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Identification of Distributed Parameter Systems Using Finite Differences

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Identification of distributed parameter systems using finite differences
by Paul Lee Collins

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY in Electrical Engineering
Montana State University
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Abstract:

A method is presented to identify the partial differential equations and associated boundary conditions of a distributed parameter system.

The technique requires that the form of the differential equation or boundary condition be known up to a set of constants. Finite differences are used to approximate derivatives. Identification is carried out by using normal operating data.

When the data is exact the identification may be performed on linear, nonlinear and time-varying systems. The accuracy of the identification depends only on approximation errors. Methods for decreasing these errors are presented. When the data is corrupted by noise additional errors are introduced and the unknown constants must be estimated. A performance index is presented which tends to minimize the combined effect of these two types of errors.

The classical least squares estimator is developed to estimate linear and nonlinear systems. When the statistics of the measurement noise are available, a modified least squares estimator is presented which is applicable to linear systems. It is shown that the modified scheme is generally more accurate than the least squares estimator.

Examples and results of digital computer simulations are given.

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IDENTIFICATION OF DISTRIBUTED PARAMETER
SYSTEMS USING FINITE DIFFERENCES

by

Paul Lee Collins, Jr.

A thesis submitted to the Graduate Faculty in partial
fulfillment of the requirements for the degree

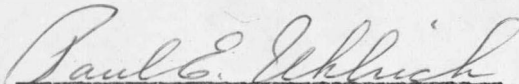
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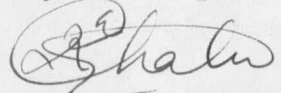
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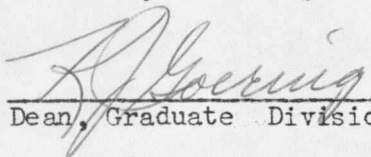
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ABSTRACT

A method is presented to identify the partial differential equations and associated boundary conditions of a distributed parameter system. The technique requires that the form of the differential equation or boundary condition be known up to a set of constants. Finite differences are used to approximate derivatives. Identification is carried out by using normal operating data.

When the data is exact the identification may be performed on linear, nonlinear and time-varying systems. The accuracy of the identification depends only on approximation errors. Methods for decreasing these errors are presented. When the data is corrupted by noise additional errors are introduced and the unknown constants must be estimated. A performance index is presented which tends to minimize the combined effect of these two types of errors.

The classical least squares estimator is developed to estimate linear and nonlinear systems. When the statistics of the measurement noise are available, a modified least squares estimator is presented which is applicable to linear systems. It is shown that the modified scheme is generally more accurate than the least squares estimator. Examples and results of digital computer simulations are given.

CHAPTER 1
INTRODUCTION

1.1 SCOPE OF THE THESIS

1.1.1 Identification of Systems without Measurement Noise

A technique is presented in Chapter 2 to identify the partial differential equation describing a distributed parameter system in the absence of measurement noise. The form of the differential equation must be known to within certain constants, which are to be identified. The constants can be changing with time, as long as their rate of change is slow with respect to the system dynamics. The technique is applicable to linear and nonlinear systems. However, the unknown coefficients must appear linearly. When the precise form of the differential equation is not known a priori extraneous terms may be included. The conditions under which the identification reveals the coefficients in the extraneous terms to be negligible are discussed.

A necessary condition for the identification using the proposed scheme is given. This condition is applicable only when the system is in a transient state. However, no special test signals are necessary and the identification can be performed on-line.

The identification of unknown parameters requires no knowledge of the boundary conditions. However, a system may have boundary conditions which contain unknown parameters. These constants can be determined using the identification scheme described herein without knowing the system partial differential equation.

The procedure uses finite differences to approximate the partial derivatives at specified points in time and space. The use of finite differences in the numerical solution of partial differential equations

is well known, [1]*, but has never been applied to the identification problem. The methods for improving the accuracy of the finite difference approximations are discussed. The improvements are applicable to the identification of distributed systems as well as the numerical solution of partial differential equations. However, the application of the finite difference approximations of higher order accuracy to the numerical solution of partial differential equations is limited because these calculations are recursive and many of the approximations result in an unstable recursive scheme. This complication does not arise in the identification problem, for the calculations at each of the stages are not dependent on the accuracy of the calculations made at any other stage.

1.1.2 Identification of Systems with Measurement Noise

Finite differences are used to approximate partial derivatives in the proposed identification technique. The data for the finite differences is obtained by performing measurements on the systems. These measurements are rarely exact, for the measurement devices have imperfections. As a result, errors due to measurement noise are introduced in the identification in addition to the errors resulting from the finite difference approximations. The identification of distributed systems in the presence of measurement noise is discussed in Chapter 3.

In the identification process, introduction of measurement noise

* Numbers in square brackets refer to the references at the end of this thesis.

suggests that the unknown parameters are functions of random variables. Thus some type of statistical estimation must be employed to determine the parameters. The least squares estimator was found to be well suited for the problem. This estimator requires no a priori knowledge of the statistics of the measurement noise as do other classical estimators. However, when the system is linear and the mean and variance of the measurement noise are known, it is possible to improve the accuracy of the least squares estimator by modifying it. The convergence of the least squares estimator and modified least squares estimator is compared and contrasted.

The identification scheme requires the calculation of the inverse of a square matrix. This matrix contains errors due to measurement noise and approximation error. Errors in identification of coefficients become very large if the matrix is ill-conditioned even if the measurement and the approximation errors are small. The origin of the ill-conditioned matrix in the identification is discussed as well as methods for detecting ill-conditioning.

1.1.3 Selection of Optimal Increments in Time and Space

The errors introduced by measurement noise and finite differences are functions of the increments in time and space selected for the finite difference formulae. The effect of measurement noise error is decreased by making the increments as large as possible. On the other hand, the error due to finite difference approximation is decreased by making these increments very small. This suggests there exists a

combination of increments in time and space which minimize the combined effect of these two errors.

A performance index which is a function of identification error due to measurement noise and finite differences is formulated. Minimization of this index is performed with respect to the time and spatial increments. The technique requires a knowledge of the statistics of the measurement noise and is not well suited to nonlinear systems. Initial estimates of the unknown parameters must be made. In spite of this, the resulting set of increments provides better results than those attainable by simply guessing.

1.2 HISTORICAL BACKGROUND

1.2.1 Derivation of Partial Differential Equations

Distributed parameter systems, that is, systems described by partial differential equations or multiple integral equations are often encountered in engineering applications. Chemical reactors, nuclear reactors, heat exchangers and physical structures are examples of systems with parameters distributed in space. Frequently, the system partial differential equation can be derived a priori from basic laws of physics. These equations usually contain constants which are unknown and must be identified. This thesis is devoted to the identification of these constants.

In this subsection the partial differential equations describing three different distributed systems are derived. The resulting equations are used exclusively in the examples appearing in this thesis. Thus, the

derivations not only demonstrate the application of physical laws to obtain the mathematical models of distributed systems, but also provide physical insight into the partial differential equations used in the examples.

Consider the flow of heat in a thin bar or wire. Assume the bar is perfectly insulated along its length so that heat flows in the x direction only. Such a system is depicted in Figure 1.1.

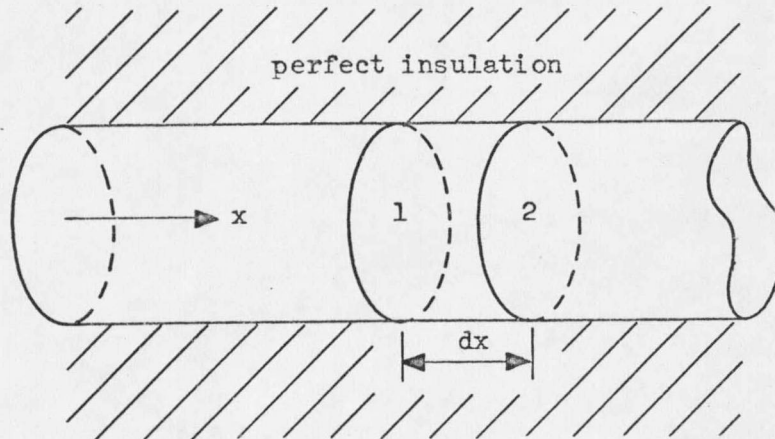


Figure 1.1 One dimensional heat flow in a thin bar.

If a segment of the bar with boundaries 1 and 2 and length dx is examined, the heat flow is governed by two basic laws of physics; namely,

$$Q = cmu \tag{1.1}$$

where

Q = heat (calories or B.T.U.'s)

c = specific heat

m = mass

u = absolute temperature

and

$$\frac{\partial Q}{\partial t} = -k A \frac{\partial u}{\partial n} \text{ cal/sec or B.T.U./sec} \quad (1.2)$$

where

A = cross-sectional area of surface in a direction perpendicular to the flow of heat

$\frac{\partial u}{\partial n}$ = gradient of temperature perpendicular to the surface

k = heat conductivity

The amount of heat contained in the segment is obtained by application of equation (1.1) with the result

$$Q = c(Adx)\rho u \quad (1.3)$$

where ρ is the density of the material and hence $Adx\rho$ is the mass of the segment. The rate at which heat is stored in this element is found by differentiating equation (1.3) with respect to time to get

$$\frac{\partial Q}{\partial t} = c(Adx)\rho \frac{\partial u}{\partial t} \quad (1.4)$$

Let Q_1 denote the heat flowing into the segment across boundary 1. The rate of heat flow across this boundary is described by equation (1.2).

