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IDEMPOTENT MULTIPLICATIONS ON SURFACES AND ASPHERICAL SPACES

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ABSTRACT. Every continuous idempotent multiplication on a space induces an idempotent comultiplication on its cohomology algebra over a commutative ring and a homomorphic idempotent multiplication on each homotopy group.

We classify all idempotent comultiplications on any graded anticommutative algebra A^* over a principal ideal domain K up to degree 2 provided the degree 1 component A^1 is torsion free and the degree 2 component A^2 is of rank 1. All algebraic possibilities can be topologically realized.

We also describe all homomorphic idempotent multiplications on arbitrary groups. This allows a complete classification up to homotopy of all idempotent multiplications on aspherical CW-complexes. For surfaces we obtain an explicit list. Notably, the Klein bottle allows infinitely many nonhomotopic idempotent multiplications, but all other surfaces with nonabelian fundamental group have only the projections as idempotent multiplications (up to homotopy).

Introduction. Idempotent multiplications on sets and topological spaces have been considered by many authors, for instance as an axiomatic approach to the averaging operation (sample: [**2, 3, 10**]).

If X denotes a connected topological space, then the existence of H-space structures places severe restrictions on the structure of X. (See, for instance, [**6**] or [**16**].) This is due to the presence of homotopy identities on both sides. If, however, one considers idempotent multiplications $\mu : X \times X \to X$, that is, multiplications which satisfy $\mu(x,x) = x$ for all $x \in X$, then no restriction follows from the presence of such multiplications, since every space X allows the two idempotent multiplications $p_1, p_2 : X \times X \to X$, $p(x,y) = x$ and $q(x,y) = y$ for all $x, y \in X$. These are the so-called *trivial multiplications*. On the other hand, the existence of *nontrivial* idempotent multiplication again forces restrictions on the space. We wish to illustrate this by discussing idempotent multiplications on suitable classes of spaces. The

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fact that idempotent multiplications restrict the structure of a space is elucidated by the fact that there exists a bijection between the homotopy classes of idempotent multiplications and the homotopy classes of multiplications with a left identity (see [**9**, 1.5]).

The testing of the restrictions imposed on a space by the presence of nontrivial idempotent multiplications is usually transferred to an algebraic problem by applying a suitable functor H from an appropriately selected category $\mathcal{T}$ of topological spaces and continuous maps, say, to a suitable category $\mathcal{A}$ of algebraic objects. Typical examples are the category of graded modules (if H is homology or cohomology), or the category of groups (if H is homotopy π_n). Now an idempotent multiplication in $\mathcal{T}$ is a continuous function $\mu : X \times X \to X$ such that, for the diagonal map $d_X : X \times X \to X$ (a map existing in any category with finite products!), we have

$$\mu d_X = \mathbf{1}_X. \tag{1}$$

The application of any functor H yields the relation

$$H(\mu)H(d_X) = \mathbf{1}_{H(X)}. \tag{2}$$

In practice we choose the category $\mathcal{A}$ so that it is equipped with a binary operation $(A, B) \mapsto A \otimes B : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ on the functorial level. Examples are the tensor product of graded algebras or the cartesian product of groups. We assume that H respects multiplication, that is, $H(X \times Y) \cong H(X) \otimes H(Y)$. Then $H(d_X) = \delta_{H(X)} : H(X) \to H(X) \otimes H(X)$ is an associative comultiplication, respectively, multiplication in the opposite category of $\mathcal{A}^{\mathrm{op}}$ of $\mathcal{A}$ (in which case the presence of a natural transformation $H(X) \otimes H(Y) \to H(X \times Y)$ suffices instead of a natural isomorphism).

We shall illustrate two instances.

Firstly, we shall consider the category $\mathcal{T}$ of compact connected spaces and continuous maps, the category $\mathcal{A}$ of graded K modules, where K is a principal ideal domain, and the functor $H = \{X \mapsto H^*(X, K)\}$ of Alexander–Čech-cohomology over K. Due to the Künneth formula, we have a natural injection

$$\alpha_{X,Y} : H(X) \otimes H(Y) \to H(X \times Y)$$

and thus a product, called *cup product*,

$$\cup : H(X) \otimes H(X) \xrightarrow{\alpha_{X,X}} H(X \times X) \xrightarrow{H(d_X)} H(X),$$

making $H(X)$ into an associated graded (anti-) commutative K-algebra $H(X) = A^0 \oplus A^1 \oplus A^2 \oplus \cdots$.

We are mostly interested in surfaces or, at least, two-dimensional spaces. At any rate, we make the assumption that $H^2(X, \mathbf{Z})$ is torsion free of rank one. Moreover, we recall that there is a natural isomorphism $H^1(X, \mathbf{Z}) \cong [X, S^1]$. Hence, the abelian group $H^1(X, \mathbf{Z})$ is always torsion free. The universal coefficient theorem yields the exact sequence

$$0 \to H^n(X, \mathbf{Z}) \otimes K \to H^n(X, K) \to \mathrm{Tor}(H^{n+1}(X, \mathbf{Z}), K) \to 0$$

in which the Tor-term vanishes for $n = 1$ if $H^2(X, \mathbf{Z})$ is torsion free. Thus, $A^1 \cong H^1(X, \mathbf{Z}) \otimes K$. Note that $F \otimes K$ is torsion free over K if F is a torsion free abelian group and K is a domain.

Furthermore, an idempotent multiplication $\mu : X \times X \to X$ gives a comultiplication

$$c^n : A^n = H^n(X) \xrightarrow{H(m)} H^n(X \times X) \xrightarrow{\alpha^{-1}_{X,X}} H^n(X) \otimes H^n(X) = A^n \otimes A^n$$

in all those degrees in which A^n is torsion free, and these satisfy the relations

$$\cup^n \circ c^n = \mathbf{1}_{A^n}. \tag{3}$$

Now a nontrivial idempotent multiplication μ yields a nontrivial comultiplication where a comultiplication c is called *trivial* if $c(x) = x \otimes 1$ or $c(x) = 1 \otimes x$ for all $x \in A^n$ for $n = 0, 1, \ldots$.

Our main results (Theorems 2.6, 2.7, Corollary 2.8, and Theorem 2.9) imply a complete classification of all nontrivial idempotent comultiplications of graded K-algebras $A^* = A^0 \oplus A^1 \oplus A^2$ where A^0 and A^2 are of rank one. If $A^1 A^1 \neq \{0\}$, then the main consequence is $\mathrm{rank}_R\, A^1 = 2$, and this will allow us to describe the comultiplications explicitly. In the most important case that $K = \mathbf{Z}$, *any* torsion free abelian group of rank 2 can occur as A^1 with nontrivial comultiplications on A^*, and all those instances come from idempotent comultiplications on compact connected spaces, notably on the underlying spaces of two-dimensional compact connected abelian groups. If $A^1A^1 = \{0\}$, then it is a bit more cumbersome to exhibit topological realizations. In order to illustrate such realizations, we discuss the case of the Klein bottle with its idempotent multiplications in some detail and some idempotent multiplications on abelian topological groups containing the classifying space $B(\mathbf{T})$ of the circle group $\mathbf{T}$.

Secondly, we shall consider the category $\mathcal{T}$ of aspherical CW-complexes and continuous maps and the category $\mathcal{A}$ of groups; this time we take for H the assignment of the fundamental group $X \mapsto \pi_1(X)$. For the multiplication $\otimes$ in $\mathcal{A}$ we consider the cartesian product since H is multiplicative because of $\pi_1(X\times Y) \cong \pi_1(X)\times\pi_1(Y)$. Then every idempotent multiplication $\mu : X \times X \to X$ produces a group homomorphism $m : H(X) \times H(X) \to H(X)$ satisfying $md_{H(X)} = \mathbf{1}_{H(X)}$. If X is aspherical, then so is $X \times X$, and, therefore, every homomorphism $m : H(X) \times H(X) \to H(X)$ is realized by a continuous map $\mu : X \times X \to X$ (see [**15**; Chap. V, Sec. 4, pp. 224-225]). So the realization is not problematic in this case. We shall show that there is a bijection from the set of idempotent multiplications on a group G to the semigroup $\mathrm{End}^*(G) \subseteq \mathrm{Hom}(G,G)$ of all endomorphisms which commute with all inner automorphisms. For Eilenberg-MacLane complexes $K(G,1)$, this yields a classification of the set of homotopy classes of idempotent multiplications. In the case of surfaces we shall present explicit results.

The Klein bottle with its idempotent multiplications is a good example for both approaches.

We remark that in both routes, the cohomological as well as the homotopical one, the functor H factors through the category of topological spaces and homotopy classes of continuous maps, the so-called *homotopy category*. Therefore, we could have considered in each case for $\mathcal{T}$ the associated homotopy category. But then the issue arises whether any homotopy idempotent multiplication $\mu : X \times X \to X$ is in fact homotopic to an idempotent one. We indicate briefly that this is true: If $\mu d_X \sim \mathbf{1}_X$, we have to find a $\nu : X \times X \to X$ with $\nu \sim \mu$ and $\nu d_X = \mathbf{1}_X$. Then we have a homotopy $f_t : X \to X$ with $f_0 = \mu d_X$ and $f_1 = \mathbf{1}_X$, and we are looking for an extension to a homotopy $F_t : X \times X \to X$ with $F_0 = \mu$ and $F_1 d_X = \mathbf{1}_X$; if we have this extension we shall set $\nu = F_1$. The existence of such an extension F, however, is guaranteed by the following fact: $d_X : X \to X \times X$ is always a cofibration, because Δ_X, the diagonal of $X \times X$ is a retract of $X \times X$, and thus $\Delta_X \times [0,1]$ is a retract of $X \times X \times [0,1]$.

1. Preliminaries on multilinear algebra. In our discussion of cohomology algebras we need certain background material on modules which we shall provide in the present section.

In all of the following, K shall denote a commutative ring with identity, frequently a principal ideal domain. Also, V shall denote a K-module. All tensor products and exterior algebras are taken over K. The module $V \otimes V$ will be denoted W. There are natural endomorphisms $\tau, \sigma, \alpha : W \to W$:

(i) $\tau(u \otimes v) = v \otimes u$,

(ii) $\sigma(u \otimes v) = u \otimes v + v \otimes u$, that is, $\sigma = \mathbf{1}_W + \tau$,

(iii) $\alpha(u \otimes v) = u \otimes v - v \otimes u$, that is, $\alpha = \mathbf{1}_W - \tau$.

Clearly, τ is an involution, that is, $\tau^2 = \mathbf{1}_W$. Consequently, $\sigma^2 = 2 \cdot \sigma$ and $\alpha^2 = 2 \cdot \alpha$. Also, $\sigma\alpha = (1+\tau)(1-\tau) = 0$.

