IDEMPOTENT ELEMENTS IN A BERNSTEIN ALGEBRA

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Introduction

A finite-dimensional commutative algebra A over a field K is called a Bernstein algebra if there exists a non-trivial homomorphism $\omega: A \to K$ (baric algebra) such that the identity $(x^2)^2 = \omega(x)^2 x^2$ holds in A (see [7]).

The origin of Bernstein algebras lies in genetics (see [**2**, **8**]). Holgate (in [**2**]) was the first to translate the problem into the language of non-associative algebras. Information about algebraic properties of Bernstein algebras, as well as their possible genetic interpretations, can be found in [**10**, Chapter 9B; **11**; **12**; **3**; **1**].

The existence of idempotent elements, that is elements e, $e \neq 0$, such that $e^2 = e$, is of interest in the study of the structure of a non-associative algebra. From the biological aspect the existence of such elements is also interesting, because the equilibria of a population which can be described by an algebra correspond to idempotent elements of this algebra.

The algebras occurring in applications usually do contain an idempotent. This occurs in Bernstein algebras (see [**10**]). With respect to an idempotent $e \in A$ (whose existence is guaranteed), A splits into the direct sum $A = \langle e \rangle \dotplus U \dotplus Z$, where

$$U = \{ey : y \in \operatorname{Ker} \omega\}, \qquad Z = \{x \in A : ex = 0\},$$

and the set of idempotents of A is given by $\{e + u + u^2 : u \in U\}$. Ljubic in [**8**] associates with each Bernstein algebra a pair of integers, the type $(r+1, d)$ of the algebra, where $r = \dim U$ and $d = \dim Z$, and he presents a classification of Bernstein algebras via their type. Another decomposition for a Bernstein algebra A with respect to one idempotent e is given by Holgate [**7**].

In this paper we intend to study the influence of the idempotent elements of a Bernstein algebra in another direction. In previous papers [**4**, **5**] we have studied an order relation in Jordan rings without non-zero nilpotent elements, in the same direction that Abian had done for reduced associative rings and Myung and Jimenez for the alternative case. When this order relation is considered only over idempotent elements we obtain the usual boolean relation: two idempotent elements e, f satisfy $e \leqslant f$ if and only if $ef = e$. We do not have generally a boolean algebra, only an ordered set.

It is natural to think about this relation for Bernstein algebras. However, it is trivial in this case, that is, if e and f satisfy $e \leqslant f$, then $e = f$. So we cannot get information about the Bernstein algebra by studying this relation between idempotents. So we define a new relation between idempotent elements that will be an equivalence relation and try to get information about the whole algebra by using the information about this relation.

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First we need to improve the characterization of Jordan–Bernstein algebras given in [**11**].

1. *Jordan–Bernstein algebras*

DEFINITION 1. Let A be a commutative K-algebra, $\operatorname{char} K \neq 2$, and $\omega: A \to K$ a non-trivial homomorphism ('weight homomorphism'). We call A a Bernstein algebra if the identity $(x^2)^2 = \omega(x)^2 x^2$ holds in A for every x in A.

OBSERVATIONS. 1. The weight homomorphism $\omega: A \to K$ in a Bernstein algebra is uniquely determined.

2. Every Bernstein algebra has at least one idempotent element $0 \neq e = e^2$.

3. The algebra A splits into a direct sum of vector subspaces

$$A = K(e) \dotplus U \dotplus Z,$$

with $U = \{ey : y \in \operatorname{Ker}\omega\} = \{x : ex = \frac{1}{2}x\}$ and $Z = \{z \in A : ez = 0\}$. Then $\operatorname{Ker}\omega = U \dotplus Z$ and if $x = \alpha e + u + z$, $\alpha \in K$, $u \in U$, $z \in Z$, then $\omega(x) = \alpha$.

In [**11**] the following properties are proved:

(1) $U^2 \subseteq Z$, $UZ \subseteq U$, $Z^2 \subseteq U$, $Z^2 U = 0$;

(2) $u^3 = 0$ for every u in U;

(3) $u(uz) = uz^2 = u^2(uz) = z^2(uz) = (uz)^2 = u^2 z^2 = 0$ for every u in U and z in Z.

Also in [**11**], the following result [**11**, Theorem 3] is proved.

