Identifiability and Identification of Chaotic Systems Based on Adaptive Synchronization

Hervé Dedieu, *Member, IEEE*, and Maciej J. Ogorzałek, *Fellow, IEEE*

Abstract—**This paper deals with the problem of synchronization of chaotic systems when the driven (slave, receiver) system has the same structure as the master (driving, emitter) system but its parameters are unknown. It is shown that the concept of synchronization provides an efficient way to find the unknown slave system parameters. Parameter mismatch between master and slave systems and high sensitivity of response to changes of these parameters were so far considered as crucial for security issues. This paper shows evidence that this claimed advantage becomes in fact a major drawback in chaos communication schemes since parameters can easily be found using adaptive synchronization and optimization tools. The general problem of identifiability of chaotic systems is defined and discussed in the context of possibilities for finding the unknown chaotic receiver parameters. Several typical systems used in experiments in chaos communication are tested for identifiability showing direct applications of the introduced concepts. In particular examples of the skew tent map, Hénon map, Markov maps and Chua's circuit are considered in detail illustrating the problems of global and local identifiability.**

Index Terms—**Communication, chaos synchronization, identifiability, identification.**

I. INTRODUCTION

A. Background

SINCE THEIR discovery in the late 1980's [11], chaos synchronization principles have been applied in different fields ranging from communications [1] to control [13]. In particular there was a considerable effort in designing simple chaotic circuits producing signals suitable for information transmission purposes, which could offer both: spread spectrum and privacy. In many applications the Pecora and Carroll synchronization scheme [11], [12] has been applied directly. This scheme relies on a suitable decomposition of the original chaotic system—later called the *master* system—into an open-loop cascade of subsystems (called *slave* system). This cascade of subsystems is forced using as driving signal one particular signal transmitted from the master. In general the slave cascade consists of stable and unstable subsystems. To achieve synchronization one has to make sure that all the conditional Lyapunov exponents associated with some variational equation

are negative [11]. The term "conditional" comes from the fact that the Lyapunov exponents depend on the driving signal. If all the conditional Lyapunov exponents are negative it can be expected that all the state variables of the slave system are forced to follow the corresponding state variables of the master system (although just one variable was used as driving signal).

In the above-mentioned applications, the decomposition is carried out in such a way that the slave subsystems are exact copies of the master subsystems (structure and parameter values) and synchronization fails if this is not the case. It has been confirmed by many authors that sensitivity to parameter mismatch is a common drawback also for other synchronization schemes like error-feedback, inverse system, active-passive decomposition etc. and one has to be very cautious when designing and exploiting chaos synchronization principles. On the other hand it is claimed that this extreme sensitivity to changes of parameters is advantageous because it enhances transmission security. As even very small parameter mismatch results in a very different dynamic behavior it was believed that it will be totally impossible to find system parameters using measurements of transmitted signals alone.

In this paper we will show that this claim not only is not true but as a matter of fact we have an opposite situation—we propose to take advantage of the property of sensitivity of chaos synchronization to parameter mismatch for parameter identification purposes. We show how this property, which is a major hindrance in design and a claimed advantage of communication schemes employing chaotic carriers, can be successfully exploited to find unknown slave system parameters. What was supposed so far to offer security will be used here to break this issue. This is achieved by controling the slave system by means of tuning its parameters until synchronization occurs.

The principles of chaotic identification have been first described by the authors in [4], [5] and developed in a series of papers [6], [7]. Independently, some of the identification aspects of synchronized chaotic systems were addressed in [3], [8], and [9].

It is supposed in [3], [7]–[9] that one knows the class of system which produced a measured time series (which can be continuous or sampled at discrete time moments), i.e., the underlying equation structure for the system under consideration is known but one is not able to measure either the system parameters or all the state variables. The

parameter identification is therefore carried out using both synchronization and optimization algorithms which tune the slave system parameters in order that synchronization occurs. Synchronization can be seen here as an elegant way to cope with the problem of sensitivity to initial conditions. Indeed if we tried to identify a chaotic oscillator directly in a closed loop, we would be faced with the problem of rebuilding the state variables and the exact values of the parameters. These two problems are difficult to solve due to the intrinsic sensitivity to initial values of the state variables. By applying synchronization principles, i.e., by trying to identify some forced open loop version of the oscillator, we can avoid the problem of sensitivity and find a simple way to rebuild the state variables of the master system.

We should remark here that synchronization reduces drastically the dimension of the minimization problem since it is not necessary to search for the state variables (these are found by synchronization!!), it is only necessary to search (optimize) for the unknown parameters.

Let us also remark that the underlying principle behind identification is linked to the sensitivity of synchronization to parameter mismatch. This sensitivity has been advocated to be the main foundation for security of transmission with chaotic carriers as it is claimed that synchronization is only feasible when slave system parameters match accurately enough the master system parameters. If the ewes dropper does not know the parameters accurately enough, he will not be able to find the message hidden within the chaotic signal.

The examples already given in [3], [7]–[9] show the opposite of this claim—it is possible to exploit the parameter mismatch sensitivity and to 'break the code' which consist of finding the values of the parameters of the master system. Synchronization itself provides a tool for parameter identification and possible break of code.

B. Scope of the Paper

In this paper we describe the principles of parameter identification for chaotic systems, given a scalar sampled output signal produced by the system and given the structure of the system that produced the signal.

At the beginning we review some basic synchronization principles. Next we carefully check if it is possible to recover the system parameters from the measurements by applying identifiability principles. This identifiablity approach is here applied to chaotic systems for the first time and gives new tools to the parameter identification of chaotic systems. It provides also a key tool for checking transmission system security. In the third part we explain in detail the identification algorithms which could be used to tune the slave system parameters. Finally we present several identification examples involving both discrete and analog systems. We present both, theoretical proofs of identifiability and results of numerical experiments. These kind of analysis of systems important in applications is also described for the first time.
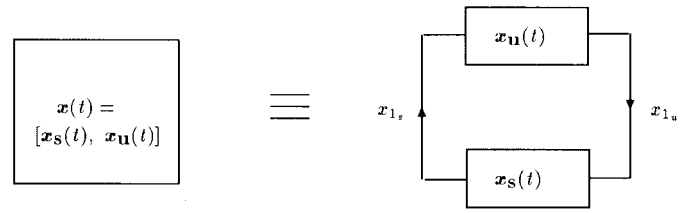

Fig. 1. Decomposition of the system into two subsystems coupled in a closed loop.

II. SYNCHRONIZATION OF CHAOTIC SYSTEMS

A. Principles of Synchronization When the Parameters are Known

Let us consider here the basic principles that ensure that a slave system will synchronize its state variables with the corresponding ones of the chaotic master system. Let us assume that the system of equations of the master is known and the exact values of the master parameters are also known. Although there are different manners to achieve synchronization, we present below the most popular scheme which was suggested by Pecora and Carroll. For more details on synchronization, the reader is referred to [1], [2], and [13].

We suppose that we are given a nonlinear dynamical system which state variables obey an equation of the form

$$\dot{x}(t) = F_{\theta}(x(t)) \tag{1}$$

where x is the state vector, $\dot{x}(t) = x(t)/dt$ and θ is the vector of parameters associated with the nonlinear system.

Suppose that the system can be partitioned into two subsystems $F_{s\theta}$ and $F_{u\theta}$ in such a way that the original equation could be rewritten as $x^T(t) = [x_s(t), x_u(t)]$

$$\dot{x}_s(t) = F_{s\theta}(x_s(t), x_{1u}(t))$$
$$\dot{x}_u(t) = F_{u\theta}(x_u(t), x_{1s}(t)). \tag{2}$$

As shown in Fig. 1 the two subsystems are coupled via the one-dimensional signals x_{1s} which is one component of x_s and x_{1u} which is one component of x_u. In general such a simple decomposition is not always possible and the coupling variables can be of greater dimension.

We will assume in what follows that the output $y(t)$ of the oscillator can be chosen as one of these two one-dimensional coupling signals in such a way that the dynamical system could be rewritten as described by (3)

$$\dot{x}_s(t) = F_{s\theta}(x_s(t), y(t))$$
$$\dot{x}_u(t) = F_{u\theta}(x_u(t), x_{1s}(t)). \tag{3}$$

The whole system is therefore depicted in Fig. 2.

The synchronization by decomposition into subsystems consists of building a slave system which is an open loop version of the master system obeying the following equations:

$$\dot{\hat{x}}_s(t) = Fs\theta(\hat{x}_s(t), y(t)) \quad \dot{\hat{x}}_u(t) = F_{u\theta}(\hat{x}_u(t), \hat{x}_{1s}(t)). \tag{4}$$

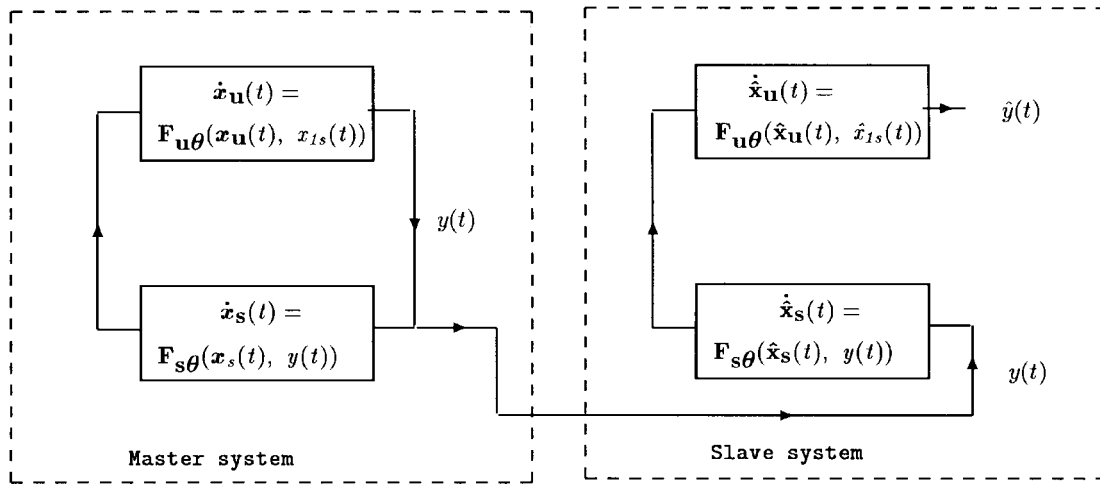This principle of synchronization is shown in Fig. 3.

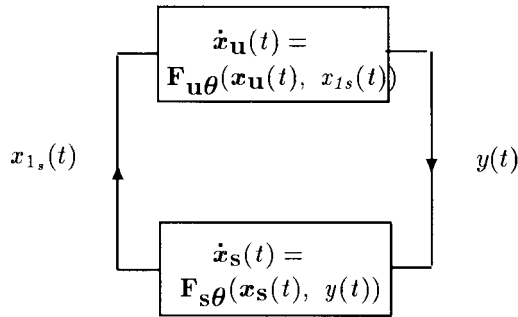Fig. 3.   General scheme for master-slave set up for synchronization by decomposition into subsystems.

Fig. 2.   Decomposition of the system.

The slave system synchronizes with the master system if

$$|\hat{y}(t) - y(t)| \to 0 \quad \text{as} \quad t \to \infty. \tag{5}$$

This situation arises when the cascade of the slave subsystems exhibits conditional Lyapunov exponents which are all negative [11], [12].