We let $\Sigma^2(V)$ denote the submodule of W generated by all elements $u \otimes u$ with $u \in V$. We recall that

$$\bigwedge^2 V = W/\Sigma^2(V). \tag{1}$$

Lemma 1.1. *If V is a direct sum of cyclic modules, then $\Sigma^2(V) = \ker \alpha$.*

Proof. Since $\alpha(u \otimes u) = 0$, the containment $\Sigma^2(V) \subseteq \ker \alpha$ is trivial. Let us prove the reverse. By hypothesis, $V = \bigoplus_{j \in J} K \cdot e_j$. Then W is the direct sum of the modules $K \cdot (e_j \otimes e_k)$, $j, k \in J$. Let $x \in \ker \alpha$. Then there are elements $r_{jk} \in K$ such that $x = \sum_{j,k \in J} r_{jk} \cdot (e_j \otimes e_k)$ with $0 = \alpha(x) = \sum_{j,k \in J} r_{jk} \cdot (e_j \otimes e_k - e_k \otimes e_j) = \sum_{j<k} (r_{jk} - r_{kj}) \cdot (e_j \otimes e_k)$, where $<$ denotes an arbitrary total order of J. Thus, $j \neq k$ implies $r_{jk} = r_{kj}$. Therefore, $x = \sum_{j<k} r_{jk} \cdot (e_j \otimes e_k + e_k \otimes e_j) + \sum_{j \in J} r_{jj} \cdot (e_j \otimes e_j)$. Since $e_j \otimes e_k + e_k \otimes e_j = (e_j + e_k) \otimes (e_j + e_k) - e_j \otimes e_j - e_k \otimes e_k$, we see that x is in $\Sigma^2(V)$ as we had to show. $\square$

Lemma 1.2. *Suppose $V = \varinjlim V_i$ is a colimit of a direct system of modules such that $\Sigma^2(V_i) = \ker \alpha_i$. Then $\Sigma^2(V) = \ker \alpha$.*

Proof. Let $f_{ij} : V_i \to V_j$ denote the maps of the direct system and $f_i : V_i \to V$ the colimit maps. Set $W_i = V_i \otimes V_i$ and denote with $F_{ij} : W_i \to W_j$ and $F_i : W_i \to W$ the induced maps $F_{ij} = f_{ij} \otimes f_{ij}$ and $F_i = f_i \otimes f_i$. Since tensor products commute with direct limits and the direct limit functor is exact, we have an infinite commutative diagram in which all vertical sequences indicate direct limit diagrams and in which the horizontal sequences are exact except possibly the bottom sequence:

$$\begin{tikzcd}
\ker \alpha_i \arrow[r, "\mathrm{incl}_i"] \arrow[d] & W_i \arrow[r, "\alpha_i"] \arrow[d, "F_{ij}"] & W_j \arrow[d, "F_{ij}"] \\
\ker \alpha_j \arrow[r, "\mathrm{incl}_j"] \arrow[d] & W_j \arrow[r, "\alpha_j"] \arrow[d, "F_j"] & W_j \arrow[d, "F_j"] \\
\ker \alpha \arrow[r, "\mathrm{incl}"'] & W \arrow[r, "\alpha"'] & W
\end{tikzcd}$$

We have to show that the bottom sequence is exact. Let us denote with X the set of all $u \otimes u \in W$ with $u \in V$, and with X_i the set of all $u \otimes u \in W_i$ with $u \in V_i$. Then $X_i \subseteq \ker \alpha_i$. Let $G_{ij} : X_i \to X_j$ denote the restriction and corestriction of F_{ij} and $G_i : X_i \to X$ the restriction and corestriction of F_i. If $u \in V$ is given, then since $V = \varinjlim V_i$, there is an index j and an element $u_j \in V_j$ such that $f_j(u_j) = u$. Then $u \otimes u = f_j(u) \otimes f_j(u) = G_j(u_j \otimes u_j)$. Thus,

$$X_i \xrightarrow{G_{ij}} X_j \xrightarrow{G_j} X$$

is a direct limit diagram in the category of sets. Now suppose that some $w \in \ker \alpha$ is given. Then, by hypothesis, there is an index j and there are elements $u_p \in V_j$, $p = 1, \dots, q$, such that $w = F_j(\sum_p u_p \otimes u_p) = \sum_p G_j(u_p \otimes u_p)$. Thus w is in the module generated by X, that is, in $\Sigma^2(V)$, and this proves the lemma. $\square$

The preceding two lemmas taken together yield immediately the following result:

Lemma 1.3. *If V is the colimit of a direct system of modules, each of which is a direct sum of cyclic modules, then* $\Sigma^2(V) = \ker \alpha$.

Proposition 1.4. *If K is a principal ideal domain, then* $\Sigma^2(V) = \ker \alpha$.

Proof. Every module is the colimit of the direct system of its finitely generated submodules and the corresponding inclusion maps. If K is a principal ideal domain, then every finitely generated module is a direct sum of cyclic ones. Thus, Lemma 1.3 establishes the claim. □

Now let us define

$$\Theta^2(V) = \operatorname{im} \alpha. \tag{2}$$

Then $\Theta^2(V) \cong W/\ker \alpha$. From (1) we recall $W/\Sigma^2(V) = \bigwedge^2 V$. Thus, we have

Remark 1.5. Whenever $\Sigma^2(V) = \ker \alpha$, then $\Theta^2(V) \cong \bigwedge^2 V$. In particular, this is the case whenever K is a principal ideal domain.

Under these circumstances, $\Theta^2(V)$ is a manifestation of $\bigwedge^2 V$ inside $W = V \otimes V$.

Now we comment on those modules V for which $\bigwedge^2 V$ is cyclic. We preface the next lemma with the remark that every flat module is the colimit of a direct system of finitely generated free modules (see [**1**; §1, p. AX14, Théorème 1]). The converse is true since free modules are flat and direct limits of flat modules are flat.

Lemma 1.6. *Suppose that V is the colimit of a direct system of finitely generated free modules and injective colimit maps and that K is a domain. If $\bigwedge^2 V$ is nonzero cyclic, then V is free of rank 2. If $V \neq \{0\}$ and $\bigwedge^2 V = \{0\}$, then* $\operatorname{rank} V = 1$.

Proof. We write $V = \varinjlim V_j$ with finitely generated free modules V_j and colimit maps $f_j : V_j \to V$. The exterior algebra functor $\bigwedge$ from the category of K-modules to the category $\mathcal{A}$ of graded anticommutative algebras is left adjoint to the functor associating with a graded anticommutative algebra $A^* = \bigoplus_{n=0}^{\infty} A^n$ the module A^1. Hence, it preserves all colimits. Since direct limits in $\mathcal{A}$ are computed by homogeneous components, the functor $A^* \mapsto A^2$ preserves direct limits. Hence, the endofunctor $U \mapsto \bigwedge^2 U$ of the category of K-modules preserves direct limits. Thus, $\bigwedge^2 V = \varinjlim \bigwedge^2 V_j$. Since $\bigwedge^2 V$ is generated by one element, there is an index j such that

$$\bigwedge^2 V = \Big(\bigwedge^2 f_j\Big)\Big(\bigwedge^2 V_j\Big). \tag{3}$$

Since all maps f_j are injective and all V_i and V are flat, also $\bigwedge^2 f_j$ is injective and, thus, an isomorphism in view of (3). Thus, $\bigwedge^2 V_j$ is cyclic and V_j is free. Thus, rank $V_j = 2$, say $V_j = K \cdot u_j \oplus K \cdot v_j$. We set $u = f_j(v_j)$ and $v = f_j(v_j)$ and $\bigwedge^2 V = K \cdot (u \wedge v)$. We claim that $V = K \cdot u + K \cdot v$. Now K is a domain; let Q denote its quotient field. As $\bigwedge^2 V$ is cyclic and nonzero, then the Q-vector space $Q \otimes \bigwedge^2 V = \bigwedge_Q^2(Q \otimes V)$ is one-dimensional, and thus $Q \otimes V$ is two dimensional. Hence, $Q \otimes V = Q \otimes u \oplus Q \otimes v$. Thus, for every element $w \in V$, there is a ring element $a \in K$ such that $a \cdot w \in K \cdot u + K \cdot v$. Hence there are elements $x, y \in K$ with $a \cdot w = x \cdot u + y \cdot u$. Since $\bigwedge^2 V = K \cdot (u \wedge v)$, there are elements $x', y' \in K$ such that $u \wedge w = y' \cdot (u \wedge v)$ and $w \wedge v = x' \cdot (u \wedge v)$. Now $(y'a) \cdot (u \wedge v) = (u \wedge a \cdot w) = (u \wedge (x \cdot u + y \cdot v)) = y \cdot (u \wedge v)$, whence $ay' = y$ since $\bigwedge^2 V$ is free. Similarly, $ax' = x$. Thus, $a \cdot w = a \cdot (x' \cdot u + y' \cdot v)$, whence $w = x' \cdot u + y' \cdot v$. This establishes the claim. Now $f_j : V_j \to V$ is injective and surjective, and thus is an isomorphism. This shows that V is free of rank 2.

Now suppose that $\bigwedge^2 V = \{0\}$. Then $\bigwedge_Q^2(Q \otimes V) = \{0\}$, and thus $\dim_Q(Q \otimes V) \leq 1$. Hence, $V = \{0\}$ or rank $V = 1$. ◻

Lemma 1.7. *Let K be a principal ideal domain and let V be nonzero torsion free. If $\bigwedge^2 V$ is nonzero of rank 1, then V is of rank* 2. *If $\bigwedge^2 V = \{0\}$, then* $\operatorname{rank} V = 1$. *If $\bigwedge^2 V$ is nonzero cyclic, then V is free.*

Proof. If K is a principal ideal domain, then every finitely generated torsion free module is free. Since every module is the colimit of the direct system of finitely generated submodules, the hypotheses of Lemma 1.6 are satisfied, and that lemma proves the assertion in case that $\bigwedge^2 V$ is cyclic. If $\bigwedge^2 V$ is of rank 2, then let V_j be any nonzero finitely generated submodule of V. Then $\bigwedge^2 V_j$ may be considered as a submodule of $\bigwedge^2 V$, and since the latter is of rank 1, then $\bigwedge^2 V_j$ is of rank 1 and finitely generated. Hence, it is cyclic nonzero and thus V_j is free of rank 2 by the preceding. Thus V is the ascending union of free submodules of rank 2, hence has rank 2. $\square$

At this point we consider two endomorphisms $f, g : V \to V$. They induce an endomorphism $f \otimes g : W \to W$ and we shall have reason in the next section to consider the morphism

$$T \stackrel{\text{def}}{=} (f \otimes g)\alpha : W \to W. \tag{4}$$

Remark 1.8. If $f \otimes g$ is injective, then $\operatorname{im} T \cong \Theta^2(V)$. In particular, if K is a principal ideal domain, then $\operatorname{im} T \cong \bigwedge^2 V$.