THEOREM. *Let $A = K(e) \dotplus U \dotplus Z$ be a Bernstein algebra over K. Then A is a Jordan algebra if and only if $Z^2 = 0$ and the following equations are satisfied for every u, u_1 in U and z, z_1, z_2 in Z:*

(i) $(uz_1)z_2 + (uz_2)z_1 = 0$,

(ii) $(u_1^2 u_2)z + 2((u_1 z)u_2)u_1 = 0$,

(iii) $((uz_1)z_2)z_1 = 0$,

(iv) $(u_1^2 u_2)u_1 = 0$,

(v) $((uz_1)z_2)u = 0$.

This theorem can be improved to the following.

THEOREM 1. *Let $A = K(e) \dotplus U \dotplus Z$ be a Bernstein algebra over K. Then A is a Jordan algebra if and only if $z^2 = 0$ and $(uz)z = 0$ for every u in U and z in Z.*

Proof. Notice first that if $\operatorname{char} K \neq 2$, the relation (i) is equivalent, by linearizing, to the relation $(uz)z = 0$ for every u in U and z in Z.

One implication is clear; if A is a Jordan algebra, then $z^2 = 0$ and $(uz)z = 0$ for every u in U and z in Z.

Reciprocally, let us suppose that $z^2 = 0 = (uz)z$ for every u in U and z in Z. Then if $x = \alpha e + u + z$, $y = \beta e + u' + z'$ are two elements in A ($\alpha, \beta \in K$, $u, u' \in U$ and $z, z' \in Z$) we can see that

$$x^2(yx) = \alpha^3\beta e + (\tfrac{1}{4}\alpha^3 u' + \tfrac{3}{4}\alpha^2\beta u + \tfrac{1}{2}\alpha^2 zu' + \tfrac{1}{2}\alpha^2 uz' + \alpha\beta uz) + (\tfrac{1}{2}\alpha^2 uu' + \tfrac{1}{2}\alpha\beta u^2) = (x^2 y)x$$

and consequently A is a Jordan algebra.

Note. To prove that $x^2(yx) = (x^2y)x$ we have used

(a) $u_1(u_2 z) + u_2(u_1 z) = 0$, the linearized form of $u(uz) = 0$,

(b) $(u_1 u_2)u_3 + (u_2 u_3)u_1 + (u_3 u_1)u_2 = 0$, the linearized form of $u^3 = 0$,

(c) $u^2u'+2(uu')u=0$, the particular case of (b) with $u_1=u_2=u$ and $u_3=u'$,
(d) $(uz_1)z_2+(uz_2)z_1=0$, the linearized form of $(uz)z=0$,
(e) $u^2(u'z)=-(u'u^2)z=2((u'u)u)z=-2(u'u)(uz)$ obtained by using (c) and (d), and where u,u',u_1,u_2,u_3 are elements of U *and* z,z',z_1,z_2 are elements of Z.

2. *Idempotent elements*

Let $A=K(e)\dot{+}U\dot{+}Z$ be a Bernstein algebra as before. It is known that the set of idempotent elements of A (which we will denote by $I(A)$) is given by $I(A)=\{e+u+u^2:u\in U\}$.

It is clear that

(a) A has no identity element, except in the trivial case when $A=K(e)$.

(b) A has only one idempotent $e\neq 0$ if and only if $A=K(e)\dot{+}Z$; that is, $ez=0$ for every z in $\operatorname{Ker}\omega$. Consequently $\operatorname{Ker}\omega$ is a zero algebra.

As we have already said, the aim of this paper is to study the set of idempotent elements $I(A)$ and to transfer the information about $I(A)$ to the whole of A. First, we shall consider the relation $\leqslant$ as in the associative case, given by $e\leqslant f$ if $ef=e$.

But if $f=e+u+u^2$, $u\in U$, and $e=ef=e+\frac{1}{2}u$, then $u=0$; that is, $e\leqslant f$ if and only if $e=f$. So we cannot get information about A by using information about this relation $\leqslant$. (In some sense, this result is not surprising, because a Bernstein algebra is close to a nil algebra and the order relation has a good behaviour in the reduced case.)

Now we are going to study the cases in which the product of two idempotent elements is again an idempotent element.

If $e,f=e+u+u^2$, $u\in U_e$, are two idempotent elements, then $ef=e+\frac{1}{2}u$ is an idempotent element if and only if $u^2=0$; that is, $f=e+u$ with $u^2=0$ and so $uf=ue=\frac{1}{2}u$, $u\in U_e\cap U_f$, where we write

$$U_e=\{u\in\operatorname{Ker}\omega:ue=\tfrac{1}{2}u\}\quad\text{and}\quad U_f=\{x\in\operatorname{Ker}\omega:xf=\tfrac{1}{2}x\}.$$

So we can define a relation R' in $I(A)$ as follows.

