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Identification of a point source in a linear advection–dispersion–reaction equation: application to a pollution source problem

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Abstract
We consider the problem of identification of a pollution source in a river. The mathematical model is a one-dimensional linear advection–dispersion–reaction equation with the right-hand side spatially supported in a point (the source) and a time-dependent intensity, both unknown. Assuming that the source becomes inactive after the time T^*, we prove that it can be identified by recording the evolution of the concentration at two points one of which is strategic.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

When we observe a river, the transparency of its water, the natural aspect of its banks and its bottom can sometimes reflect its quality. However, to be sure of this quality, we have to analyse the composition of the water and the quality of sediments the river transports.

By the quality of water we mean its physical, chemical and biological properties which can be estimated by measuring, for example, the quantity of organic matter contained in water.

By organic matter we mean a set of organic substances the degradation of which implies consumption of the oxygen dissolved in the water with direct consequences on the aquatic life. These substances are contained in the discharges of human and agricultural origin and in numerous industrial discharges. The importance of this pollution is estimated by the measures of the so-called BOD (biologic oxygen demand) and COD (chemical oxygen demand); see [12, 16] for more details.

In this paper, we are concerned with the problem of identifying the location and intensity of a pollution point source from the measurements of BOD or COD at some points in the river. The portion (of the river) under surveillance is assimilated to a segment of a line. The governing equations and the problem statement are specified in section 2. We then prove, in section 3, that the pollution source is identifiable with measurements at two points, one upstream, the other downstream from the source, provided that one of them is *strategic* in a sense that will be described later. We establish, in section 4, a local Lipschitz stability result. Section 5 is devoted to two numerical algorithms to recover this source. Some experiments comparing their efficiency are given in section 6.

2. Governing equations and problem statement

The pollutant concentration u that we consider here (BOD or COD) is governed by the following one-dimensional linear advection–dispersion–reaction equation,

$$L[u](x,t) = F(x,t), \qquad 0 < x < \ell, \quad 0 < t < T \tag{2.1}$$

where

$$L[u](x,t) = \partial_t u(x,t) - D\partial_{xx}u(x,t) + V\partial_x u(x,t) + Ru(x,t)$$

with u the concentration of pollutant, V the velocity of the river, D a dispersion coefficient, R a reaction coefficient, and F the source term.

For more information one can see [12] or [16] where detailed derivations and discussions of the governing equations for flow and transport on surface water systems are available.

As usual, to the evolution equation (2.1) one has to append initial and boundary conditions. For the first one, there is no restriction to start the time interval at some moment where no pollution has occurred yet. For the second one, physical considerations indicate that things are different at the two extreme points of the observed portion of the river. Indeed, in most situations of interest, transport is unidirectional in nature. It means that, there is no significant transport upstream. Therefore, the null concentration at some point situated upstream can be used as the boundary condition. On the other hand, there are two options for modelling the downstream boundary: a zero gradient or a zero concentration assumption. The first option corresponds better to the transport physics, however if the downstream point is far enough from the source, the second one seems reasonable as well. To simplify the presentation, we will only consider here the first option. This corresponds to the following initial–boundary conditions:

$$\begin{aligned} &u(x,0) = 0 \qquad \text{for} \quad 0 < x < \ell \\ &u(0,t) = 0 \qquad \text{and} \qquad \partial_x u(\ell,t) = 0 \qquad \text{for} \quad 0 < t < T. \end{aligned} \tag{2.2}$$

The operator L is merely a linear parabolic partial differential operator, and it is well known that, under various assumptions on F, the problem (2.1), (2.2) has a unique solution, denoted here by $u = u(x,t;F)$, which is smooth enough so that it makes sense to talk about its point values. Then for $0 < a < b < \ell$, one can define the observation operator

$$B[F] := \{u(a,t;F), u(b,t;F), 0 < t < T\}.$$

This is the so-called direct problem. The inverse problem we are concerned with is as follows:

ISP. Given the records $\{d_1(t), d_2(t), 0 < t < T\}$ of the concentration u at two observation points $a, b \in (0,\ell)$, find the source of the pollutant; that is, to find F such that

$$B[F] = \{d_1(t), d_2(t), 0 < t < T\}. \tag{2.3}$$

Several questions arise in such inverse problem: do the available data (the RHS of (2.3)) uniquely determine F (identifiability) and if so, how do the source parameters depend on the measurements (stability)? Is there a constructive algorithm for determining this source (identification)?

These questions will be treated in the following sections, but we can say here and now that the identifiability of function sources is not true in general. This non-identifiability is shown by the following example.

Let $f \in \mathcal{C}_c^\infty(a, b)$. Then for real $\sigma \neq \rho$, and $F(x, t) = L[(e^{\sigma t} - e^{\rho t}) f(x)]$, it is apparent that $u(x, t; F) = (e^{\sigma t} - e^{\rho t}) f(x)$ satisfies (2.1) and (2.2) as well as $B[F] = \{0, 0\}$. Thus, two function sources F_1, F_2 cannot be distinguished from measurements at two points a and b.

To overcome this difficulty, people generally assume that some *a priori* information on the sources is available. For example, *time-independent* sources $F(x, t) = f(x)$ are treated by Cannon [2] using spectral theory, and by Engl, Scherzer and Yamamoto [9] using the approximated controllability of the heat equation. The results of this last paper are generalized by Yamamoto [19, 20] to the sources of the form $F(x, t) = \alpha(t) f(x)$, $f \in L^2$, where the time part function $\alpha \in C^1[0, T]$ is known and satisfying the condition $\alpha(0) \neq 0$. Recently, Hettlich and Rundell [10] considered a $2D$ problem for the heat equation with the sources of the form $F(x, t) = \chi_D(x)$, where D is a subset of a disc. They proved that the set D can be identified with the measures of the flow at two different points on the boundary, and gave a numerical method to identify it. Finally, the non-linear source problem, where F is dependent on the solution of the equation, i.e. $F(x, t) = G(u(x, t))$, is considered in the papers of DuChateau and Rundell [6], and Cannon and DuChateau [3].

3. Identifiability

Following the usual modelling of point sources in physics, we assume that F is of the form

$$F(x, t) = \lambda(t)\delta(x - S), \tag{3.4}$$

with $0 < S < \ell$ and $\lambda \in L^2(0, T)$.