B. Principles of Synchronization for a Slave System with Unknown Parameters

Let us consider the case when we do not know the parameter vector $\boldsymbol{\theta}$ beforehand and we would like to find an algorithm to recover $\boldsymbol{\theta}$. The basic principle is to relay on the sensitivity of synchronization to parameter mismatch. Let us define a slave system with time variant parameters $\hat{\boldsymbol{\theta}}$. Synchronization occurs asymptotically when $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$ and fails when $\hat{\boldsymbol{\theta}} \neq \boldsymbol{\theta}$. The idea is therefore to build a cost function measuring the synchronization quality and to derive the algorithm which provides a recursive estimate of $\boldsymbol{\theta}$ converging toward the minimum of the cost function. This process is called recursive identification [21]. In order to find the best estimate we will design an adaptive procedure (Fig. 4) in such a way that the error of synchronization $|\hat{y}_{\hat{\boldsymbol{\theta}}}(t) - y(t)|$ will tend to zero as t tends to infinity. In its sampled (discrete time) version the adaptive algorithm works as follows: Suppose that ΔT is the sampling period of the waveform $y(t)$ and let us denote by $y(n)$ the sample of $y(t)$ measured at the time

$t = n\Delta T$. Our goal is to find $\hat{\boldsymbol{\theta}}$ that minimizes the cost function $V(\hat{\boldsymbol{\theta}})$

$$V(\hat{\boldsymbol{\theta}}) = E[e^2(n)] \tag{6}$$

where

$$e(n) = y(n) - \hat{y}_{\hat{\boldsymbol{\theta}}}(n). \tag{7}$$

The goal is therefore to find $\hat{\boldsymbol{\theta}}$ that minimizes (6). A local minimum of $V(\hat{\boldsymbol{\theta}})$ is located at $\hat{\boldsymbol{\theta}}$ such that

$$\frac{\partial V(\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}} = -E\left[\frac{\partial y(n)}{\partial \boldsymbol{\theta}} e(n)\right] = 0. \tag{8}$$

Unfortunately the solution of (8) requires knowledge of the probability distributions of all the pairs $(e_i(n), \partial y_i(n)/\partial \theta_j)$ for $i = 1 \cdots M$ and $j = 1 \cdots N$, these probability distributions are not known beforehand.

Let us describe now some possible approaches for finding solutions to the above-defined optimization problem.

The underlying idea behind stochastic approximation (which has been developed as a special branch of the sequential parameter estimation in the statistical literature) is to build a sequence of estimates $\hat{\boldsymbol{\theta}}(n)$ in such a way that

$$\lim_{n \to \infty} \hat{\boldsymbol{\theta}}(n) = \boldsymbol{\theta} \tag{9}$$

where $\boldsymbol{\theta}$ is the global minimum of $V(\hat{\boldsymbol{\theta}})$. Observe that in our case $V(\boldsymbol{\theta}) = 0$ provided we get perfect synchronization.

A simple method to find a sequential estimate is to move the current vector of parameters in the direction opposite to the gradient, i.e.,

$$\hat{\boldsymbol{\theta}}(n) = \hat{\boldsymbol{\theta}}(n-1) - \mu(n)\nabla_{\boldsymbol{\theta}} V[\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(n-1)] \tag{10}$$

where $\mu(n)$ is a small positive gain chosen in a convenient way. Equation (10) can be rewritten

$$\hat{\boldsymbol{\theta}}(n) = \hat{\boldsymbol{\theta}}(n-1) + \mu(n)E\left[\frac{\partial y(n)}{\partial \boldsymbol{\theta}} e(n)\right] \tag{11}$$

How to compute $E[\partial y(n)/\partial \boldsymbol{\theta} e(n)]$? Solution proposed by Robbins and Monro [21] assumes that for a $\mu(n)$ chosen small
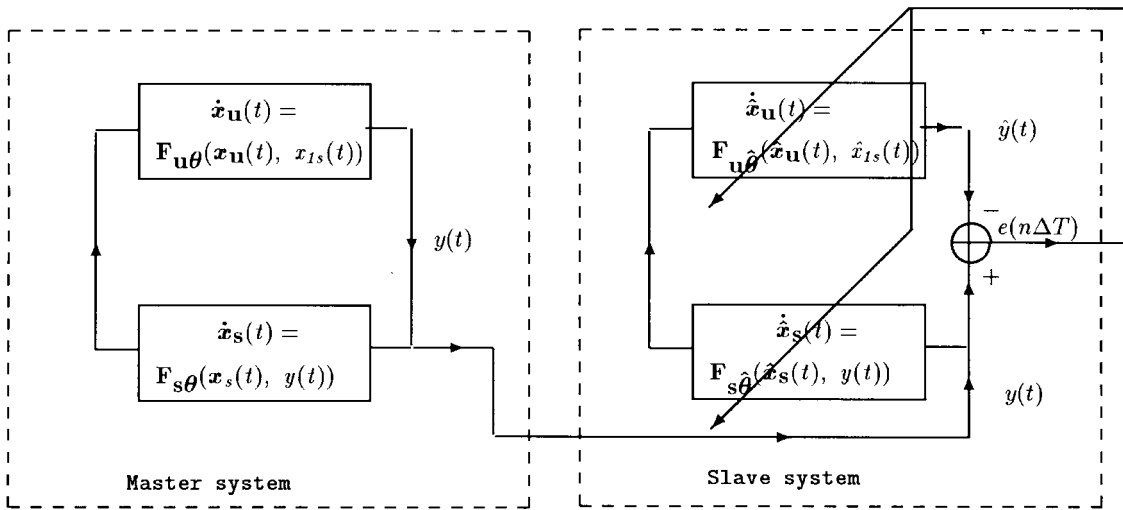
Fig. 4. General scheme for adaptive synchronization.

enough (in the original version of the Robbins and Monro scheme $\mu(n)$ is a sequence of positive scalars which tends to zero) the expectation can be removed in (11). In other terms, if $\boldsymbol{\theta}$ varies slowly in time, 'on the average' the adjustments are made in the negative gradient direction. The gradient algorithm is therefore defined by

$$\hat{\boldsymbol{\theta}}(n) = \hat{\boldsymbol{\theta}}(n-1) + \mu(n)\boldsymbol{\psi}(n)\boldsymbol{e}(n) \quad (12)$$

where $\boldsymbol{\psi}(n)$ is the vector of sensitivities (the reaction of the output to an infinitesimal change of the parameters) which plays a key role for optimization, i.e.,

$$\boldsymbol{\psi}(n) = \frac{\partial y(n)}{\partial \boldsymbol{\theta}}. \quad (13)$$

A major drawback of the stochastic gradient method is its slow convergence rate. The reason for this poor performance is obvious—on the average the method gives the direction to the nearest local minimum but nothing is said about the distance (or the average distance) to this nearest local optimum.

To improve the convergence rate second-order methods are used. They guarantee faster convergence rates because at every iteration both, the direction and the distance to the nearest local optimum are taken into account.

The underlying idea behind second-order methods is to make a second-order Taylor expansion around the current vector of parameters $\boldsymbol{\theta}(n-1)$, i.e.,

$$V(\hat{\boldsymbol{\theta}}(n-1) + \Delta\hat{\boldsymbol{\theta}})$$
$$= V(\hat{\boldsymbol{\theta}}(n-1)$$
$$+ \frac{\partial V(\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(n-1))}{\partial \boldsymbol{\theta}}\Delta\hat{\boldsymbol{\theta}}$$
$$+ \frac{1}{2}\Delta\hat{\boldsymbol{\theta}}^T \frac{\partial^2 V(\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(n-1))}{\partial \boldsymbol{\theta}^2}\Delta\hat{\boldsymbol{\theta}} \quad (14)$$

and to choose an update $\Delta\boldsymbol{\theta}^*$ which ensures the best descent for the performance criterion. This is achieved when choosing a $\Delta\boldsymbol{\theta}^*$ for which the derivatives at the right-hand side (RHS) of (14) are equal 0. Thus we obtain an equation for the optimal

update value

$$\Delta\boldsymbol{\theta}^* = \hat{\boldsymbol{\theta}}(n) - \hat{\boldsymbol{\theta}}(n-1)$$
$$= -\left[\frac{\partial^2 V(\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(n-1))}{\partial \boldsymbol{\theta}^2}\right]^{-1} \frac{\partial V(\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(n-1))}{\partial \boldsymbol{\theta}} \quad (15)$$

and further the equation which defines the so-called *Newton method*

$$\hat{\boldsymbol{\theta}}(n) = \hat{\boldsymbol{\theta}}(n-1)$$
$$- \left[\frac{\partial^2 V(\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(n-1))}{\partial \boldsymbol{\theta}^2}\right]^{-1} \frac{\partial V(\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(n-1))}{\partial \boldsymbol{\theta}}. \quad (16)$$

It is worthwhile noting that taking $\Delta\boldsymbol{\theta}^*$ as an update we obtain for the value of the performance criterion at iteration n [by substituting (15) into (14)]

$$V(\hat{\boldsymbol{\theta}}(n)) = V(\hat{\boldsymbol{\theta}}(n-1))$$
$$- \frac{\partial V(\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(n-1))}{\partial \boldsymbol{\theta}}^T \left[\frac{\partial^2 V(\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(n-1))}{\partial \boldsymbol{\theta}^2}\right]^{-1}$$
$$\cdot \frac{\partial V(\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(n-1))}{\partial \boldsymbol{\theta}}. \quad (17)$$

Equation (17) has an important consequence in practice since it shows that if the Hessian matrix

$$\boldsymbol{H}(n) = \frac{\partial^2 V(\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(n-1))}{\partial \boldsymbol{\theta}^2} \quad (18)$$

is positive-definite we are sure that $V(\hat{\boldsymbol{\theta}}(n))$ will point downhill with respect to $V(\hat{\boldsymbol{\theta}}(n-1))$. Close to the optimum the quadratic approximation (14) is generally a good model of V and the Hessian is therefore a positive definite matrix near the optimum since V has a bowl shaped form (upward concavity). The quadratic model is in general not valid far from the local minimum, this is why it is often preferred to replace the Hessian by a *guaranteed positive semidefinite approximation*. The methods in which the Hessian is positive semidefinite are referred to as *quasi-Newton methods*.

Quasi-Newton methods ensure that even far from the optimum the parameter search is carried out "downhill."

$$\boldsymbol{H}(n) = E\left[\frac{1}{2}\frac{\partial^2 e^2(n)}{\partial \boldsymbol{\theta}^2}\right]$$
$$= E\left[\frac{\partial e(n)}{\partial \boldsymbol{\theta}}\frac{\partial^T e(n)}{\partial \boldsymbol{\theta}} + e(n)\frac{\partial^2 e(n)}{\partial \boldsymbol{\theta}^2}\right] \quad (19)$$

is usually replaced by the positive semidefinite approximation

$$\hat{\boldsymbol{H}}(n) = E\left[\frac{\partial e(n)}{\partial \boldsymbol{\theta}}\frac{\partial^T e(n)}{\partial \boldsymbol{\theta}}\right] = E[\boldsymbol{\psi}(n)\boldsymbol{\psi}^T(n)] \quad (20)$$

This choice has also the advantage that only first order derivatives (i.e., sensitivities) are required. The so-called *Gauss-Newton method* is defined by the following equation:

$$\hat{\boldsymbol{\theta}}(n) = \hat{\boldsymbol{\theta}}(n-1) + \hat{\boldsymbol{H}}^{-1}(n)E[e(n)\boldsymbol{\psi}(n)], \quad (21)$$

Calculation of formula (21) requires knowledge of two kinds of expectations for which the correspondent probability distributions are not available. This is why as in the Robbins and Monro scheme it is proposed to remove the expectations in (21). The stochastic equivalent of (21) is then given by

$$\hat{\boldsymbol{\theta}}(n) = \hat{\boldsymbol{\theta}}(n-1) + \hat{\boldsymbol{H}}^{-1}(n)e(n)\boldsymbol{\psi}(n). \quad (22)$$

If $\hat{\boldsymbol{\theta}}(n)$ is slowly time varying the recursivity of the algorithm will provide on average a good estimate of $E[e(n)\boldsymbol{\psi}(n)]$. One problem left is to find an estimate of $\hat{\boldsymbol{H}}(n)$ since this quantity is also an expectation. One possible choice often made is to average over a time window the product $\boldsymbol{\psi}(n)\boldsymbol{\psi}^T(n)$ thus giving more weight to the current and recent products and discarding (forgetting) the older estimate. For instance the estimation

$$\hat{\boldsymbol{H}}(n) = \lambda\hat{\boldsymbol{H}}(n-1) + \boldsymbol{\psi}(n)\boldsymbol{\psi}^T(n) \quad (23)$$

where λ is a constant close to, but less than 1, is often chosen as a good and convenient recursive estimate. The factor λ is referred to as *the forgetting factor*. This factor fixes the tradeoff between the rate of convergence and the precision of the algorithm in the steady state. The closer λ is to 1, the slower is the convergence but the better is the precision (variance of $\hat{\boldsymbol{\theta}}$ in the steady state).