Proof. Clearly, if $f \otimes g$ is injective, then $\operatorname{im} T \cong \operatorname{im} \alpha$, whence the first assertion follows from Definition (2). The second claim is then a consequence of Remark 1.5. $\square$

The following lemma will utilize this information. It will be applied in the next section.

Lemma 1.9. *Assume the following hypotheses*:

(i) *K is a principal ideal domain.*

(ii) *V is torsion free.*

(iii) *f and g are injective.*

(iv) *The image of T satisfies*

$$\operatorname{rank} \operatorname{im} T \leq 1. \tag{5}$$

Then the following conclusions hold: *If* $T \neq 0$, *then* V *is of rank* 2. *If* $T = 0$, *then* $V = \{0\}$ *or* V *is of rank* 1.

Moreover, if $\operatorname{im} T$ *is nonzero cyclic, then* V *is free* (*of rank* 2).

Proof. Since V is torsion free, V is flat and thus (iii) implies that $f \otimes g$ is injective. Now Remark 1.8 applies and shows that $\operatorname{im} T \cong \bigwedge^2 V$. The present lemma now follows from Lemma 1.7. $\square$

We retain our assumptions on f and g but no longer assume that $f \otimes g$ is injective. In exchange for that, however, we now assume that

$$f + g = \mathbf{1}_V. \tag{6}$$

In particular, we continue to assume that V is flat so that, for any submodule V' of V, we may identify $V \otimes V'$ and $V' \otimes V$ with submodules of W.

Lemma 1.10. *If* V *is flat and* (6) *is satisfied, then*

$$T(V \otimes \ker g) = \ker g \otimes \operatorname{im} g \tag{7}$$

and

$$T(\ker f \otimes V) = \operatorname{im} f \otimes \ker f. \tag{8}$$

Proof. We prove (7); the proof of (8) is analogous. First we note that an element $w \in V$ is in $\ker g$ if and only if $0 = g(w) = (\mathbf{1} - f)(w)$, that is, if and only if $f(w) = w$. Next we observe that $\alpha(V \otimes \ker g)$ in W is the submodule generated by the elements $v \otimes w - w \otimes v$ with $v \in V$ and $w \in \ker g$. But for any such element we have $(f \otimes g)(v \otimes w - w \otimes v) = -f(w) \otimes g(v) = -w \otimes g(v)$ in view of our first observation. Thus, $T(V \otimes \ker g) = (f \otimes g)\alpha(V \otimes \ker g) = \ker g \otimes \operatorname{im} g$ which is the asserted identity (7). $\square$

We summarize the results of this section in the following proposition:

Proposition 1.11. *Let* V *be a* K*-module with two endomorphisms* f *and* g *and assume the following hypotheses*:

(i) *K is a principal ideal domain.*

(ii) *V is torsion free.*

(iii′) $f + g = \mathbf{1}_V$.

(iv) *The image of T satisfies*

$$\operatorname{rank} \operatorname{im} T \leq 1. \tag{5}$$

Then one of the following conclusions holds:

(a) *$f = 0$ and $g = \mathbf{1}$, or else $f = \mathbf{1}$ and $g = 0$.*

(b) *$f \neq 0$ and $g \neq 0$ and*

(b1) *if $T \neq 0$, then V is of rank 2, and if, in addition,* $\operatorname{im} T$ *is cyclic, then V is free*,

(b2) *if $T = 0$ and $V \neq \{0\}$, then V is of rank* 1,

(b3) *$V = \{0\}$.*

Moreover, Case (*b*2) *can occur only if f and g are both injective.*

Proof. Lemma 1.10 applies, and then (5), (7) and (8) show that each of $\operatorname{im} f \otimes \ker f$ and $\ker g \otimes \operatorname{im} g$ is of rank 1 and indeed cyclic if $\operatorname{im} T$ is cyclic, since K is a principal ideal domain. Now we have a case distinction: Either we are in Case (a) or in Case (b). If we are in case (b), then both $\operatorname{im} f \otimes \ker f$ and $\ker g \otimes \operatorname{im} g$ can be zero only if $\ker f$ and $\ker g$ are zero. Then (b1), (b2) and (b3) follow by Lemma 1.9. We now assume that $\ker f \neq \{0\}$. (The case that $\ker g \neq \{0\}$ is treated analogously.) But also $f \neq 0$ since we are not in Case (a). Hence, the fact that $\operatorname{im} f \otimes \ker f$ is of rank 1, respectively cyclic, implies that both of the torsion free modules $\operatorname{im} f$ and $\ker f$ are of rank 1, respectively cyclic. In the cyclic case, they are free of rank 1. Then the exact sequence

$$0 \to \ker f \xrightarrow{\text{incl}} V \xrightarrow{\text{quot}} \operatorname{im} f \to 0$$

shows that V is of rank 2, respectively free of rank 2, and so we are in Case (b1). $\square$

2. Idempotent comultiplications. In this section we consider graded anticommutative algebras over a principal ideal domain K. Such algebras arise as the cohomology algebras of compact spaces.

Since we shall be interested eventually in two-dimensional spaces, we concentrate on low dimensions and require that the degree 2 component of the algebra be cyclic.

Specifically, we shall deal with the following data: As before, K shall be a principal ideal domain. We consider a graded K-algebra

$$A = A^0 \oplus A^1 \oplus A^2 \oplus \cdots$$

with $A^p A^q \subseteq A^{p+q}$; further, $x^p y^q = (-1)^{pq} y^q x^p$ for arbitrary elements $x^p \in A^p$ and $y^q \in A^q$ and $x^p x^p = 0$ for p odd. As is customary in this context, we assume A^0 is a free module generated by the identity element 1, that is, $A^0 = K \cdot 1$ and $k \mapsto k \cdot 1$ is an isomorphism $K \to A^0$. In addition, however, we make the assumption

$$\text{(Z)} \qquad A^2 = K \cdot e \quad \text{for some element } e,$$

that is, A^2 is cyclic.

Now consider a comultiplication $c : A \to A \otimes A$, that is, a morphism of graded K-algebras. We denote with $m : A \otimes A \to A$ the multiplication of the algebra, that is, we have $ab = m(a \otimes b)$. Our basic assumption is the hypothesis that the comultiplication c be *idempotent*, specifically, that the following condition is satisfied:

$$\text{(I)} \qquad mc = \mathbf{1}_A.$$

For simplicity, we denote the K-module A^1 with V, we write $W = V \otimes V$, and we use the algebra multiplication and hypothesis (Z) in order to define an antisymmetric bilinear from $b : V \times V \to K$ by the following condition:

$$\text{(1)} \qquad uv = b(u, v) \cdot e \quad \text{for } u, v \in V.$$

The bilinear form b corresponds to a linear form $b' : W \to K$ such that $b(u, v) = b'(u \otimes v)$. The comultiplication c defines module morphisms in degrees 1 and 2:

$$c^1 : V \to V \otimes \mathbf{1} \oplus \mathbf{1} \otimes V,$$
$$c^2 : K \cdot e \to K \cdot e \otimes \mathbf{1} \oplus W \oplus \mathbf{1} \otimes K \cdot e.$$

This allows us to introduce the following data.

Definition 2.1. There are well-defined K-module morphisms $f, g : V \to V$, well-defined elements $r, s \in K$ and a well-defined tensor $t \in W$ such that

$$c^1(v) = f(v) \otimes \mathbf{1} + \mathbf{1} \otimes g(v), \quad \text{for all } v \in V, \tag{2}$$

$$c^2(e) = r \cdot (e \otimes \mathbf{1}) + t + s \cdot (\mathbf{1} \otimes e). \tag{3}$$

Lemma 2.2. *If the comultiplication c is idempotent, then*

$$f + g = \mathbf{1}_V, \tag{4}$$

$$r + b'(t) + s = 1. \tag{5}$$

Proof. The idempotency (I) of c implies $m^1 \circ c^1 = \mathbf{1}_V$ in degree 1, and this condition, in view of (2), is equivalent to (4). Likewise, we have $m^2 \circ c^2 = \mathbf{1}_{K \cdot e}$ and, in view of (3), this means (5). $\square$

In the next lemma we use the function α of the introduction to Section 1 and the morphism $T : W \to W$ introduced in (4) of Section 1.

Lemma 2.3. *For any comultiplication c, the data $b, f, g, r, s,$ and t are connected by the following identities*:

$$b(f(u), f(v)) = rb(u, v) \quad \text{for all } u, v \in V. \tag{6}$$

$$b(g(u), g(v)) = sb(u, v) \quad \text{for all } u, v \in V. \tag{7}$$

$$\begin{aligned} b(u, v) \cdot t &= (f \otimes g)(u \otimes v - v \otimes u) \\ &= (f \otimes g)\alpha(u \otimes v) \\ &= T(u \otimes v) \quad \text{for all } u, v \in V. \end{aligned} \tag{8}$$

Proof. Since c is a morphism of K-algebras, we have the relation

$$c(uv) = c(u)c(v) \quad \text{for all } u, v \in V. \tag{$*$}$$

Because of (1), the left side of $(*)$ is $c(uv) = c(b(u,v)\cdot e) = b(u,v)\cdot c(e)$ which equals $b(u,v)\cdot(r\cdot(e\otimes 1)+t+s\cdot(1\otimes e))$ in view of (3). Because of (2), the right-hand side of $(*)$ is $c(u)c(v) = (f(u)\otimes 1 + 1\otimes g(u))(f(v)\otimes 1 + 1\otimes g(v)) = f(u)f(v)\otimes 1 + f(u)\otimes g(v) - f(v)\otimes g(u) + 1\otimes g(u)g(v)$, where we pay careful attention to the anticommutativity of A. Now $f(u)f(v)\otimes 1 = b(f(u),f(v))\cdot(e\otimes 1)$ by (1). Similarly, $1\otimes g(u)g(v) = b(g(u),g(v))\cdot(1\otimes e)$. Finally, $f(u)\otimes g(v) - f(v)\otimes g(u) = (f\otimes g)(u\otimes v - v\otimes u)$. If we finally evaluate $c(u)c(v)$ and compare the components in $K\cdot(e\otimes 1)$, W, and $K\cdot(1\otimes e)$, respectively, on both sides, we find precisely the equations (6), (8), and (7), respectively. $\square$

We shall now apply the results of Proposition 1.11. For this purpose, we need a definition.

Definition 2.4. (i) The multiplication of a graded algebra A is called *trivial up to degree* 2 if $A^1A^1 = \{0\}$.