In this case, it is known (see, for example, [14]) that the problem (2.1), (2.2) has a unique solution which belongs to the functional space

$$L^2(0, T; H^1(0, \ell)) \cap C([0, T]; L^2(0, \ell)).$$

Thus, by imbedding Sobolev theorem, one can define, as claimed in the previous section, the values of u at any point (x, t) in $(0, \ell) \times (0, T)$ and the problem (ISP) makes sense again. To determine the parameters λ and S of the source, and to emphasize their role, we will denote the solution in the following as $u = u(x, t; \lambda, S)$ and the observation operator as

$$B[\lambda, S] = \{u(a, t; \lambda, S), u(b, t; \lambda, S)\ 0 < t < T\}.$$

It is also worth noting that the dependence of u and B on the couple (λ, S) is no more linear.

In [7], for general two-dimensional (2D) and three-dimensional (3D) space domains, we have considered the problem of detecting pointwise sources with boundary measurements. Here, in the one-dimensional (1D) case, as stated in ISP, the measurements are made at two *interior* points. We also assume that, from some *a priori* knowledge of the localization of the source, one of the two points is chosen upstream and the other downstream with respect to the source, that is $0 < a < S < b < \ell$. The main result (theorem 1 below) is then slightly different from that in [7] and appeals to the concept of *strategic point* introduced by El Jai and Pritchard in [8] in a control problem.

Definition 1. *Let $\{\psi_n\}$ be a complete orthonormal family of continuous functions in $L^2(0, \ell)$. Then a point $b \in (0, \ell)$ is said to be a 'strategic point' relative to the family $\{\psi_n\}$ if*

$$\psi_n(b) \neq 0 \quad \forall\, n. \tag{3.5}$$

In particular, let $\{\psi_n\}$ denote the complete orthonormal family of eigenfunctions for the following Sturm–Liouville problem, in which the parameters, D, V, R are those from the operator $L[u]$,

$$\begin{aligned} &-D\psi(x)'' + \varrho\psi(x) = \mu\psi(x), \qquad 0 < x < \ell \\ &\psi(0) = \psi'(\ell) - \alpha\psi(\ell) = 0, \end{aligned} \tag{3.6}$$

where

$$\alpha = \frac{-V}{2D} \qquad \text{and} \qquad \varrho = \alpha^2 D + R. \tag{3.7}$$

One can easily verify that $\psi_n(x) = c_n \sin(\beta_n x)$ where $(\beta_n), n \geqslant 0$ are the positive solutions of equation $\beta l \cot(\beta l) = \alpha l$ listed in increasing order, and c_n a normalization coefficient. The associated eigenvalues μ_n to ψ_n are

$$\mu_n = \varrho + D\beta_n^2.$$

Thus

$$\beta_n = (2n+1)\frac{\pi}{2} - \epsilon_n \quad \text{with} \quad 0 < \epsilon_n < \pi/2 \quad \text{and} \quad \lim_{n\to\infty} \epsilon_n = 0, \tag{3.8}$$

so that

$$\varrho < \mu_n < \mu_{n+1}, \quad \mu_n \sim \frac{D^2}{\ell^2}\pi^2 n^2 \text{ at infinity}. \tag{3.9}$$

Then we have the following:

Lemma 1. *Let $T^* < T$ and $b \in (0, \ell)$ be a strategic point relative to the family $\{\psi_n\}$, and suppose that $w = w(x, t)$ satisfies*

$$\begin{aligned} &L[w](x,t) = 0, \qquad 0 < x < \ell, \quad T^* < t < T, \\ &w(0,t) = \partial_x w(\ell, t) = 0, \qquad \textit{for} \quad T^* < t < T, \\ &w(.,T^*) \in L^2(0,\ell). \end{aligned} \tag{3.10}$$

Then

$$w(b,t) = 0, \qquad \forall t \in]T^*, T[\Longrightarrow w(., T^*) = 0 \quad \textit{in} \quad L^2(0,\ell).$$

Proof. Let

$$z(x,t) = \mathrm{e}^{\alpha x} w(x,t) \quad \text{with } \alpha \text{ given in (3.7)}. \tag{3.11}$$

Then, w is the solution of (3.10) if and only if z is the solution of the following heat equation:

$$\begin{aligned} &\partial_t z - D\partial_{xx} z + \varrho z = 0, \qquad 0 < x < \ell, \quad T^* < t < T \\ &z(0,t) = \partial_x z(\ell,t) - \alpha z(\ell,t) = 0, \qquad \text{for} \quad T^* < t < T, \\ &z(.,T^*) \in L^2(0,\ell). \end{aligned} \tag{3.12}$$

Thus, z is given by the Fourier expansion in L^2 sense,

$$z(x,t) = \sum_{n=0}^{\infty} \langle z(.,T^*), \psi_n \rangle_{L^2} \psi_n(x)\, \mathrm{e}^{-\mu_n(t-T^*)}, \tag{3.13}$$

where $\langle f, g\rangle_{L^2}$ denotes the $L^2(0,\ell)$ inner product $\int_0^\ell f(x)g(x)\,\mathrm{d}x$.

Actually, from (3.9), one sees that the expansion on the RHS of (3.13) is uniformly convergent for all $t \geqslant t_0 > T^*$ and represents a real analytic function of $t \in]T^*, \infty[$ for every $x \in (0,\ell)$. This gives a sense for $z(b,t)$ and $w(b,t)$ for $t > T^*$.

Now, since

$$z(b,t)=0, \qquad \forall\, t \in]T^*, T[,$$

by analytic continuation, we conclude that

$$\sum_{n=0}^{\infty} \langle z(.,T^*), \psi_n\rangle_{L^2(0,\ell)} \psi_n(b)\, \mathrm{e}^{-\mu_n(t-T^*)} = 0 \qquad \forall\, t \in\,]T^*, +\infty[. \tag{3.14}$$

Therefore, using again (3.9), one can successively deduce that all the coefficients of $\mathrm{e}^{-\mu_n(t-T^*)}$ in the series (3.14) are null:

$$\langle z(.,T^*), \psi_n\rangle_{L^2} \psi_n(b) = 0, \qquad \forall\, n \in \mathbb{N}.$$

Since b is strategic, one has

$$\langle z(.,T^*), \psi_n\rangle_{L^2} = 0, \qquad \forall\, n \in \mathbb{N}.$$

Thus, $z(.,T^*) = 0$ in $L^2(0,\ell)$ and therefore $w(.,T^*) = 0$ in $L^2(0,\ell)$. This is the desired conclusion. □