In general convergence of the nonlinear optimization process (22) toward its optimum is not guaranteed, which is usual for all nonlinear adaptive optimization schemes, since the cost function can exhibit multiple local minima. In many cases this means that the algorithm should be restarted with different initial conditions.

In our studies the identification process has been carried out using second quasi-Newton techniques described by (22) and (23).

It should be noted that the cost function (6) will certainly exhibit several local minima specially in the case in which the system to be identified is of great dimension. One possibility to avoid remaining stuck on a local minimum is to find an additional way to monitor the search process. This additional mechanism can be in principle implemented by adding a constraint term in the cost function (6). This constraint is however dependent on a certain *a priori* knowledge we could have on the range of certain parameters. Once this constraint term is added to the cost function the "instantaneous" gradient of the cost function can be used as usual in the stochastic approximation technique.

III. STRUCTURAL IDENTIFIABILITY

Until now we have described one possible technique for identification. We have not formulated the fundamental question if it does really make sense to perform such a procedure. The output $y(t)$ of our chaotic oscillator is a 1-D (scalar) signal which is defined by a nonlinear mapping of state variables which are typically of greater dimension. The fundamental problem consists therefore of determining if the signal $y(t)$ carries enough information to allow rebuilding of the parameters. A different way to pose this problem is to check, given a structure of identification, if there exists only one set of parameters which can produce the same signal $y(t)$. For instance if there exists an infinite number of parameter sets which explain the same $y(t)$ it is certain that we cannot retrieve the particular set of parameters which was used in the master system.

In order to check if it is possible to identify the parameters of the master system we have to study *the structural identifiability* [23], [22] of our model. The following definitions are borrowed from Walter and Pronzato [23], [22].

A. Definitions—Structural Identifiability

Let us suppose first that for the master system the parameters of which we want to identify we can build a slave system (which is a cascade of two subsystems as before) which exhibits synchronization when driven with the master output signal $y(t)$, for an unknown set of parameters $\boldsymbol{\theta}$ (i.e., $\lim|y_{\boldsymbol{\theta}}(t) - y(t)| \to 0$ as $t \to \infty$).

Let us suppose next that we drive an identical structure with the same input $y(t)$ and that this structure has a set of parameters $\hat{\boldsymbol{\theta}}$. Under the above conditions it is always possible to tune the parameters of the model $\hat{\boldsymbol{\theta}}$ in such a way that

$$\lim|y_{\boldsymbol{\theta}}(t) - y_{\hat{\boldsymbol{\theta}}}(t)| \to 0 \quad \text{as} \quad t \to \infty. \quad (24)$$

This experimental setup is shown in Fig. 5.

If (24) holds we shall denote by

$$M(\hat{\boldsymbol{\theta}}) = M(\boldsymbol{\theta}) \quad (25)$$

the equivalence of the two models of identification.

Let us define as P the prior feasible parameter space. The parameter θ_i is said to be *structurally globally identifiable* (s.g.i.) [22], [23] if for almost any $\hat{\boldsymbol{\theta}} \in P, M(\hat{\boldsymbol{\theta}}) = M(\boldsymbol{\theta}) \Rightarrow \hat{\theta}_i = \theta_i$.

The model $M(\cdot)$ is said to be *structurally globally identifiable* if all its parameters θ_i are s.g.i.

The parameter θ_i is said to be *structurally locally identifiable* s.l.i. [22], [23] if for almost any $\hat{\boldsymbol{\theta}} \in P$, there exists a neighborhood $v(\hat{\boldsymbol{\theta}})$ such that $\hat{\boldsymbol{\theta}} \in v(\hat{\boldsymbol{\theta}})$ and $M(\hat{\boldsymbol{\theta}}) = M(\boldsymbol{\theta}) \Rightarrow \hat{\theta}_i = \theta_i$. The model $M(\cdot)$ is said to be *structurally locally identifiable* if all its parameters θ_i are s.l.i.
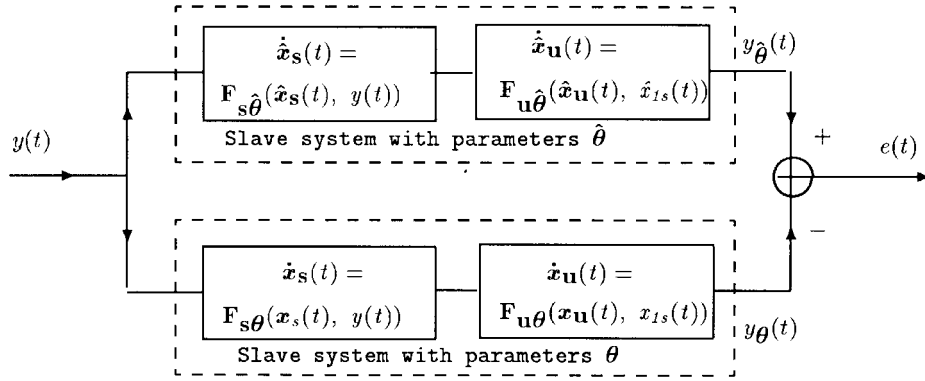
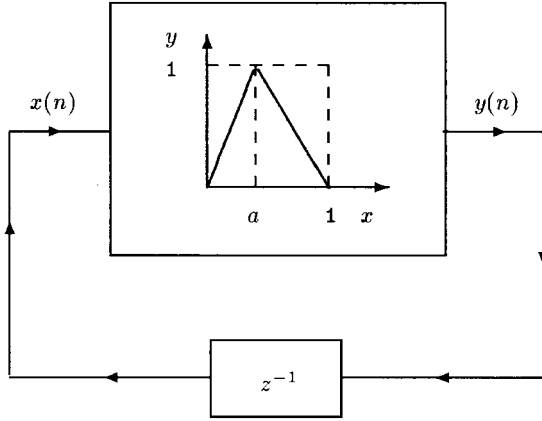Fig. 5. Framework for structural identifiability study.

Fig. 6. Skew tent map system.

The parameter θ_i is said to be *structurally nonidentifiable* (s.n.i.) if for almost any $\hat{\boldsymbol{\theta}} \in P$, there is no neighborhood $v(\hat{\boldsymbol{\theta}})$ such that $\hat{\boldsymbol{\theta}} \in v(\hat{\boldsymbol{\theta}})$ and $M(\hat{\boldsymbol{\theta}}) = M(\boldsymbol{\theta}) \Rightarrow \hat{\theta}_i = \theta_i$.

The model $M(\cdot)$ is said to be *structurally nonidentifiable* if at least one of its parameters is s.n.i.

In order to check if it makes sense to identify the parameters of the chaotic system our goal will be to check the structural identifiability property of the slave system. Examples will be given in the following.

IV. EXAMPLES

A. The Skew Tent Map

This 1-D map is defined by the following equations:

$$y(n) = \begin{cases} \dfrac{1}{a}x(n) & \text{if } 0 \leq x(n) \leq a \\ \dfrac{1}{a-1}x(n) - \dfrac{1}{a-1} & \text{if } a \leq x(n) \leq 1 \end{cases}$$
$$x(n) = y(n-1). \tag{26}$$

The block diagram of the skew ten map system is given by Fig. 6.

1) Structural Identifiability of the Skew Tent Map System: Next we investigate the question of identifiability of the skew tent map system. The identifiability setup depends on the identification structure we will choose. As $y(n)$ is our observed signal, our slave system will be a cascade of two subsystems, a stable one which will be a simple delay followed by a

nonlinearity of the skew tent map type. To check if the system is identifiable by synchronization, one has to check the number of realizations of the parameter $\hat{a}$ which will ensure that $y_{\hat{a}}(n) = y(n)$ in Fig. 7.

Let $f_a(\cdot)$ denote the skew tent map, where a is its parameter. We have

$$y_{\hat{a}}(n) = y(n) \Rightarrow f_{\hat{a}}(x) = f_a(x) \text{for} x \in [0,1] \Rightarrow \hat{a} = a. \tag{27}$$

This means that the model to be identified by synchronization is globally structurally identifiable. In the next section we will construct an identification algorithm to find the unknown parameter a of the slave system.

2) Identification of the Skew Tent Map System: The identification of the skew tent map parameter is not a difficult problem since examining the dependence between $y(n)$ and $y(n-1)$ would give us directly the skew tent map parameter a. However let us consider this identification example in order to check the efficiency of the proposed identification procedure. The setup for identification is shown in Fig. 8. Starting from an initial guess for the parameter $\hat{a}(0)$ we can build a sequential estimate by applying (22), i.e.,

$$\hat{a}(n) = \hat{a}(n-1) + \frac{e(n)}{R(n)}\phi(n) \tag{28}$$

where

$$e(n) = y(n) - f_{\hat{a}(n-1)}(y(n-1)) \tag{29}$$

where $R(n)$ is given by [applying (23)]

$$R(n) = \lambda R(n-1) + \phi^2(n). \tag{30}$$

The sensitivity $\phi(n)$ is computed as follows:

$$\psi(n) = \begin{cases} \dfrac{\partial \hat{y}(n)}{\partial \hat{a}} = \dfrac{-x(n)}{\hat{a}^2(n-1)} \\ \quad \text{if } 0 \leq x(n) \leq \hat{a}(n-1) \\ \dfrac{\partial \hat{y}(n)}{\partial \hat{a}} = \dfrac{-x(n)}{(\hat{a}(n-1)-1)^2} + \dfrac{1}{(\hat{a}(n-1)-1)^2} \\ \quad \text{if } \hat{a}(n-1) < x(n) \leq 1 \end{cases} \tag{31} \tag{32}$$