(ii) A comultiplication c of a graded algebra A is called *trivial up to degree* 2 if $c(x) = x\otimes 1$ for all $x\in A^n$ or $c(x) = 1\otimes x$ for all $x\in A^n$ for $n = 0,1,2$.

Remark 2.5. Under our general hypotheses on A, the multiplication is trivial up to degree 2 if and only if $b = 0$, and the comultiplication is trivial up to degree 2 if and only if one of the following two possibilities occurs:

$$(f,g,r,s,t) = (\mathbf{1}_V, 0_V, 1, 0, 0) \quad \text{or} \quad (f,g,r,s,t) = (0_V, \mathbf{1}_V, 0, 1, 0).$$

Now we can formulate the following theorem:

Theorem 2.6. *Let A^* be a graded anticommutative algebra over a principal ideal domain K such that A^0 and A^2 are cyclic and A^1 torsion free. Suppose further that there is an idempotent comultiplication. Then exactly one of the following two possibilities occurs*:

(i) *The multiplication and the comultiplication are not trivial up to degree* 2 *and A^1 is free of rank* 2.

(ii) *The multiplication is trivial up to degree* 2 *and the following cases arise*:

(iia) $A^1 \neq \{0\}$ *and* $c(v) = v \otimes 1$ *for all* $v \in A^1$*, respectively,* $c(v) = 1 \otimes v$ *for all* $v \in V$*, and* $c(e) = r \cdot (e \otimes 1) + t + (1-r) \cdot (1 \otimes e)$ *with arbitrary elements* $t \in A^1 \otimes A^1$ *and* $r \in K$.

(iib) $\operatorname{rank} A^1 = 1$ *and* $c(v) = f(v) \otimes 1 + 1 \otimes (\mathbf{1}_V - f)(v)$ *for all* $v \in A^1$ *and* $c(e) = r \cdot (e \otimes 1) + (1-r) \cdot (1 \otimes e)$ *with an arbitrary endomorphism* $f \neq 0$, $\mathbf{1}_V$ *of the rank* 1*-module* A^1 *and an arbitrary* $r \in K$.

(iic) $A^1 = \{0\}$ *and* $c(e) = r \cdot (e \otimes 1) + (1-r) \cdot (1 \otimes e)$ *with an arbitrary* $r \in K$.

Proof. First assume that the multiplication and the comultiplication are not trivial up to degree 2. Then the assertion follows from Proposition 1.11 unless $f = 0$ or $f = \mathbf{1}_V$. Let us now suppose that $f = 1_V$; the case that $f = 0$, that is $g = \mathbf{1}_V$, is treated analogously. Then (7) implies $s = 0$ since $b \neq 0$. As $g = 0$, equation (8) implies that $t = 0$. Either (6) or (5) now implies $r = 1$. Then Remark 2.5 says that the comultiplication is trivial up to degree 2, contradicting our assumption.

Now we assume that the multiplication is trivial up to degree 2. Then (5) implies

$$r + s = 1. \tag{5*}$$

Again, we have to distinguish the case that $f = 0$ or $f = \mathbf{1}_V$ or its negation. Once more we shall treat the case that $f = \mathbf{1}_V$. This is Case (iia). Now suppose that $f \neq 0, \mathbf{1}_V$. We apply Proposition 1.11(b). Since multiplication is trivial up to degree 2, Case 1.11 (b1) is ruled out. In Case 1.11(b2) we have rank $A^1 = 1$ obtaining our Case (iib), and in Case 1.11(b3) we get our Case (iic). $\square$

We note that an endomorphism of a rank 1 module over a principal ideal domain K amounts to a scalar multiplication with a suitable element of the quotient field of K.

It remains to discuss in detail the Case (i) of nontrivial multiplication and comultiplication where A^1 is free of rank 2.

We begin this discussion with a remark on skew symmetric forms on a free K-module V of rank 2. Two forms b_1 and b_2 are equivalent if there is an automorphism $\varphi : V \to V$ such that $b_2(u, v) = b_1(\varphi(u), \varphi(v))$. Each skew symmetric form $b : V \times V \to K$ is uniquely determined by a linear form $b'' : \bigwedge^2 V \to K$ via $b(u, v) = b''(u \wedge v)$. If φ is an automorphism of V, then $b(\varphi(u), \varphi(v)) = b''((\bigwedge^2 \varphi)(u \wedge v)) = \det(\varphi) b''(u \wedge v) = \det(\varphi) v(u, v)$ by the definition of the determinant. It follows that b_1 and b_2 are equivalent if and only if there is an automorphism φ of V such that $b_2 = \det(\varphi) b_1$. Let K^* denote the group of units of K. Since $\det : \mathrm{Aut}(V) \to K^*$ is surjective, the equivalence classes of such forms under automorphisms of V correspond bijectively to the cosets of the multiplicative semigroup of K modulo K^*. If we fix a basis of V, then each skew symmetric bilinear form is equivalent to one which is expressed by the matrix

$$a \cdot H, \quad \text{where } H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{9}$$

and where $a \in K$ is uniquely determined modulo K^* by the equivalence class.

Theorem 2.7. *Let A^* be a graded anticommutative algebra over a principal ideal domain K such that A^0 and A^2 are cyclic and A^1 is torsion free. Suppose further that there is an idempotent comultiplication which is not trivial up to degree 2 and that the multiplication is not trivial up to degree 2. Then A^1 is a free K-module V of rank 2 equipped with a nondegenerate skew symmetric bilinear map $b : V \times V \to K$. Moreover, there exists a basis e_1, e_2 of V, an endomorphism $f \neq 0, \mathbf{1}_V$ of V, and ring elements $a, m_{jk} \in K$, $j, k = 1, 2$, such that $b(e_1, e_2) = a$ and that the comultiplication c in degrees 1 and 2 is given by*

$$\begin{aligned} c^1(v) &= f(v) \otimes 1 \oplus 1 \otimes (\mathbf{1}_V - f), \\ c^2(e) &= \det f \cdot (e \otimes 1) \\ &\quad + \sum_{j,k=1,2} m_{jk} \cdot (e_j \otimes e_k) \\ &\quad + (1 - \operatorname{tr} f + \det f) \cdot (1 \otimes e). \end{aligned}$$

The matrix coefficients a_{jk} of f and the m_{jk} are related as follows:

$$
\begin{aligned}
am_{11} &= -a_{12}, \\
am_{22} &= a_{21}, \\
am_{12} &= a_{11} - \det f, \\
am_{21} &= -a_{22} + \det f.
\end{aligned} \tag{8†}
$$

The coefficients a_{jk} satisfy the following congruence relations:

$$
\begin{aligned}
&a_{12} \equiv a_{21} \equiv 0 \pmod{a}, \\
&a_{11} \equiv a_{22} \pmod{a}, \\
&a_{11}^2 \equiv a_{11} \pmod{a}.
\end{aligned} \tag{10}
$$

Proof. We select a basis of V so that b has the matrix representation (9) with $a \neq 0$. We may now express an element u of V as a pair (x, y) with $x, y \in K$ and identify $V \otimes V$ with the space $M_2(K)$ of 2×2-matrices in such a way that the tensor $u \otimes v$ of $u = (x, y)$ and $v = (x', y')$ is given by the matrix

$$
\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x' & y' \end{pmatrix} = \begin{pmatrix} xx' & xy' \\ yx' & yy' \end{pmatrix}.
$$

Then we have

$$
\begin{aligned}
\alpha(u \otimes v) &= \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x' & y' \end{pmatrix} - \begin{pmatrix} x' \\ y' \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} \\
&= \begin{pmatrix} 0 & xy' - x'y \\ x'y - xy' & 0 \end{pmatrix} = \det(u, v) \cdot H,
\end{aligned}
$$

where $\det(u, v) = xy' - x'y$.

Let us write the matrix of f as

$$
F = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},
$$

whence $\mathbf{1}_V - f$ has the matrix

$$
E_2 - F = \begin{pmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{pmatrix}.
$$

Condition (6) now translates into the matrix form $a \cdot FHF^{\mathrm{T}} = ra \cdot H$, where $(\cdot)^{\mathrm{T}}$ denotes transposition. Since $a \neq 0$ and K is a domain, this is equivalent to

$$FHF^{\mathrm{T}} = r \cdot H. \tag{6*}$$

An explicit matrix calculation shows $FHF^{\mathrm{T}} = \det f \cdot H$, whence (6*) and thus (6) is equivalent to

$$r = \det f. \tag{6**}$$

Analogously, (7) is equivalent to $s = \det(\mathbf{1}_V - f)$. An explicit computation shows that $\det(\mathbf{1}_V - f) = 1 - \operatorname{tr} f + \det f$ where $\operatorname{tr} f$ is the trace of f. Thus, in the presence of (6**), condition (7) is equivalent to

$$1 + r - s = \operatorname{tr} f. \tag{7**}$$

In degree 2, the given comultiplication yields via (3) a tensor t which corresponds to a matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

The linear map $b' : M_2(K) \to K$ is determined by its values on the basis elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

With the aid of the formula $b((x,y),(x',y')) = b' \left(\begin{smallmatrix} xx' & xy' \\ yx' & yy' \end{smallmatrix}\right)$ we calculate these values to be $0, 0, a$, and $-a$, respectively, since b has the matrix $a \cdot H$. Thus,

$$b'(t) = a(m_{12} - m_{21}). \tag{11}$$

Now (5), in view of (6**), (7**) and (11) is equivalent to

$$a(m_{12} - m_{21}) = \operatorname{tr} f - 2 \det f. \tag{5**}$$

Finally, we shall exploit (8). First we write $g = \mathbf{1}_V - f$ as before, and thus also $G = E_2 - F$. Now we calculate $(f \otimes (\mathbf{1}_V - f)) \circ \alpha(u \otimes v) =$

$f(u)\otimes g(v)-f(v)\otimes g(u) = F\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)(x' \quad y')G^{\mathrm{T}} - F\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right)(x \quad y)G^{\mathrm{T}}$, but this matrix equals $F\left(\begin{smallmatrix} 0 & xy'-x'y \\ x'y-xy' & 0 \end{smallmatrix}\right)G^{\mathrm{T}} = \det(u,v)\cdot FH(\mathbf{1}_V - F)^{\mathrm{T}}$. On the other hand, $b(u,v)\cdot t = (x \quad y)(a\cdot H)\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right)M = a\det(u,v)\cdot M$. Hence, (8) is equivalent to

$$a\cdot M = FH(E_2 - F)^{\mathrm{T}}. \tag{8*}$$

An explicit calculation, however, yields

$$FH(E_2-F)^{\mathrm{T}} = (F - \det f\cdot E_2)H.$$

Thus (8) is equivalent to

$$a\cdot M = (F-\det f\cdot E_2)H. \tag{8**}$$

Translated into coefficients, this last relation is equivalent to the following set of linear equations for the coefficients m_{jk} of M:

$$\begin{aligned} am_{11} &= -a_{12}, \\ am_{22} &= a_{21}, \\ am_{12} &= a_{11} - \det f \\ &= a_{11}(1-a_{22}) + a_{12}a_{21}, \\ am_{21} &= -a_{22} + \det f \\ &= a_{22}(a_{11}-1) - a_{12}a_{21}. \end{aligned} \tag{8$\dagger$}$$

We note from this that (5^{**}) is a consequence of (8†).

