From assumption we get immediately $i \circ \varphi$ is null homotopic in BSO_{k+1}. But BSO_k is a bundle over BSO_{k+1} with fibre S^k, therefore by the lifting homotopy property, φ is homotopic to a map

$$\varphi'\colon M_1^m \to S^k \subseteq BSO_k,$$

and hence φ restricted to the $(k-1)$-skeleton of M_1 is null-homotopic and hence $\sigma_i = 0$ for $i \leqslant k-1$.

COROLLARY. *If M_1^m and M_2^m are two 1-connected closed manifolds which are tangentially equivalent then*:

(1) *If* $5 \leqslant m \leqslant 15$, *then* $M_1^m \times D^k \equiv M_2^m \times D^k$, $k \geqslant \frac{1}{2}(m+4)$,

(2) $M_1^m \times D^k \equiv M_2^m \times D^k$, $k \geqslant m-2$, $m \geqslant 5$.

Proof. The corollary follows from the theorem, Proposition 3 and the remark.

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Idempotent generated algebras and Boolean pairs

by

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Abstract. Let R be a commutative ring with unit. In this paper we introduce the category of Boolean R-pairs and obtain a full faithful functor $\mathcal{B}$ from the category of idempotent generated R-algebras to the category of Boolean R-pairs. We also determine an adjoint for $\mathcal{B}$. Results are given to point out some applications of these functors.

0. Introduction. If A is a commutative algebra over a commutative ring R, then it is well-known that the set of idempotents of A can be made into a Boolean ring. However this functor is not full. We consider the category, R-IGA, of commutative idempotent generated R-algebras and obtain a full faithful functor $\mathcal{B}$ on this category to the category of Boolean R-pairs (defined below) which contains full subcategories isomorphic to the category of Boolean rings. This result is then applied to the recent problem of finding categories in which the objects are determined (up to isomorphism) by monoids of endomorphisms. For related results on this problem see [3], [4], [5], [7], [8], [9], [10].

For the particular case of torsion free idempotent generated rings, George M. Bergman (see [1]) indicates a left adjoint for the functor $\mathcal{B}$. That is, given any Boolean ring B he constructs an idempotent generated ring $Z[B]$ with torsion free additive group such that the Boolean ring of idempotents of $Z[B]$ is isomorphic to B. Here we present the construction of such an adjoint for the category R-IGA of commutative idempotent generated R-algebras. Upon restricting to a certain subcategory of pairs, we obtain an equivalence with R-IGA which contains results of McCrea [5] and Stringall [10] as special cases.

Conventions. In this paper, all rings will be associative, commutative, with unit and all algebras will be unitary. For an R-algebra A, let

$$\mathrm{End}_R A = \{f\colon A \to A \mid (a+b)f = af+bf,\ (ab)f = afbf,\ (raf) = r(af),\ a, b \in A,\ r \in R\}.$$

We give a short outline of the paper. In Section 1, we show that the Boolean ring of any R-algebra is determined by $\mathrm{End}_R A$. In Section 2, the category of Boolean R-pairs is defined and the functor $\mathcal{B}$ is constructed. Applications of these results give the results of McCrea [5], Smith and Luh [8], and Stringall [10]. In Section 3, the left adjoint of $\mathcal{B}$ is constructed and in Section 4 an equivalence between R-IGA and a certain subcategory of pairs is obtained which generalizes the work of McCrea [5] and Stringall [10].

1. $\mathrm{End}_R A$ determines A. Let $\mathscr{B}(A)$ denote the set of idempotents of the R-algebra $A \equiv \langle A, +, \cdot \rangle$. (Recall that A is a commutative ring.) For e, f in $\mathscr{B}(A)$, if we define $e \oplus f = e+f-2ef$, then $\mathscr{B}(A) \equiv \langle \mathscr{B}(A), \oplus, \cdot \rangle$ is a Boolean ring. We now show that $\mathrm{End}_R A$ determines $\mathscr{B}(A)$ in the sense that whenever $\mathrm{End}_R A_1$ and $\mathrm{End}_R A_2$ are semigroup isomorphic, $\mathscr{B}(A_1)$ and $\mathscr{B}(A_2)$ are ring isomorphic.

Let $E = \{\varphi \in \mathrm{End}_R A | \varphi^2 = \varphi\}$. Partially order E by $\varphi \leqslant \psi$ if $\varphi\psi = \psi\varphi = \varphi$. For $e \in \mathscr{B}(A)$, define $\varphi_e\colon A \to A$ by $a\varphi_e = ae$. Then $e \to \varphi_e$ is a monomorphism and hence an order preserving map of $\mathscr{B}(A)$ into E. Clearly, φ_0 and φ_1 are respectively the minimum and maximum elements of E. For $\varphi \in E$ let $C(\varphi)$ denote $\{\psi \in E | \psi\varphi = \varphi\psi\}$.