where in (31) and (32) $x(n) = y(n-1)$. We simulated the adaptive synchronization of the skew tent map system with parameter to be identified set to $a = 0.47$. Initializing $\hat{a}(0)$ to 0.1, $R(0) = 10$ and fixing the forgetting factor λ to 0.98 we obtained a sequence of estimates for the parameter $\hat{a}$
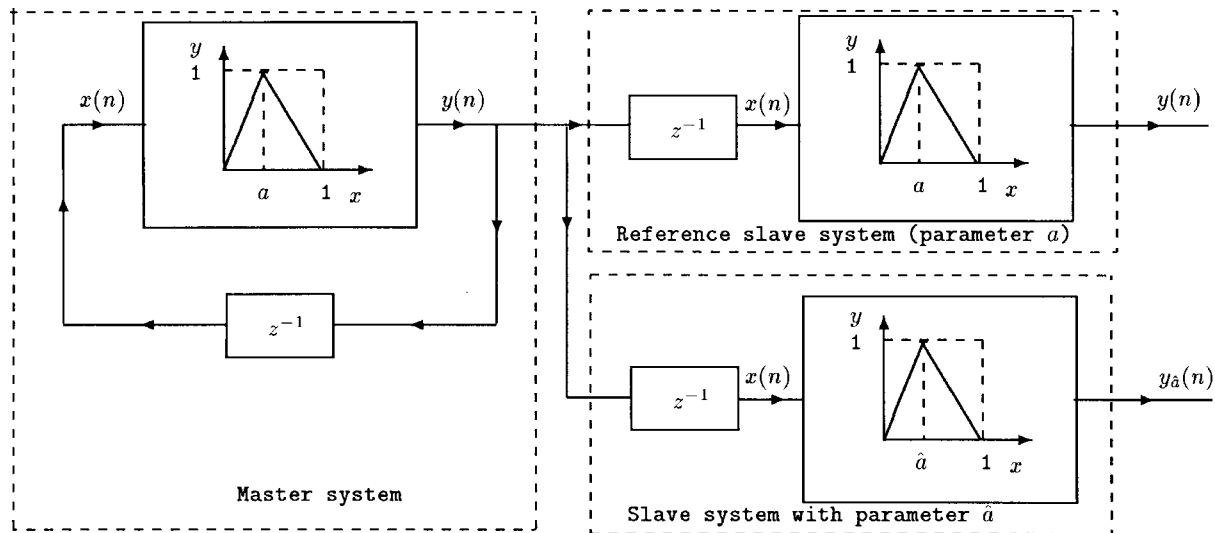
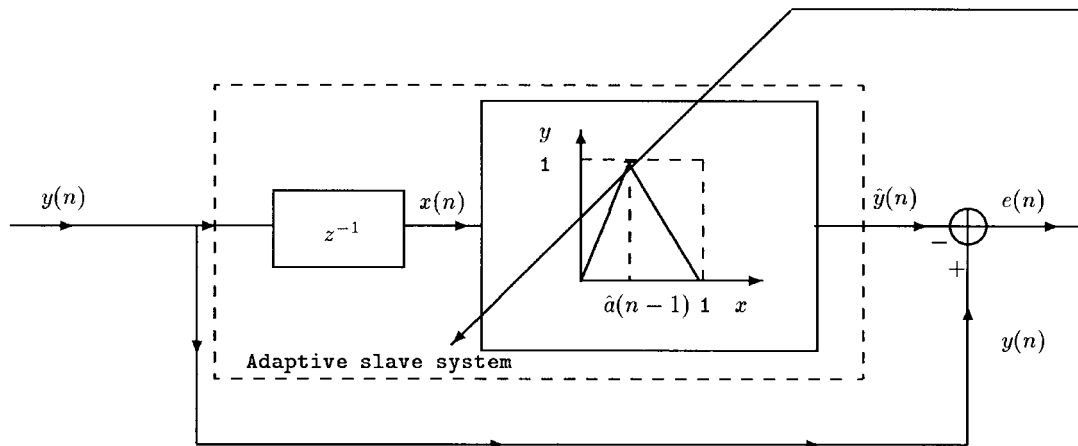Fig. 7.   Identifiability framework for the skew tent map system.

Fig. 8.   Setup for identification of the skew tent map system.

which is presented in Fig. 9. After 400 iterations the adaptive synchronization converged perfectly. Both the synchronization error and the parameter mismatch $\hat{a} - a$ are 0. Repeating the adaptive synchronization with different values of the initial guess for $\hat{a}(0)$ showed that the adaptive synchronization occurred independently of this initial guess.

B. Hénon Map System

This two-dimensional system is described by the following equations:

$$\begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} = \begin{pmatrix} 1 - ax_1^2(n-1) + bx_2(n-1) \\ x_1(n-1) \end{pmatrix}. \quad (33)$$

By observing the signal $y(n) = x_1(n)$, one possible setup for adaptive synchronization is to build a forced system of the form

$$\begin{pmatrix} \hat{x}_1(n) \\ \hat{x}_2(n) \end{pmatrix} = \begin{pmatrix} 1 - \hat{a}x_1^2(n-1) + b\hat{x}_2(n-1) \\ x_1(n-1) \end{pmatrix} \quad (34)$$
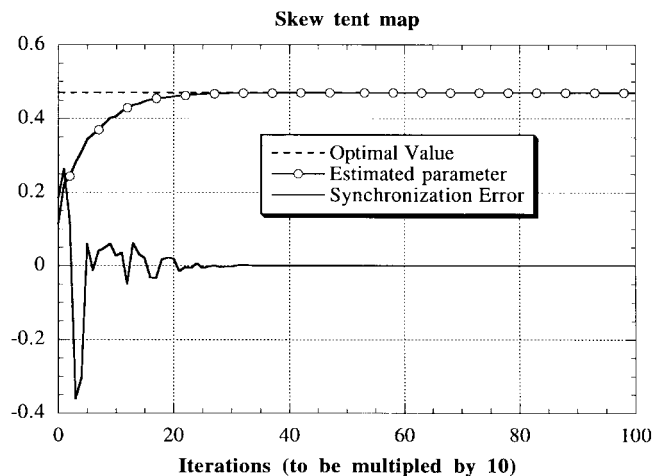
in which $x_1(n-1)$ is the forcing signal.

Fig. 9.   Sequential estimate of parameter $\hat{a}$ (Skew tent map system).

1) Identifiability of the Hénon Map System: The identifiability study is carried out by inspecting the set of the parameters $(\hat{a}, \hat{b})$ which is governed by the following
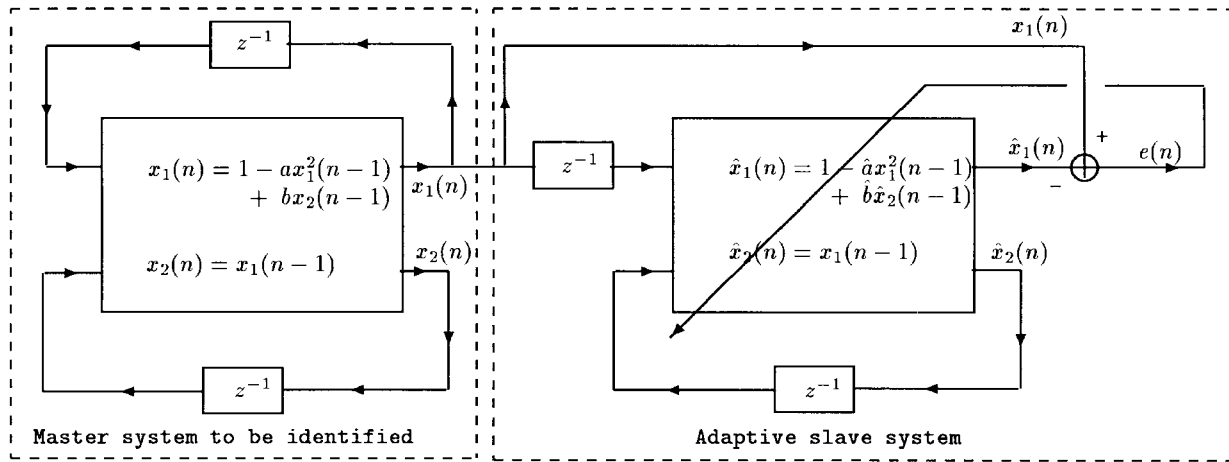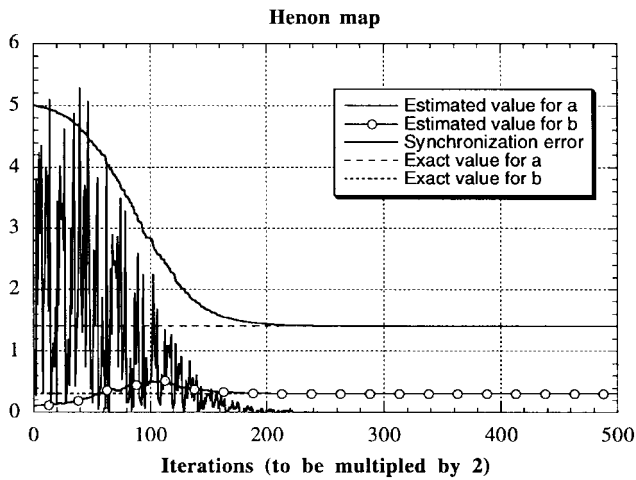
Fig. 10.   Block diagram of the identification setup for Hénon system.

Fig. 11.   Sequential estimate of parameters $\hat{a}$ and $\hat{b}$ (Hénon map system).

Fig. 12.   Decomposition of $[-1, +1]^2$.

equations:

$$1 - \hat{a}x_1^2(n-1) + \hat{b}\hat{x}_2(n-1)$$
$$= 1 - ax_1^2(n-1) + bx_2(n-1) \qquad (35)$$
$$\hat{x}_2(n)$$
$$= x_1(n-1). \qquad (36)$$

From (36) $\hat{x}_2(n-1) = x_1(n-2)$ and from (33) $x_2(n-1) = x_1(n-2)$, therefore (35) becomes

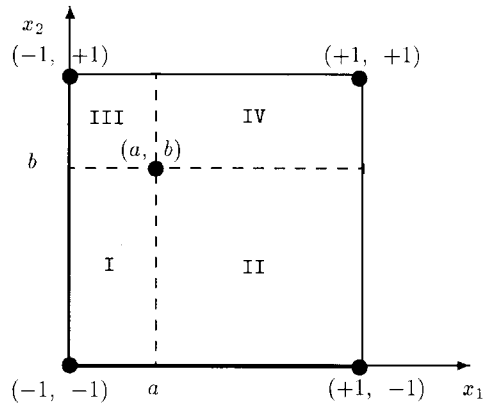$$1 - \hat{a}x_1^2(n-1) + \hat{b}x_1(n-2) = 1 - ax_1^2(n-1) + bx_1(n-2). \qquad (37)$$

Let us define $\Delta\hat{a} = \hat{a} - a$ and $\Delta b = \hat{b} - b$, from (37) we obtain

$$-\Delta\hat{a}x_1^2(n-1) + \Delta\hat{b}x_1(n-2) = 0. \qquad (38)$$

Since from the (33) it is clear that $x_1(n-1)$ depends on $x_1(n-2)$ and $x_1(n-3)$, the only solution for (38) is

$$\Delta\hat{a} = \Delta\hat{b} = 0 \qquad (39)$$

which shows that the Hénon map system is structurally globally identifiable.

2) Identification of the Hénon Map System: The identification setup for the Hénon system is shown in Fig. 10. Applying (22) the recursive estimate of $\hat{a}$ and $\hat{b}$ is given as

$$\begin{bmatrix} \hat{a}(n) \\ \hat{b}(n) \end{bmatrix} = \begin{bmatrix} \hat{a}(n-1) \\ \hat{b}(n-1) \end{bmatrix} + e(n)\boldsymbol{H}^{-1}(n)\boldsymbol{\psi}(n) \qquad (40)$$

where

$$e(n) = x_1(n) - \hat{x}_1(n) \qquad (41)$$
$$\boldsymbol{H}(n) = \lambda\boldsymbol{H}(n-1) + \boldsymbol{\psi}(n)\boldsymbol{\psi}^T(n) \quad \boldsymbol{H}(0) = R_0\boldsymbol{I} \qquad (42)$$

and

$$\boldsymbol{\psi}(n) = \begin{bmatrix} \dfrac{\partial\hat{x}_1(n)}{\partial\hat{a}} \\[2mm] \dfrac{\partial\hat{x}_1(n)}{\partial\hat{b}} \end{bmatrix} = \begin{bmatrix} -x_1^2(n-1) \\ \hat{x}_2(n-1) \end{bmatrix} = \begin{bmatrix} -x_1^2(n-1) \\ x_1(n-2) \end{bmatrix}. \qquad (43)$$