We claim that

$$\begin{aligned} a_{12} &\equiv a_{21} \equiv 0 \pmod{a}, \\ a_{11} &\equiv a_{22} \pmod{a}, \\ a_{11}^2 &\equiv a_{11} \pmod{a}. \end{aligned} \tag{10}$$

By (8†), a divides a_{12}, a_{21}, and $a_{22}-a_{11}$, hence also $a_{11}(a_{22}-1)$; since $a_{22}\equiv a_{11} \pmod{a}$, this proves the claim. □

Corollary 2.8. *Let $A^* = A^0 \oplus A^1 \oplus A^2$ be a graded anticommutative algebra over a principal ideal domain K such that A^0 and A^2 are cyclic* *and A^1 is torsion free. Suppose further that the multiplication is not trivial. The following are necessary and sufficient conditions for a nontrivial idempotent comultiplication c to exist on A^*.*

(i) *The degree 1 component A^1 is a free K-module V of rank 2 equipped with a nondegenerate skew symmetric bilinear map $b : V \times V \to K$.*

(ii) *There exist a basis e_1, e_2 of V and ring elements $a, a_{jk}, m_{jk} \in K$, $j, k = 1, 2$, such that*

(iia) *$b(e_1, e_2) = a$ and*

(iib) *with the endomorphism $f \neq 0, \mathbf{1}_V$ of V whose matrix with respect to the basis e_1, e_2 is*

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

the following system of equations is satisfied;

$$\begin{aligned} am_{11} &= -a_{12}, \\ am_{22} &= a_{21}, \\ am_{12} &= a_{11} - \det f, \\ am_{21} &= -a_{22} + \det f. \end{aligned} \tag{8†}$$

Under these circumstances, the comultiplication c is given by

$$\begin{aligned} c^1(v) &= f(v) \otimes 1 \oplus 1 \otimes (\mathbf{1}_V - f), \\ c^2(e) &= \det f \cdot (e \otimes 1) \\ &\quad + \sum_{j,k=1,2} m_{jk} \cdot (e_j \otimes e_k) \\ &\quad + (1 - \operatorname{tr} f + \det f) \cdot (1 \otimes e). \end{aligned}$$

The coefficients a_{jk} satisfy the following congruence relations:

$$\begin{aligned} a_{12} &\equiv a_{21} \equiv 0 \pmod{a}, \\ a_{11} &\equiv a_{22} \pmod{a}, \\ a_{11}^2 &\equiv a_{11} \pmod{a}. \end{aligned} \tag{10}$$

Proof. By Theorem 2.7 we know that the stated conditions are necessary. Its proof, however, shows that they are also sufficient. □

We shall now relax the requirement that K is a principal ideal domain and that A^2 is cyclic. We shall assume merely that K is a domain and that A^2 has K-rank one. In this case we consider the quotient field Q of K and the ground ring extension $A^*_Q = A^0_Q \oplus A^1_Q \oplus \cdots$, where $A^n_Q = Q \otimes A^n$. We may assume that A^* is a graded K-subalgebra of A^*_Q (when considered as a K-algebra). Since the comultiplication of A^*_Q remains nontrivial if that of A^* is nontrivial, then Theorems 2.6 and 2.7 and Corollary 2.8 apply at once to A^*_Q. If the multiplication of A^* is trivial up to degree 2, then the same is true for A^*_Q. This yields the information contained in Theorem 2.6(ii) for A^*_Q with Q in place of K. Scalar multiplications with $r, 1-r \in Q$, however, will preserve A^1. If the multiplication in A^* is not trivial up to degree 2, then the same holds for A^*_Q and by Theorem 2.6(i), the K-module $V = A^1$ is of rank 2. Furthermore, Theorem 2.7 applies to A_Q with Q in place of K and with V_Q in place of V. We select the basis $\{e_1, e_2\}$ of V_Q in V. Then the elements a, m_{jk}, $j, k = 1, 2$, persist to be in K. However, the matrix coefficients a_{jk} need not be all in K; yet equations (8†) show that a_{12}, a_{21} and the trace of f, namely, $a_{11} + a_{22}$ are all in K. The equations (10) are without relevance over the field Q. For easy reference we formulate these conclusions at least in the case that $A^n = \{0\}$ for $n \geq 3$ (the situation of Corollary 2.8). It is understood that for a K-module V the ground field extension $Q \otimes V$ is denoted V_Q with V identified with a submodule of V_Q and that f_Q for an endomorphism f of V denotes the extension to V_Q.

Theorem 2.9. *Let $A^* = A^0 \oplus A^1 \oplus A^2$ be a graded anticommutative algebra over a domain K such that A^0 is cyclic, A^2 is of K-rank one, and A^1 is torsion free. Suppose further that the multiplication is not trivial. Let Q denote the quotient field of K. If c is a nontrivial idempotent comultiplication c on A^*, then the following conclusions hold*:

(i) *The degree 1 component A^1 is a K-module V of rank 2 equipped with a nondegenerate skew symmetric bilinear map $b : V \times V \to K$.*

(ii) *There exist a free subset* $\{e_1, e_2\}$ *of* V, *ring elements* $a, m_{jk} \in K$, $j, k = 1, 2$, *and an endomorphism* f *of* V *different from* $0, \mathbf{1}_V$ *such that*

(iia) $b(e_1, e_2) = a$ *and*

(iib) *the endomorphism* f_Q *of* V_Q *has a matrix over* Q *with respect to the basis* e_1, e_2 *of the form*

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

such that the following system of equations is satisfied:

$$\begin{aligned} am_{11} &= -a_{12}, \\ am_{22} &= a_{21}, \\ am_{12} &= a_{11} - \det f, \\ am_{21} &= -a_{22} + \det f. \end{aligned} \tag{8†}$$

In particular, a_{12}, a_{21}, *and* $\operatorname{tr} f = a_{11} + a_{22}$ *are contained in* K. *Under these circumstances, the comultiplication* c *is given by*

$$\begin{aligned} c^1(v) &= f(v) \otimes 1 \oplus 1 \otimes (\mathbf{1}_V - f)(v), \\ c^2(e) &= \det f \cdot (e \otimes 1) \\ &\quad + \sum_{j,k=1,2} m_{jk} \cdot (e_j \otimes e_k) \\ &\quad + (1 - \operatorname{tr} f + \det f) \cdot (1 \otimes e). \qquad \square \end{aligned}$$

We remark that there are further conditions imposed on the matrix coefficients a_{11} and a_{22} of f_Q due to the fact that the K-submodule V of V_Q is invariant under f_Q and $\mathbf{1}_{V_Q} - f_Q$, and that the scalar multiplications with $\det f$ and $1 - \operatorname{tr} f + \det f$ leave A^2 invariant as a consequence of the particular form of $c^2(e)$ above.

3. Multiplications on groups. In the next section we shall need information on idempotent homomorphic multiplications on groups. Here an *idempotent homomorphic multiplication on a group* G is a homomorphism $m : G \times G \to G$ such that $m(x, x) = x$ for all $x \in G$. The set of all idempotent multiplications on G will be denoted $\operatorname{idhom}(G)$.

Definition 3.1. Let G be a group. For any function $f : G \to G$ we define $m_f : G \times G \to G$ by $m_f(g,h) = f(g)f(h)^{-1}h$. For each function $m : G \times G \to G$ we define $f_m : G \to G$ by $f(g) = m(g,1)$.

Remark 3.2. For any f the function m_f satisfies $m_f(g,g) = g$ and $f_{m_f} = f$.

In a group G we shall write $[g,h]$ for the commutator $ghg^{-1}h^{-1}$ of two elements.

Lemma 3.3. *Let f be an endomorphism of a group G. Then the following conditions are equivalent*:

(1) $$[f(g), f(h)^{-1}h] = 1 \quad \text{for all } g,h \in G.$$

(2) $m_f : G \times G \to G$ *defines an idempotent homomorphic multiplication of G.*

Proof. The idempotency of m_f is clear by Remark 3.1. First assume (1). A direct computation will show that m_f is a homomorphism: $m_f(gg', hh') = f(gg')f(hh')^{-1}hh' = f(g)f(g')f(h')^{-1}(f(h)^{-1}h)h' = f(g)f(h)h^{-1}f(g')f(h')^{-1}h'$ by (1). This last term is $m_f(g,h)m_f(g',h')$, and thus (2) is proved. Conversely, assume (2). Then $f(h)^{-1}hf(g) = m_f(1,h)m_f(g,1) = m_f(g,h)$ by (2). But $m_f(g,h) = f(g)f(h)^{-1}h$. This implies (1).

Lemma 3.4. *If $m : G \times G \to G$ is an idempotent homomorphic map, then the $f_m : G \to G$ given by $f_m(g) = m(g,1)$ is an endomorphism satisfying $m = m_{f_m}$. In particular, f_m satisfies* (1) *above.*

Proof. Trivially, $f_m(g)f_m(h) = m(g,1)m(h,1) = m(gh,1) = f_m(gh)$. Further,

$$\begin{aligned} m_{f_m}(g,h) &= f_m(g)f_m(h)^{-1}h = m(g,1)m(h,1)^{-1}h \\ &= m(gh^{-1},1)h = m(gh^{-1},1)m(h,h) = m(g,h) \end{aligned}$$

in view of the idempotency of m. This proves the lemma.

Lemma 3.5. *Condition* (1) *of Lemma* 3.3 *implies that* $[g, f(g)] = 1$ *for all* g *and is equivalent to the following*:
(3)
f *commutes with all inner automorphisms* $I_g = (x \mapsto gxg^{-1})$ *of* G.

Proof. If, in condition (1) one takes $h = g$, one sees $g = f(g)(f(g)^{-1}g) = (f(g)^{-1}g)f(g)$ which is equivalent to $[g, f(g)^{-1}] = 1$, and this is equivalent to $[g, f(g)] = 1$. Now we prove the implication (1) $\Rightarrow$ (3):

$$\begin{aligned}(f \circ I_g)(x) &= f(gxg^{-1}) = f(g)f(x)f(g)^{-1} \\ &= gg^{-1}f(g)f(x)f(g)^{-1} = gf(x)g^{-1}f(g)f(g)^{-1}, \qquad \text{(by (1))} \\ &= gf(x)g^{-1} = (I_g \circ f)(x)\end{aligned}$$

for all $x \in G$, which is (2). Now we show (3) $\Rightarrow$ (1): In view of (3) we compute

$$\begin{aligned}f(g)f(h)^{-1} &= f(gh^{-1}) = f(h)^{-1}f(I_h g) \\ &= f(h)^{-1}I_h f(g) = f(h)^{-1}hf(g)h^{-1}\end{aligned}$$

which implies (1). $\square$

Definition 3.6. Let $\mathrm{End}^*(G)$ denote the set of all endomorphisms f of G satisfying the equivalent conditions (1) and (3).