Therefore

$$\frac{\partial Q_1}{\partial t} = -k A \frac{\partial u}{\partial x} \quad (1.5)$$

The rate at which the heat, Q_2 , flows out boundary 2 can be found by considering the Taylor's series expansion of $\frac{\partial Q_1}{\partial t}$. If $\frac{\partial Q_1}{\partial t}$ has all

orders of derivatives with respect to x , the expansion of $\frac{\partial Q_1}{\partial t}$ about x is given by

$$\frac{\partial Q_1(x + dx)}{\partial t} = \frac{\partial Q_1(x)}{\partial t} + \frac{\partial}{\partial x} \left(\frac{\partial Q_1(x)}{\partial t} \right) dx + \frac{\partial^2}{\partial x^2} \left(\frac{\partial Q_1(x)}{\partial t} \right) \frac{dx^2}{2!} + \dots \quad (1.6)$$

where ! denotes the factorial. When dx is very small in absolute value compared to unity, the terms of the series involving high powers of dx may be neglected. Thus a first order approximation for equation (1.6) is

$$\frac{\partial Q_1(x + dx)}{\partial t} = \frac{\partial Q_1(x)}{\partial t} + \frac{\partial}{\partial x} \left(\frac{\partial Q_1(x)}{\partial t} \right) dx \quad (1.7)$$

Notice that the term on the left hand side of equation (1.7) is the heat flow from boundary 2. Therefore

$$\frac{\partial Q_2}{\partial t} = \frac{\partial Q_1}{\partial t} + \frac{\partial}{\partial x} \left(\frac{\partial Q_1}{\partial t} \right) dx \quad (1.8)$$

Substituting equation (1.5) into the result of equation (1.8) yields

$$\frac{\partial Q_2}{\partial t} = -kA \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} \left(kA \frac{\partial u}{\partial x} \right) dx \quad (1.9)$$

The rate at which heat is stored in the element is found by subtracting the rate of heat flow out from the rate of heat flow in. This is expressed mathematically by

$$\frac{\partial Q}{\partial t} = \frac{\partial Q_1}{\partial t} - \frac{\partial Q_2}{\partial t} \quad (1.10)$$

Substituting equations (1.4), (1.5), (1.9) into (1.10) gives

$$cA\rho \frac{\partial u}{\partial t} dx = \frac{\partial}{\partial x} \left(kA \frac{\partial u}{\partial x} \right) dx \quad (1.11)$$

Assume A is independent of x . Then after dividing both sides of equation (1.11) by $cA\rho dx$ there results

$$\frac{\partial u}{\partial t} = \frac{1}{c\rho} \frac{\partial}{\partial x} \left(k \frac{\partial u}{\partial x} \right) \quad (1.12)$$

Finally, when k is independent of x equation (1.12) can be written

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad (1.13)$$

where

$$\alpha = \frac{k}{c\rho} \quad (1.14)$$

Equation (1.13) is known as the diffusion equation. It is a linear partial differential equation containing one constant which may or may not be known. A simple extension of the above problem leads to a linear partial differential equation with two constants.

Consider once again the bar shown in Figure 1.1. Assume the bar moves in the x direction with a uniform velocity, v . The next derivation leads to the partial differential equation describing the heat flow in the moving bar with respect to a stationary coordinate system.

Let the moving coordinate system be denoted by \bar{x} and \bar{t} . The heat flow with respect to the moving system is given by equation (1.13). Hence

$$\frac{\partial u}{\partial \bar{t}}(\bar{x}, \bar{t}) = \alpha \frac{\partial^2 u}{\partial \bar{x}^2}(\bar{x}, \bar{t}) \quad (1.15)$$

This equation can be rewritten in terms of a stationary coordinate system by changing the independent variables. First note that the stationary and moving systems are related by the equation

$$x = vt + \bar{x} + x_0$$

where x, t denote space and time in the stationary system and x_0 is the separation between the origin of the stationary and moving systems at time zero.

The change of variables is defined to be

$$t = t(\bar{x}, \bar{t}) \quad (1.17)$$

$$= \bar{t} \quad (1.18)$$

$$x = x(\bar{x}, \bar{t})$$

$$= v\bar{t} + \bar{x} + x_0 \quad (1.19)$$

$$= v\bar{t} + \bar{x} + x_0 \quad (1.20)$$

Consider the term on the left hand side of equation (1.15). Application of the chain rule yields

$$\frac{\partial u}{\partial \bar{t}} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \bar{t}} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial \bar{t}} \quad (1.21)$$

But from equations (1.18) and (1.20)

$$\frac{\partial x}{\partial \bar{t}} = v$$

and

$$\frac{\partial t}{\partial \bar{t}} = 1$$

Hence equation (1.21) can be written

$$\frac{\partial u}{\partial \bar{t}} = \frac{\partial u}{\partial x} v + \frac{\partial u}{\partial t} \quad (1.22)$$

Next consider the derivative appearing in the right hand side of equation (1.15). The first derivative is found by use of the chain rule to be

$$\frac{\partial u}{\partial \bar{x}} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \bar{x}} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial \bar{x}} \quad (1.23)$$

From equations (1.18) and (1.20) there results the relationships

$$\frac{\partial x}{\partial \bar{x}} = 1 \quad (1.24)$$

and

$$\frac{\partial \bar{t}}{\partial \bar{x}} = 0 \quad (1.25)$$

Therefore, equation (1.23) simplifies to read

$$\frac{\partial u}{\partial \bar{x}} = \frac{\partial u}{\partial x} \quad (1.26)$$

Differentiating equation (1.26) once more with respect to x and making use of equations (1.24) and (1.25) leads to the expression

$$\frac{\partial^2 u}{\partial \bar{x}^2} = \frac{\partial^2 u}{\partial x^2} \quad (1.27)$$

Substituting for equation (1.15) from equations (1.22) and (1.27) yields

$$\frac{\partial u}{\partial \bar{t}} + v \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2} \quad (1.28)$$

Notice that equations (1.18) and (1.20) can be solved for \bar{x} and \bar{t} to get

$$\bar{x} = \bar{x}(x, t) \quad (1.29)$$

$$\bar{t} = \bar{t}(x, t) \quad (1.30)$$

The substitution of equations (1.29) and (1.30) into $u(\bar{x}, \bar{t})$ produces a function of x and t , say

$$u = U(x, t) \quad (1.31)$$

Hence equation (1.28) can be written

$$\alpha \frac{\partial^2 U}{\partial x^2} - v \frac{\partial U}{\partial x} = \frac{\partial U}{\partial t} \quad (1.32)$$

This linear partial differential equation contains two constants which may or may not be known a priori. The third derivation results in a non-linear partial differential equation containing several coefficients.

Many unrelated physical phenomena give rise to the same equation if

variables and constants are appropriately interpreted. Such a situation arises in the derivation of the concentration dependent diffusion equation. Crank [2] shows that when C is the concentration of the diffusing substance and D the diffusion coefficient

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left(D \frac{\partial C}{\partial x} \right) \quad (1.33)$$

Notice the similarity between this result and equation (1.12). In many systems, for example the interdiffusion of metals or the diffusion of organic vapours in high polymer substances, D depends on the concentration of the diffusing substance. Let this dependence be given by the equation

$$D = \frac{2 - C^{1/2}}{4C} \quad (1.34)$$

Differentiating this expression with respect to C gives

$$\frac{\partial D}{\partial C} = \frac{C^{1/2} - 4}{8C^2} \quad (1.35)$$

Substitution of (1.34) and (1.35) into equation (1.33) yields

$$\frac{1}{2C} \frac{\partial^2 C}{\partial x^2} + \frac{1}{8C^{3/2}} \left(\frac{\partial C}{\partial x} \right)^2 = \frac{\partial C}{\partial t} + \frac{1}{4C^{1/2}} \frac{\partial^2 C}{\partial x^2} + \frac{1}{2C^2} \left(\frac{\partial C}{\partial x} \right)^2 \quad (1.36)$$

The derivations leading to equations (1.13), (1.32) and (1.36) are representative of arguments required to develop the mathematical model for distributed parameter systems. The next subsection concerns the control of such systems.

1.2.2 Adaptive Control of Distributed Parameter Systems.

In many distributed systems the parameters which characterize the

systems dynamics and environmental conditions under which the system operates are changing with time. For example, in a chemical reactor a catalyst may be slowly deteriorating and the chemical concentration of the reactants may also be changing, and these changes affect the system dynamics. Optimum control of such a changing system requires the use of an adaptive control system which is able to identify the system dynamics during a given interval of time, and make appropriate adjustments in the controller to compensate for changes in system dynamics.

Adaptive control of linear lumped parameter systems with slowly varying parameters has been considered [3,4,5] quite extensively in the literature. There has been some attempt to obtain adaptive control of distributed systems using approximate representations of system dynamics [6]. However, this area needs a more detailed study, using more accurate mathematical models of distributed parameter system dynamics [7,8]. The first requisite of an adaptive control is system identification. Unlike lumped parameter systems, little work has been done in the identification of distributed systems.

1.2.3 Identification of Distributed Parameter Systems

Identification of distributed parameter systems poses a few difficulties due to mathematical complexities of partial differential equations and the associated boundary conditions. Jones [9] and Douglas and Jones [10] identified a time-varying coefficient $\alpha(t)$ in the diffusion equation with the following boundary conditions:

$$\begin{aligned}
 u(x,0) &= 0 & 0 \leq x < \infty \\
 u(0,t) &= f(t) & 0 \leq t < T, f(0) = 0 \\
 -\alpha(t) \frac{\partial u}{\partial x}(0,t) &= g(t) & 0 < t < T
 \end{aligned} \tag{1.37}$$

Note that $\alpha(t)$ appears in the boundary condition at $x = 0$ as well as in the differential equation. This is necessary because the authors restrict themselves to measuring the state variable u at the boundary $x = 0$. Hence this problem, in essence, reduces to identification of the boundary condition.