LEMMA 1.1. *For $\varphi \in E$ the following are equivalent*:

(1) *there exists $e \in \mathscr{B}(A)$ such that $\varphi = \varphi_e$*;

(2) *there exists $\psi \in E$ such that $C(\varphi) = C(\psi)$, $\varphi_0 = \mathrm{g.l.b.}\{\varphi, \psi\}$ in E and $\varphi_1 = \mathrm{l.u.b.}\{\varphi, \psi\}$ in E.*

Proof. Suppose (1) holds. It is easily verified that $\varphi_{1-e} = \psi$ satisfies (2). To see that (2) implies (1), let $e = 1\varphi$ and $f = 1\psi$. From the fact that $C(\varphi) = C(\psi)$ we find $\psi\varphi \leqslant \varphi$ and $\psi\varphi \leqslant \psi$ which in turn gives $\psi\varphi = \varphi\psi = \varphi_0$. From this we obtain $\varphi, \psi \leqslant \varphi+\psi$ which implies that $\varphi+\psi = \varphi_1$. Hence $e+f = 1$. It is easily verified that $\varphi+\varphi_f$ is in E and $\varphi+\varphi_f \geqslant \varphi, \psi$. Thus $\varphi+\varphi_f = \varphi_1$ and $\varphi = \varphi_e$.

If $\Phi\colon \mathrm{End}_R A_1 \to \mathrm{End}_R A_2$ is a semigroup isomorphism and $H_i = \{\varphi_e | e \in \mathscr{B}(A_i)\}$ $i = 1, 2$, then we see from the above characterization of the H_i that the restriction of Φ to H_1 is a semigroup isomorphism onto H_2. Thus $\mathscr{B}(A_1)$ and $\mathscr{B}(A_2)$ are semigroup isomorphic and hence are ring isomorphic (see [4]).

THEOREM 1.2. *For R-algebras A_1 and A_2, if $\mathrm{End}_R A_1 \cong \mathrm{End}_R A_2$ then $\mathscr{B}(A_1) \cong \mathscr{B}(A_2)$.*

The converse of this theorem is not true. In fact, for $A_1 = Z_2 \times Z_2$ and $A_2 = Z_3 \times Z_2$ we have $\mathscr{B}(A_1) \cong \mathscr{B}(A_2)$ but $\mathrm{End}_Z A_1 \ncong \mathrm{End}_Z A_2$. Moreover, as is well-known, $\mathscr{B}(A_1) \cong \mathscr{B}(A_2)$ does not in general imply $A_1 \cong A_2$. In the next section, we consider this problem further and do find classes $\mathscr{C}$ of algebras such that for $A_1, A_2 \in \mathscr{C}$, from $\mathrm{End}_R A_1 \cong \mathrm{End}_R A_2$ we obtain $A_1 \cong A_2$.

2. The category of Boolean R-pairs. An R-algebra A is said to be *idempotent generated* if for each $a \in A$, $a = \sum_{i=1}^{n} r_i e_i$, $r_i \in R$, $e_i \in \mathscr{B}(A)$. Recall that a set $\{f_1, f_2, \ldots, f_n\}$ of idempotents is pairwise orthogonal if $f_i f_j = 0$, $i \neq j$. Thus if a has representation, $a = \sum_{i=1}^{n} r_i e_i$, then

$$a = a\Big(\prod_{j=1}^{n}(e_j+e'_j)\Big) = \sum_{i=1}^{n} r_i e_i\Big(\prod_{j=1}^{n}(e_j+e'_j)\Big),$$

where $e'_j = 1-e_j$. Consequently a can be represented as a linear combination of pairwise orthogonal idempotents which we call an *orthogonal representation* for a.

We let R-IGA denote the category of idempotent generated R-algebras. We now introduce the category of Boolean R-pairs.

Let $\mathscr{L}(R) \equiv \langle \mathscr{L}(R), \subseteq \rangle$ denote the collection of ideals of R under the inclusion relation. Let $B \equiv \langle B, \oplus, \cdot \rangle$ be a Boolean ring and let $\delta\colon B \to \mathscr{L}(R)$ be a function such that for $e, f \in B$,

(i) $(e \vee f)\delta = e\delta \cap f\delta$ where $e \vee f = e \oplus f \oplus ef$,

(ii) $(ef)\delta \supseteq e\delta + f\delta$ ('$+$' denotes sum of ideals),

(iii) $e\delta = R$ if and only if $e = 0$,

$\langle B, \delta \rangle$ is called a *Boolean R-pair*.

