $S = \mathbb{R}_+$ or $S = \mathbb{Z}_+$. If v and v' are fixed, then $\widehat{\pi}_{v',v} \in S \cup \{\infty\}$.

PROPOSITION 3.1. *Let A be a linear operator in V, $\pi_s = exp(sA)$, and $\dim V < \infty$. Then the set of all dequantizations $\{\widehat{\pi}_{v',v}\}$ coincides with the set of real parts of all eigenvalues of A.*

There are generalizations of this result for the case $\dim V = \infty$.

Suppose that for every $v' \in V'$ there exists a number $r > 0$ such that the set $\{r^{-s}\cdot(|\, \langle v', \pi_s v\rangle\,|), s \in S\}$ is bounded for every $v \in V$ and the number r does not depend on $v' \in V'$. Then the representation π is called *exponential*. Note, that if V is a Banach space and π is weakly continuous, then π is exponential. In the general case the *spectral radius* ρ_π of π is

defined by the formula:

$$\rho_\pi = \inf\{r \mid r^{-s}\pi_s v \to 0 \text{ weakly for every } v \in V \text{ as } s \to +\infty\}$$

PROPOSITION 3.2. *If A is a bounded linear operator in a Banach space V, $S = \mathbb{Z}_+$, $\pi = A^s$, then $\rho_\pi = \rho(A) = \lim_{s\to\infty} \|A^s\|^{1/s}$, i.e. ρ_π is the traditional spectral radius of A.*

THEOREM 3.1. *If π is exponential, then*

$$\log \rho_\pi = \sup\{\widehat{\pi}_{v',v} \mid v' \in V', v \in V\}.$$

THEOREM 3.2. *Suppose that A is a compact operator and $\pi_s = A^s$, where $s \in S = \mathbb{Z}_+$. Then the set $\{\widehat{\pi}_{v',v}\}$ of all dequantizations of π coincides with the set of all numbers of the form $\log(|\ \lambda\ |)$, where λ runs the spectrum of A.*

4. Dequantization of set functions on metric spaces

Let M be a metric space, S its arbitrary subset with a compact closure. It is well-known that a Euclidean d-dimensional ball B_ρ of radius ρ has volume

$$\mathrm{vol}_d(B_\rho) = \frac{\Gamma(1/2)^d}{\Gamma(1+d/2)}\rho^d,$$

where d is a natural parameter. By means of this formula it is possible to define a volume of B_ρ for any *real* d [9]. Cover S by a finite number of balls of radii ρ_m. Set

$$v_d(S) := \lim_{\rho\to 0} \inf_{\rho_m<\rho} \sum_m \mathrm{vol}_d(B_{\rho_m}).$$

Then there exists a number D such that $v_d(S) = 0$ for $d > D$ and $v_d(S) = \infty$ for $d < D$. This number D is called the *Hausdorff-Besicovich dimension* (or *HB-dimension*) of S [9]. Note that a set of non-integral HB-dimension is called a fractal in the sense of B. Mandelbrot.

Denote by $\mathcal{N}_\rho(S)$ the minimal number of balls of radius ρ covering S. Then

$$D(S) = \underline{\lim}_{\rho\to+0} \log_\rho(\mathcal{N}_\rho(S)^{-1}),$$

where $D(S)$ is the HB-dimension of S. Set $\rho = e^{-s}$, then

$$D(S) = \underline{\lim}_{s\to+\infty}(1/s)\cdot\log\mathcal{N}_{exp(-s)}(S).$$

So the HB-dimension $D(S)$ can be treated as a result of a dequantization of the set function $\mathcal{N}_\rho(S)$.

Let μ be a set function on M (e.g., a probability measure) and suppose that $\mu(B_\rho) < \infty$ for every ball B_ρ. Let $B_{x,\rho}$ be a ball of radius ρ having the point $x \in M$ as its center. Then define $\mu_x(\rho) := \mu(B_{x,\rho})$ and let

$$D_{x,\mu} := \underline{\lim}_{s\to+\infty} - (1/s) \cdot \log(|\mu_x(e^{-s})|).$$

This number could be treated as a dimension of M at the point x with respect to the set function μ. There are many dequantization procedures of this type in different mathematical areas. In particular, V.P. Maslov's negative dimension [10] can be treated similarly.