Recently, Perdreauxville and Goodson [11] identified unknown coefficients of a system described by partial differential equations by using a technique similar to that used by Shinbrot [12] for identification of lumped parameter systems. This identification technique is limited to a class of partial differential equations which can be reduced to an algebraic equation by an integral transformation.

Sanathanan [13] discusses the identification of the transfer function of a distributed parameter system by obtaining a harmonic response function. The method is limited to linear systems and is not suitable for an on-line identification scheme.

CHAPTER 2

IDENTIFICATION OF DISTRIBUTED PARAMETER

SYSTEMS WITHOUT MEASUREMENT NOISE

2.1 INTRODUCTION

In this chapter the identification of distributed systems in the absence of measurement noise is considered. It is assumed that the partial differential equation is known up to a set of constant parameters which must be determined. The differential equation may be nonlinear in the state variables and their derivatives but must be linear in the unknown coefficients. When the precise form of the equation is not known a priori, extraneous terms may be included. The conditions under which the identification yields negligible values for the coefficients of these extraneous terms is discussed.

The necessary conditions for identification are developed before introducing the finite difference approximations. Techniques for reducing the error due to these approximations are presented. It is shown that the identification of the differential equations does not depend on the boundary conditions. Also, when the boundary conditions contain an unknown parameter, it too can be identified. Examples and results of digital computer simulations are given.

2.2 THEORY

2.2.1 Problem Statement

Let the dynamics of a distributed parameter system be described by a partial differential equation of the form

$$F\left(\frac{\partial^Q u}{\partial t \partial x \partial y \partial z}, \dots, u, t, x, y, z, c_1, c_2, \dots, c_M\right) = f_\Omega(t, x, y, z) \quad (2.1)$$

where $u = u(t, x, y, z)$ is the state variable, t denotes time, and $x, y,$ and

z denote spatial variables in a Cartesian coordinate system. F can be a nonlinear function of t, x, y, z and the state variable, but is linear with respect to c_1, c_2, \dots, c_M . Notice that a system with a nonlinear constant can generally be made linear by simply substituting a new linear constant for the nonlinear one. The differential equation is of order $Q = T + X + Y + Z$. The constants c_1, c_2, \dots, c_M are the coefficients appearing in the M terms of F and only N of these constants are unknown. N may be less than or equal to M . f_Ω is a distributed control input where Ω denotes the spatial domain over which the state variable is defined. It is assumed that only u can be measured directly.

Given measurements on the state variable and f_Ω , the problem is to find the unknown coefficients c_m for $m = 1, \dots, N$. Here for simplicity, u, F , and f_Ω are considered to be scalar. The identification of a set of partial differential equations containing more than one state variable is discussed in Appendix B.

2.2.2 Solution

Since F is linear in the coefficients c_1, c_2, \dots, c_M , it is convenient to write equation (2.1) in the form

$$c_1 h_1 + c_2 h_2 + \dots + c_M h_M = f_\Omega \quad (2.2)$$

where h_m ($m = 1, \dots, M$) are functions of u, t, x, y, z and derivatives of u . It can be assumed, without loss of generality, that the terms containing the unknown coefficients are the first N terms appearing in equation (2.2). Transferring the terms containing the known constants

to the right, equation (2.2) may be written in more convenient form

$$c_1 h_{11} + c_2 h_{22} + \dots + c_N h_{NN} = f_\Omega - c_{N+1} h_{N+1} - \dots - c_M h_{MM} \quad (2.3)$$

This is a linear equation in the N unknown coefficients and N such equations are required to solve for them. Since equation (2.3) is valid at all points in Ω and for all t , these N equations may be generated by evaluating equation (2.3) at N different points in time and/or space as follows.

For simplicity of notation define

$$h_{im} = h_m(t_i, x_i, y_i, z_i) \quad m = 1, 2, 3, \dots, N \quad (2.4)$$

$$g_i = f_\Omega(t_i, x_i, y_i, z_i) - c_{N+1} h_{N+1}(t_i, x_i, y_i, z_i) - \dots - c_M h_{MM}(t_i, x_i, y_i, z_i) \quad (2.5)$$

where (x_i, y_i, z_i) denotes a point in Ω and t_i denotes a specific value of t . At point (t_i, x_i, y_i, z_i) equation (2.3) becomes

$$c_1 h_{i1} + c_2 h_{i2} + \dots + c_N h_{iN} = g_i \quad (2.6)$$

When equation (2.6) is evaluated at N different points ($i = 1, \dots, N$) the following matrix equation is obtained

$$\underline{H}\underline{c} = \underline{g} \quad (2.7)$$

where

$$\underline{H} = \begin{bmatrix} h_{11} & \dots & h_{1N} \\ \vdots & & \vdots \\ h_{N1} & \dots & h_{NN} \end{bmatrix} \quad (2.8)$$

$$\underline{c} = [c_1 \quad c_2 \quad \dots \quad c_N]' \quad (2.9)$$

$$\underline{g} = [g_1 \quad g_2 \quad \dots \quad g_N]^T \quad (2.10)$$

with T denoting the transpose. If H is nonsingular and if \underline{g} is not a null vector the constants c_1, c_2, \dots, c_N are uniquely determined by the equation

$$\underline{c} = H^{-1} \underline{g} \quad (2.11)$$

The proposed identification scheme, equation (2.11), requires that H be nonsingular. This requirement is necessary for identifying the partial differential equation using the proposed method; that is, if H is nonsingular the constant vector \underline{c} can be determined.

At present no rigorous statement can be made as to which systems possess a nonsingular H matrix. However, if a partial differential equation describing the system has a unique solution and if the solution depends continuously on the initial and boundary conditions, it will probably have a nonsingular H matrix [14].

In order to evaluate the inverse of H it is necessary to measure u and its derivatives which appear in H . This introduces measurement noise. Also it is not always convenient or possible to measure the partial derivatives so it becomes necessary to approximate them. Thus errors are introduced in the H matrix. Now, the existence of the inverse of the H matrix does not guarantee that the matrix which approximates H has an inverse. If H has an inverse and if the errors have little effect on the value of the inverse, the process of taking the inverse is said to be well-conditioned [15]. For a distributed system, circumstances under which the inversion process is ill-conditioned due to approximation error

and noise is discussed in section 3.5.

Herein, it is assumed that the matrix H and the vector \underline{g} are either known or can be measured directly, unless the elements of H or \underline{g} contain derivatives of the state variables. However, since the derivatives are to be evaluated at specific points in time and space, they can be approximated by finite differences. This finite difference technique, which has been applied extensively in the numerical solution of partial differential equations, utilizes values of u in the neighborhood of the desired point in time and space to approximate the derivative. The feasibility of the proposed identification scheme using finite differences is demonstrated in the next section.

2.3 EXAMPLES AND RESULTS OF DIGITAL COMPUTER SIMULATIONS

In this section unknown coefficients in the partial differential equations derived in Chapter 1 are determined by applying equation (2.11). Partial derivatives are approximated by finite differences. Though each of the examples presented is a form of the parabolic equation, the technique is general and is not limited to this class of equations. Numerical results obtained by digital computer simulations are given.

2.3.1 Identification of a Single Constant

Consider the problem of identifying the diffusivity constant α in the diffusion equation derived in subsection 1.2.1

$$\alpha \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial u}{\partial t}(x,t) \quad (2.12)$$

For comparison, this problem is first solved by approximating derivatives

by central differences [16] and is then solved using high accuracy formulae suggested by Mitchell and Pearce [17] for obtaining a numerical solution of the diffusion equation.

Since there is only one unknown coefficient, α , it suffices to evaluate equation (2.12) at a single point, say (x_i, t_j) ; thus

$$\alpha \frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{\partial u}{\partial t}(x_i, t_j) \quad (2.13)$$

Assuming that only u can be measured it is necessary to approximate the partial derivatives by using a finite difference approximation. For $\frac{\partial u}{\partial t}$ consider the central difference approximation

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta t} \quad (2.14)$$

where

$$u_{i \pm m, j \pm n} = u(x_i \pm m\Delta x, t_j \pm n\Delta t); m, n = 1, 2, \dots$$

$$\Delta x = x_{i+1} - x_i$$

$$\Delta t = t_{j+1} - t_j \quad (2.15)$$

The approximation of expression (2.14) has an error of order Δt^2 . In principle, the accuracy of this approximation may be improved by decreasing Δt . However, if Δt is decreased indefinitely the error due to measurement noise becomes much larger than the improvement in the accuracy of equation (2.14). This problem is discussed in detail in Chapter 3. Also, an extremely small Δt requires a very high rate of sampling u which may not always be feasible or desirable. Yet, the accuracy of the approximation for $\frac{\partial u}{\partial t}$ can be improved without decreasing

At by including in the approximation the values of $u_{i,j+2}$ and $u_{i,j-2}$.

Thus the approximation

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{-u_{i,j+2} + 8u_{i,j+1} - 8u_{i,j-1} + u_{i,j-2}}{12\Delta t} \quad (2.16)$$

has an error of order Δt^4 . In general, the first derivative of u may be approximated to within an order Δt^m by taking m measurements.

A similar argument applies for improving the accuracy of the approximation for spatial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ etc. For example, the approximation

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \quad (2.17)$$

has an error of order Δx^2 . The order of error may be reduced to Δx^4 by the expression

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{-u_{i+2,j} + 16u_{i+1,j} - 30u_{i,j} + 16u_{i-1,j} - u_{i-2,j}}{12\Delta x^2} \quad (2.18)$$

which requires measurements at five spatial points each separated by Δx .

There are three basic limitations in improving the accuracy of the approximation of the spatial derivative: (1) Δx cannot be very small because it is difficult to mount the transducers very close to each other; (2) the error due to measurement noise becomes much larger than the improvement in the accuracy of equation (2.18); and (3) for a fixed Δx an improvement in the accuracy requires the measurement of the state variable at a greater number of spatial points (2.18) which, of course, increases the number of transducers needed.