Theorem 1. *Suppose $F_j(x,t) = \lambda_j(t)\delta(x - S_j)$ where $\lambda_j \in L^2(0,T)$ is such that for $j = 1, 2$, $\lambda_j(t) \geqslant 0$ with $\lambda_j(t) = 0$ for $T^* < t < T$, and $S_j \in (a,b)$, $j = 1, 2$. If at least one of the points a or b is strategic with respect to the family $\{\psi_n\}$, then $B[\lambda_1, S_1] = B[\lambda_2, S_2]$ implies $\lambda_1(t) = \lambda_2(t)$, almost everywhere in $(0,T)$, and $S_1 = S_2$.*

Proof. Let u_i, $i = 1, 2$ be the solutions of

$$\begin{aligned}
&L[u_j](x,t) = \lambda_j(t)\delta_j(x - S_j), \qquad 0 < x < \ell, \quad 0 < t < T,\\
&u_j(0,t) = \partial_x u_j(\ell,t) = 0 \qquad \text{for} \quad 0 < t < T,\\
&u_j(x,0) = 0 \qquad \text{for} \quad 0 < x < \ell.
\end{aligned}$$

Consider the difference $v = u_2 - u_1$, which is the solution of

$$\begin{aligned}
&L[v](x,t) = \lambda_2(t)\delta_2(x - S_2) - \lambda_1(t)\delta_1(x - S_1), \qquad 0 < x < \ell, \quad 0 < t < T,\\
&v(0,t) = \partial_x v(\ell,t) = 0 \qquad \text{for} \quad 0 < t < T,\\
&v(x,0) = 0 \qquad \text{for} \quad 0 < x < \ell,
\end{aligned} \tag{3.15}$$

while $B[\lambda_1, S_1] = B[\lambda_2, S_2]$ means that

$$v(a,t) = v(b,t) = 0 \qquad \text{for} \quad 0 < t < T. \tag{3.16}$$

In the first step, we consider v in $]0,\ell[\times]T^*, T[$. Since, $\lambda_j(t) = 0$ for $T^* < t < T$, one gets

$$\begin{aligned}
&L[v](x,t) = 0, \qquad 0 < x < \ell, \quad T^* < t < T,\\
&v(0,t) = \partial_x v(\ell,t) = 0 \qquad \text{for} \quad T^* < t < T.
\end{aligned} \tag{3.17}$$

Since $v(b,t) = 0$ in $]T^*, T[$, lemma 1 implies that

$$v(x,T^*) = 0 \qquad \text{for} \quad 0 < x < \ell. \tag{3.18}$$

In the second step, we consider v in $]0,\ell[\times]0,T^*[$. For more clarity, let us rewrite (3.18) here:

$$\begin{aligned}
&L[v](x,t) = \lambda_2(t)\delta_2(x - S_2) - \lambda_1(t)\delta_1(x - S_1), \qquad 0 < x < \ell, \quad 0 < t < T^*,\\
&v(0,t) = \partial_x v(\ell,t) = 0, \qquad \text{for} \quad 0 < t < T^*,\\
&v(x,0) = 0 \qquad \text{for} \quad 0 < x < \ell.
\end{aligned} \tag{3.19}$$

Since $0 < a < S_i < b < \ell$, $j = 1, 2$, we deduce from (3.19)

$$v = 0 \quad \text{in} \quad]0, a[\times]0, T^*[\cup]b, \ell[\times]0, T^*[.$$

Therefore

$$\partial_x v(a, t) = \partial_x v(b, t) = 0 \qquad \text{for} \quad 0 < t < T^*. \tag{3.20}$$

Let now r_i, $i = 1, 2$ be the solutions of the characteristic equation $-Dr^2 - Vr + R = 0$ and $v_i(x) = \mathrm{e}^{r_i x}$.

Multiplying the first equation of (3.19) by v_i and, integrating with respect to x and t, one gets

$$\int_a^b \int_0^{T^*} L[v](x, t) v_i(x)\, \mathrm{d}t\, \mathrm{d}x = \mathrm{e}^{r_i S_2} \int_0^{T^*} \lambda_2(t)\, \mathrm{d}t - \mathrm{e}^{r_i S_1} \int_0^{T^*} \lambda_1(t)\, \mathrm{d}t. \tag{3.21}$$

Since $v \in L^2(0, T; H^1(0, \ell)) \cap C(0, T; L^2(0, \ell))$, using Green's formula and according to (3.16), (3.18) and (3.20), the LHS of (3.21) vanishes, so that

$$\overline{\lambda}_2\, \mathrm{e}^{r_1 S_2} = \overline{\lambda}_1\, \mathrm{e}^{r_1 S_1}$$
$$\overline{\lambda}_2\, \mathrm{e}^{r_2 S_2} = \overline{\lambda}_1\, \mathrm{e}^{r_2 S_1}$$

where $\overline{\lambda} = \int_0^{T^*} \lambda(t)\, \mathrm{d}t$. Since $r_1 \neq r_2$ and $\overline{\lambda}_i > 0$, one has $S_1 = S_2$ and $\overline{\lambda}_1 = \overline{\lambda}_2$.

Let us now set $S = S_1 = S_2$. Let $y(x, t) = \mathrm{e}^{\alpha x} v(x, t)$ withα given in (3.7). Then, v is the solution of (3.15) if and only if y is the solution of the following heat equation,

$$\begin{aligned} &\partial_t y - D\partial_{xx} y + \varrho y = \mathrm{e}^{\alpha S} [\lambda_2(t) - \lambda_1(t)]\, \delta(x - S), \qquad 0 < x < \ell, \quad 0 < t < T \\ &y(0, t) = \partial_x y(\ell, t) - \alpha y(\ell, t) = 0 \qquad \text{for} \quad 0 < t < T \\ &y(x, 0) = 0 \qquad \text{for} \quad 0 < x < \ell \end{aligned} \tag{3.22}$$

which is given by the Fourier expansion

$$y(x, t) = \sum_n a_n(t) \psi_n(x)$$

with $a_n(t) = \int_0^\ell y(x, t)\psi_n(x)\, \mathrm{d}x$ and $a_n(0) = 0$. Now, since $\frac{\mathrm{d}}{\mathrm{d}t}\langle y(x, t), \psi_n(x)\rangle = \left\langle \frac{\partial}{\partial t} y(x, t), \psi_n(x)\right\rangle$ ([5], chapter 18), one obtains