DEFINITION 2. Two idempotent elements $e,f\in I(A)$ are related by R', $e\,R'f$, if ef is idempotent (that is, $f=e+u$, $u^2=0$, $u\in U_e$).

It is clear that this relation is reflexive and symmetric, but is not transitive (in general) as the following example shows.

Let A be the 4-dimensional algebra with basis $\{e,u,u',uu'\}$ and product given by

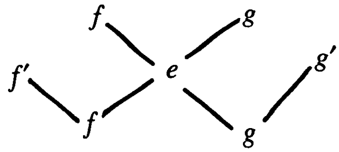
$$ee=e,\quad eu=\tfrac{1}{2}u,\quad eu'=\tfrac{1}{2}u',\quad e(uu')=0,$$
$$uu=u'u'=0=u(uu')=u'(uu')=(uu')(uu').$$

This algebra is a Bernstein algebra (it is even a Jordan–Bernstein algebra), $A=K(e)\dot{+}U\dot{+}Z$, where $U=K(u,u')$ and $Z=K(uu')$. It is easy to see that $e\,R'\,e+u$ and $e\,R'\,e+u'$, but $e+u\not\!R'\,e+u'$ since $(e+u)(e+u')=e+\frac{1}{2}(u+u')+uu'$ is not idempotent because $(\frac{1}{2}(u+u'))^2=\frac{1}{2}uu'\neq uu'$.

Note. If $e,f\in I(A)$ and $e\,R'f$, then $f=e+u$ with $u^2=0$ and $eu=\frac{1}{2}u$. Consequently the idempotent element $g=ef=e+\frac{1}{2}u$ satisfies $e\,R'g$ and $f\,R'g$. So if we consider the relation $\bar{R}$ given by $e\,\bar{R}f$ if there is an element $g\in I(A)$ with $e=fg$, this relation $\bar{R}$ is the same as the above relation R'. In fact, if $e\,R'f$, then $f=e+u$ with $eu=\frac{1}{2}u$ and

$u^2 = 0$. Then the element $g = e-u$ is also idempotent and $e = fg$. Reciprocally, if $e\,\bar{R}f$, then there is $g \in I(A)$ with $e = fg$. But then $f R' g$ and the above comment ensures that $e R' f$ and $e R' g$.

Consequently the relation $\bar{R}$ is symmetric in $I(A)$; that is, if e and f are idempotent elements and there exists another idempotent element g such that $e = fg$, then there are idempotent elements f' and g' such that $g = eg'$ and $f = ef'$.



3. *An equivalence relation*

The aim now is to modify somehow the above definition to get an equivalence relation:

Definition 3. Let $e,f \in I(A)$. Then $e R f$ if $f = e+u$, $u \in U_e$ and $uU_e = 0$.

In order to see that R is an equivalence relation we shall give first another, equivalent, definition.

Proposition. *If $e,f \in I(A)$, then $e R f$ if and only if $U_e = U_f$.*

Proof. Suppose that $U_e = U_f$ and $f = e+u+u^2$ as usual. Then $ue = uf = \frac{1}{2}u$ by hypothesis; that is,

$$\tfrac{1}{2}u = uf = ue+u^2+u^3 = \tfrac{1}{2}u+u^2$$

and consequently $u^2 = 0$, $f = e+u$. If u' is an element of U_e, then $u' \in U_f$. So $\frac{1}{2}u' = u'f = u'e+u'u = \frac{1}{2}u'+u'u$. Then $u'u = 0$; that is, $uU_e = 0$ and $e R f$.

Reciprocally, let $e R f$, that is, $f = e+u$, with $uU_e = 0$. Then for every u' in U_e, $u'f = u'e+u'u = u'e = \frac{1}{2}u'$; that is, $u' \in U_f$ and so $U_e \subseteq U_f$. Let $x \in U_f$, $x = \bar{u}+\bar{z}$, $\bar{u} \in U_e$, $\bar{z} \in Z_e$. Then

$$\tfrac{1}{2}x = xf = xe+xu = \bar{u}e+\bar{z}e+\bar{u}u+\bar{z}u = \tfrac{1}{2}\bar{u}+0+0+\bar{z}u.$$

So $\frac{1}{2}\bar{u}+\frac{1}{2}\bar{z} = \frac{1}{2}\bar{u}+\bar{z}u$. This implies that $\frac{1}{2}\bar{z} = \bar{z}u \in Z_e \cap U_e = 0$. Then $x = \bar{u} \in U_e$ and $U_f \subseteq U_e$.