When the central differences in equations (2.14) and (2.17) are

substituted for the derivatives in equation (2.13), the following estimate for α is obtained.

$$\tilde{\alpha} = \frac{\Delta x^2 (u_{i,j+1} - u_{i,j-1})}{2\Delta t (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})} \quad (2.19)$$

Notice that equation (2.19) identifies the unknown coefficient α independent of the boundary conditions. Results of digital computer simulations of (2.19) are given later in this subsection.

A high accuracy formula developed by Mitchell and Pearce for numerical solution of the diffusion equation is now investigated for its application to the identification problem. Their high accuracy discrete formula is

$$\begin{aligned} & au_{i,j} + bu_{i,j+1} + cu_{i,j-1} + d(u_{i+1,j} + u_{i-1,j}) \\ & + e(u_{i+1,j+1} + u_{i-1,j+1}) + f(u_{i+1,j-1} + u_{i-1,j-1}) = 0 \end{aligned} \quad (2.20)$$

where the coefficients are given by

$$\begin{aligned} a &= 16p^4 - p^2 + \frac{313}{6,300} \\ b, c &= 4p^4 \pm 5p^3 - \frac{1}{10} p^2 \mp \frac{23}{84} p - \frac{313}{12,600} \\ d &= -8p^4 + \frac{1}{2} p^2 - \frac{313}{12,600} \\ e, f &= -2p^4 \pm \frac{1}{2} p^3 + \frac{1}{20} p^2 \mp \frac{11}{840} p + \frac{13}{25,200} \end{aligned} \quad (2.21)$$

with

$$p = \alpha \frac{\Delta t}{\Delta x^2}$$

When the constants from equation (2.21) are substituted in equation (2.20) a quartic equation in α results. This equation yields four possible values of α and the correct one must be selected from this set.

The values of α can be calculated on a digital computer by solving the quartic explicitly or by using a search routine. In either case, if an a priori estimate of α is available, the correct one can be selected from the set. When no a priori estimate is available it is necessary to use a central difference expression, such as equation (2.19) to estimate α and then apply equation (2.20) to obtain higher accuracy.

A less accurate formula is derived in Appendix D using the Mitchell-Pearce approach. Equation (2.20) in this case has the coefficients

$$\begin{aligned}
 a &= 200p^2 - 90p - 35 \\
 b &= 104p^2 + 127p + 17.5 \\
 c &= -4p^2 - 37p + 17.5 \\
 d &= -100p^2 - 9p - 2 \\
 e &= -52p^2 + p + 1 \\
 f &= 2p^2 + 8p + 1
 \end{aligned} \tag{2.22}$$

Substitution of equation set (2.22) into equation (2.20) results in a quadratic in α . The loss in accuracy using the coefficients of (2.22) rather than those of (2.21) is offset by the comparative ease in solving a quadratic vice a quartic equation.

An analytic solution of equation (2.12) with $\alpha = \frac{1}{\pi^2}$ and boundary conditions $u(x,0) = \frac{\pi}{4}$, $u(0,t) = u(1,t) = 0$ is given by

$$u = \sum_{n=1}^{\infty} \frac{1}{2n} \sin(2n\pi x) e^{-(2n)^2 t} \tag{2.23}$$

Using equation (2.23), values of u at $x = 0.0, 0.05, 0.10, \dots$ and $t = 0.0, 0.025, 0.05, \dots$ were calculated on an IBM 1620 II digital computer with

the results shown in Figure 2.1. Data obtained from this calculation was used to identify α . Note that the data for the state variable u for this example and those to follow are generated by using an analytic solution. The finite difference methods for solution of the boundary value problems was avoided in generation of data in order to isolate the effects of finite difference approximation on the accuracy of the proposed identification technique. This, of course, has restricted the examples considered to the class of boundary value problems for which analytic solutions were readily available. However, this does not imply limitations for the applications of the proposed identification scheme.

The central difference expression of equation (2.19) with $\Delta t = 0.025$ and $\Delta x = 0.05$, produced the value of $\hat{\alpha}$ plotted in Figure 2.2. The identification of α is very accurate when the rate of change is not very large or very small. It is interesting to note that when equation (2.19) is applied to obtain a numerical solution of the diffusion equation, the solution is unstable [1]. Since the identification scheme presented here is a single stage process, stability is of no concern. Furthermore, α itself may be varying in time and space and still be identified, as long as the change in α over one identification interval, $2\Delta t$ by $2\Delta x$, is negligible.

When the high accuracy formula in equation (2.20) with the coefficients given in equation (2.21) was used to identify α , the data shown in Figure 2.3 resulted. Figure 2.4 shows the results of the identification using the coefficients given in equation (2.22). In both cases, accuracy

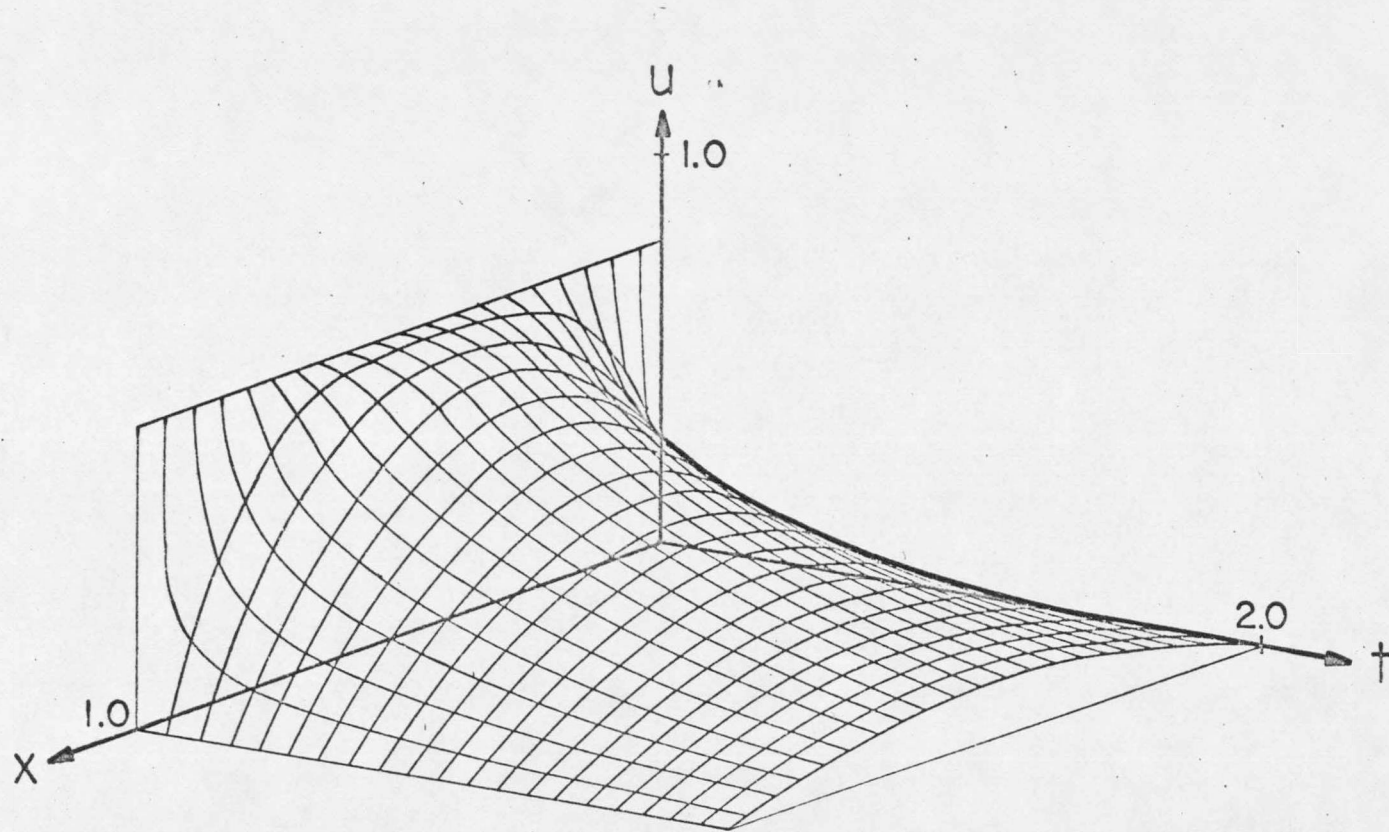


Figure 2.1 A solution of the diffusion equation.