$$\begin{aligned} a_n'(t) &= \int_0^\ell [D\partial_{xx} y(x, t) - \varrho y(x, t)]\psi_n(x)\, \mathrm{d}x + \mathrm{e}^{\alpha S} [\lambda_2(t) - \lambda_1(t)]\, \psi_n(S) \\ &= -\mu_n a_n(t) + \mathrm{e}^{\alpha S} [\lambda_2(t) - \lambda_1(t)]\, \psi_n(S) \end{aligned}$$

and therefore

$$v(x, t) = \sum_{n=0}^{\infty} \psi_n(S)\psi_n(x)\, \mathrm{e}^{-\alpha(x-S)} \int_0^t (\lambda_2 - \lambda_1)(\zeta)\, \mathrm{e}^{-\mu_n(t-\zeta)}\, \mathrm{d}\zeta$$

that is

$$v(x, t) = \int_0^t (\lambda_2 - \lambda_1)(\xi)\Phi(x, t - \zeta)\, \mathrm{d}\zeta$$

where

$$\Phi(x, t) = \sum_{n=0}^{\infty} \psi_n(S)\psi_n(x)\, \mathrm{e}^{-\alpha(x-S)}\, \mathrm{e}^{-\mu_n t}.$$

The above inversion of integration and summation is justified by Lebegues's theorem of dominated convergence. This is seen from the estimate

$$\begin{aligned}\sum_{n_0}^{\infty} |\psi_n(S)\psi_n(x)\,\mathrm{e}^{-\alpha(x-S)}|\,\mathrm{e}^{-\mu_n t} &\leqslant C\sum_{n_0}^{\infty} \mathrm{e}^{-Kn^2 t}\\ &\leqslant C\int_0^{\infty} \mathrm{e}^{-Kt\xi^2}\,\mathrm{d}\xi\\ &= C_1\frac{1}{\sqrt{t}} \end{aligned}\qquad (3.23)$$

where C, K, C_1 are constant.

The first inequality of (3.23) is obtained according to (3.9), for a sufficiently large n_0. Now, since $v(b,t) = 0$ (or $v(a,t) = 0$), $0 < t < T$, one gets

$$\int_0^t (\lambda_2 - \lambda_1)(\xi)\Phi(b, t-\xi)\,\mathrm{d}\xi = 0, \qquad \forall\, t \in]0, T[.$$

According to Titchmarsh's theorem on convolution of L^1 functions [18], the functions $(\lambda_2 - \lambda_1)$ and Φ must vanish identically at least in intervals $]0, T'[$ and $]0, T''[$ respectively, with T' and T'' such that $T' + T'' \geqslant T$.

Now, if $\Phi = 0$ in $]0, T''[$ with $T'' > 0$, by analytic continuation one has $\Phi = 0$ in $]0, +\infty[$ and therefore $\psi_n(S)\psi_n(b) = 0, \forall\, n$.

Since b is strategic, we obtain $\psi_n(S) = 0\,\forall\, n$, which is impossible according to (3.8). Thus

$$\lambda_2 = \lambda_1 \quad \text{in} \quad]0, T[.$$

This ends the proof of the theorem. □

Remark 1.

(1) The point $c \in (0, \ell)$ such that $\psi_n(c) = 0$ satisfies $c = \frac{m\pi}{\beta_n} < \ell$, for m an integer; i.e., these points are countable, hence an arbitrarily chosen point $b \in (0, \ell)$ has probability 1 of being strategic.

(2) The assumption that $\lambda(t) = 0\;\forall\, t \in (T^*, T)$ where $0 < T^* < T$ corresponds to the case of an accidental pollution stopped at time T^*, while the recording of the concentration u is continued until a later time T.

(3) For the sources of the form $F(x,t) = \lambda(t)\sum_{i=1}^{m} \alpha_i \delta_{S_i}$, where $\lambda \in C^1[0, T]$ is *known* and satisfying the condition $\lambda(0) \neq 0$, one can use an appropriate change of functions to prove that the source identification problem is equivalent to the identification of initial data. This was made by Yamamoto in [19] in a wave source problem. However, his proof can easily be adapted to our parabolic equation. In this case, the number m, the values α_i, and the locations S_i are uniquely determined by one pointwise measurement situated at a strategic point.

4. Stability

Stability, with which we are concerned here, means continuous dependence of the source F on the measurements $B[F]$. Stability is a crucial issue for the numerical applications and it has concerned many authors in other situations. In this section, we prove a local Lipschitz stability result derived from the Gâteaux differentiability, by establishing that the Gâteaux derivative is not zero.

First, consider the set

$$G(T^*) = \{(\lambda, S) \in L^2(0,T) \times (a,b) : \lambda(t) \geqslant 0 \text{ with } \lambda(t) = 0 \qquad \text{for} \quad t \geqslant T^*\}.$$

If (λ, S) and (μ, τ) both belong to $G(T^*)$, then for all $h \neq 0$ sufficiently small, $(\lambda + h\mu, S + \tau h)$ also belongs to $G(T^*)$, we therefore define the corresponding source

$$F^h(x,t) = (\lambda(t) + h\mu(t))\delta(x - (S + h\tau))$$

and, moreover, Taylor expansion with respect to h shows that there exists a point S_h satisfying $|S - S_h| < |\tau h|$ such that

$$F^h(x,t) = F(x,t) + h\widehat{F}(x,t) + h^2\widetilde{F}(x,t)$$

where

$$\widehat{F}(x,t) = \mu(t)\delta(x - S) - \lambda(t)\tau\delta'(x - S)$$

and

$$\widetilde{F}(x,t) = -\mu(t)\delta'(x - S) + \tfrac{1}{2}\lambda(t)\tau^2\delta''(x - S_h).$$

Hence,

$$u(x,t;F^h) = u(x,t;F) + hu(x,t;\widehat{F}) + h^2u(x,t;\widetilde{F})$$

where $u(x,t;\widehat{F})$ and $u(x,t;\widetilde{F})$ are, respectively, solutions of (2.1) and (2.2) with $\widehat{F}$ and $\widetilde{F}$ as source terms and, therefore,

$$\lim_{h \longrightarrow 0} \frac{B[F^h] - B[F]}{h} = \{u(a,t;\widehat{F}), u(b,t;\widehat{F})\}.$$

Furthermore, since λ and μ both belong to $L^2(0,\ell)$, one can prove using transposition method (see [14]) that the function $u(.,.;\widehat{F})$ belongs to $L^2((0,\ell) \times (0,T))$, and therefore it also belongs to $C([0,T]; H^{-1}(0,\ell))$. We need this regularity result to justify the integration by parts below.