5. Dequantization of the Fourier-Laplace transform

It was noticed by V.P. Maslov (see, e.g., [1-4]) that the Legendre (or Legendre-Fenchel) transform can be treated as an idempotent (or tropical) version of the Fourier-Laplace transform. It seems to be interesting to note that the Legendre transform can be constructed from the Fourier-Legendre transform directly by means of the Maslov dequantization.

6. Dequantization of geometry

An idempotent version of real algebraic geometry was discovered in the report of O. Viro for the Barcelona Congress [11]. Starting from the idempotent correspondence principle [2], O. Viro constructed a piecewise-linear geometry of polyhedra of a special kind in finite dimensional Euclidean spaces as a result of the Maslov dequantization of real algebraic geometry. He indicated important applications in real algebraic geometry (e.g., in the framework of Hilbert's 16th problems) and relations to complex algebraic geometry and amoebas in the sense of I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. Then complex algebraic geometry was dequantized by G. Mikhalkin and the result turned out to be the same; now the new geometry is called *tropical algebraic geometry*. In particular, tropical varieties are results of a dequantization procedure (generated by the Maslov dequantization) applied to algebraic varieties. There are many applications, see, e.g., [11-13,5].

7. Remark

It would be nice to find new dequantization procedures related to the Maslov dequantization.

Bibliography

[1] V. P. Maslov,*On a new superposition principle for optimization problems*, Uspekhi Mat. Nauk, [Russian Math. Surveys], **42**, no. 3 (1987), 39–48.

[2] G. L. Litvinov and V. P. Maslov, *Correspondence principle for idempotent calculus and some computer applications*, (IHES/M/95/33), Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, 1995. Also: [3], p. 420 – 443, and arXiv:math.GM/0101021.

[3] J. Gunawardena (Ed.), *Idempotency*, Publ. of the Newton Institute, Vol. **11**, Cambridge University Press, Cambridge, 1998.

[4] G.L. Litvinov, *The Maslov dequantization, idempotent and tropical mathematics: a brief introduction,* Journal of Mathematical Sciences, **140** no. 3 (2007) 426–444. E-print: arXiv:math.GM/0507014, 2005 (http://arXiv.org).

[5] G. L. Litvinov and V. P. Maslov (Eds.), *Idempotent Mathematics and Mathematical Physics,* Contemporary Mathematics, Vol. **377**, AMS, Providence, RI, 2005.

[6] G. L. Litvinov and G. B. Shpiz, *The dequantization transform and generalized Newton polytopes.* — In [5], p. 181–186.

[7] A. Rashkovskii, *Newton numbers and residual measures of plurisubharmonic functions,* Ann. Polon. Math. **75** no. 3 (2000) 213–231.

[8] A. Rashkovskii, *Tropical analysis on plurisubharmonic singularities.* – In this volume (part 2).

[9] Yu.I. Manin, *The notion of dimension in geometry and algebra.* E-print arXiv:math.AG/0502016, 2005.

[10] V.P. Maslov, *A general notion of topological spaces of negative dimension and quantization of their densities.* Math. Notes (Mat. Zametki), **81** no. 1 (2007) 157–160 (in Russian).

[11] O. Viro, *Dequantization of real algebraic geometry on a logarithmic paper.* — In: 3rd European Congress of Mathematics, Barcelona, 2000, vol. I , Birkhäuser, Basel, 2001, p. 135–146. Also arXiv:math.AG/0005163.

[12] G. Mikhalkin, *Tropical geometry and its applications.* Proceedings of the Madrid ICM, 2006. Also arXiv:math.AG/06011041.

[13] I. Itenberg, V. Kharlamov, and E. Shustin, *Tropical geometry and enumeration of real rational curves.* – In this volume.