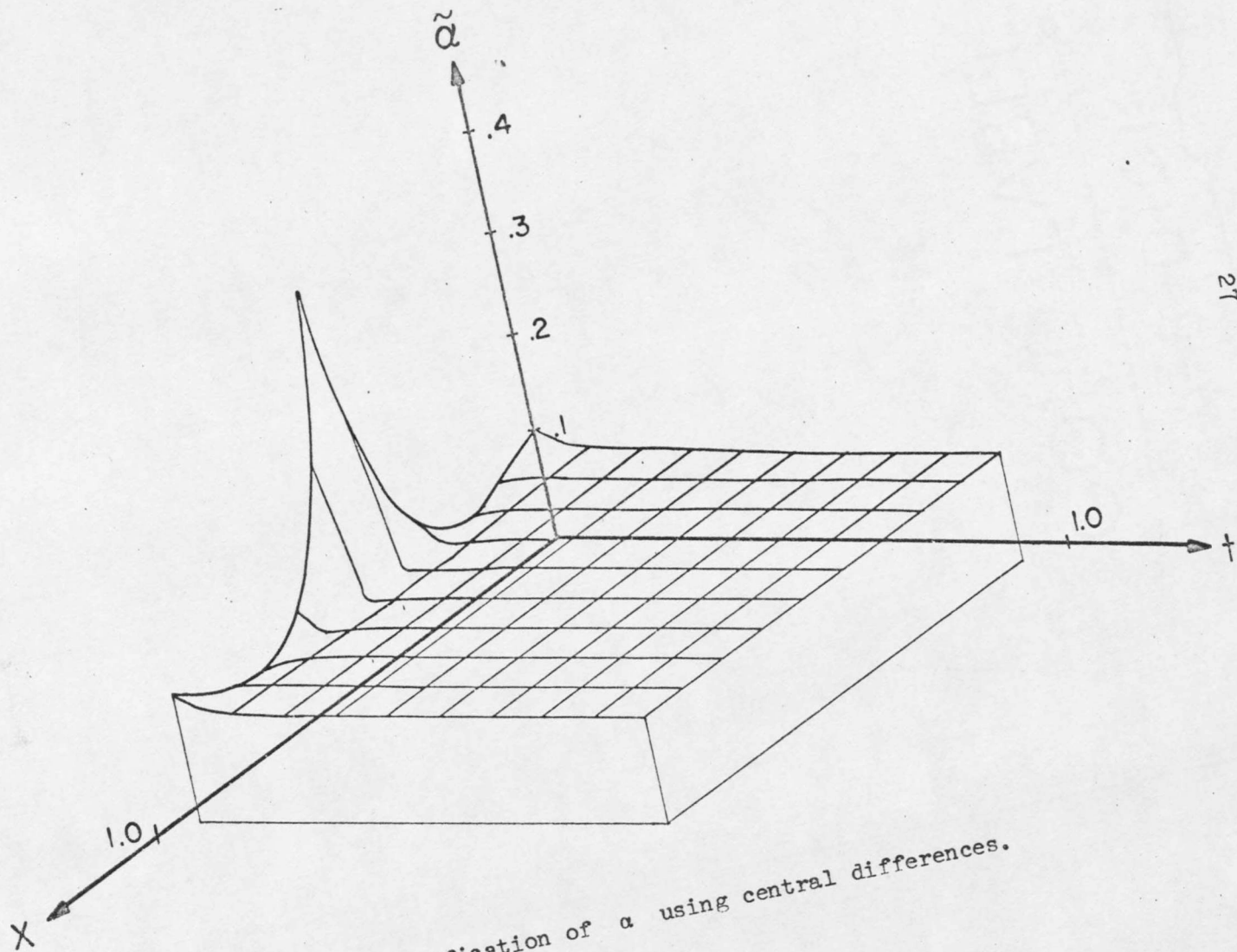


Figure 2.2 Identification of α using central differences.

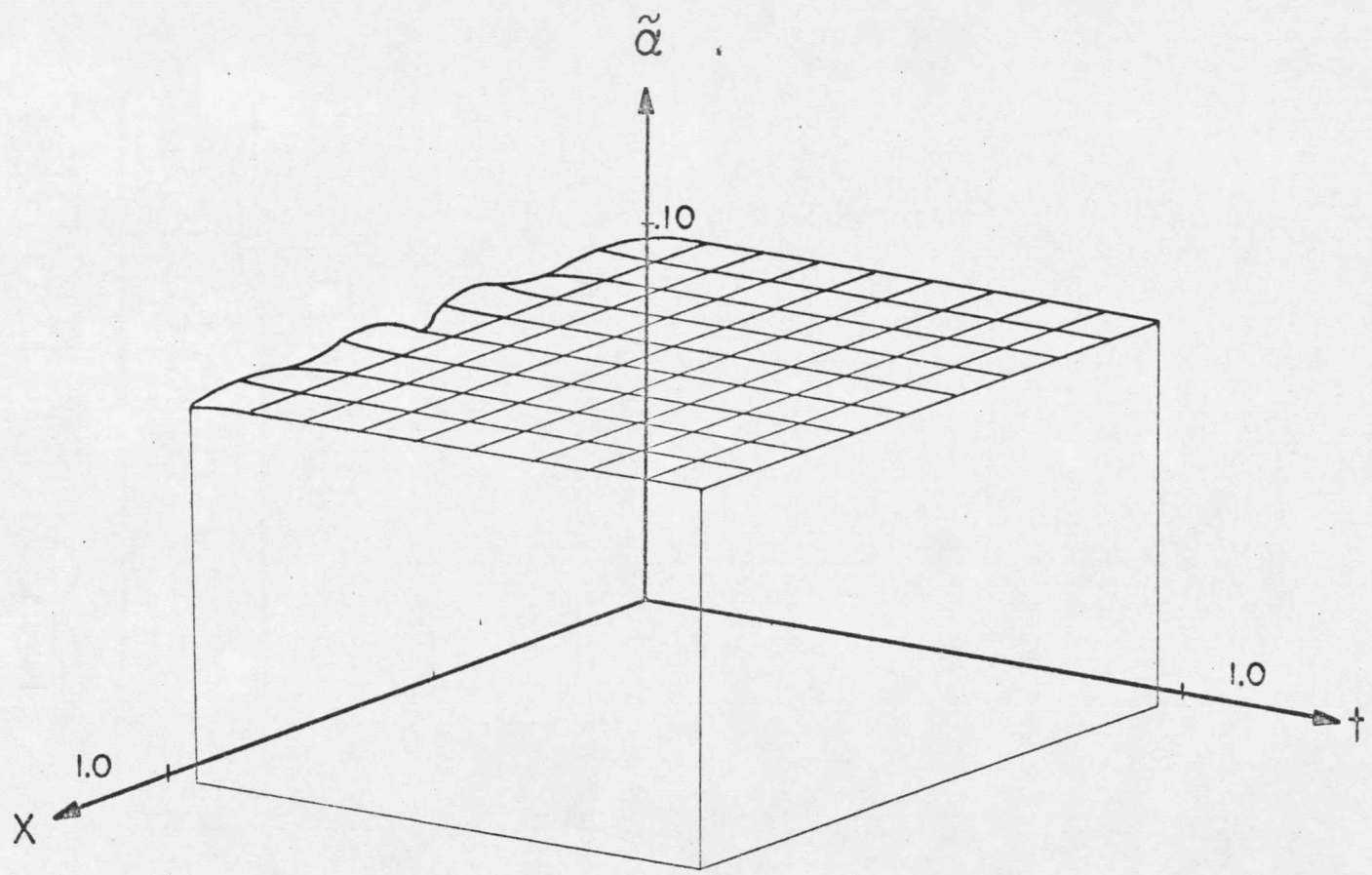


Figure 2.3 Identification of α using the quartic high accuracy formula.

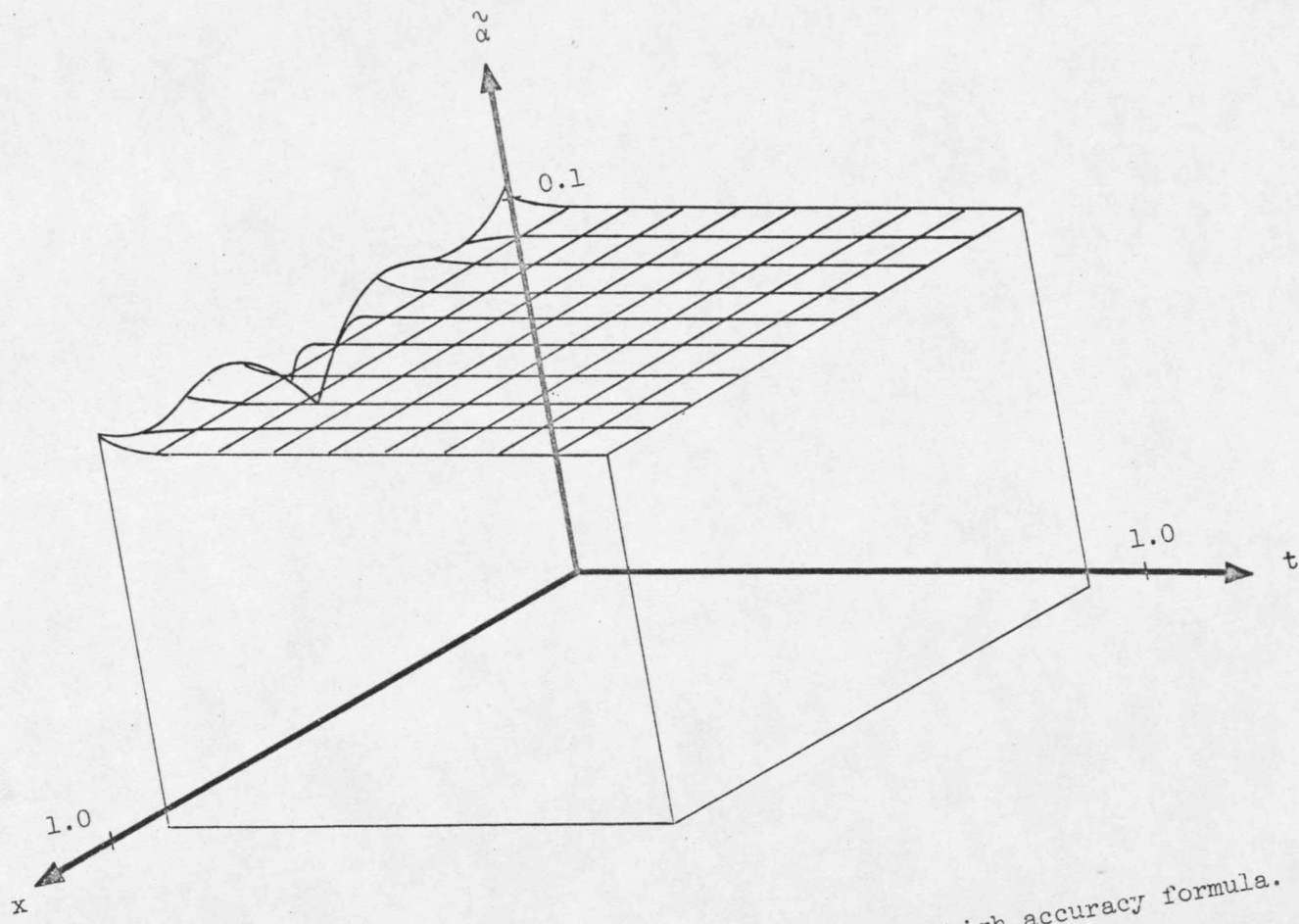


Figure 2.4 Identification of α using the quadratic high accuracy formula.

is considerably improved over that of equation (2.19) even though the same number of spatial and time measurements are used.