For a fixed R, we consider the collection of all Boolean R-pairs as objects of category in which the morphisms $\varphi\colon \langle B_1, \delta_1 \rangle \to \langle B_2, \delta_2 \rangle$ are those Boolean ring morphisms $\varphi\colon B_1 \to B_2$ such that $\delta_1 \subseteq \varphi\delta_2$. Because of the similarity of this category to the "comma categories" of MacLane [2], we denote our category of Boolean R-pairs by $\langle \mathscr{B} \downarrow R \rangle$.

$\langle B_1, \delta_1 \rangle$ and $\langle B_2, \delta_2 \rangle$ are said to be *isomorphic* in $\langle \mathscr{B} \downarrow R \rangle$ if there are morphisms $\varphi\colon \langle B_1, \delta_1 \rangle \to \langle B_2, \delta_2 \rangle$, $\psi\colon \langle B_2, \delta_2 \rangle \to \langle B_1, \delta_1 \rangle$ such that $\varphi\psi = 1_{B_1}$, $\psi\varphi = 1_{B_2}$. This is equivalent to $\varphi\colon B_1 \to B_2$ being a Boolean ring isomorphism with $\delta_1 = \varphi\delta_2$.

Let $A \in R$-IGA. For $e \in \mathscr{B}(A)$ we let $e\delta = \mathrm{ann}(e) = \{r \in R | re = 0\}$ and obtain a Boolean R-pair $\langle \mathscr{B}(A), \delta \rangle$ associated to A. For an algebra morphism $\Phi\colon A_1 \to A_2$ we obtain a morphism $\varphi = \Phi/\mathscr{B}(A_1)\colon \langle \mathscr{B}(A_1), \delta_1 \rangle \to \langle \mathscr{B}(A_2), \delta_2 \rangle$ and consequently a functor $\mathscr{B}\colon R\text{-IGA} \to \langle \mathscr{B} \downarrow R \rangle$. We now show that $\mathscr{B}$ is full.

Let $\varphi\colon \langle \mathscr{B}(A_1), \delta_1 \rangle \to \langle \mathscr{B}(A_2), \delta_2 \rangle$ and let $x \in A_1$ have orthogonal representation $x = \sum_{i=1}^{n} r_i e_i$. We now show that $\varphi\colon x \to \sum_{i=1}^{n} r_i e_i \varphi$ is a well-defined algebra morphism.

LEMMA 2.1. $\forall r, s \in R$, $\forall e, f \in \mathscr{B}(A_1)$, *if $re = sf$ then $r(e\varphi) = s(f\varphi)$.*

Proof. From $re = sf$, we find $ref = sf = re = sfe$ which in turn gives $(r-s)ef = 0$. Since $\delta_1 \subseteq \varphi\delta_2$, $(r-s)(ef)\varphi = 0$ or $r(ef\varphi) = s(ef\varphi)$. Further, $r(e-ef) = 0$ implies $r((e-ef)\varphi) = 0$ and similarly $s((f-ef)\varphi) = 0$. Since $e = ef \oplus (e-ef)$, $r(e\varphi) = r(ef\varphi)$ and similarly $s(f\varphi) = s(ef\varphi)$.

LEMMA 2.2. *Φ is well-defined.*

Proof. We must show that Φ does not depend upon the orthogonal representation used for $x \in A$. Suppose x has orthogonal representations $\sum_{i=1}^{n} r_i e_i$ and $\sum_{j=1}^{m} s_j f_j$.

Let $e = \sum_{i}^{n} e_i$, $f = \sum_{j}^{m} f_j$ and note (*) $ex = x$, $fx = x$; (**) $r_i e_i = xe_i = fxe_i = fr_i e_i$ for all i, and $s_j f_j = xf_j = exf_j = es_j f_j$, for all j: (***) $r_i e_i f_j = s_j e_i f_j$, for all i, j.