Now we are able to state our local stability result.

Theorem 2 (local Lipschitz stability)**.** *If* $(\mu, \tau) \neq (0,0)$ *for* $0 < t < T$*, then*

$$\lim_{h \longrightarrow 0} \frac{B[F^h] - B[F]}{h} \neq 0.$$

Proof. By the same technique used to show identifiability, we will proof that

$$\{u(a,t;\widehat{F}), u(b,t;\widehat{F})\} \neq (0,0), \qquad 0 < t < T.$$

In the first step, as for the identifiability issue, one can prove that

$$u(x,T^*;\widehat{F}) = 0 \qquad \text{for} \quad 0 < x < \ell$$

and

$$\partial_x u(a,t;\widehat{F}) = \partial_x u(b,t;\widehat{F}) = 0 \qquad \text{for} \quad 0 < t < T^*.$$

In the second step, we consider an infinitely differentiable function $\xi \in \mathcal{D}(0,\ell)$ such that $\xi = 1$ in a neighbourhood of S.

Then, multiplying the first equation of (2.1) with $\widehat{F}$ as a source term by ξv_i and, integrating with respect x over $(0,\ell)$, one gets

$$\langle \partial_t u(x,t;\widehat{F}), \xi(x)\,\mathrm{e}^{r_i x}\rangle_{H^{-1},H_0^1} = \mu(t)\,\mathrm{e}^{r_i S} + \lambda(t)\tau r_i\,\mathrm{e}^{r_i S}, \qquad 0 < t < T^*,$$

where $\langle .,.\rangle_{H^{-1},H_0^1}$ denotes the duality bracket between H^{-1} and H_0^1.

Since

$$u(x, T^*; \widehat{F}) = u(x, 0; \widehat{F}) = 0 \qquad \text{for} \quad 0 < x < \ell,$$

and

$$\partial_t u(x, t; \widehat{F}) \in L^1(0, T; H^{-1}(0, \ell)) \qquad \text{and} \qquad \xi\, e^{r_i x} \in H_0^1(0, \ell)$$

one has

$$\int_0^{T^*} \langle \partial_t u(x, t; \widehat{F}), \xi(x)\, e^{r_i x} \rangle_{H^{-1}, H_0^1}\, dt = \left\langle \int_0^{T^*} \partial_t u(x, t; \widehat{F})\, dt, \xi(x)\, e^{r_i x} \right\rangle_{H^{-1}, H_0^1} = 0.$$

Thus, we get

$$\overline{\mu} + \overline{\lambda}\tau r_1 = 0$$
$$\overline{\mu} + \overline{\lambda}\tau r_2 = 0$$

where $\overline{\mu} = \int_0^{T^*} \mu(t)\, dt$ and $\overline{\lambda} = \int_0^{T^*} \lambda(t)\, dt$. Since $r_1 \neq r_2$ and $\overline{\lambda} > 0$, one has $\tau = 0$ and $\overline{\mu} = 0$.

Now using the same techniques to prove identifiability, we arrive at $\mu = 0$. This ends the proof of theorem 2. □

5. Identification

The method we will present requires the knowledge of $u(x, T^*)$ for $0 < x < \ell$, $\partial_x u(a, t)$ and $\partial_x u(b, t)$ for $0 < t < T^*$.

In the first step, we will use the data $\{d_1(t), d_2(t), T^* < t < T\}$ to determine the function u_{T^*} by using the system

$$L[u](x, t) = 0, \qquad 0 < x < \ell, \quad T^* < t < T$$
$$u(0, t) = \partial_x u(\ell, t) = 0 \qquad \text{for} \quad T^* < t < T,$$
$$u(x, T^*) = u_{T^*}(x) \qquad \text{for} \quad 0 < x < \ell,$$

where the right-hand side of the first equation is null because $\lambda(t) = 0$ for $T^* < t < T$. It is a classical observability problem for which we use a least-squares regularized method to identify u_{T^*}.

In the second step, we use the data $\{d_1(t), 0 < t < T^*\}$ to determine $\partial_x u(a, t)$ by solving the following direct problem

$$L[u](x, t) = 0, \qquad 0 < x < a, \quad 0 < t < T^*,$$
$$u(0, t) = 0 \qquad \text{for} \quad 0 < t < T^*,$$
$$u(a, t) = d_1(t) \qquad \text{for} \quad 0 < t < T^*,$$
$$u(x, 0) = 0 \qquad \text{for} \quad 0 < x < a,$$

where the right-hand side of the first equation is null because $a < S$.

Finally, as above, we use the data $\{d_2(t), 0 < t < T^*\}$ to identify $\partial_x u(b, t)$ by solving the following direct problem

$$L[u](x, t) = 0, \qquad b < x < \ell, \quad 0 < t < T^*,$$
$$u(b, t) = d_2(t) \qquad \text{for} \quad 0 < t < T^*,$$
$$\partial_x u(\ell, t) = 0, \qquad \text{for} \quad 0 < t < T^*,$$
$$u(x, 0) = 0 \qquad \text{for} \quad b < x < \ell,$$

since $S < b$.

5.1. Recovering the location S

We suppose that $u(x, T^*)$, $\partial_x u(a, t)$ and $\partial_x u(b, t)$ are completely determined in their respective domains and consider the system

$$L[u](x,t) = \lambda(t)\delta(x - S), \qquad 0 < x < \ell, \quad 0 < t < T^*,$$
$$u(0,t) = \partial_x u(\ell, t) = 0 \qquad \text{for} \quad 0 < t < T^*,$$
$$u(x,0) = 0 \qquad \text{for} \quad 0 < x < \ell.$$

Multiplying the first equation by v_i and integrating with respect x and t over $]0, \ell[\times]0, T^*[$, by using Green's formula, one has

$$\overline{\lambda}\, \mathrm{e}^{r_i S} = -D\, \mathrm{e}^{r_i b} \int_0^{T^*} \partial_x u(b,t)\, \mathrm{d}t + D\, \mathrm{e}^{r_i a} \int_0^{T^*} \partial_x u(a,t)\, \mathrm{d}t + \int_0^b u(x, T^*)\, \mathrm{e}^{r_i x}\, \mathrm{d}x, \qquad i = 1, 2$$

which allows us to determine S.