2.3.2 Identification of Boundary Conditions

There are circumstances when the nature of the boundary conditions of a system are known but the values of the coefficients involved are unknown. The procedure for identifying a constant in a boundary condition is illustrated using the following boundary value problem.

Let a heat transfer system be described by the partial differential equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < L; t > 0 \quad (2.24)$$

and the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = -\frac{q(t)}{k} \quad (2.25)$$

$$\frac{\partial u}{\partial x}(L,t) = -\beta u(L,t) \quad (2.26)$$

$$u(x,0) = 0 \quad (2.27)$$

where $q(t)$ is the rate of heat input at $x = 0$, k is heat conductivity and $\beta = \frac{h}{k}$ with h being the film heat transfer coefficient. It is assumed that the value of β is unknown and must be identified.

Identification of β is carried out by employing a procedure similar to that used earlier for identifying α . From equation (2.26) it is seen that β is a function of $\frac{\partial u}{\partial x}(L,t)$ and $u(L,t)$. It is assumed that $\frac{\partial u}{\partial x}(L,t)$ cannot be measured directly and therefore must be approximated. Central differences cannot be applied to approximate $\frac{\partial u}{\partial x}(L,t)$ for a specified x and t , because the derivative must be evaluated at the boundary.

Backward differences [16], on the other hand, are well suited for this purpose even though they require measurements of u at a greater number of spatial points in order to attain the same order of accuracy as central differences. Substituting a backward difference expression for the derivative in equation (2.26), the following estimate for β , denoted by β^v , is obtained.

$$\beta^v = \frac{25u_{L,j} - 48u_{L-1,j} + 36u_{L-2,j} - 16u_{L-3,j} + 3u_{L-4,j}}{(12\Delta x)u_{L,j}} \quad (2.28)$$

This expression requires five spatial measurements of the state variable to estimate β to within an error of order Δx^4 .

It may be noted that the identification of α by using equation (2.19) and the identification of β by using equation (2.28) are independent of each other. In order to identify α it is not necessary to know the boundary conditions and in order to identify the boundary conditions, it is not necessary to know the partial differential equation. If α and β are both unknown, it is possible to identify them simultaneously by using equations (2.19) and (2.28).

The diffusion equation with the boundary conditions of equations (2.25), (2.26) and (2.27) was solved using the method of finite cosine transforms discussed in Tranter [18]. The resulting analytic solution with a forcing function $q(t) = \sin \omega t$ is

$$u = \frac{2}{k} \sum_{i=1}^{\infty} \frac{\cos(p_i x)}{L + \frac{\beta}{p_i^2 + \beta^2}} \frac{(p_i \alpha)^2}{(p_i^2 \alpha)^2 + \omega^2} \left[\sin \omega t - \frac{\omega}{p_i^2 \alpha} (\cos \omega t - e^{-p_i^2 \alpha t}) \right] \quad (2.29)$$

where the p_i are the consecutive roots of the expression $\tan(Lp) = \beta$.

The parameters selected were $\alpha = \omega = 1$, $\beta = 2/117$ and $L = 1/2$. From equation (2.29) values of u were calculated on the digital computer to generate the necessary data. β was identified from this data using equation (2.28) with $\Delta x = 0.05$. The resulting identification of β , having an error of order Δx^4 , is plotted in Figure 2.5. This figure also includes estimates of β when backward differences with errors of order Δx^3 and Δx^2 were used. Since the dynamics of the system were initiated at time zero from the boundary at $x = 0$, significant data near $x = L$ was not obtained until approximately $t = 0.1$. This emphasizes the fact that there have to be changes in the state variable in order to be able to identify the equations that describe the dynamics of the system.

2.3.3 Identification of Two Constants

The identification technique presented in subsection 2.2.2 is now applied to differential equations with two unknown coefficients. The first example is linear in the state variable and was derived in Chapter 1. The second example, also presented in the first chapter, illustrates the application of the identification of parameters in a nonlinear partial differential equation. The identification of a system with two unknown constants when one of the constants is the coefficient of an extraneous term is demonstrated in the third example.

2.3.3.1 Linear System

Consider the linear partial differential equation

$$c_1 \frac{\partial^2 u}{\partial x^2} + c_2 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0 \quad (2.30)$$

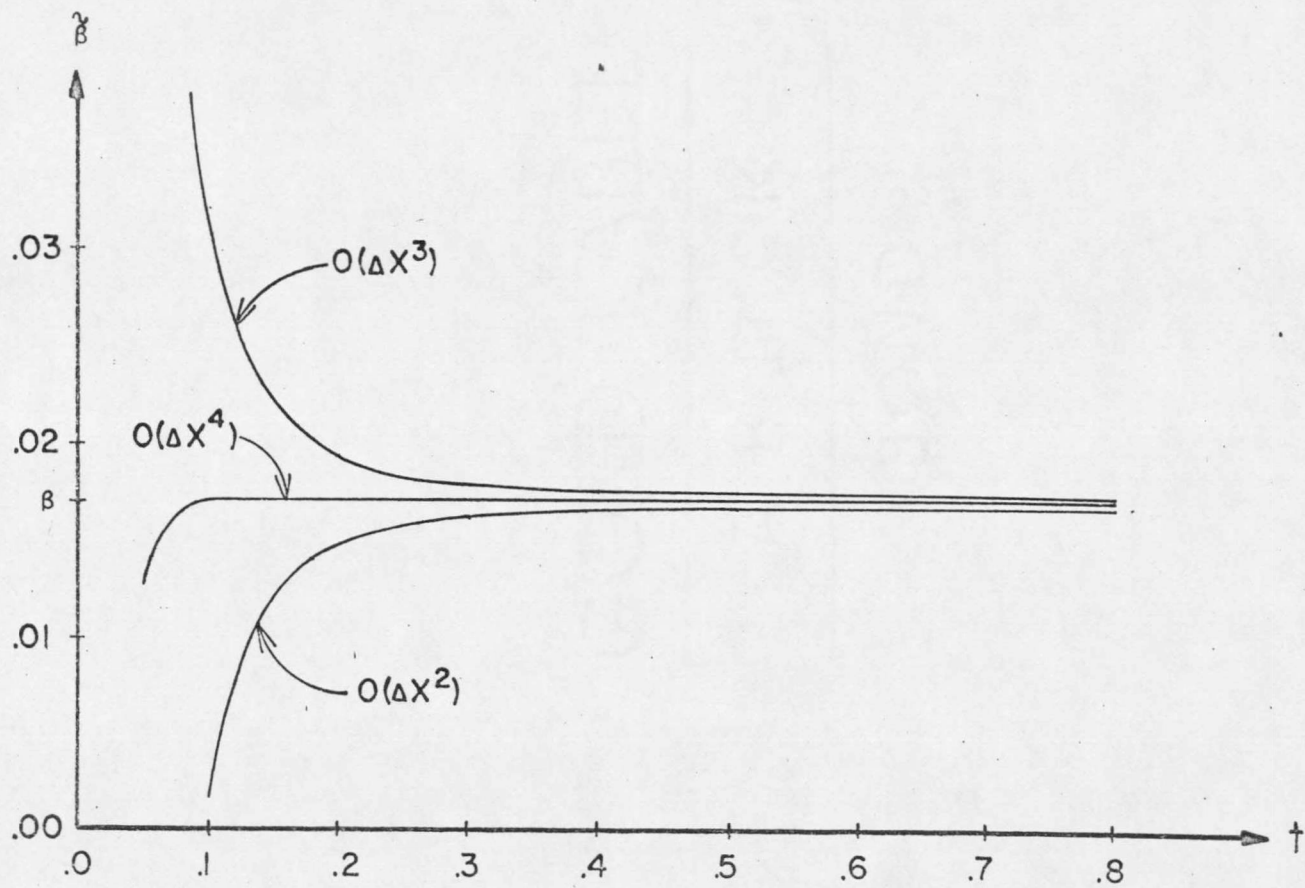


Figure 2.5 Identification of β in the boundary conditions.

where c_1 and c_2 are unknown coefficients that must be identified.