From (***) and Lemma 2.1, $r_i((e_i f_j))\varphi = s_j((e_i f_j))\varphi$ or $r_i e_i \varphi f_j \varphi = (e_i\varphi) s_j (f_j\varphi)$. Summing over j gives

$$r_i e_i \varphi \sum^{m} f_j \varphi = e_i \varphi \sum^{m} s_j (f_j \varphi)$$

and now summing over i gives

$$\sum^{n} r_i e_i\varphi \sum^{m} f_j\varphi = \sum^{n} e_i\varphi \sum^{m} s_j f_j .$$

Since the e_i (and f_j) are orthogonal, $(e_1\oplus\ldots\oplus e_n)\varphi = (e_1+\ldots+e_n)\varphi = e\varphi$ (and $f\varphi = \sum^{m} f_j\varphi$) and therefore $f\varphi \sum^{n} r_i e_i\varphi = e\varphi \sum^{m} s_j f_j\varphi$. From (**), $r_i(e_i\varphi) = r_i(e_i f)\varphi = (f\varphi)r_i(e_i\varphi)$. Now summing over i gives

$$\sum^{n} r_i(e_i\varphi) = f\varphi \sum^{n} r_i(e_i\varphi) .$$

Similarly $\sum^{m} s_j(f_j\varphi) = e\varphi \sum^{m} s_j(f_j\varphi)$. Hence we have $\sum^{n} r_i(e_i\varphi) = \sum^{m} s_j(f_j\varphi)$ as desired.

LEMMA 2.3. *Φ is an algebra morphism.*

Proof. Let $x = \sum^{n} r_i e_i$, $y = \sum^{m} s_j f_j$ be orthogonal representations of x and y in A_1. Then $xy = \sum^{n}\sum^{m} r_i s_j e_i f_j$ is an orthogonal representation for xy and it is clear that $(xy)\Phi = x\Phi y\Phi$. Also for $r\in R$, $(rx)\Phi = r(x\Phi)$. As above let $e = \sum^{n} e_i$ and $f = \sum^{m} f_j$. Then

$$x = xf + x(1-f) = \Big(\sum^{n} r_i e_i\Big)\sum^{m} f_j + \sum^{n} r_i e_i(1-f) = \sum^{n}\sum^{m} r_i e_i f_j + \sum^{n} r_i e_i(1-f) .$$

Similarly

$$y = ye + y(1-e) = \sum^{n}\sum^{m} s_j e_i f_j + \sum^{m} s_j f_j(1-e) .$$

Thus

$$x+y = \sum^{n}\sum^{m} (r_i+s_j)e_i f_j + \sum^{n} r_i e_i(1-f) + \sum^{m} s_j f_j(1-e)$$

is an orthogonal representation of $x+y$ and we find that $(x+y)\Phi = x\Phi + y\Phi$.

THEOREM 2.4. *$\mathscr{B}$: R-IGA$\to\langle\mathscr{B}\downarrow R\rangle$ is a full faithful functor.*

Proof. This follows from the above lemmas and the observation that $\Phi/\mathscr{B}(A_1) = \varphi$.

COROLLARY 2.5. *For $A_1, A_2\in R$-IGA, $A_1\cong A_2 \Leftrightarrow \langle\mathscr{B}(A_1),\delta_1\rangle \cong \langle\mathscr{B}(A_2),\delta_2\rangle$.*

We say $A\in R$-IGA is R-torsion free if for $a\in A$, $\operatorname{ann}(a) = (0)$. Thus, in the associated Boolean R-pair $\langle\mathscr{B}(A),\delta\rangle$, $0\delta = R$ and $e\delta = (0)$ for $e\neq 0$. Combining Theorem 1.2 and the above corollary gives the next result.

THEOREM 2.6. *Let $A_1, A_2\in R$-IGA. If A_1 and A_2 are R-torsion free, then $\mathrm{End}_R A_1 \cong \mathrm{End}_R A_2 \Leftrightarrow A_1\cong A_2$.*

The above theorem generalizes results in [5], [7], [8] and [10]. The special cases can be recovered by specifying the rings R.

3. Construction of the left adjoint of $\mathscr{B}$. Consider a Boolean R-pair $\langle B,\delta\rangle$ and denote the collection of maximal ideals of the Boolean ring B by $\mathscr{M}(B)$. For

each $m\in\mathscr{M}(B)$, define $R(m) = \{r\in R \mid r\in e\delta$, for some $e\notin m\}$. Using the properties of δ and m, we find that $R(m)$ is an ideal of R.

For each $m\in\mathscr{M}(B)$, $B/m\cong Z_2$ and so there is a Boolean ring embedding ϱ_m: $B/m\to R/R(m)$. We then have a Boolean ring monomorphism ψ: $\prod_m (B/m) \to \prod_m (R/R(m))$ where $\prod$ denotes the product of the indicated rings. (For $f\in\prod_m(B/m)$, $(f)\psi$: $m\to(mf)\varrho_m$.) Since B can be embedded in $\prod_m(B/m)$ ($e\to\bar{e}$ where $(m)\bar{e} = e+m$) we obtain a Boolean ring monomorphism φ: $B\to\prod_m(R/R(m))$. Let A be the R-subalgebra of $\prod_m(R/R(m))$ generated by the set of idempotents $B\varphi$.