5.2. Recovering the function λ

The problem that we have to solve is the following. Given $\{d_2(t), 0 < t < T\}$, determine λ such that

$$d_2(t) = \int_0^t \lambda(\zeta)\Phi(b, t - \zeta)\, \mathrm{d}\zeta, \qquad 0 < t < T.$$

We will present two methods for solving numerically this problem. The first one consists in replacing the above convolution equation by its approximated version. Set

$$h = \frac{T^*}{M}, \quad t_m = mh, \quad m = 1 \ldots, M, \quad 0 < t_1 < \cdots < t_k < \cdots < t_{m-1} < t_m.$$

Denote

$$y_m = d_2(t_k) \qquad \text{and} \qquad \lambda^k = \lambda(t_k).$$

In each interval $]t_k, t_{k+1}[$ we approximate the integral $\int_{t_k}^{t_{k+1}} \lambda(\zeta)\Phi(b, t - \zeta)\, \mathrm{d}\zeta$ by the trapezoidal rule, that is

$$\frac{h}{2}(\Phi(b, t_{m-k_1})\lambda^{k+1} + \Phi(b, t_{m-k})\lambda^k).$$

Thus

$$y_m = \frac{h}{2} \sum_{k=0}^{m-1} (\Phi(b, t_{m-k_1})\lambda^{k+1} + \Phi(b, t_{m-k})\lambda^k) \qquad m = 1, \ldots, M,$$

which leads to a linear system

$$A\Lambda = Y. \tag{5.24}$$

The second method consists in decomposing the function λ on a finite Fourier basis l_k

$$\lambda(t) = \sum_{k=1}^{m} \theta_k l_k(t).$$

This method also leads to a linear system

$$A\Theta = Y \tag{5.25}$$

where the coefficients of A are

$$\int_0^{t_m} l_k(\zeta)\Phi(b, t_m - \zeta)\,\mathrm{d}\zeta.$$

6. Numerical results

The numerical results are obtained in the case of a portion of the river of length $\ell = 1000$ m and during a period T = 4 h, with $T^* = 3$ h, $R = 1.01 \times 10^{-5}$ s, $V = 0.66$ m s^{-1} and $D = 29$ m^2 s^{-1} [17]. The source is located at $S = 600$ m with the intensity

$$\lambda(t) = \sum_1 3\alpha_i\, \mathrm{e}^{-\beta_i(t-\tau_i)^2}$$

where $\alpha_1 = 1.2, \alpha_2 = 0.4, \alpha_3 = 0.6, \beta_1 = 1 \times 10^6, \beta_2 = 5 \times 10^5, \beta_3 = 1 \times 10^6, \tau_1 = 4500\,\mathrm{s}$, $\tau_2 = 6500\,\mathrm{s}$, $\tau_3 = 9000\,\mathrm{s}$.

The purpose of this numerical work is to identify the location S and intensity λ according to the method proposed in section 5. We have reduced the domain of study $]0, \ell[\times]0, T[$ to $]0, 1[\times]0, 1[$ and considered the following undimensioned system

$$\begin{aligned} &L_1[u](x, t) = T\lambda(t)\delta(x - S_1), \qquad 0 < x < 1, \quad 0 < t < 1,\\ &u(0, t) = \partial_x u(1, t) = 0, \qquad \text{for} \quad 0 < t < 1,\\ &u(x, 0) = 0, \qquad \text{for} \quad 0 < x < 1 \end{aligned}$$

with $D_1 = 0.417$, $V_1 = 9.504$, $R_1 = 0.145$, $S_1 = \frac{S}{\ell}$, $a_1 = \frac{a}{\ell}$, $b_1 = \frac{b}{\ell}$, $T_1^* = \frac{T^*}{T}$ and $\lambda_1(t) = \lambda(tT)$. Here $L_1[u](x, t) = \partial_t u(x, t) - D_1\partial_{xx}u(x, t) + V_1\partial_x u(x, t) + R_1 u(x, t)$.

The numerical results are presented in the initial domain of study $]0, \ell[\times]0, T[$.

First step. The unknown function u_{T^*} has been determined by using Tikhonov regularization method with a regularization parameter $\varepsilon = 0.1$; at first by using the measurements $d_1(t)$ and then $d_2(t)$ for $T^* < t < T$.

Figure 1 is obtained by using upstream measurements $d_1(t)$ for $T^* < t < T$ at $a = 300$ m. The second with downstream measurements $d_2(t)$ for $T^* < t < T$ at $b = 800$ m.

Second step. The location S_1 is given by

$$S_1 = \frac{1}{r_1 - r_2} \ln\left(\frac{c_1}{c_2}\right)$$

where

$$c_i = -D_1\,\mathrm{e}^{r_i b_1} \int_0^{T_1^*} \partial_x u(b_1, t)\,\mathrm{d}t + D_1\,\mathrm{e}^{r_i a_1} \int_0^{T_1^*} \partial_x u(a_1, t)\,\mathrm{d}t + \int_0^{b_1} [u(x, T_1^*)\,\mathrm{e}^{r_i x} - g(x)]\,\mathrm{d}x.$$

The normal derivatives $\partial_x u(a_1, t)$ and $\partial_x u(b_1, t)$ are approximated by the finite differences method and the above integrals by the trapezoidal rule. We then obtain $S_1 = 0.853$ and therefore $S = 583$.

Last step. In this step, we determine the intensity λ by two different methods by using the numerical results obtained in both steps above.

Figures 3 and 4 compare the exact solution λ with that obtained by solving the system (5.24) by using SVD regularization method. In both cases we chose $m = 60$.

Figures 5 and 6, respectively figures 7 and 8, compare the exact solution λ with that obtained by solving the system (5.25), by using the least-squares method with SVD regularization, with $m = 10$ for 6, 7 and $m = 15$ for 7, 8.