According to condition (3), the set $\mathrm{End}^*(G)$ is the centralizer of the group $\mathrm{Inn}(G)$ of all inner automorphisms of G in the semigroup $\mathrm{End}(G)$ of all endomorphisms and is, in particular, a subsemigroup. The elements of $\mathrm{End}^*(G)$ are sometimes called the normal operators on G.

Proposition 3.7. *For any group the function* $f \mapsto m_f : \mathrm{End}^*(G) \to \mathrm{idhom}(G)$ *is a bijection with inverse* $m \mapsto f_m$. *The identity and the zero of* $\mathrm{End}^*(G)$ *yield the two trivial multiplications.*

It is clear now that we have to classify the endomorphisms satisfying (1). For any function $f : G \to G$ on a group G we define $f^* : G \to G$

by $f^*(g) = f(g)^{-1}g$, that is, by $f(g)f^*(g) = 1$ for all $g \in G$. With this notation, condition (1) is equivalent to

$$(1') \qquad [f(g), f^*(h)] = 1 \quad \text{for all } g, h \in G.$$

Moreover, $f^{**}(g) = f^*(g)^{-1}g = (f(g)^{-1}g)^{-1}g = I_{g^{-1}}f(g)$.

Lemma 3.8. *If $f \in \mathrm{End}^*(G)$, then $f^* \in \mathrm{End}^*(G)$ and $f^{**} = f$.*

Proof. If f satisfies (1), then straightforward calculation shows that, due to $[g, f(g)] = 1$, the function f^* is an endomorphism. Also, (3) implies $I_{g^{-1}}f(g) = f(g)$, whence $f^{**} = f$. Now $(1')$ is symmetric in f and f^*, whence $f^* \in \mathrm{End}^*(G)$. $\square$

We notice that an element x is in the kernel of f if and only if it is left fixed by f^* and vice versa.

Lemma 3.9. *If $f \in \mathrm{End}^*(G)$ and if we set $A = \mathrm{im}\, f$, $B = \mathrm{im}\, f^*$, then*

(i) $[A, B] = \{1\}$ *and* $A \cap B \subseteq Z(G)$,

(ii) $G = AB$, *and*

(iii) $G \cong (A \times B)/D$ *with* $D = \{(a, a^{-1}) \mid a \in A \cap B\} \cong A \cap B$.

Proof. The proof of (i) is immediate from $(1')$, that of (ii) from $f(g)f^*(g) = 1$ and from (i). $\square$

Proposition 3.10. *If $Z(G) = \{1\}$, then the following statements are equivalent*:

(i) $f \in \mathrm{End}^*(G)$.

(ii) $f^2 = f$ *and $f(G)$ is normal, that is, G is the direct product of* $\mathrm{im}\, f$ *and* $\ker f$.

Proof. The implication (ii) $\Rightarrow$ (i) is immediate. We shall now prove (i) $\Rightarrow$ (ii). Let A and B be as in Lemma 3.9. Since $Z(G) = \{1\}$ the group G is the direct product of the subgroups A and B. If $a \in A$

and $b \in B$, (3) implies $af(b)a^{-1} = f(aba^{-1}) = f(b)$, since a and b commute. Thus $[A, f(B)] = \{1\}$. But $Z(A) = \{1\}$; hence $f(B) = \{1\}$. Also, $f(A) = f(AB) = f(G) = A$. Since $\ker f = \operatorname{fix} f^* \subseteq B$, we know that $\varphi = f \mid A : A \to A$ is an automorphism commuting with all inner automorphisms. By (1′) we have $\varphi^*(A) \subseteq Z(A) \subseteq Z(G) = \{1\}$, whence $\varphi = \mathrm{id}_A$. This means that $f^2 = f$, $\operatorname{im} f = A$. The proof is of the equivalence of (i), and (ii) is complete.

Under these circumstances, the members of $\operatorname{End}^*(G)$ are called decomposition endomorphisms (see [**11**; p. 238, 2.3.5]). From loc.cit., p. 238, Theorem 14, we know that $\operatorname{End}^*(G)$ is abelian. In other words, $\operatorname{End}^*(G)$ is a semilattice. $\square$

We discuss an example which is relevant for the classification of the idempotent multiplications on the Klein bottle. We let $\eta : \mathbf{Z} \to \{0, 1\}$ denote the nontrivial group homomorphism

$$\eta(z) = \frac{1}{2}(1 - (-1)^z).$$

Proposition 3.11. *Let G denote the semidirect product $\mathbf{Z} \rtimes_\delta \mathbf{Z}$ with the unique nonconstant homomorphism $\delta : \mathbf{Z} \to \operatorname{Aut} \mathbf{Z}$. Then*

(i) *for each endomorphism φ of G there are integers p, r, s such that*

$$\begin{aligned}\varphi(x, y) &= \begin{cases} (py, sy), & \text{if } s \equiv 0 \pmod 2; \\ (rx + \eta(y)p, sy), & \text{if } s \equiv 1 \pmod 2 \end{cases} \\ &= (r\eta(s)x + p(\eta(s)\eta(y) + \eta(1-s)y), sy).\end{aligned}$$

(ii) *The endomorphism φ is in $\operatorname{End}^*(G)$ if and only if φ agrees with φ_s, $s \in \mathbf{Z}$ given by*

$$\varphi_s(x, y) = (\eta(s)x, sy).$$

Proof. We begin by observing that the multiplication in G is given by

$$(x, y)(x', y') = (x + (-1)^y x', y + y'). \tag{4}$$

(i). We consider an endomorphism φ of G. The commutator group G' is $2\mathbf{Z} \times \{0\}$, and thus $\varphi((1,0)^2) = \varphi(2,0) \in G'$. Set $\varphi(1,0) = (a,b)$. Then $(a + (-1)^b a, 2b) = \varphi(1,0)^2 \in G'$, and thus $b = 0$. Hence, $\varphi(\mathbf{Z} \times \{0\}) \subseteq \mathbf{Z} \times \{0\}$. Since φ induces an endomorphism on the factor group $G/(\mathbf{Z} \times \{0\})$ we can write $\varphi(x,y) = (\alpha(x,y), sy)$ with an integer s and a function $\alpha : \mathbf{Z}^2 \to \mathbf{Z}$ which we have yet to determine. A quick calculation shows the functional equation

$$\alpha(x + (-1)^y x', y + y') = \alpha(x,y) + (-1)^{sy} \alpha(x', y'). \tag{5}$$

We first set $x' = y = 0$ and deduce $\alpha(u,v) = \beta(u) + \gamma(v)$ with $\beta(u) = \alpha(u,0)$ and $\gamma(v) = \alpha(0,v)$. If in (5) we set $y = y' = 0$, we obtain $\beta(x + x') = \beta(x) + \beta(x')$. Thus, there is an integer r such that $\beta(u) = ru$. Finally, in (5) we set $x = x' = 0$ and derive $\gamma(y+y') = \gamma(y) + (-1)^{sy}\gamma(y')$. That is, γ is a cocycle and $y' = 0$ shows $\gamma(0) = 0$. Now we apply (5) with $x = y' = 0$, $x' = y = 1$ and derive $(-1)r + \gamma(1) = \gamma(1) + (-1)^s r$, that is, $(1 + (-1)^s)r = 0$. Thus, $s \equiv 0 \pmod 2$ implies $r = 0$.

If $s \equiv 0 \pmod 2$, then γ is an endomorphism, hence there is an integer p such that $\gamma(y) = py$. In this case, $\varphi(x,y) = (py, sy)$.

Now suppose that $s \equiv 1 \pmod 2$. We set $\gamma(1) = p$. Then $\gamma(1+y) = p - \gamma(y)$. Inductively, we obtain

$$\gamma(y) = \begin{cases} 0, & \text{if } y \equiv 0 \pmod 2, \\ p, & \text{if } y \equiv 1 \pmod 2. \end{cases} \tag{6}$$

But (6) is equivalent to

$$\gamma(y) = \frac{1}{2}(1 - (-1)^y)p = \eta(y)p. \tag{7}$$

Now we proceed to the proof of (ii). Let $I_{(p,t)}$ denote the inner automorphism implemented by (p,t). We note that $(p,t)^{-1} = (-(-1)^t p, -t) = ((-1)^{t+1}p, -t)$. Hence,

$$\begin{aligned} I_{(p,t)}(x,y) &= (p,t)(x,y)(p,t)^{-1} \\ &= (p,t)(x + (-1)^{y+t+1}p, y - t) \\ &= (p + (-1)^t x + (-1)^{y+1}p, t + y - t) \\ &= ((-1)^t x + (1 - (-1)^y)p, y). \end{aligned}$$

In order for the endomorphism φ to commute with all inner automorphisms, it is necessary and sufficient that φ commute with $I_{(0,1)}$, $I_{(1,0)}$ since (0,1) and (1,0) generate G. Now suppose that $\varphi(x,y) = (rx + (1-(-1)^y)p/2, sy)$ with $s \equiv 1 \pmod 2$. Then $\varphi I_{(0,1)}(x,y) = \varphi(-x,y) = (-rx + (1-(-1)^y)p/2, y)$, while $I_{(0,1)}\varphi(x,y) = (-rx - (1-(-1)^y)p/2, sy) = (-p,s)$. The equality of these two expressions for all (x,y) is equivalent to $p = 0$. We now assume $p = 0$. Similarly, $\varphi I_{(1,0)}(x,y) = \varphi((x + (1-(-1)^y), y), 1) = (rx + r(1-(-1)^y), sy)$ and $I_{(0,1)}\varphi(x,y) = I_{(0,1)}(rx, sy) = (rx + (1-(-1)^{sy}s), s)$. The equality of these two expressions for all x, y is equivalent to $r = 1$. Thus, if $s \equiv 1 \pmod 2$, then $\varphi \in \mathrm{End}^*(G)$ if and only if $\varphi(x,y) = (x, sy)$.