Fig. 11 shows the convergence of the parameters $\hat{a}$ and $\hat{b}$ toward their optimal values. The parameters ab in the master system were set to $a = 1.4$, $b = 0.3$. The initial values for $\hat{a}$ and $\hat{b}$ were set to $\hat{a}(0) = 5$ and $\hat{b}(0) = 0.1$. The matrix $\boldsymbol{H}(0)$ was initialized to $\boldsymbol{H}(0) = 1000\boldsymbol{I}$. The forgetting factor λ was set to 0.98. Perfect synchronization occurred after 600 iterations. The parameters were recovered exactly. We
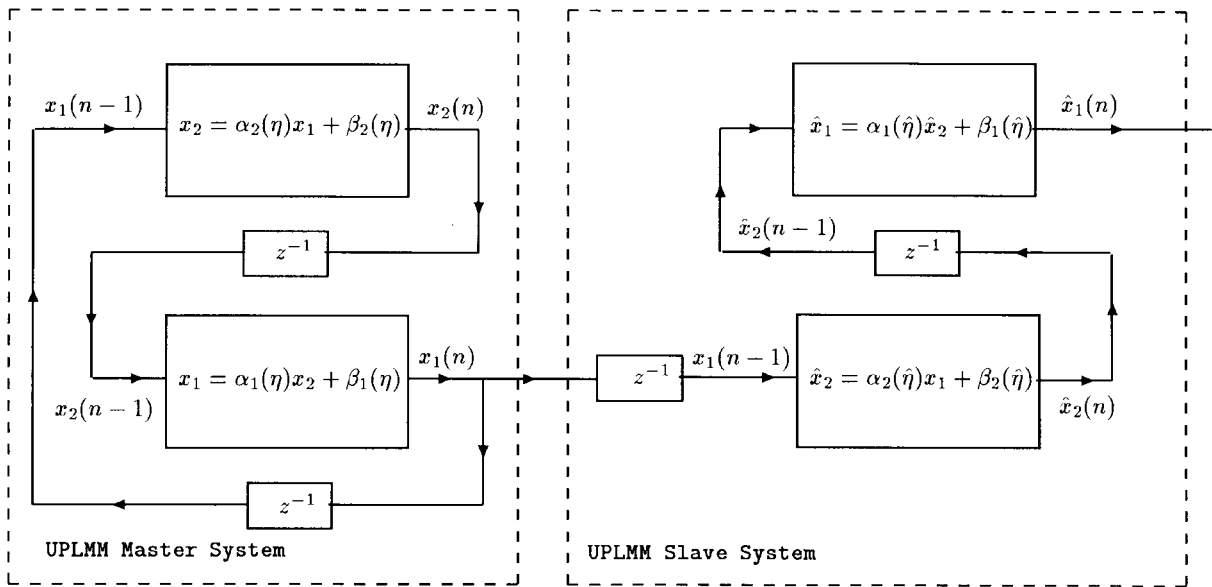
Fig. 13. Synchronization setup for the second-order UPLMM system.

obtained similar behavior for various initializations of $\hat{a}(0)$ and $\hat{b}(0)$.

C. The Second-Order Markov Map System

We consider in the next a special class of chaotic maps—the so-called Uniform Piecewise Linear Markov Maps (UPLMM's). Order N UPLMM have the very interesting feature of exhibiting a uniform density on the unit cube of $\mathbf{R}^N$. These types of maps have attracted attention both in the mathematical [26] and the engineering literature [27]–[29]. Applications of Markov maps have been proposed for different modulation-demodulation schemes for transmission of chaotic signals.

A piecewise linear transformation F of $[-1, +1]^N$ onto itself is a UPLMM if $[-1, +1]^N$ can be decomposed as

$$[-1, +1]^N = \bigcup_{j=1}^{J} S_j$$

such that F is affine on each S_j and maps it onto the whole $[-1, +1]^N$. The constant density on $[-1, +1]^N$ is the unique invariant probability density of any UPLMM.

We will restrict the presentation to an order 2 UPLMM example, for more details on higher-order maps and their applications the reader is referred to [27].

Let us consider a second-order UPLMM in which $[-1, +1]^2$ has been decomposed into four rectangular regions as shown in Fig. 12.

The four rectangles which are parametrized by a and b are uniformly expanded onto $[-1, +1]^2$ using four linear maps F_1, F_2, F_3 and F_4 which are such that they provide an expansion parallel to each of the coordinate axes and that

$$F_1\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad F_1\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} +1 \\ +1 \end{pmatrix}$$

$$F_2\begin{pmatrix} +1 \\ -1 \end{pmatrix} = \begin{pmatrix} +1 \\ -1 \end{pmatrix}, \quad F_2\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ +1 \end{pmatrix} \quad (44)$$

$$F_3\begin{pmatrix} -1 \\ +1 \end{pmatrix} = \begin{pmatrix} -1 \\ +1 \end{pmatrix}, \quad F_3\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} +1 \\ -1 \end{pmatrix},$$

$$F_4\begin{pmatrix} +1 \\ +1 \end{pmatrix} = \begin{pmatrix} +1 \\ +1 \end{pmatrix}, \quad F_4\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ -1. \end{pmatrix} \quad (45)$$

Using (44)–(45) we obtain the general solution

$$F_\eta\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha_1(\eta)x_2 + \beta_1(\eta) \\ \alpha_2(\eta)x_1 + \beta_2(\eta) \end{pmatrix} \quad \eta = 1 \cdots 4 \quad (46)$$

where η stands for the region number in which (x_1, x_2) falls. More precisely

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} \dfrac{2x_2 + (1-b)}{1+b} \\ \dfrac{2x_1 + (1-a)}{1+a} \end{pmatrix} \\ \quad \text{for } -1 \le x_1 < a \text{ and } -1 \le x_2 < b \\ \quad (\eta = 1) \\ \begin{pmatrix} \dfrac{-2x_2 - (1-b)}{1+b} \\ \dfrac{-2x_1 + (1+a)}{1-a} \end{pmatrix} \\ \quad \text{for } a \le x_1 < +1 \text{ and } -1 \le x_2 < b \\ \quad (\eta = 2) \\ \begin{pmatrix} \dfrac{-2x_2 + (1+b)}{1-b} \\ \dfrac{-2x_1 - (1-a)}{1+a} \end{pmatrix} \\ \quad \text{for } -1 \le x_1 < a \text{ and } b \le x_2 < +1 \\ \quad (\eta = 3) \\ \begin{pmatrix} \dfrac{2x_2 - (1+b)}{1-b} \\ \dfrac{2x_1 - (1+a)}{1-a} \end{pmatrix} \\ \quad \text{for } a \le x_1 < +1 \text{ and } b \le x_2 < +1 \\ \quad (\eta = 4). \end{cases} \quad (47)$$

We assume that we are given the signal $x_1(n)$ as output of the UPLMM system. The synchronization setup we will

consider is shown in Fig. 13, where $\hat{\eta}$ stands for the identifier of the region in which falls $(x_1(n-1), \hat{x}_2(n-1))$.

It has been shown in [27] that starting from arbitrary initial conditions for $\hat{x}_1(0)$ and $\hat{x}_2(0)$ the system of Fig. 13 exhibits almost certain synchronization. Therefore there exists an n_0 depending on the initial conditions of both, the master and the slave systems, such that

$$\text{for} \quad n > n_0 \quad \begin{array}{l} \hat{x}_1(n) = x_1(n) \\ \hat{x}_2(n) = x_2(n). \end{array}$$

1) Identifiability of the Second-Order Markov Map System:
We assume in this subsection that there exists $n_1 > n_0$ such that the state variables of the UPLMM slave system are such that

$$\hat{x}_1(n) = \hat{\alpha}_1(n-1)\hat{x}_2(n-1) + \hat{\beta}_1(n-1)$$
$$\hat{x}_2(n) = \hat{\alpha}_2(n-1)x_1(n-1) + \hat{\beta}_2(n-1) \quad (48)$$

and that

$$\hat{x}_1(n) = x_1(n) \quad (49)$$

The parameters $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1$ and $\hat{\beta}_2$ are linked to $\hat{a}$ and $\hat{b}$ by the following relationships, as shown in (50) at the bottom of the page. We would like to identify the set of parameters $(\hat{a}, \hat{b})$ for which the (49) is satisfied. Using (48) we obtain the input-output relationship of the slave system with parameters $\hat{a}$ and $\hat{b}$,

$$\hat{x}_1(n) = \hat{\alpha}_1(n-1)\hat{\alpha}_2(n-2)x_1(n-2)$$
$$+ \hat{\alpha}_1(n-1)\hat{\beta}_2(n-2) + \hat{\beta}_1(n-1). \quad (51)$$

This expression must be compared to the corresponding one for the synchronized slave system having the parameters a and b, i.e.,

$$x_1(n) = \alpha_1(n-1)\alpha_2(n-2)x_1(n-2)$$
$$+ \alpha_1(n-1)\beta_2(n-2) + \beta_1(n-1). \quad (52)$$

Using (49), (51), and (52) we obtain for $n > n_1$ the relationships

$$\hat{\alpha}_1(n-1)\hat{\alpha}_2(n-2)$$
$$= \alpha_1(n-1)\alpha_2(n-2) \quad (53)$$
$$\hat{\alpha}_1(n-1)\hat{\beta}_2(n-2) + \hat{\beta}_1(n-1)$$
$$= \alpha_1(n-1)\beta_2(n-2) + \beta_1(n-1). \quad (54)$$

For $(a,b) \in\,]-1,+1[$ and $(\hat{a}, \hat{b}) \in\,]-1,+1[$ let us define as T_1 the rectangular region which is the intersection between regions $\eta = 1$ and $\hat{\eta} = 1$, i.e.,

$$T_1 = [-1, \inf(a, \hat{a})] \times [-1, \inf(b, \hat{b})]$$

This intersection is nonempty and therefore there is a nonzero probability that there exists $n_2 > n_1$ such, that two consecutive points (x_1, x_2) belong to T_1, i.e.,

$$\begin{pmatrix} x_1(n_2-1) \\ x_2(n_2-1) \end{pmatrix} \in T_1 \quad \text{and} \quad \begin{pmatrix} x_1(n_2-2) \\ x_2(n_2-2) \end{pmatrix} \in T_1.$$

Therefore we obtain

$$\alpha_1(n_2-1)\alpha_2(n_2-2) = \frac{4}{(1+b)(1+a)}. \quad (55)$$

Observe that for $(a,b) \in\,]-1,+1[$ the RHS term in (55) is always positive. It is easy to show that the points $(x_1(n_2-1), \hat{x}_2(n_2-1))$ and $(x_1(n_2-2), \hat{x}_2(n_2-2))$ can be only such, that they consecutively belong either to the region $\hat{\eta} = 1$ or to the region $\hat{\eta} = 3$. The combinations $\hat{\eta}(n_2-2) = 1$, $\hat{\eta}(n_2-1) = 3$ or $\hat{\eta}(n_2-2) = 3$, $\hat{\eta}(n_2-1) = 1$ are impossible since they give a product $\hat{\alpha}_1(n_2-1)$ $\hat{\alpha}_2(n_2-2)$ which is always negative.