Applying $\mathscr{B}$ to the above constructed algebra, we obtain the Boolean R-pair $\langle\mathscr{B}(A),\bar{\delta}\rangle$. We now show $\varphi\bar{\delta} = \delta$.

LEMMA 3.1. *For $e\in B$, $e\varphi\bar{\delta} = \bigcap_{\substack{m\\ e\notin m}} R(m)$.*

Proof. Since $e\varphi = \bar{e}\psi$, $e\varphi\bar{\delta} = \operatorname{ann}(\bar{e}\psi)$. Recall that if $e\notin m$, $(m)(\bar{e}\psi) = 1+R(m)$, while if $e\in m$, $(m)(\bar{e}\psi) = R(m)$. Now $r\in\operatorname{ann}(\bar{e}\psi)$ if and only if, $r[(m)\bar{e}\psi] = R(m)$ for all m. But this is equivalent to $r\in R(m)$ for all m such that $e\notin m$.

LEMMA 3.2. *For b_i, e_i in B, $i = 1,2,\ldots,n$, let $e = b_1e_1\oplus\ldots\oplus b_ne_n$, then $e\delta\supseteq\bigcap_i(b_ie_i)\delta$.*

Proof. For a and c in B, $(a\oplus c)(a\vee c) = a\oplus c$ which in turn implies $(a\oplus c)\delta\supseteq(a\vee c)\delta = a\delta\cap c\delta$. The desired result now follows by induction.

LEMMA 3.3. *For $e\in B$, $e\delta = \bigcap_{\substack{m\\ e\notin m}} R(m)$.*

Proof. It is clear that $e\delta\subseteq R(m)$ for all m such that $e\notin m$. Conversely, for $r\in\bigcap_{\substack{m\\ e\notin m}} R(m)$, there exists $e(m)$ in B such that $e(m)\notin m$ and $r\in e(m)\delta$. Let $K_0(r)$ denote the collection of all such idempotents; i.e., $K_0(r) = \{e_\alpha \mid e_\alpha\in B, e_\alpha\notin m$ for some m in $\mathscr{M}(B)$ such that $e\notin m$, and $r\in e_\alpha\delta\}$. Let K be the ideal of B generated by $K_0(r)$. If $e\in K$, then $e = b_1e_{\alpha_1}\oplus\ldots\oplus b_ne_{\alpha_n}$, $b_i\in B$. From (3.2) $e\delta\supseteq\bigcap_i(b_ie_{\alpha_i})\delta$ and since $(b_ie_{\alpha_i})\delta\supseteq b_i\delta+e_{\alpha_i}\delta$ we have $r\in e\delta$. If, on the other hand, $e\notin K$ then using Zorn's lemma we obtain a maximal ideal m_0 of B such that $K\subseteq m_0$ and $e\notin m_0$. Consequently, $r\in R(m_0)$ and thus there exists $e(m_0)\notin m_0$ such that $r\in e(m_0)\delta$. But this means that $e(m_0)$ is in $K_0(r)$ and thus in K which is impossible. Hence $e\in K$ and $r\in e\delta$.

THEOREM 3.4. *For every Boolean R-pair $\langle B,\delta\rangle$ there exists an algebra A in R-IGA and a Boolean ring monomorphism φ: $B\to\mathscr{B}(A)$ such that $\varphi\bar{\delta} = \delta$ where $\langle\mathscr{B}(A),\bar{\delta}\rangle$ is the Boolean R-pair associated with A.*

For $\langle B,\delta\rangle$ in $\langle\mathscr{B}\downarrow R\rangle$ let $\mathscr{A}(B)$ denote the idempotent generated R-algebra given by the above construction. As in Theorem 2.4, we find that a morphism

$\varphi\colon \langle B_1, \delta_1\rangle \to \langle B_2, \delta_2\rangle$ of Boolean R-pairs extends to an algebra morphism $\mathscr{A}(B_1)\to\mathscr{A}(B_2)$. Thus $\mathscr{A}\colon \langle \mathscr{B}\downarrow R\rangle \to R$-IGA is a functor. (Note that $\mathscr{B}(\mathscr{A}(B))\supseteq B$. None-the-less, B (better an isomorphic copy) generates $\mathscr{A}(B)$.) For $A\in R$-IGA and $\langle B, \delta\rangle \in \langle \mathscr{B}\downarrow R\rangle$, a map Φ in $\mathrm{Hom}(\mathscr{A}(B), A)$ induces a map φ in $\mathrm{Hom}(\langle B, \delta\rangle, \langle \mathscr{B}(A), \bar{\delta}\rangle)$ and conversely a map $\sigma\in\mathrm{Hom}(\langle B, \delta\rangle, \langle\mathscr{B}(A), \bar{\delta}\rangle)$ gives rise to a map $\sum \in \mathrm{Hom}(\mathscr{A}(B), A)$. Since these processes are inverses of each other and natural in both A and $\langle B, \delta\rangle$ we have the following result.