In order to identify c_1 and c_2 , equation (2.30) must be evaluated at two points in time and/or space. In this example, the two points chosen are separated in time by $4\Delta t$ and utilize measurements taken from the same points in space. When equation (2.30) is evaluated at the two points the results may be cast into matrix notation to conform with equation (2.7); thus

$$\begin{bmatrix} \frac{\partial^2 u}{\partial x^2}(x_i, t_j) & \frac{\partial u}{\partial x}(x_i, t_j) \\ \frac{\partial^2 u}{\partial x^2}(x_i, t_{j+4}) & \frac{\partial u}{\partial x}(x_i, t_{j+4}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial t}(x_i, t_j) \\ \frac{\partial u}{\partial t}(x_i, t_{j+4}) \end{bmatrix} \quad (2.31)$$

Since the derivatives of u cannot be measured directly, central differences are used to approximate equation (2.31). Let the first and second order derivatives be approximated by the central difference formulae given in (2.16) and (2.18) respectively. Then equation (2.31) may be modified to read

$$\begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{21} & \tilde{h}_{22} \end{bmatrix} \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix} = \begin{bmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{bmatrix} \quad (2.32)$$

where

$$\tilde{h}_{11} = \frac{1}{12\Delta x^2}(-u_{i+2,j} + 16u_{i+1,j} - 30u_{i,j} + 16u_{i-1,j} - u_{i-2,j})$$

$$\tilde{h}_{12} = \frac{1}{12\Delta x}(-u_{i+2,j} + 8u_{i+1,j} - 8u_{i-1,j} + u_{i-2,j})$$

$$\tilde{h}_{21} = \frac{1}{12\Delta x^2}(-u_{i+2,j+4} + 16u_{i+1,j+4} - 30u_{i,j+4} + 16u_{i-1,j+4} - u_{i-2,j+4})$$

$$\tilde{h}_{22} = \frac{1}{12\Delta x}(-u_{i+2,j+4} + 8u_{i+1,j+4} - 8u_{i-1,j+4} + u_{i-2,j+4})$$

$$\tilde{\epsilon}_1 = \frac{1}{12\Delta t}(-u_{i,j+2} + 8u_{i,j+1} - 8u_{i,j-1} + u_{i,j-2})$$

$$\tilde{\epsilon}_2 = \frac{1}{12\Delta t}(-u_{i,j+6} + 8u_{i,j+5} - 8u_{i,j+3} + u_{i,j+2})$$

with \tilde{c}_1 and \tilde{c}_2 denoting the approximate values of c_1 and c_2 resulting from finite difference errors.

The data necessary for the identification of c_1 and c_2 is obtained by solving equation (2.30) with the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 1, \quad u(x,0) = 0 \quad (2.33)$$

The resulting analytic solution is

$$u(x,t) = \frac{e^{-c_2 x/c_1} - 1}{e^{-c_2/c_1} - 1} + 2 \sum_{i=1}^{\infty} (-1)^i \frac{i\pi}{(i\pi)^2 + (c_2/2c_1)^2} \cdot \{e^{-c_2(x-1)/2c_1} \sin(i\pi x) e^{-[i^2\pi^2 c_1 + c_2^2/4c_1]t}\} \quad (2.34)$$

In this particular example, c_1 and c_2 were chosen to be $1/\pi^2$ and $2/\pi^2$. After these values were substituted into equation (2.34), $u(x,t)$ was calculated on the digital computer and plotted in Figure 2.6. Thus data which would be obtained by measurements from actual physical systems was provided for the solution of \tilde{c}_1 and \tilde{c}_2 in equation (2.32). Δx and Δt were selected to be 0.05 and 0.025 respectively. When the matrix expression is solved for c_1 and c_2 using Cramer's rule, the array shown in Figure 2.7 results.

2.3.3.2 Nonlinear System

The identification technique is now applied to the nonlinear equation

$$\frac{1}{8u} \frac{1}{3/2} \left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2u} \frac{\partial^2 u}{\partial x^2} - \frac{1}{2u^2} \left(\frac{\partial u}{\partial x}\right)^2 - \frac{1}{4u} \frac{1}{1/2} \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \quad (2.35)$$

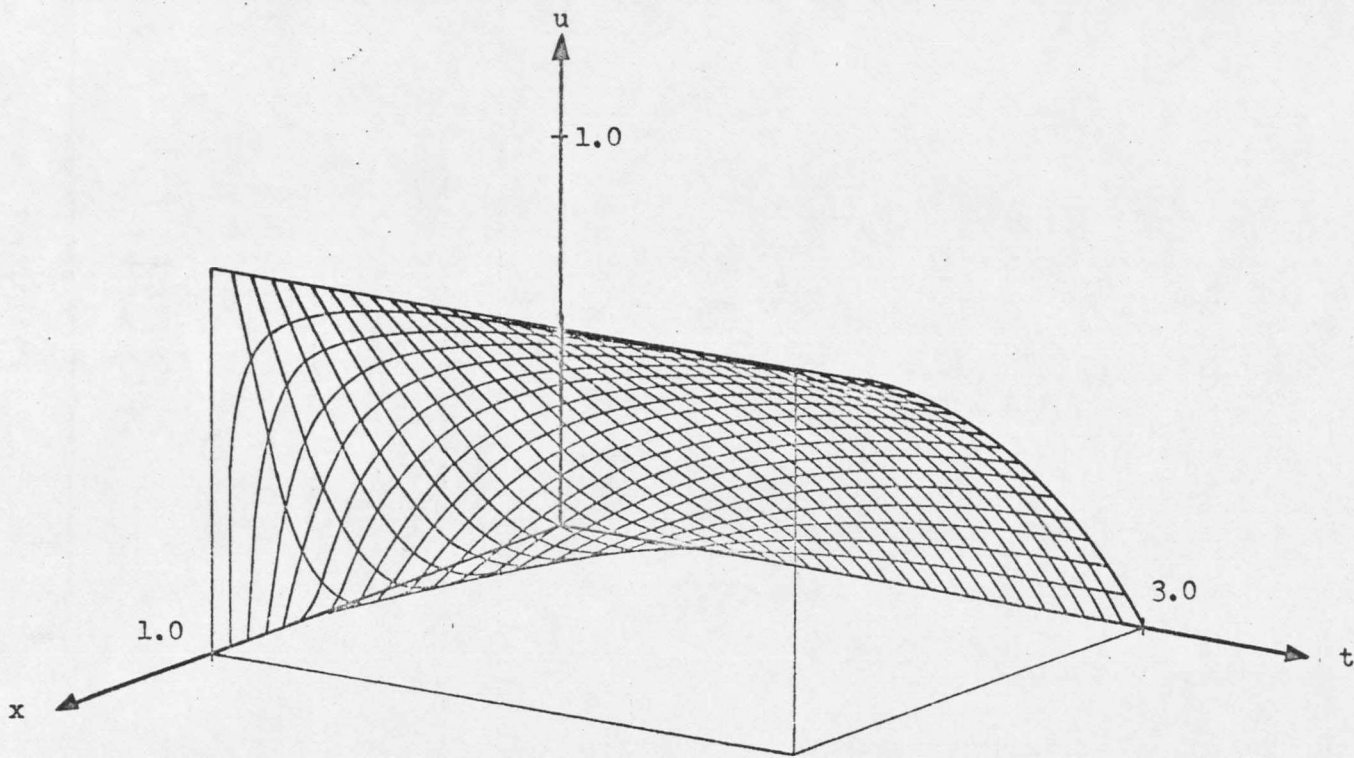


Figure 2.6 A solution of the linear system with two coefficients.

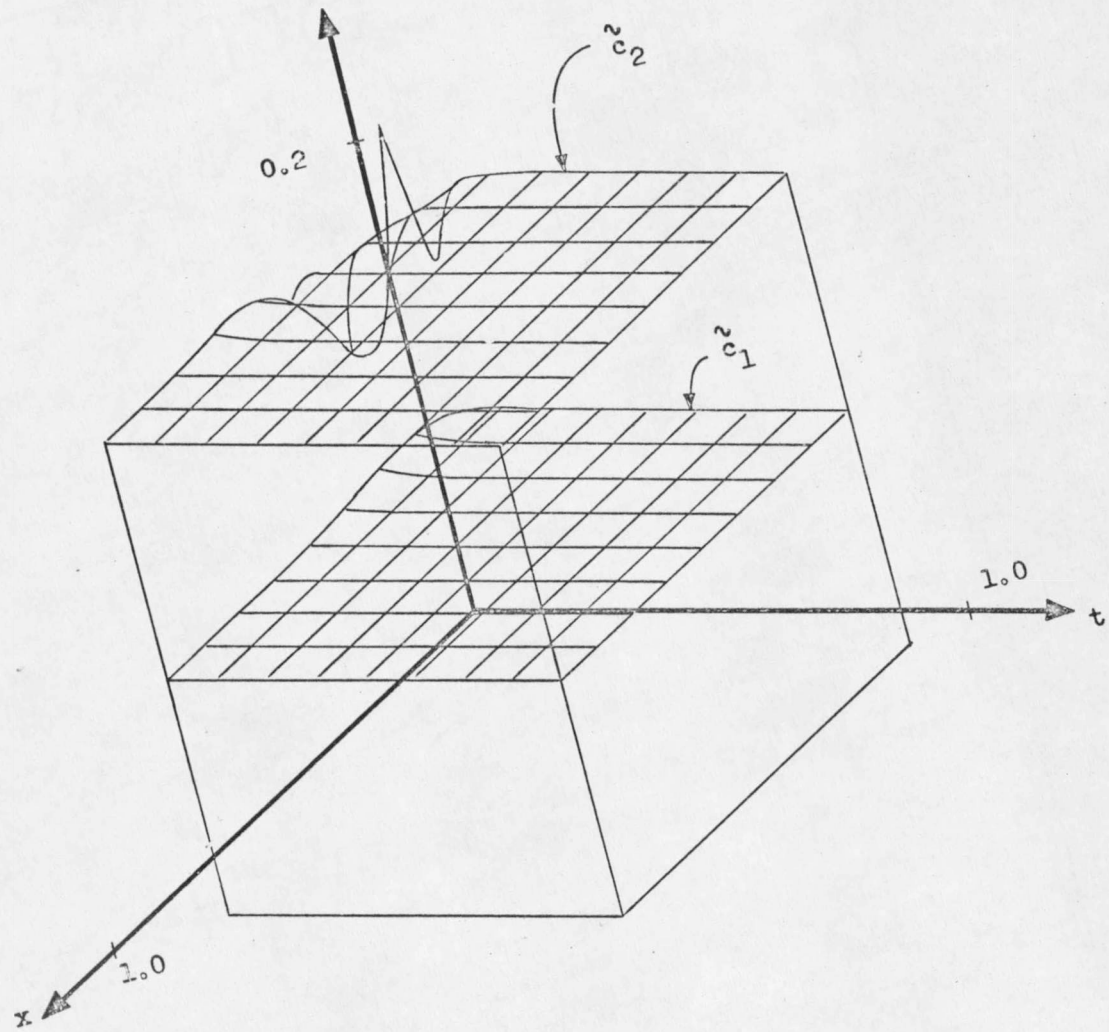


Figure 2.7 Identification of c_1 and c_2 in the linear system.