If we suppose that $\hat{\eta}(n_2-1) = 1$ and $\hat{\eta}(n_2-2) = 1$, (50), (53) and (55) give

$$1 + a + b + ab = 1 + \hat{a} + \hat{b} + \hat{a}\hat{b}. \quad (56)$$

If we suppose that $\hat{\eta}(n_2-1) = 3$ and $\hat{\eta}(n_2-2) = 3$, we obtain

$$1 + a + b + ab = 1 + \hat{a} - \hat{b} - \hat{a}\hat{b}. \quad (57)$$

$$\begin{cases} \hat{\alpha}_1 = \dfrac{2}{1+\hat{b}} \quad \hat{\beta}_1 = \dfrac{1-\hat{b}}{1+\hat{b}} \qquad \text{for } -1 \leq x_1 < \hat{a} \text{ and } -1 \leq \hat{x}_2 < \hat{b} \qquad (\hat{\eta} = 1) \\[2mm] \hat{\alpha}_2 = \dfrac{2}{1+\hat{a}} \quad \hat{\beta}_2 = \dfrac{1-\hat{a}}{1+\hat{a}} \\[2mm] \hat{\alpha}_1 = \dfrac{-2}{1+\hat{b}} \quad \hat{\beta}_1 = -\dfrac{1-\hat{b}}{1+\hat{b}} \qquad \text{for } \hat{a} \leq x_1 < +1 \text{ and } -1 \leq \hat{x}_2 < \hat{b} \qquad (\hat{\eta} = 2) \\[2mm] \hat{\alpha}_2 = -\dfrac{2}{1-\hat{a}} \quad \hat{\beta}_2 = \dfrac{1+\hat{a}}{1-\hat{a}} \\[2mm] \hat{\alpha}_1 = \dfrac{-2}{1-\hat{b}} \quad \hat{\beta}_1 = \dfrac{1+\hat{b}}{1-\hat{b}} \qquad \text{for } -1 \leq x_1 < \hat{a} \text{ and } \hat{b} \leq \hat{x}_2 < +1 \qquad (\hat{\eta} = 3) \\[2mm] \hat{\alpha}_2 = \dfrac{-2}{1+\hat{a}} \quad \hat{\beta}_2 = -\dfrac{1-\hat{a}}{1+\hat{a}} \\[2mm] \hat{\alpha}_1 = \dfrac{2}{1-\hat{b}} \quad \hat{\beta}_1 = -\dfrac{1+\hat{b}}{1-\hat{b}} \qquad \text{for } \hat{a} \leq x_1 < +1 \text{ and } \hat{b} \leq \hat{x}_2 < +1 \qquad (\hat{\eta} = 4). \\[2mm] \hat{\alpha}_2 = \dfrac{2}{1-\hat{a}} \quad \hat{\beta}_2 = -\dfrac{1+\hat{a}}{1-\hat{a}} \end{cases} \quad (50)$$
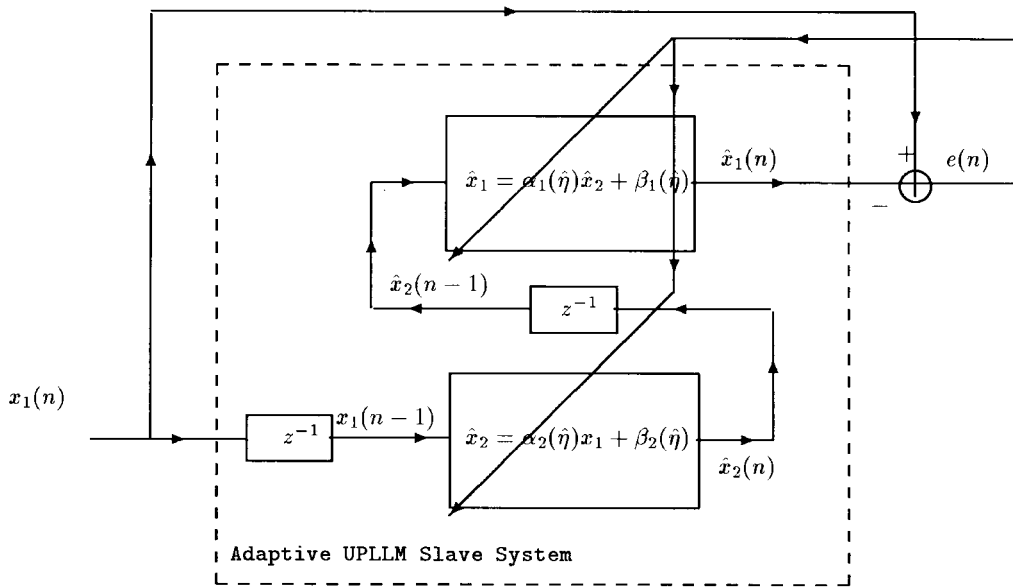
Fig. 14. Block diagram of the identification setup for the UPLMM system.

Let us define as T_3 the rectangular region which is the intersection between region $\eta = 3$ and $\hat{\eta} = 3$, i.e.,

$$T_3 = [-1, \inf(a, \hat{a})] \times [\sup(b, \hat{b}), +1].$$

Using the same argument as above we can show now by considering two consecutive points $(x_1(n_3 - 1), x_2(n_3 - 1))$ and $(x_1(n_3 - 2), x_2(n_3 - 2)) \in T_3$ that the points $(x_1(n_3 - 1), \hat{x}_2(n_3 - 1))$ and $(x_1(n_3 - 2), \hat{x}_2(n_3 - 2))$ can be only such that they belong consecutively either to the region $\hat{\eta} = 1$ or the region $\hat{\eta} = 3$. This leads to the following equations:

$$1 + a - b - ab = 1 + \hat{a} + \hat{b} + \hat{a}\hat{b} \tag{58}$$

$$1 + a - b - ab = 1 + \hat{a} - \hat{b} - \hat{a}\hat{b}. \tag{59}$$

Combining (56)–(59) shows that for almost every $(a, b) \in]-1, +1[$ the only compatible equation pairs are (56)–(59) and (57)–(58). Solving these compatible sets of equations gives

$$\hat{a} = a \quad \hat{b} = \pm b. \tag{60}$$

Inspecting these two pairs of solutions shows that the UPLMM system is **locally identifiable** when $x_1(n)$ is chosen as the observed (output) signal. The solution $\hat{a} = a\,\hat{b} = \pm b$ are such that there exist m_0 and m_1 such that

$$\hat{a} = a \text{ and } \hat{b} = b \Rightarrow \text{ for } n > m_0$$
$$\hat{x}_1(n) = x_1(n) \quad \hat{x}_2(n) = x_2(n)$$
$$\hat{a} = a \text{ and } \hat{b} = -b \Rightarrow \text{ for } n > m_1$$
$$\hat{x}_1(n) = x_1(n) \quad \hat{x}_2(n) = -x_2(n). \tag{61}$$

This paper shows that two different sets of parameters could be chosen in order to produce the same chaotic output of the system. In other terms the identification of the second-order Markov map parameters could result in the identification of either (a, b) or $(a, -b)$.

2) Identification of the Second-Order Markov Map System: As the Markov map system is a piecewise linear system a specific identification scheme could be designed by observing the dependence between $x_1(n)$ and $x_1(n-2)$. However let us consider this example in order to test our general procedure of identification. The identification setup of the UPLMM system is shown in Fig. 14. The key parameters to be found are the sensitivities of $\hat{x}_1(n)$ with respect to the parameters $\hat{a}$ and $\hat{b}$. Computing these parameters gives (62), as shown at the bottom of the page, where $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1$ and $\hat{\beta}_2$ are given in (50).

The update equations for the time variant parameters in the adaptive procedure $\hat{a}(n)$ and $\hat{b}(n)$ are similar to those given in (40)–(42) for the Hénon map.

Due the discontinuous nature of the map convergence of the algorithm was very sensitive to the choice of the initial values of the parameters. For certain initial values we did not observe convergence in a reasonable number of iterations (10 000).

We have plotted the results of two experiments in which the optimal master system parameters were set to $a = -0.48$ and $b = 0.56$. Fig. 15 shows the exact convergence in about 600 iterations. The forgetting factor was set to $\lambda = 0.98$, the initial values for $\hat{a}$ and $\hat{b}$ were $\hat{a}(0) = -0.7$ and $\hat{b}(0) = 0.72$. The matrix $\boldsymbol{H}(0)$ was initialized to $\boldsymbol{H}(0) = 10\,000\boldsymbol{I}$. Exact

$$\psi(n) = \begin{pmatrix} \dfrac{\partial \hat{x}_1(n)}{\partial \hat{a}} \\[2mm] \dfrac{\partial \hat{x}_1(n)}{\partial \hat{b}} \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_1(\hat{b}, n-1)x_1(n-2)\dfrac{\partial \hat{\alpha}_2(\hat{a}, n-2)}{\partial \hat{a}} + \hat{\alpha}_1(\hat{b}, n-1)\dfrac{\partial \hat{\beta}_2(\hat{a}, n-2)}{\partial \hat{a}} \\[4mm] \hat{x}_2(n-1)\dfrac{\partial \hat{\alpha}_1(\hat{b}, n-1)}{\partial \hat{b}} + \dfrac{\partial \hat{\beta}_1(\hat{b}, n-1)}{\partial \hat{b}} \end{pmatrix}. \tag{62}$$
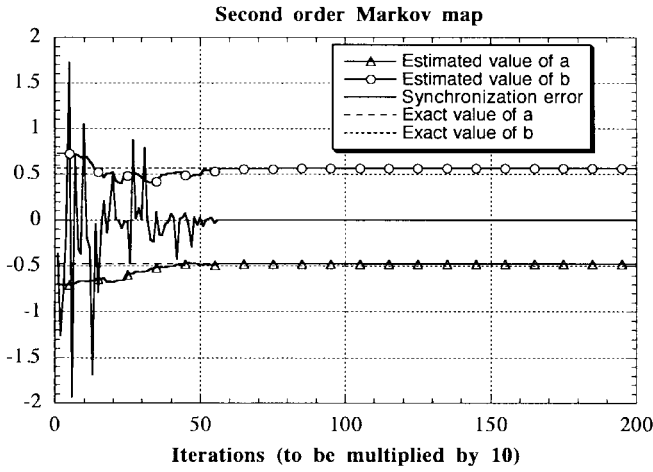
Fig. 15. Identification of the UPLMM system, convergence toward the master system parameters.
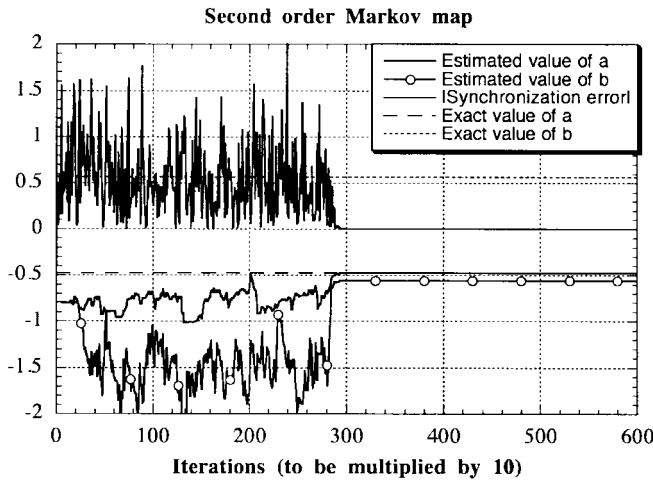

Fig. 16. Identification of the UPLMM system, verification of the identifiability defect.

synchronization occurred after 600 iterations. The parameters were recovered perfectly.

Fig. 16 shows the parameter variability for a different initialization—$\hat{a}(0) = -0.8$ and $\hat{b}(0) = 0.8$. In this case we obtain perfect synchronization after about 3000 iterations and we recover exactly the parameters $(a, -b)$ instead of (a, b). This example shows how important it is to study the identifiability problem before the identification process—the considered system is only locally identifiable, thus the results of identification procedure depend on the initial values chosen at the beginning of the procedure.

D. Chua's Circuit

Chua's circuit is a piecewise linear circuit (Figs. 17 and 18) whose state equation is of the form

$$\frac{d\boldsymbol{x}(\tau)}{d\tau} = \boldsymbol{A}(\eta)\boldsymbol{x}(\tau) + \boldsymbol{b}(\eta) \tag{63}$$

in which η refers to one of the three linear regions of the nonlinear resistor of Fig. 18, $\tau = t/RC_2$ is the normalized time. The vector $\boldsymbol{x}$ is composed of the normalized variables
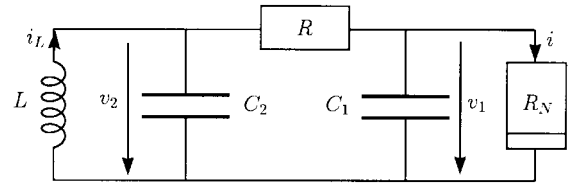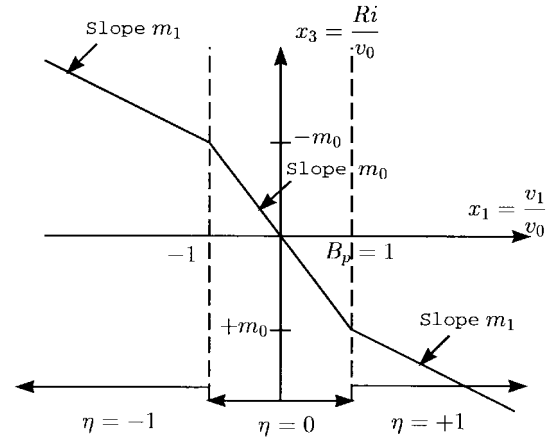

Fig. 17. Chua's circuit.