THEOREM 3.5. *$\mathscr{A}$ is a left adjoint of $\mathscr{B}$.*

4. Equivalence results. In this section we obtain an equivalence between a subcategory of $\langle \mathscr{B}\downarrow R\rangle$ and R-IGA. Restricting our ring R of scalers, we obtain some new (and some old) special cases.

For $A\in R$-IGA and $m\in\mathscr{M}(\mathscr{B}(A))$ let $\overline{m} = \sum_{e\in m} Ae$. $\overline{m}$ is an ideal of A and for each $e\in\mathscr{B}(A)-m$, $e\equiv 1 \operatorname{modulo} \overline{m}$. In fact, if $e\notin m$, $(1-e)\in m$ and therefore $a-ae = a(1-e)\in\overline{m}$, for each $a\in A$.

LEMMA 4.1. *$A/\overline{m}$ has only the trivial idempotents.*

Proof. If $x+\overline{m}$ is an idempotent of $A/\overline{m}$ then $x^2-x\in\overline{m}$. Hence $x^2-x = \sum_{i=1}^{k} a_i e_i$, $a_i\in A$, $e_i\in m$. Thus $1-e_i = e_i'\notin m$. Consequently, if $e = \prod_{i=1}^{k} e_i'$, then $e\notin m$ and $(x^2-x)e = 0$. If we let $y = xe$ then y is an idempotent in A and $x-y = x(1-e)\in\overline{m}$. Since $y\in\mathscr{B}(A)$ we have the desired result.

It is clear that $A/\overline{m}\in R$-IGA and since $A/\overline{m}$ has only one non-zero idempotent, we have $A/\overline{m}\cong R(\bar{1})$ and $R(\bar{1})\cong R/\mathrm{ann}(\bar{1})$ where $\bar{1} = 1+\overline{m}$.

LEMMA 4.2. $\mathrm{ann}(\bar{1}) = \overline{R(m)}$ *where* $\overline{R(m)} = \{r\in R\mid r\in\mathrm{ann}(e)$ *for some* $e\notin m\}$.

Proof. If $r\in\mathrm{ann}(\bar{1})$ then $r\cdot\bar{1} = \overline{m}$ which in turn give $r\cdot 1\in\overline{m}$. Hence $r\cdot 1 = \sum_{i=1}^{k} r_i e_i$. If we again let $e = \prod_{i=1}^{k} e_i'$ then $e\notin m$ and $0 = (r\cdot 1)e = re$ which shows that $r\in\overline{R(m)}$. On the other hand, if $re = 0$ for some $e\notin m$ then $1-e\in m$ and $r\cdot 1 = r(1-e)\in\overline{m}$; i.e., $r\in\mathrm{ann}(\bar{1})$.

Combining these two lemmas we obtain the next result.

COROLLARY 4.3. *For $A\in R$-IGA and $m\in\mathscr{M}(\mathscr{B}(A))$, $R/\overline{R(m)}$ has only the trivial idempotents.*

Let $\langle B, \delta\rangle$ be a Boolean R-pair and construct $\mathscr{A}(B)$. For each $e\in B$, let $\mathscr{N}(e) = \{m\in\mathscr{M}(B)\mid e\notin m\}$. It is well-known that these sets determine a topology on $\mathscr{M}(B)$ in such a manner that $\mathscr{M}(B)$ is a Boolean space (compact totally disconnected Hausdorff space). We note that $\mathscr{A}(B)$ is a set of functions from $\mathscr{M}(B)$ to D where D denotes the disjoint union of the quotient rings $R/R(m)$, $m\in\mathscr{M}(B)$. In order to define a topology on D so that each f in $\mathscr{A}(B)$ becomes a continuous map, we need the following analogue of a result of Pierce ([6], Lemma 4.3, p. 16).