On the other hand, if $\varphi(x,y) = (py, sy)$ with $s \equiv 0 \pmod 2$, then $I_{(0,1)}\varphi(0,1) = I_{(0,1)}(p,s) = (-p,s)$ and $\varphi I_{(0,1)}(0,1) = \varphi(0,1) = (p,s)$. Hence, $p = 0$. One verifies quickly that $\varphi(x,y) = (0, sy)$ yields a φ in $\mathrm{End}^*(G)$. A recombination of the two cases yields exactly the assertion (ii). $\square$

We derive a complete list of homomorphic idempotent multiplications on G:

Corollary 3.12. *The idempotent homomorphic multiplications* $m : G \times G \to G$ *on the group* $G = \mathbf{Z} \rtimes_\delta \mathbf{Z}$ *are as follows*:
(8)
$$\begin{aligned} m((u,v),(x,y)) &= \begin{cases} ((-1)^{v-y}x, sv + (1-s)y), & \text{if } s \equiv 0 \pmod 2; \\ (u, sv + (1-s)y), & \text{if } s \equiv 1 \pmod 2 \end{cases} \\ &= ((-1)^{v-y}\eta(1-s)x + \eta(s)u, sv + (1-s)y). \end{aligned}$$

Proof. By Proposition 3.7, every idempotent multiplication on G is of the form m_φ with some $\varphi \in \mathrm{End}^*(G)$. By Definition 3.1,

$$m_\varphi((u,v),(x,y)) = \varphi(u,v)\varphi(x,y)^{-1}(x,y). \tag{9}$$

From Proposition 3.11, we know that $\varphi \in \mathrm{End}^*(G)$ implies $\varphi(a,b) = (ra, sb)$ with $r = (1-(-1)^s)/2$. Then

$$\begin{aligned} m_\varphi((u,v),(x,y)) &= (ru,sv)(rx,sy)^{-1}(x,y) \\ &= (ru,sv)(-(-1)^{sy}rx,-sy)(x,y) \\ &= (ru,sv)(-(-1)^{sy}rx+(-1)^{-sy}x,-sy+y) \\ &= (ru+(-1)^{sv}((-1)^{-sy}(1-r)x),su-sy+y), \end{aligned}$$

whence

$$m_\varphi((u,v),(x,y)) = (ru+(1-r)(-1)^{s(v-y)}x, sv+(1-s)y). \tag{10}$$

However, (10) and (8) are equivalent statements. □

4. Aspherical CW-complexes. We recall that a connected space X is called *aspherical* if $\pi_n(X)$ for all $n \geq 2$. If $\mu : X \times X \to X$ is an idempotent multiplication, then, upon identifying $\pi_1(X \times X)$ with $\pi_1(X) \times \pi_1(X)$, on the group $G = \pi_1(X)$ we obtain a group homomorphism $m = \pi_1(\mu)$, $m : G \times G \to G$ with $md_G = \mathbf{1}_G$. Such an m is *a homomorphic idempotent multiplication on the group* G. If X is an aspherical connected CW-complex, then every such m is induced by a continuous function $\mu : X \times X \to X$ (see [**15**; Chap. V, Sec. 4, pp. 224-225]) which, by a remark in our introduction is homotopic to an idempotent multiplication (inducing, of course, the same m). In short, *the set* $\mathrm{id}(X)$ *of homotopy classes of idempotent multiplications of* X *is mapped bijectively under the function* $[\mu] \mapsto \pi_1(\mu)$ *onto the set* $\mathrm{idhom}(G)$ *of homomorphic idempotent multiplications of* $G = \pi_1(X)$. Therefore, we have classified the idempotent homomorphic multiplications of an arbitrary group G up to homotopy in Section 3. This yields, in particular, a classification of the homotopy classes of nontrivial idempotent multiplications on all Eilenberg–MacLane complexes $K(G,1)$. (See, for example, [**14**, p. 95].)

The simplest examples of aspherical Eilenberg–MacLane complexes $K(G,1)$ are the surfaces with the exception of the sphere S^2 and the projective plane P^2. In the case of aspherical surfaces we shall now make the classification of idempotent multiplications up to homotopy explicit. According to [**4**], the aspherical surfaces fall into three classes:

(1) Abelian fundamental group.

  (a) the plane,

(b) the cylinder,

(c) the Möbius band, and

(d) the 2-torus.

For this class we have $\mathrm{End}^*(\pi_1(X)) = \mathrm{End}(\pi_1(X))$ and *every* endomorphism of $\varphi_1(X)$ yields an idempotent homomorphic multiplication of $\pi_1(X)$ by Proposition 3.7.

(2) Nonabelian fundamental group with nontrivial center.

(e) The Klein bottle.

Corollary 3.12 has classified all idempotent homomorphic multiplications on $\pi_1(X) \cong \mathbf{Z} \rtimes_\delta \mathbf{Z}$.

(3) Nonabelian fundamental group with trivial center.

(f) All other surfaces.

The remaining task is to determine $\mathrm{idhom}(\pi_1(X))$.

Lemma 4.1. *If X is a surface, then $G = \pi_1(X)$ cannot be a product AB of two subgroups satisfying $[A, B] = \{1\}$ and $A \not\subseteq B$ such that B is nonabelian.*

Proof. Assume the contrary. Let Z denote the cyclic subgroup of A generated by some element $a \notin B$. Then $[Z, B] = \{1\}$ and, thus, ZB Is a subgroup with Z in its center. Hence, we find a covering surface Y of X such that $\pi_1(Y) \cong ZB$, and in view of the classification of surfaces, Y is a Klein bottle. Then $ZB \cong \pi_1(Y) \cong \mathbf{Z} \rtimes_\delta \mathbf{Z}$. The center of $\mathbf{Z} \rtimes_\delta \mathbf{Z}$ is $\{0\} \times 2\mathbf{Z}$, and the image of Z must be a subgroup of this group. Since no nontrivial central subgroup of $\mathbf{Z} \rtimes_\delta \mathbf{Z}$ splits, the central subgroup $Z \cap B$ must be nontrivial. Hence, there is a natural number $n \geq 1$ such that

$$(Z/(Z \cap B)) \times (B/(Z \cap B)) \cong ZB/(Z \cap B) \cong \mathbf{Z} \rtimes_{\delta'} \mathbf{Z}(2n),$$

where δ' is induced by δ. The center of $\mathbf{Z} \rtimes_{\delta'} \mathbf{Z}(2n)$ is $\{0\} \times \mathbf{Z}(4n)$, and thus no nontrivial central subgroup splits. This implies $a \in Z \subseteq B$; hence, $a \in B$, a contradiction. $\square$

Proposition 4.3. *If, on an aspherical surface X, there is an idempotent multiplication which is not homotopic to one of the two* *projections, then X is a cylinder, a Möbius band, a 2-torus or a Klein bottle.*

Proof. Lemma 4.1 and Proposition 3.10 rule out the surfaces whose fundamental group has trivial center. Thus, the classification of surfaces proves the assertion. □

5. Cohomology 2-surfaces.

Definition 5.1. Let K be an integral domain. A K-*cohomology surface* is a Hausdorff space X such that the Čech-cohomology groups $H^n(X;K)$ over K vanish for $n > 2$ and $H^2(X,K)$ is either zero or torsion free of rank 1, that is, $H^2(X,K)$ may be isomorphically embedded into the additive group of the quotient field of K.

Remark 5.2. All compact abelian groups of topological dimension 2 are K-cohomology surfaces for all K.

Proof. If X is a compact abelian group, then its character group $\widehat{X}$ is naturally isomorphic to $H^1(X,\mathbf{Z}) \cong [X,S^1]$. Moreover, $H^n(X,\mathbf{Z}) \cong \bigwedge^n \widehat{X}$ (see, e.g., [**5**]). The space X is topologically two-dimensional if and only if rank $\widehat{X} = 2$. Thus, every compact two-dimensional group is a $\mathbf{Z}$-cohomology surface. Since the $\mathbf{Z}$-cohomology groups are torsion free in dimensions $n > 0$, the universal coefficient theorem shows that they are K-cohomology surfaces for each domain K. □

We note that, in particular, *every* torsion free abelian group F of rank 2 is the first integral cohomology group of a cohomology surface, namely $\widehat{F}$.

An idempotent multiplication $m : X \times X \to X$ on a cohomology surface induces an idempotent comultiplication on the cohomology ring $A = H^*(X,K)$ since $H^2(X,K)$ is torsion free.

Then Theorems 2.6 and 2.7, Corollary 2.8, and Theorem 2.9 classify the cohomology classes $H^*(m;K)$ of the possible comultiplications. In particular, we have the following consequence.

Proposition 5.3. *Consider a K-cohomology manifold X for which the cup product does not induce the zero map $H^1(X,K)\otimes H^1(X,K)\to H^2(X,K)$. If there is an idempotent multiplication m on X which is not cohomologous to one of the two projections, then $H^1(X,K)$ has rank* 2.

We wish to observe in the following remarks that the algebraic situations described in these results actually do arise from idempotent comultiplications of compact cohomology surfaces.

Remark 5.4. Let F be any torsion-free abelian group of rank 2. Then $A=\bigwedge F$ is an algebra satisfying the hypotheses of Theorem 2.9. Every endomorphism $f:F\to F$ gives an endomorphism $\hat{f}$ of the compact abelian group $X=\widehat{F}$. If $m:X\times X\to X$ is given by $m(x,y)=\hat{f}(x)+(1-\hat{f})(y)$, then m induces a comultiplication on A with the properties described in Theorem 2.9. Thus, in Theorem 2.6 the case (i) is topologically realized.

We shall note that the cases listed under (ii) in Theorem 2.6 are topologically realized.