Fig. 18. Characteristic of the nonlinear resistor R_N.

$x_1 = v_1/v_0, x_2 = v_2/v_0$, $x_3 = R(i_L/v_0), v_0$ being the 'breakpoint voltage' of the nonlinear resistor.

Taking as parameters $\alpha = C_2/C_1$ and $\beta = R^2(C_2/L)$, we obtain for matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$ of (63)

$$\boldsymbol{A}(\eta) = \begin{pmatrix} -\alpha(1+m_{|\eta|}) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}$$

$$\boldsymbol{b}(\eta) = \begin{pmatrix} -\eta\alpha(m_0 - m_1) \\ 0 \\ 0 \end{pmatrix} \tag{64}$$

In (64) $\eta = 0$ stands for the linear region $|x_1| < 1$ while $\eta = 1$ for $x_1 \geq 1$ and $\eta = -1$ for $x_1 \leq -1$. We assume that the parameters are chosen such that the system operates in a chaotic mode.

Suppose that $y(t) = x_1(t)$ is the observed output of Chua's circuit. We can build a] synchronization setup by applying the Pecora and Carroll scheme, i.e., by cascading two appropriate subsystems. The first subsystem which is forced by $y(t)$ is described by

$$\begin{pmatrix} \dot{x}_{2r}(\tau) \\ \dot{x}_{3r}(\tau) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -\beta & 0 \end{pmatrix}\begin{pmatrix} x_{2r}(\tau) \\ x_{3r}(\tau) \end{pmatrix} + \begin{pmatrix} y(\tau) \\ 0. \end{pmatrix} \tag{65}$$

The parameter β in the state-matrix in (65) is such that this matrix is a stable one, therefore $x_{2r}(\tau)$ and $x_{3r}(\tau)$ exhibit an exponential convergence toward $x_2(\tau)$ and $x_3(\tau)$, respectively.

The second subsystem is described by

$$\dot{x}_{1r}(\tau) = -\alpha(1+m_{|\eta_r|})x_{1r}(\tau) + \alpha x_{2r}(\tau) - \eta_r\alpha(m_0 - m_1). \tag{66}$$
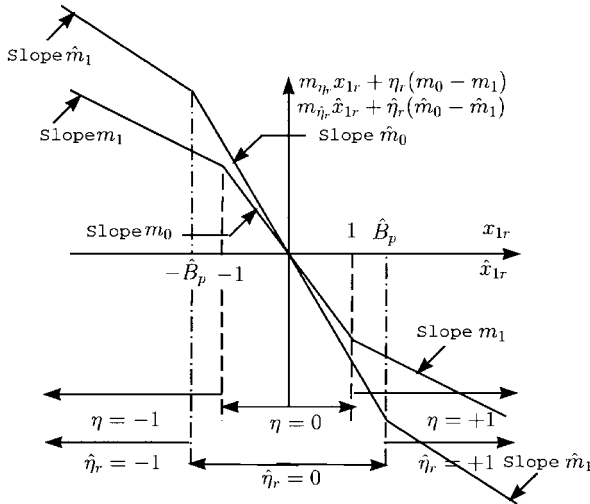
Fig. 19. Nonlinear resistor for $M(\theta)$ and $M(\hat{\theta})$.

For the study of the convergence of $x_{1r}(\tau)$ toward $x_1(\tau)$ the reader is referred to the papers by Tesi *et al.* [24], [25], we will assume in the following that $x_{1r}(\tau)$ converges asymptotically toward $x_1(\tau)$.

1) Structural Identifiability of Chua's Circuit: The slave system built as a cascade of two subsystems [(65) and (66)] can be rewritten as

$$\begin{pmatrix} \dot{x}_{1r}(\tau) \\ \dot{x}_{2r}(\tau) \\ \dot{x}_{3r}(\tau) \end{pmatrix} = \begin{pmatrix} -\alpha(1+m_{|\eta_r|}) & \alpha & 0 \\ 0 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} \begin{pmatrix} x_{1r}(\tau) \\ x_{2r}(\tau) \\ x_{3r}(\tau) \end{pmatrix}$$
$$+ \begin{pmatrix} -\eta_r \alpha(m_0 - m_1) \\ y(\tau) \\ 0 \end{pmatrix} \qquad (67)$$

Let θ be the set of parameters $\alpha, \beta, m_0, m_1, B_p = 1$. The system (67) can be therefore described as

$$\frac{d\boldsymbol{x}_r(\tau)}{d\tau} = \boldsymbol{A}_r(\boldsymbol{\theta}, \eta_r)\boldsymbol{x}_r(\tau) + \boldsymbol{b}_r(\boldsymbol{\theta}, \eta_r, y(\tau)). \qquad (68)$$

The output of the system denoted by $y_r(\tau)$ is defined as

$$y_r(\tau) = \boldsymbol{c}_r^T \boldsymbol{x}_r(\tau) \quad \text{with} \quad \boldsymbol{c}_r^T = [1, 0, 0]. \qquad (69)$$

Let $\hat{\boldsymbol{\theta}}$ be the set of parameters $\hat{\alpha}, \hat{\beta}, \hat{m}_0, \hat{m}_1, \hat{B}_p$. These parameters are explained in Fig. 19 where $\hat{\eta}_r = 0$ stands for the linear region $\hat{x}_{1r} \leq \hat{B}_p, \hat{\eta}_r = 1$ stands for $\hat{x}_{1r} > \hat{B}_p$ and $\hat{\eta}_r = -1$ stands for the linear region $\hat{x}_{1r} < -\hat{B}_p$.

Let $M(\hat{\boldsymbol{\theta}})$ be the slave system with state space $\hat{\boldsymbol{x}}_r$ which is governed by

$$\frac{d\hat{\boldsymbol{x}}_r(\tau)}{d\tau} = \boldsymbol{A}_r(\hat{\boldsymbol{\theta}}, \hat{\eta}_r)\hat{\boldsymbol{x}}_r(\tau) + \boldsymbol{b}_r(\hat{\boldsymbol{\theta}}, \hat{\eta}_r, y(\tau)). \qquad (70)$$

We assume that $\hat{\boldsymbol{\theta}}$ is such that there exists a time τ_0 such that

$$\hat{x}_{1r}(\tau) = x_{1r}(\tau) = x_1(\tau) \quad \text{for} \quad \tau > \tau_0 \qquad (71)$$

and want to find all the sets of parameters $\hat{\boldsymbol{\theta}}$ for which (71) is satisfied. Having this goal in mind we will first show that for $\tau > \tau_0$ we have $\eta_r = \hat{\eta}_r$ (i.e., $\hat{B}_p = B_p = 1$).

Suppose that there exist two time intervals $[\tau_1, \tau_2]$ and $]\tau_2, \tau_3](\tau_3 > \tau_2 > \tau_1 > \tau_0)$ such that $x_{1r}(\tau)$ is in region $\eta_r = 0$

if $\tau \in [\tau_1, \tau_2]$ and $x_{1r}(\tau)$ is in region $\eta_r = -1$ if $\tau \in]\tau_2, \tau_3]$. According to (68) the dynamics of $x_{1r}(\tau)$ is slowed down by switching m_{η_r} from m_0 to m_1, this switching occurs at time τ_2 when $x_{1r}(\tau_2) = +1$. According to (70) and (71) the dynamics of $\hat{x}_{1r}(\tau)$ should be also slowed down at time τ_2 because if not it would not possible for $\hat{x}_{1r}(\tau)\tau > \tau_2$ to follow the new dynamic of $x_{1r}(\tau)$. Therefore this switching occurs for $\hat{x}_{1r}(\tau_2) = x_{1r}(\tau_2) = +1$ which indicates that $\hat{B}_p = 1$ and that $\hat{\eta}_r(\tau) = \eta_r(\tau)$ for any $\tau > \tau_0$.

Having shown that the two systems $M(\boldsymbol{\theta})$ and $M(\hat{\boldsymbol{\theta}})$ have to operate in the same linear region at the same time simplifies the identifiability study since we can apply the similarity transformation approach in each linear region. Let us define

$$\boldsymbol{z}_r(\tau) = \boldsymbol{T}\boldsymbol{x}_r(\tau) \qquad (72)$$

where $\boldsymbol{T}$ is a constant invertible square matrix. The dynamical system $M(\boldsymbol{\theta})$ is then described as

$$\frac{d\boldsymbol{z}_r(\tau)}{d\tau} = \boldsymbol{T}\boldsymbol{A}_r(\boldsymbol{\theta}, \eta_r)\boldsymbol{T}^{-1}\boldsymbol{z}_r(\tau) + \boldsymbol{T}\boldsymbol{b}_r(\boldsymbol{\theta}, \eta_r, y(\tau)),$$
$$y_r(\tau) = \boldsymbol{c}_r^T(\boldsymbol{\theta})\boldsymbol{T}^{-1}\boldsymbol{z}_r(\tau). \qquad (73)$$

2) Identifiability in the Region $\eta_r = 0$: If we can find parameters $\hat{\boldsymbol{\theta}}$ such that

$$\boldsymbol{A}_r(\hat{\boldsymbol{\theta}}, \eta_r = 0) = \boldsymbol{T}\boldsymbol{A}_r(\boldsymbol{\theta}, \eta_r = 0)\boldsymbol{T}^{-1} \qquad (74)$$
$$\boldsymbol{b}_r(\hat{\boldsymbol{\theta}}, \eta_r = 0, y(\tau)) = \boldsymbol{T}\boldsymbol{b}_r(\boldsymbol{\theta}, \eta_r = 0, y(\tau)) \qquad (75)$$
$$\boldsymbol{c}_r^T(\hat{\boldsymbol{\theta}}) = \boldsymbol{c}_r^T(\boldsymbol{\theta})\boldsymbol{T}^{-1} \qquad (76)$$

then (73) describes the system $M(\hat{\boldsymbol{\theta}})$ and $M(\hat{\boldsymbol{\theta}}) = M(\boldsymbol{\theta})$.

Let $\boldsymbol{T} = [t_{ij}]$. Using (76) we can show that

$$t_{11} = 1, \quad t_{12} = 0, \quad t_{13} = 0. \qquad (77)$$

By substituting (77) in (75) we obtain

$$t_{22} = 1, \quad t_{32} = 0. \qquad (78)$$

Then using (77) and (78) in (74) we obtain

$$t_{21} = 0, \quad t_{23} = 0, \quad t_{31} = 0, \quad t_{33} = 1$$
$$\hat{\alpha} = \alpha, \quad \hat{\beta} = \beta, \quad \hat{m}_0 = m_0 \qquad (79)$$

which shows that

$$\boldsymbol{T} = \boldsymbol{I}, \quad \hat{\alpha} = \alpha, \quad \hat{\beta} = \beta, \quad \hat{m}_0 = m_0. \qquad (80)$$

3) Identifiability in the Region $\eta_r = 1$: Application of the similarity transformation approach $\boldsymbol{T}$ in region $\eta_r = 1$ shows that

$$\boldsymbol{T} = \boldsymbol{I}, \quad \hat{\alpha} = \alpha, \quad \hat{\beta} = \beta, \quad \hat{m}_1 = m_1. \qquad (81)$$

Concluding, (80) and (81) show that Chua's circuit is structurally globally identifiable since there is only one set of parameters $\boldsymbol{\theta}$ which produces the same output $y_r(\tau)$.
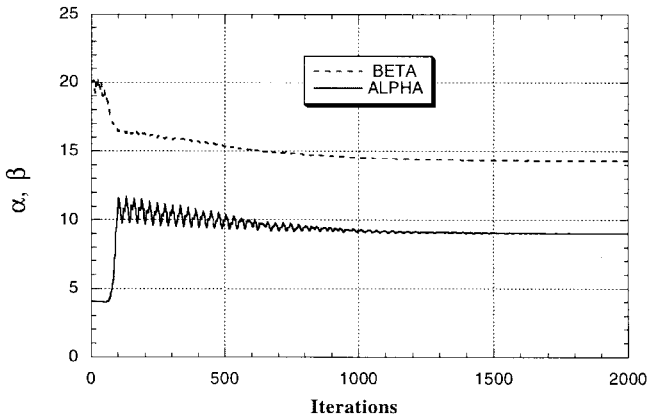
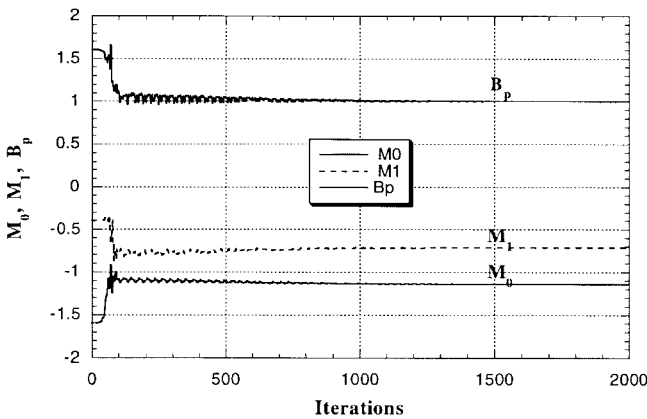Fig. 20. Convergence of $\hat{\alpha}$ and $\hat{\beta}$.