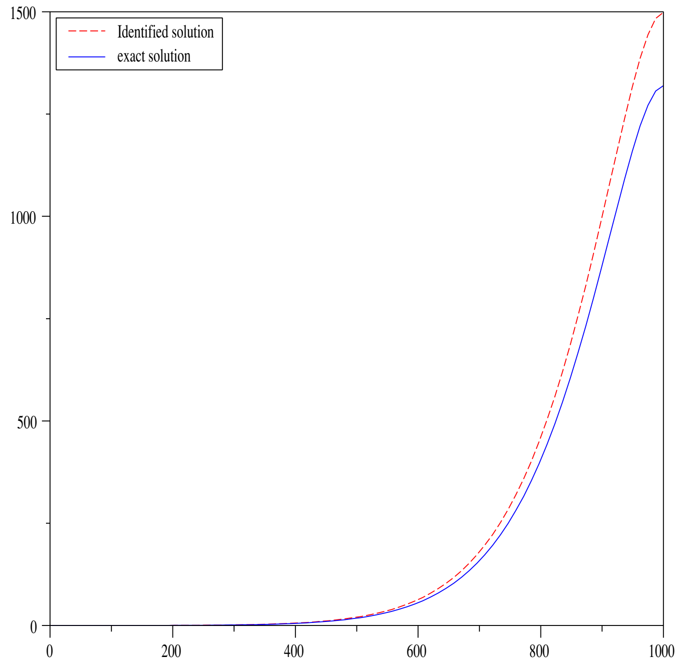

Figure 1. Relative error 0.13.

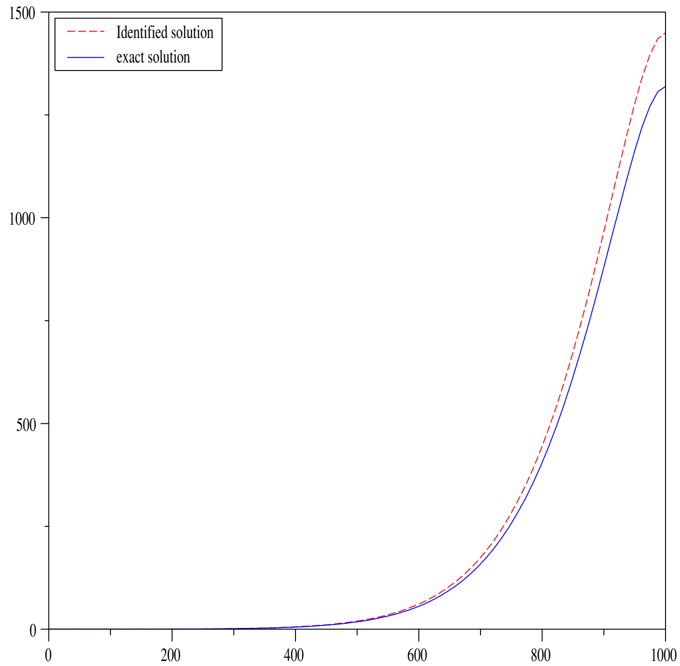

Figure 2. Relative error 0.09.

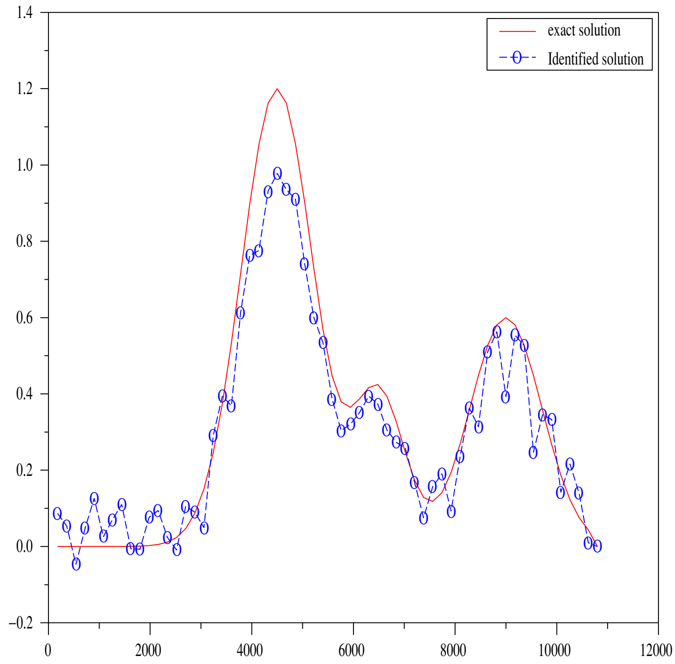

Figure 3. Noise 3%, relative error 0.25.

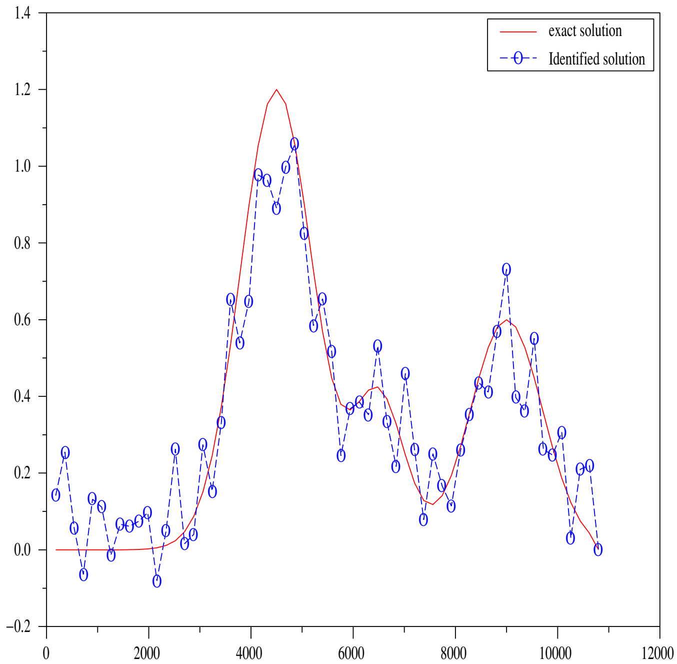

Figure 4. Noise 5%, relative error 0.32.