LEMMA 4.4. *Let $a, b\in\mathscr{A}(B)$. If there exists $m\in\mathscr{M}(B)$ such that $(m)a = (m)b$ then there exists an e in B such that $m\in\mathscr{N}(e)$ and $(h)a = (h)b$ for all $h\in\mathscr{N}(e)$.*

Proof. Let $a = \sum_{i=1}^{n} r_i\bar{e}_i$, $b = \sum_{i=1}^{n} s_i\bar{e}_i$, $r_i, s_i\in R$, $\bar{e}_i, \bar{e}_j\in B\varphi$ be orthogonal representations of a and b respectively. Suppose $m\bar{e}_i = \bar{0}$ for all i. Then for all i, $e_i\in m$ and consequently $e_0 = \prod_{i=1}^{n} e_i\notin m$. Then $m\in\mathscr{N}(e_0)$ and for $h\in\mathscr{N}(e_0)$, $e_0\notin h$ implies $e_i\in h$, $i = 1, 2, \ldots, n$. Thus, $(h)a = (h)b$. If for some i, $m\bar{e}_i\neq\bar{0}$, then $(m)a = r_i+R(m) = s_i+R(m) = (m)b$. Thus $r_i-s_i\in R(m)$ and so by definition, $r_i-s_i\in f\delta$ for some $f\in B$, $f\notin m$. Now $m\in\mathscr{N}(e_i f)$ since $e_i\notin m$ and $f\notin m$. If $h\in\mathscr{N}(e_i f)$, $e_i f\notin h$ which means that $e_i\delta+f\delta\subseteq(e_i f)\delta\subseteq R(h)$ or $r_i-s_i\in R(h)$. Since $e_i\notin h$, $(h)a = r_i+R(h) = s_i+R(h) = (h)b$.

Following Pierce ([6], pp. 16–18) we are now able to define a topology on D so that the functions in $\mathscr{A}(B)$ are continuous and have compact support ([6], p. 37) (Support of $f\equiv\operatorname{Supp} f = \{m\in\mathscr{M}(B)\mid mf\neq\bar{0}\}$.)

LEMMA 4.5. *If $f\in\mathscr{B}(\mathscr{A}(B))$ is such that $mf\in\{\bar{0}, \bar{1}\}$ for each $m\in\mathscr{M}(B)$ then $f\in B\varphi$. (Recall that $\varphi\colon B\to\mathscr{B}(\mathscr{A}(B))$ is the Boolean ring monomorphism constructed above.)*

Proof. Since $f\in\mathscr{A}(B)$, $f = \sum_{i=1}^{n} r_i\bar{e}_i$, $r_i\in R$ and the $\bar{e}_i$ are orthogonal idempotents in $B\varphi$ and thus it suffices to show for arbitrary i, $1\leqslant i\leqslant n$, $r\bar{e}_i$ is in $B\varphi$. For $m\in\operatorname{Supp}(r\bar{e}_i)$, $(m)r\bar{e}_i = 1+R(m)$ which is equivalent with $1-r\in R(m)$. Thus there exists $e_\alpha\notin m$ such that $1-r\in e_\alpha\delta$. Now, $m\in\mathscr{N}(e_i e_\alpha)$ and for $h\in\mathscr{N}(e_i e_\alpha)(h)r\bar{e}_i = (h)\overline{e_i e_\alpha}$ since $1-r\in e_\alpha\delta\subseteq(e_i e_\alpha)\delta\subseteq R(h)$. Since $\operatorname{Supp}(r\bar{e}_i)$ is compact, there is a finite subcollection $\mathscr{N}(e_i e_{\alpha_1}), \ldots, \mathscr{N}(e_i e_{\alpha_t})$ with $\operatorname{Supp}(r\bar{e}_i) = \bigcup_{i=1}^{t}(e_i e_{\alpha_j})$. But then $r\bar{e}_i = \overline{e_i e_{\alpha_1}}\vee\ldots\vee\overline{e_i e_{\alpha_t}}$ and since each $\overline{e_i e_{\alpha_j}}$ is in $B\varphi$ so is $r\bar{e}_i$.

THEOREM 4.6. *For a Boolean R-pair $\langle B, \delta\rangle$, $B\varphi = \mathscr{B}(\mathscr{A}(B))$ if and only if $R/R(m)$ has only the trivial idempotents, for each m in $\mathscr{M}(B)$.*

Proof. Every maximal ideal of $B\varphi$ has the form $m\varphi$ where $m\in\mathscr{M}(B)$. If $B\varphi = \mathscr{B}(\mathscr{A}(B))$ then from Corollary 4.3, $R/\overline{R(m\varphi)}$ has only the trivial idempotents. However, $r\in\overline{R(m\varphi)}$ if and only if $r\in\bar{e}\bar{\delta}$ for some $\bar{e}\notin m\varphi$; i.e., if and only if $r\in e\varphi\bar{\delta}$ for some $e\notin m$. Using the fact that $\varphi\bar{\delta} = \delta$, this last condition is equivalent to $r\in R(m)$. Hence $R/R(m)$ has only the trivial idempotents. Conversely, we have $B\varphi\subseteq\mathscr{B}(\mathscr{A}(B))$. For $f\in\mathscr{B}(\mathscr{A}(B))$, $m\in\mathscr{M}(B)$, mf is an idempotent in $R/R(m)$ and consequently $mf\in\{\bar{0}, \bar{1}\}$. By the previous lemma, $f\in B\varphi$.