Fig. 21. Convergence of $\hat{m}_0, \hat{m}_1$ and $\hat{\beta}$.

4) Identification of the Chua's Circuit: The identification of such a system has been already performed in [3], [7], [9] by exploiting synchronization and by using different optimization techniques. In particular in [9] a deterministic optimization exploiting the Powell's method has been shown to be efficient. Using the Gauss-Newton procedure described in Section II we were also able [7] to match the parameters with a precision better than 0.01%. However in order to get a perfect synchronization in a reasonable amount of time we had to run the optimization using different initializations of the parameter set $\hat{\alpha}, \hat{\beta}, \hat{m}_0, \hat{m}_1, \hat{B}_p$. This shows the limitation of our optimization technique which is local in nature. Figs. 20 and 21 show the convergence of $\hat{\alpha}, \hat{\beta}, \hat{m}_0, \hat{m}_1, \hat{B}_p$. The true value of these parameters were $\alpha = 9$, $\beta = \frac{100}{7}, m_0 = -\frac{8}{7}$, $m_1 = -\frac{5}{7}$, these values were set in such a way that the system operates in chaotic mode.

V. CONCLUSIONS

We proposed a general method to find unknown slave (receiver) system parameters in common chaotic synchronization schemes. The method is based on adaptive control methods used in combination with synchronization principles. The synchronization allows a dimension reduction of the minimization problem which we have to solve and provides an elegant way to cope with the sensitivity to initial conditions.

Although a deeper study should be carried out to develop better optimization schemes there is at least experimental evidence that it is possible to find the unknown parameters of the slave system using the mixed approach we propose.

We have also used the notion of structural identifiability which enables us to address the problem of whether it is possible at all to find the unknown system parameters given only measured output signals from the system.

In communication applications the recovery of the parameters can be explained in other terms indicating that transmission secret keys can easily be found. From this perspective security claims in transmission systems using chaotic carriers should be handled with great care since for low order systems the knowledge of the structure of the chaotic coder (master system) is sufficient in most cases to identify its parameters (secret key) thus breaking the code and finding the coded information.

However the proposed identification method will not work properly when the dynamical system to be identified is a high order system. The reason for this lack of efficiency is obvious since synchronization will occur when the initial parameters are not too faraway from the true parameters. Starting from an arbitrary initial condition there is indeed a great probability for the parameters to be trapped at some local minimum of the synchronization cost function. This means that some monitoring of the search process should be forseen whenever possible. This monitoring could be in principle done by injecting some constraints or *a priori* knowledge in the cost function. For instance the identification of the Markov map system could be done easily in high dimension since the parameter range is known beforehand and since some correlation technique could give some rough estimation of the parameter values. For some specific systems it is possible to design a rigourous adaptive synchronization scheme whose synchronization error is guaranteed to converge toward zero. For instance feedback linearizable systems such as those studied in [30] allow the design of Lyapunov functions such that adaptive control of the parameters ensures at each step a monotonic decrease of the Lyapunov function.

REFERENCES

[1] M. Hasler, "Synchronization principles and applications," in *ISCAS'94 Tutorials,* C. Toumazou, Ed., London, England, 1994, pp. 314–327.
[2] ——, "Engineering chaos for encryption and broadband communication," *Phil. Trans. Royal Soc. Lond.,* vol. 352, pp. 1–12, 1995.
[3] R. Caponetto, L. Fortuna, G. Manganaro, and M. G. Xibilia, "Synchronization-based nonlinear chaotic circuit identification," in *Chaotic Circuits for Communications,* J. M. H. Elmirghani, Ed., Philadelphia, PA, *SPIE Proc.,* vol. 2612, pp. 48–56, Oct. 1995.
[4] H. Dedieu and M. Ogorzałek, "Finding initial conditions of a chaotic oscillator in order to reproduce transient target signals," in *Proc. NDES'94,* Krakow, 1994, pp. 123–128.
[5] ——, "Identification and control on nonlinear dynamics fro signal codin and compression," in *Proc. ECCTD'95,* Istanbul, Turkey, vol. 2, 1995, pp. 1137–1140.
[6] ——, "Signal coding and compression based on chaos control techniques," in *Proc. 1995 IEEE ISCAS,* pp. 1191–1194.
[7] ——, "Identification and control of a particular class of chaotic systems," in *Chaotic Circuits for Communications,* J. M. H. Elmirghani, Ed., Philadelphia, PA, *SPIE Proc.,* vol. 2612, pp. 148–156, Oct. 1995.
[8] U. Parlitz, "Estimating model parameters from time series by autosynchronization," *Phys. Rev. Lett.,* vol. 76, no. 8, pp. 1232–1235, Feb. 19, 1996.

[9] U. Parlitz, L. Junge, and L. Kocarev, "Synchronization-based parameter estimation from time series," *Phys. Rev. E,* vol. 54, no. 6, pp. 6253–6259, Dec. 1996.

[10] G. Chen and X. Dong "From chaos to order—perspectives and methodologies in controlling chaotic nonlinear dynamical systems," *Int. J. Bifurc. Chaos,* vol. 3, 1993 (in press).

[11] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.,* vol. 64, no. 8, pp. 821–824, Feb. 1990.

[12] T. L. Carroll and L. M. Pecora, "Synchronizing chaotic circuits," *IEEE Trans. Circuits Syst.,* vol. 38, pp. 453–456, Apr. 1991.

[13] M. J. Ogorzałek, "Taming chaos: Part I—Synchronization," *IEEE Trans. Circuits Syst.,* vol. 40, pp. 693–699, Oct. 1993.

[14] ——, "Taming chaos: Part II—Control," *IEEE Trans. Circuits Syst.,* vol. 40, pp. 700–706, Oct. 1993.

[15] M. J. Ogorzałek and H. Dedieu, "Chaos cont techniques for signal processing," in *Proc. IEEE Workshop Nonlinear Signal and Image Processing,* Neos Marmaras, vol. I, 1995, pp. 133–136.

[16] M. Vidyasagar, "Nonlinear systems analysis," Englewood Cliffs, NJ: Prentice-Hall, 1978.

[17] L. O. Chua, M. Komuro, and T. Matsumoto, "The double scroll family—Parts I and II," *IEEE Trans. Circuits Syst.,* vol. CAS-33, pp. 1072–1118, Nov. 1986.

[18] R. N. Madan, Guest Ed., "Chua's circuit: A paradigm for chaos," *J. Circuits, Syst. Comput.—Part I,* vol. 3, no. 1, Mar. 1993; Part II, vol. 3, no. 2, 1993.

[19] H. Dedieu and M. Ogorzałek, "Controlling chaos in Chua's circuit via sampled inputs," *Int. J. Bifurc. Chaos,* vol. 4, no. 2, pp. 447–455, Apr. 1994.

[20] M. P. Kennedy, "Chaos in the Colpitts oscillator," *IEEE Trans. Circuits Syst. I,* vol. 41, pp. 771–774, Nov. 1994.

[21] L. Ljung and T. Söderström, *Theory and Practice of Recursive Identification.* Cambridge, MA: MIT Press, 1987.

[22] E. Walter and L. Pronzato, "Identifiabilities and nonlinearities" in *Nonlinear Systems. Vol. 1. Modelization and Estimation,* A. J. Fossard and D. Normand-Cyrot, Eds. London, England: Chapman and Hall, 1995, ch. 3.

[23] ——, "Qualitative and quantitative experiment design for phenomenological models—A survey," *Automatica,* vol. 26, no. 2, pp. 195–213, 1990

[24] A. Tesi, A. De Angeli, and R. Genesio, "On the system decomposition for synchronizing chaos," Tech. Rep. RT 12/93, Universita di Firenze, Florence, Italy.

[25] A. De Angeli, A. Tesi, and R. Genesio, "Self-synchronizing continuous and discrete chaotic systems: Stability and dynamic performance analysis of several schemes," in *Proc. NDES'94,* Krakow, Poland, pp. 109–114.

[26] V. Baladi, "Infinite kneading matrices and weighted zeta functions on interval maps," *J. Funct. Anal.,* vol. 128, pp. 226–244.

[27] M. Hasler, M. Delgado-Restituto, and A. Rodríguez-Vázquez, "Markov maps for communications with chaos," in *Proc. Nonlinear Dynamics in Electron. Syst.,* NDES'96, Seville, Spain, June 1996, pp. 161–166.

[28] M. Hasler, "Synchronization of iterated Markov maps and information transmission by predictive control," in *Proc. Int. Symp. Nonlinear Theory and its Applicat. (NOLTA' 96),* Katsurahama-so, Kochi, Japan, Oct. 1996, pp. 189–192.

[29] M. Goetz, K. Kelber, and T. Kilias, "Statistical analysis of two-dimensional discrete-time systems using Frobenius-Perron operator," in *Proc. Workshop on Nonlinear Dynam. Electron. Syst.,* NDES'95, Dublin, Ireland, June 1995, pp. 25–28.

[30] D. Djukic, M. Hasler, and H. Dedieu, "Identification and synchronization of the discrete iterated maps by adaptive control tools," *Proc. Workshop on Nonlinear Dynam. Electron. Syst.,* NDES'96, Seville, Spain, June 1996, pp. 357–362.

Hervé Dedieu (M'92) received the Ph.D. degree from the Institut National Polytechnique de Toulouse, France, in 1988.

From 1984 to 1988 he worked with the Signal Processing Laboratory, Ecole Nationale Supérieure d'Electronique, d'Electrotechnique et d'Hydraulique de Toulouse where his research was concentrated on adaptive filtering. During the same period he worked also as a contractual engineer for the French Spatial Agency (CNES). In 1989 he joined the Circuits and Systems Group, Swiss Federal Institute of Technology, Lausanne, where his research interests include identification of nonlinear systems, applications of nonlinear dynamics in signal processing and speech recognition. He coauthored two books, with C. Dehollain, M. Hasler and J. Neirynck, *Electrical Filters* and *Nonlinear Effects in Digital Filters*, with R. Boite and M. Hasler, Ed., in French, (Presses Polytechniques Romandes). He was also the Editor of the book *Selected Topics in Circuits and Systems* (New York: Elsevier, 1993). He is currently lecturer at the Swiss Federal Institute of Technology, Lausanne.

Maciej J. Ogorzałek (M'87–SM'93–F'97), for a photograph and biography, see this issue, p. 855.