Note. It is known that in the finite-dimensional case, $\dim U_e = \dim U_f$ for any pair of idempotent elements e,f. So $U_e \subseteq U_f$ implies that $U_e = U_f$. However, the above result is true without any restriction over $\dim_K A$.

Corollary. *The above relation R is an equivalence relation.*

Suppose that there is only one equivalence class; that is, $e R f$ for every two idempotent elements $e,f \in I(A)$. Then for every u in U_e, $e R e+u+u^2$ and consequently $uU_e = 0$; that is, $U_e^2 = 0$. Clearly the converse is true.

Theorem 2. *There is only one equivalence class of idempotent elements by R if and only if $U_e^2 = 0$ for every idempotent e and this is the case if and only if U_e is a subalgebra of A and if and only if U_e is an ideal of A.*

Note. The above result implies that when $U_e^2 = 0$ for one idempotent element, then $U_f^2 = 0$ for every idempotent element. As we shall see later, this is not the case for Z_e.

Suppose now that there are as many equivalence classes as idempotent elements; that is, each equivalence class has only one element.

In this case, if $z \in Z_e$, then $u = z^2$ satisfies that $uU_e = 0$, so $e\,R\,e + z^2$ for every z in Z_e. By the hypothesis $e = e + z^2$; that is $z^2 = 0$ for every z in Z_e. Also, for u in U_e and z in Z_e the element $(uz)z$ satisfies that $((uz)z)U_e = 0$ $(u'(z(uz)) = -(uz)(u'z) = 0)$ for all $u' \in U_e$. So again $e\,R\,e + z(zu)$ for every u in U and z in Z. With the present hypothesis $(uz)z = 0$ for every u in U_e and z in Z_e.

Consequently we have proved the following.

THEOREM 3. *If each equivalence class has only one element, then A is a Jordan–Bernstein algebra.*

The converse is not true as it is shown by any Jordan–Bernstein algebra with $U \neq 0$ and $U^2 = 0$.

If we consider an idempotent element e, then we obtain a decomposition $A = K(e) \dotplus U_e \dotplus Z_e$, where

$$U_e = \{u \in \operatorname{Ker}\omega : ue = \tfrac{1}{2}u\} \quad \text{and} \quad Z_e = \{z \in \operatorname{Ker}\omega : ze = 0\}.$$

For another idempotent element f the decomposition is $A = K(f) \dotplus U_f \dotplus Z_f$, where

$$U_f = \{u' \in \operatorname{Ker}\omega : u'f = \tfrac{1}{2}u'\}, \qquad Z_f = \{z' \in \operatorname{Ker}\omega : z'f = 0\}.$$

If $f = e + \bar{u} + \bar{u}^2$ with $\bar{u} \in U_e$, then the relation between U_e and U_f and Z_e and Z_f is given by:

$$U_f = \{u + 2u\bar{u} : u \in U_e\}, \qquad Z_f = \{-2(z\bar{u} + z\bar{u}^2) + z : z \in Z_e\}.$$

We shall write

$$U_e^0 = \{u \in U_e : uU_e = 0\} \quad \text{and} \quad U_f^0 = \{u' \in U_f : u'U_f = 0\}.$$

By the definition of R, the equivalence class of e is given by $[e] = \{e + u : u \in U_e^0\}$. Similarly $[f] = \{f + u' : u' \in U_f^0\}$.

LEMMA. (i) *$U_e^0 = U_f^0$ for every pair of idempotent elements e, f.*

(ii) *Two equivalence classes have the same cardinal.*

Proof. It suffices to prove (i), because if $U_e^0 = U_f^0$ we can define clearly a bijection between $[e]$ and $[f]$.

By symmetry it suffices to prove that $U_e^0 \subseteq U_f^0$.

Let $u^0 \in U_e^0$, $u' \in U_f$. Then there exists $u \in U_e$ with $u' = u + 2u\bar{u}$, using the same notation as above. First notice that $u^0 = u^0 + 2u^0\bar{u} \in U_f$ and

$$u^0u' = u^0u + 2u^0(u\bar{u}) = 0 + 2u^0(u\bar{u}).$$

But $u^0(u\bar{u}) + (u^0u)\bar{u} + (u^0\bar{u})u = 0$ and $u^0u = u^0\bar{u} = 0$. So $u^0(u\bar{u}) = 0$ and $u^0u' = 0$; that is, $u^0 \in U_f^0$.