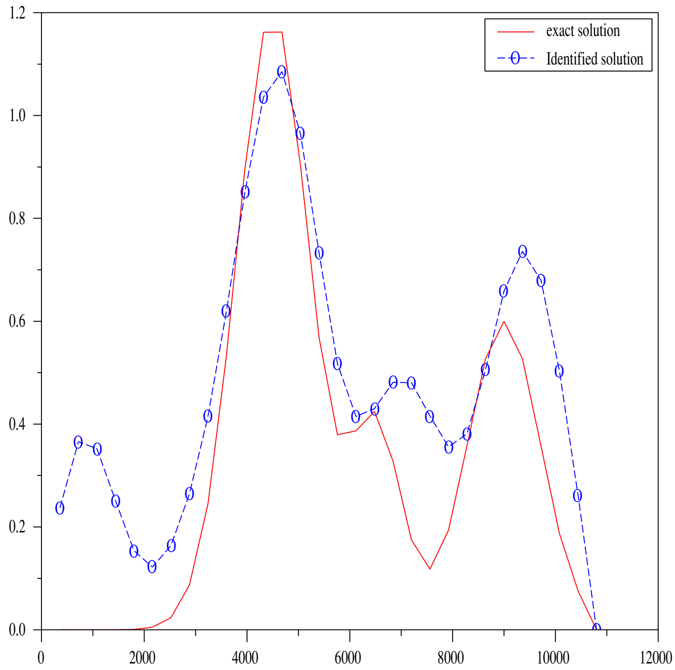

Figure 5. Noise 3%, relative error 0.31.

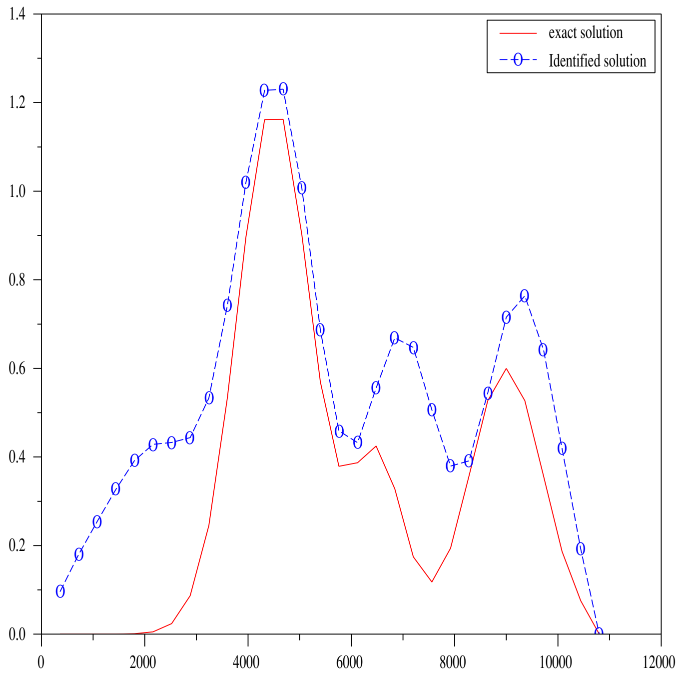

Figure 6. Noise 5%, relative error 0.40.

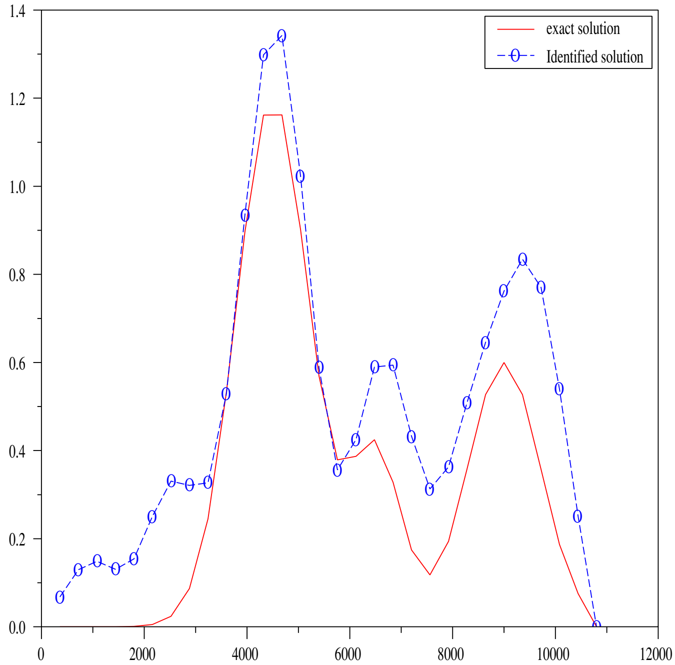

Figure 7. Noise 3%, relative error 0.35.

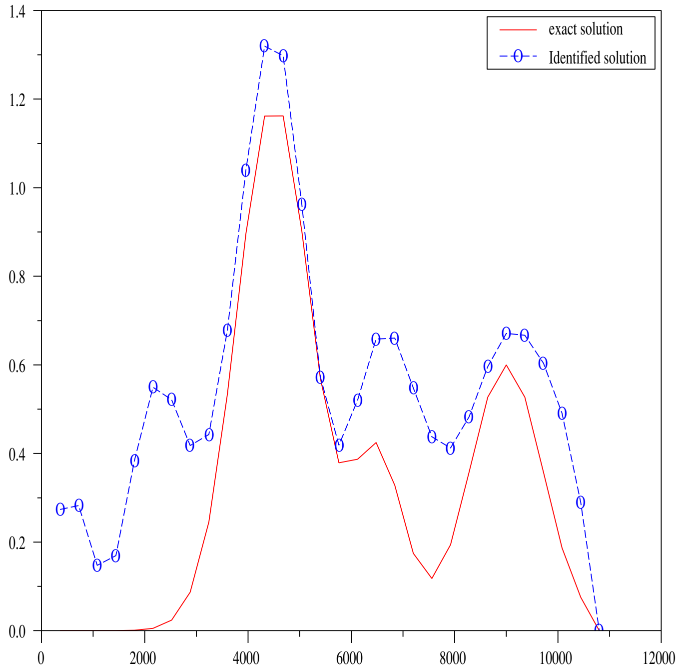

Figure 8. Noise 5%, relative error 0.46.

7. Conclusion

The localization for a pollution point source (location and intensity) in a river has been studied by two pointwise measurements situated one upstream and another downstream with respect to the source. Identifiability and stability results are established when at least one of the two points is strategic.

Assuming that the source becomes inactive after some time T^*, which corresponds to an accidental pollution stopped at time T^*, the measurements after T^* first permit us to obtain the knowledge of the state at T^*. This information is then used to identify the source by a variational method, without any iterative procedure. Finally, the intensity can be determined numerically by a deconvolution or a Fourier method. Some numerical results are presented.

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Endnotes

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