We say a Boolean R-pair is *reduced* if it satisfies the conditions of the above theorem and we denote the subcategory of reduced Boolean R-pairs by ${}_r\langle\mathscr{B}\downarrow R\rangle$.

THEOREM 4.7. *R-IGA is equivalent to ${}_r\langle\mathscr{B}\downarrow R\rangle$.*

Proof. We first note that for $A\in R$-IGA, $\langle\mathscr{B}(A), \delta\rangle$ is reduced (Corollary 4.3). Thus $\mathscr{B}(\mathscr{A}(\mathscr{B}(A)))$ is isomorphic to $\mathscr{B}(A)$ and this isomorphism induces (as in

Theorem 2.4) an isomorphism between $\mathscr{A}(\mathscr{B}(A))$ and A. That is, $\mathscr{A}\mathscr{B}$ is naturally isomorphic to the identity functor on R-IGA. On the other hand, for a reduced pair $\langle B, \delta\rangle$, $\mathscr{B}(\mathscr{A}(B)) = B\varphi$ and $\bar{\delta} = \varphi\delta$. Thus $\mathscr{B}\mathscr{A}$ is naturally isomorphic to the identity functor on $\langle\mathscr{B}\downarrow R\rangle$.

We conclude by considering some special cases. In particular, the collection of Boolean R-pairs $\langle B, \delta_0\rangle$ where $e\delta_0 = 0$ if $e \neq 0$ and $0\delta_0 = R$ are the objects of a full subcategory of $\langle\mathscr{B}\downarrow R\rangle$. Since this subcategory is isomorphic to the category of Boolean rings, we denote it by Borng. If R has only the trivial idempotents then $\langle B, \delta_0\rangle$ is reduced. The following corollary is a generalization of a result of McCrea for the special case in which R is the ring of integers.

COROLLARY 4.8. *If R is a ring with only the trivial idempotents then the category of R-torsion free idempotent generated R-algebras is equivalent to the category of Boolean rings.*

COROLLARY 4.9. *If R is a field, R-IGA is equivalent to Borng.*

Thus for fields F_1 and F_2, F_1-IGA and F_2-IGA are equivalent. In particular, for prime integers p, we obtain the result of Stringall [10] that the categories of p-rings are equivalent.

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3-dimensional AR's which do not contain 2-dimensional ANR's

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Abstract. There exists an upper semicontinuous decomposition G of 3-dimensional cell B^3 such that the decomposition space B^3/G is a 3-dimensional AR which does not contain any 2-dimensional ANR.

1. Introduction and terminology. By an AR (ANR) we understand a compact metric absolute retract (compact absolute neighborhood retract). One may consult [9] for additional information on AR's (ANR's) and related terminology.

If G is an upper semicontinuous decomposition of a topological space X we denote the associated decomposition space by X/G and by $p\colon X\to X/G$ the canonical projection, unless otherwise stated. For more information on upper semicontinuous decompositions see [21]. A survey of results on upper semicontinuous decompositions can be found in [2] and [21].

Let n denote a positive integer. By E^n we shall always mean an n-dimensional Euclidean space, by B^n the closed ball of unit radius, and by S^{n-1} the boundary sphere of B^n. By a disc we shall always understand a space homeomorphic to B^2. All maps will be continuous.

A family (collection, sequence) C of subsets of metric space X is called a *null family* (*collection, sequence*) provided that for each $\varepsilon>0$ at most a finite number of elements of C are of diameter greater than ε.

The purpose of this paper is to provide an affirmative answer to the following question which arises in Bing and Borsuk [8] and Armentrout [4]:

Do there exist 3-dimensional AR's which do not contain 2-dimensional AR's or even 2-dimensional ANR's?

In [8], Bing and Borsuk described an upper semicontinuous decomposition G of B^3 whose nondegenerate elements form a countable null family of arcs such that the decomposition space B^3/G is a 3-dimensional AR which does not contain any disc. They asked whether their 3-dimensional AR B^3/G contained any 2-dimensional AR. Armentrout [4] described an upper semicontinuous decomposition G of B^3 similar to the one described by Bing and Borsuk [8] such that B^3/G is a 3-dimensional AR which does not contain any disc but does contain 2-dimensional AR's.

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