If we call $I = U^0 = \{u \in U_e : uU_e = 0\}$ for some idempotent element e, the above result asserts that U^0 is independent of the idempotent e; that is, $U^0 = U_e^0$ for every e in $I(A)$.

I is an ideal of A (for all $z \in Z_e$, $u^0 \in U^0 = I$, $u \in U_e$ we have $(u^0 z)u = -(uz)u^0 = 0$) satisfying $I^2 = 0$, $AI = I$. Clearly I is the biggest ideal of A contained in U_e (for every idempotent element e).

COROLLARY. *The following affirmations are equivalent.*

(i) *Every equivalence class of R has only one idempotent element.*
(ii) $U^0 = 0$.
(iii) *There is one idempotent element e with $[e] = \{e\}$.*
(iv) *There is one idempotent element e such that A has no non-zero ideals I with $I \subseteq U_e$.*

THEOREM 4. *Let A be a Bernstein algebra. Then A is Jordan–Bernstein if and only if $Z_e^2 = 0$ for every idempotent element e.*

Proof. Let $e \in I(A)$, and U_e and Z_e be as usual.

If $f \in I(A)$, then there is an element $\bar{u} \in U_e$ such that $f = e + \bar{u} + \bar{u}^2$ and $Z_f = \{-2(z\bar{u} + z\bar{u}^2) + z : z \in Z_e\}$.

Clearly if A is Jordan then $Z_f^2 = 0$. (The proof that $Z_e^2 = 0$ is independent of the idempotent element. Also it is clear using the expression of Z_f and the characterization of Jordan algebras given in Theorem 1.)

Reciprocally, suppose that $Z_e^2 = 0$ for every $e \in I(A)$. Let $e \in I(A)$ and $u \in U_e$ (arbitrary). Then $f = e + u + u^2 \in I(A)$ and by hypothesis $Z_f^2 = 0$. But $zu^2 \in Z_e^2 = 0$, for every $z \in Z_e$, and so $Z_f = \{-2zu + z : z \in Z_e\}$. Consequently

$$(-2uz + z)(-2uz + z) = 4(zu)(zu) - 4z(zu) + zz = 0 - 4z(zu) + 0 = 0$$

for every $z \in Z_e$. Then $z(zu) = 0$ for all $u \in U_e$ and $z \in Z_e$ and A is a Jordan–Bernstein algebra.

Note. There are Bernstein algebras (not Jordan) with $Z_e^2 = 0$ for one idempotent element e, but $Z_f^2 \neq 0$ for other $f \in I(A)$. It suffices to consider the 3-dimensional algebra with basis $\{e, u, z\}$ and products given by

$$ee = e, \quad eu = \tfrac{1}{2}u, \quad ez = 0, \quad uu = zz = 0, \quad uz = u.$$

Clearly A is a Bernstein algebra, $U_e = K(u)$, $Z_e = K(z)$ and $Z_e^2 = 0$. If we consider $f = e + u$, $f \in I(A)$ (even $f R e$) and $Z_f = \{-2z'u + z' : z' \in Z_e\} = K(z - 2u)$. But $(z-2u)(z-2u) = -4u \neq 0$ and so $Z_f^2 \neq 0$.

COROLLARY. *Let A be a Jordan–Bernstein algebra and $e \in I(A)$. Then $U_e Z_e = 0$ if and only if $Z_e = Z_f$ for all $f \in I(A)$.*

Proof. Suppose that $U_e Z_e = 0$. Let f be an element in $I(A)$. Then there is $u \in U_e$ such that $f = e + u + u^2$. By the hypothesis $uz \in U_e Z_e = 0$, $u^2 z \in Z_e^2 = 0$ for all $z \in Z_e$. So

$$Z_f = \{-2(zu + zu^2) + z : z \in Z_e\} = Z_e.$$

Reciprocally, if $Z_e = Z_f$ for all $f \in I(A)$, this means that

$$Z_{e+u+u^2} = \{-2uz + z : z \in Z_e\} = Z_e$$

for all $u \in U_e$. So $uz = 0$ for all $z \in Z_e$; that is, $U_e Z_e = 0$.

COROLLARY. *Let A be a Jordan–Bernstein algebra. Then if $U_e Z_e = 0$ for one idempotent element e, then $U_f Z_f = 0$ for every idempotent element f.*

Proof. If $U_e Z_e = 0$ for one idempotent element e, Corollary 1 implies that $Z_e = Z_f$ for every idempotent element f. If $f \in I(A)$, then $Z_f = Z_g$ for all $g \in I(A)$. By Corollary 1 again this implies that $U_f Z_f = 0$.

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