We shall make repeated reference to the universal coefficient theorem which provides two exact sequences (split in a nonnatural fashion):

(U1)
$$0\to H^n(X;\mathbf{Z})\otimes K\to H^n(X;K)\to \mathrm{Tor}(H^{n+1}(X,\mathbf{Z}),K)\to 0,\qquad n=0,1,\dots,$$

and
(U2)
$$0\to \mathrm{Ext}(H_{n-1}(X),K)\to H^n(X;K)\to \mathrm{Hom}(H_n(X;\mathbf{Z});K)\to 0,\qquad n=1,2,\dots.$$

In particular, for a path-connected space X, the singular integral homology group $H_0(X)$ is cyclic and, therefore, the group $\mathrm{Ext}(H_0(X),K)$ vanishes. The exact sequence

$$0\to\pi_1(X)'\to\pi_1(X)\xrightarrow{h}H^1(X)\to 0$$

yields a natural isomorphism

$$\alpha_X: H^1(X,K)\to \mathrm{Hom}(H_1(X);K)\xrightarrow{\mathrm{Hom}(h,K)}\mathrm{Hom}(\pi_1(X),K).$$

Hence, a continuous function $f : X \to Y$ between pathwise connected spaces induces a diagram of homomorphisms in which the vertical maps are natural isomorphisms.

$$
\begin{tikzcd}
H^1(Y,K) \arrow[rr, "H^1(f,K)"] \arrow[d, "\alpha_Y"'] & & H^1(X,K) \arrow[d, "\alpha_X"] \\
\mathrm{Hom}(\pi_1(Y),K) \arrow[rr, "\mathrm{Hom}(\pi_1(f),K)"'] & & \mathrm{Hom}(\pi_1(X),K).
\end{tikzcd}
\tag{1}
$$

Let us first consider the Klein bottle $\mathbf{K}$. Then the first singular integral homology group, as the abelianization of the fundamental group, is $\mathbf{Z}(2) \times \mathbf{Z}$. The second one is zero since $\mathbf{K}$ is nonorientable. The universal coefficient theorem (U2) shows that the integral cohomology groups are $H^1(\mathbf{K};\mathbf{Z}) = \mathbf{Z}$ and $H^2(\mathbf{K};\mathbf{Z}) = \mathbf{Z}(2)$, respectively $H^1(\mathbf{K}, GF(2)) = \mathbf{Z}(2)^2$ and $H^2(\mathbf{K}, GF(2)) = \mathbf{Z}(2)$. Since a rank 1 abelian group does not support a nonzero skew-symmetric bilinear form, the cup product is trivial on $A = H^*(\mathbf{K};\mathbf{Z})$. For different reasons, the cup product on $A = H^*(\mathbf{K}, GF(2))$ is trivial, too. (Indeed, if $H^1(\mathbf{K}; GF(2)) \otimes H^1(\mathbf{K}, GF(2)) \to H^2(\mathbf{K}; GF(2))$ is nonzero, then we consider the double covering $\mathbf{T}^2 \to \mathbf{K}$ of the Klein bottle by the 2-torus and conclude that the map $H^1(\mathbf{K}; GF(2)) \otimes H^1(\mathbf{K}, GF(2)) \to H^1(\mathbf{T}^2; GF(2)) \otimes H^1(\mathbf{T}^2; GF(2)) \xrightarrow{\cup} H^2(\mathbf{T}^2; GF(2))$ is nonzero. We can write $H^1(\mathbf{T}^2; GF(2)) = GF(2)^2$ and identify the cup product with

$$
\begin{aligned}
u \otimes v \to u \wedge v : H^1(\mathbf{T}^2; GF(2)) &\otimes H^1(\mathbf{T}^2, GF(2)) \\
&\to \bigwedge^2 GF(2)^2 \cong H^2(\mathbf{T}^2; GF(2)).
\end{aligned}
$$

Now the homomorphism $\eta : H^1(\mathbf{K}; GF(2)) \to H^1(\mathbf{T}^2; GF(2))$ is equivalent to the homomorphism

$$\mathrm{Hom}(\pi_1(\mathbf{K}), \mathbf{Z}(2)) \to \mathrm{Hom}(\pi_1(\mathbf{T}^2), \mathbf{Z}(2))$$

induced by $\mathbf{Z}^2 \cong \pi_1(\mathbf{T}^2) \to \pi_1(\mathbf{K}) \cong \mathbf{Z} \rtimes_\delta \mathbf{Z}$ given by $(m,n) \mapsto (m, 2n)$. It follows that $\operatorname{im} \eta$ has $GF(2)$-dimension 1. But then $\eta(x) \wedge \eta(y) = 0$ for all $x, y \in H^1(\mathbf{K}; GF(2))$. This is a contradiction which proves the claim.) Thus, in both cases, the algebra A belongs to type (ii) in Theorem 2.6.

The idempotent multiplications $m : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$ were completely characterized by the homomorphic multiplications $\mu : \pi_1(\mathbf{K}) \times \pi_1(\mathbf{K}) \to \pi_1(\mathbf{K})$ which we characterized in Corollary 3.12:

(2)
$$\mu((u,v),(x,y)) = \begin{cases} ((-1)^{v-y}x, sv + (1-s)y), & \text{if } s \equiv 0 \pmod 2, \\ (u, sv + (1-s)y), & \text{if } s \equiv 1 \pmod 2, \end{cases}$$

for all homotopy classes $u, v, x, y \in \pi_1(\mathbf{K})$. The comultiplication induced by $H^1(m; K)$ on the level of the first cohomology groups

$$H^1(\mathbf{K}; K) \to (H^1(\mathbf{K}; K) \otimes K) \oplus (K \otimes H^1(\mathbf{K}; K)) \cong H^1(\mathbf{K}; K) \oplus H^1(\mathbf{K}; K),$$

in view of (1), is equivalent to

$$\begin{aligned} M : \mathrm{Hom}(\pi_1(\mathbf{K}), K) &\xrightarrow{\mathrm{Hom}(\mu, K)} \mathrm{Hom}(\pi_1(\mathbf{K}) \times \pi_1(\mathbf{K}), K) \\ &\xrightarrow{\cong} \mathrm{Hom}(\pi_1(\mathbf{K}), K) \oplus \mathrm{Hom}(\pi_1(\mathbf{K}), K). \end{aligned} \tag{3}$$

Thus, if $\Phi \in \mathrm{Hom}(\pi(\mathbf{K}), K)$, we have $M(\Phi) = M_1(\Phi) + M_2(\Phi)$ with $M_k(\Phi) = \Phi \circ \mu \circ \mathrm{pr}_k$, $k = 1, 2$, with the two projections pr_k onto the factors of $\pi_1(\mathbf{K})^2$. Thus, if μ is given by $\mu(g, h) = \varphi(g)\varphi(h)^{-1}h$ with a $\varphi = \varphi_s$ as in Proposition 3.11, then

$$\Phi(\mu(g, h)) = (\Phi \circ \varphi)(g) + (1 - (\Phi \circ \varphi))(h),$$

whence $M_1(\Phi) = \Phi \circ \varphi$ and $M_2(\Phi) = 1 - (\Phi \circ \varphi)$.

From the fact that $\mathrm{Hom}(H_2(\mathbf{K}), K) = \mathrm{Hom}(\{0\}, K) = \{0\}$, the universal coefficient theorem gives us a natural isomorphism

$$\mathrm{Ext}(H_1(\mathbf{K}), K) \to H^2(\mathbf{K}, K).$$

We recall $H_1(\mathbf{K}) \cong \mathbf{Z}(2) \oplus \mathbf{Z}$. When the characteristic of K is different from 2 we have $\mathrm{Ext}(\mathbf{Z}, K) = \{0\}$. If the characteristic is 2, then $H^2(\mathbf{K}, K)$ is nonzero. In particular, this group contains exactly one nonzero element e if $K = GF(2)$. In this case the endomorphism φ_s of $\pi_1(\mathbf{K})$ induces on $\mathrm{Ext}(\mathbf{H}_1(\mathbf{K}), K)$, hence on $H^2(\mathbf{K}, K)$, the zero-morphism if $s \equiv 0 \pmod 2$ and the identity morphism if $s \equiv 1 \pmod 2$. This means that the induced homomorphism is just scalar multiplication with s on $\mathbf{Z}(2)$. The resulting comultiplication in dimension 2 is given by $c(e) = s \cdot (e \otimes 1) + (1 - s) \cdot (1 \otimes e)$.

If $K = \mathbf{Z}$, then the rank of $A^1 = H^1(\mathbf{K}, \mathbf{Z}) \cong \mathbf{Z}$ is 1. Then the Klein bottle $\mathbf{K}$ and its idempotent multiplications topologically realize the case (iib) of Theorem 2.6. (In this case the scalar r of the formula for $c(e)$ in (iib) equals the present scalar s, and thus r and $f = (\Phi \mapsto \Phi \circ \varphi_s)$ are not independent.)

If $K = GF(2)$, then $H^2(\mathbf{K}, K) = \{0\}$, and the endomorphisms φ_s of $\pi_1(\mathbf{K})$ induce on $H^1(\mathbf{K}, GF(2)) = \mathbf{Z}(2)^2$ the zero morphism if $s \equiv 0 \pmod 2$ and the identity morphism if $s \equiv 1 \pmod 2$. This gives us a topological realization of case (iia) in Theorem 2.6 with $t = 0$.

For the projective plane P^2, the algebra $A = H^*(P^2; \mathbf{Z})$ satisfies $A^1 = \{0\}$ and $A^2 = \mathbf{Z}(2)$. Multiplication on $\mathbf{Z}(2)$ with any integral scalar is either zero or the identity. The comultiplication on the level of dimension 2 in case (iic) is then automatically trivial; it is then realized by the trivial multiplications on P^2.

We summarize:

Remark 5.5. In the listing of comultiplications on graded algebras in Theorem 2.6, case (ii) is topologically realized as follows:

(iia) $A^* = H^*(\mathbf{K}, GF(2))$ with the idempotent multiplications of the Klein bottle. In this realization, the tensor t always vanishes.

(iib) $A^* = H^*(\mathbf{K}, \mathbf{Z})$ with the idempotent comultiplications of the Klein bottle. In this realization, the scalar r depends on the endomorphism f.

(iic) $A^* = H^*(\mathbf{P}^2, GF(2))$ with the trivial multiplications on P^2.

We do not know whether all topological realizations of the case (iia) in Theorem 2.6 must have $t = 0$.

We remark that there are simple examples of topological realizations of cases (iib) and (iic) which are not cohomology 2-surfaces but do have the desired properties up to degree 2. If $\mathbf{T}$ denotes the circle group and $B(\mathbf{T})$ the classifying space of the circle group according to Milgram [**8**] (see also [**13**] and [**7**]), then $B(\mathbf{T})$ is a topological abelian group whose cohomology ring $H^*(B(\mathbf{T}), \mathbf{Z})$ is the polynomial ring $\mathbf{Z}[e]$ in one generator with $\deg(e) = 2$. Also, the cohomology ring $H^*(\mathbf{T}, \mathbf{Z})$ of $\mathbf{T}$ is the exterior algebra $\Lambda_{\mathbf{Z}}[x]$ with $\deg(x) = 1$. If we set $X = \mathbf{T} \times B(\mathbf{T})$, then $H^*(X, \mathbf{Z}) = \Lambda_{\mathbf{Z}}[x] \otimes \mathbf{Z}[e]$. The endomorphism $\varphi : X \to X$ given by $\varphi(a, b) = (na, rb)$ with $n, r \in \mathbf{Z}$, $n \neq 0, 1$, gives a continuous idempotent multiplication m_φ on X by Definition 3.1. The comultiplication c induced on $A^* = H^*(X, \mathbf{Z})$ is of the type described in (iib) in Theorem 2.6 with $f(v) = nv$.

If we set $X = B(\mathbf{T})$, then $A^* = H^*(X, \mathbf{Z}) = \mathbf{Z}[e]$. We define a continuous idempotent multiplication m_φ on X with $\varphi(b) = rb$ with $r \in \mathbf{Z}$, and obtain a topological realization of case (iic) in Theorem 2.6.

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