with the boundary conditions

$$u(x,0) = 0, \quad u(0,t) = 1, \quad \text{and} \quad \frac{\partial u}{\partial x}(0,t) = -2t^{-1/2} \quad (2.36)$$

It is shown by Philip [19] that the solution of this boundary value problem is given by

$$u = \frac{t}{(x+t)^{1/2}} \quad (2.37)$$

For purposes of illustration it is assumed that the constants in the first two terms of equation (2.35) are unknown. Letting c_1 and c_2 denote these constants and rewriting (2.35) in the same form as equation (2.3) one gets

$$c_1 \frac{1}{u^{3/2}} \left(\frac{\partial u}{\partial x} \right)^2 + c_2 \frac{1}{u} \frac{\partial^2 u}{\partial x^2} = \frac{1}{2u^2} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{4u^{1/2}} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \quad (2.38)$$

As was demonstrated in example 2.3.3.1, c_1 and c_2 may be determined by evaluating equation (2.38) at two different points.

The coefficients c_1 and c_2 of equation (2.38) were estimated using data obtained by calculating equation (2.37). A plot of the state variable u over a limited range is shown in Figure 2.8. The derivatives in equation (2.38) were approximated using the central difference expressions in equations (2.16) and (2.18) with $\Delta t = 0.1$ and $\Delta x = 0.05$. Equation (2.38) was evaluated at the same time but at two points in space separated by a distance Δx . The resulting identification of c_1 and c_2 is shown in Figure 2.9.

2.3.3.3 Mathematical Model Containing an Extraneous Term

It is shown in Appendix C that when the assumed system partial differential equation contains an extraneous term, the coefficient of this term

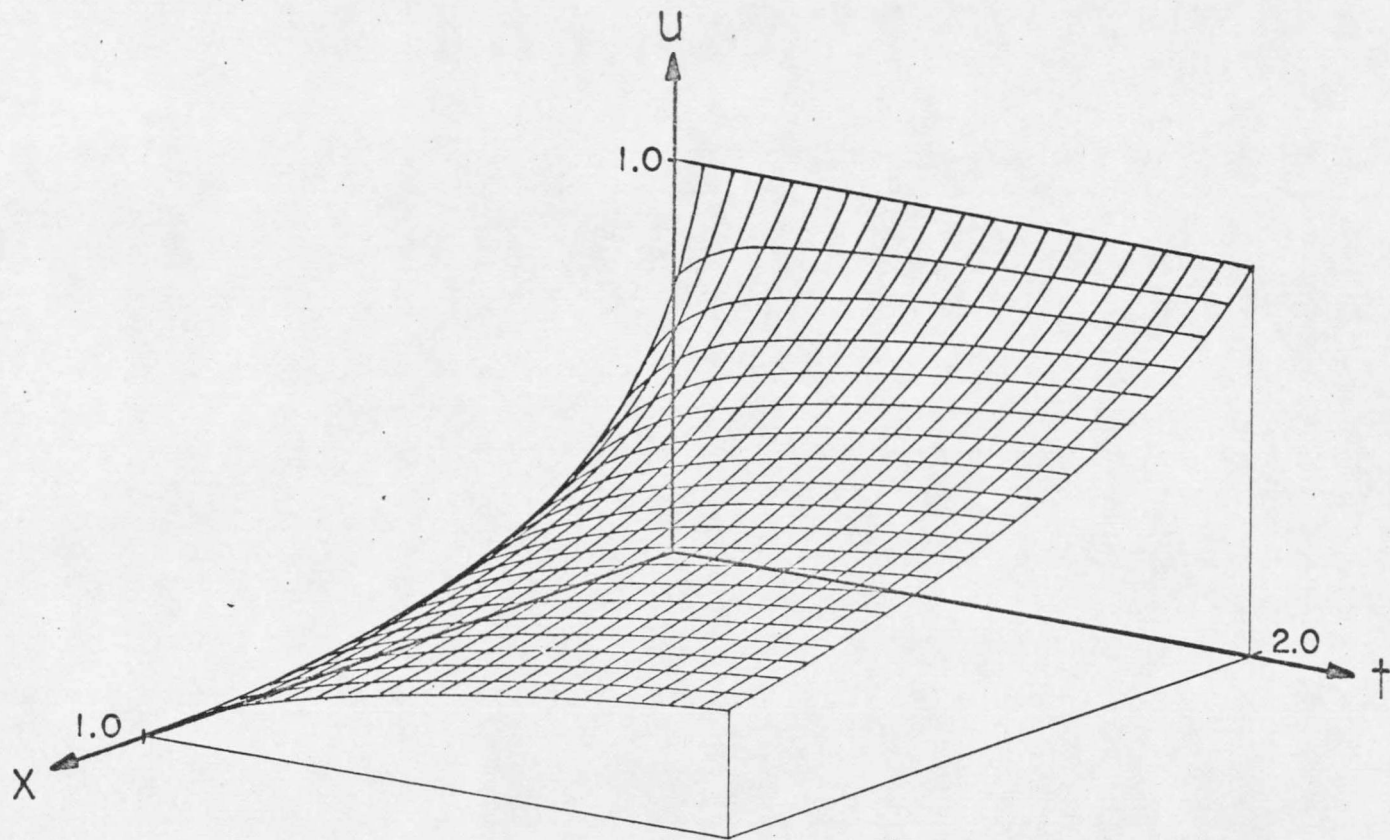


Figure 2.8 Solution of the nonlinear system.

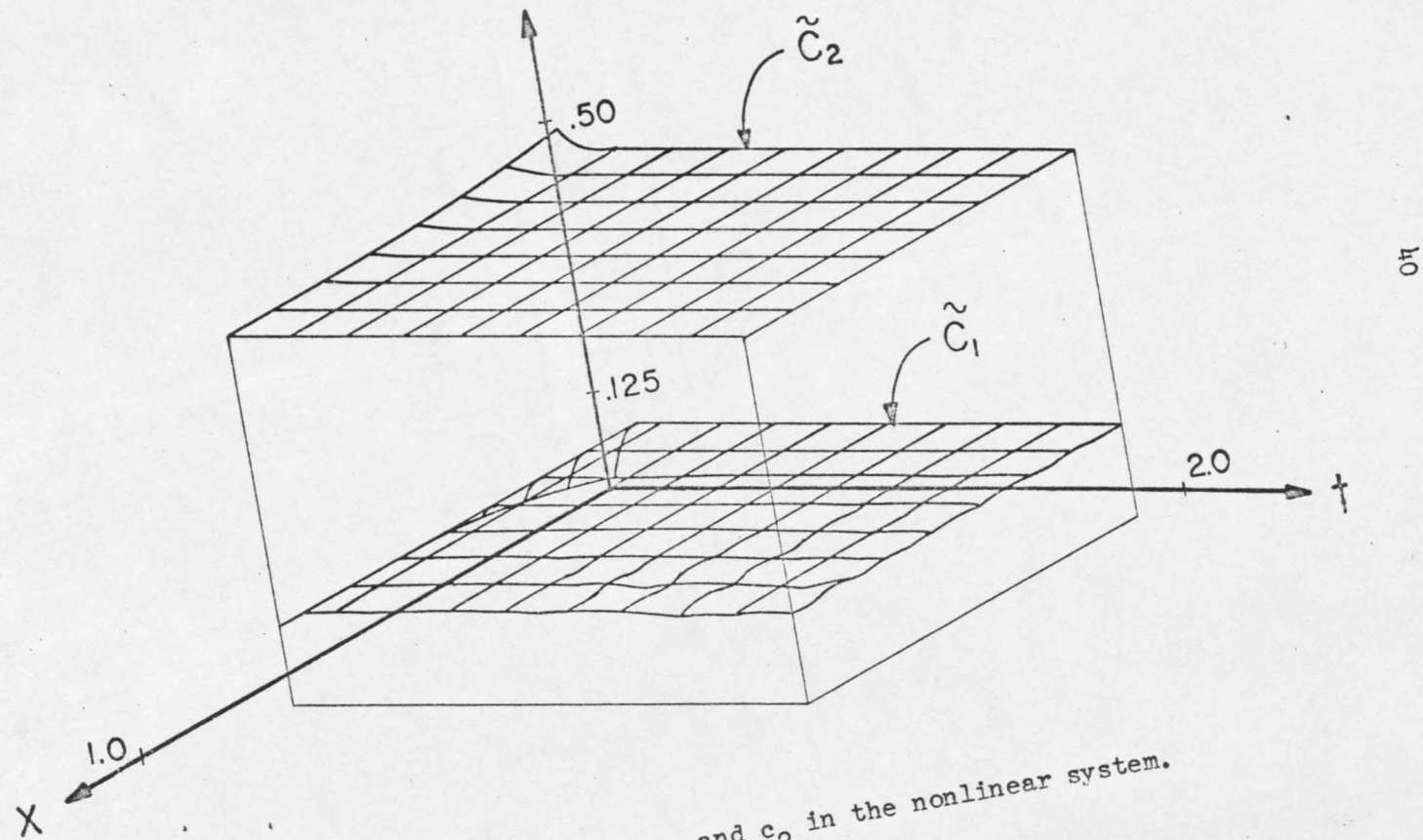


Figure 2.9 Identification of c_1 and c_2 in the nonlinear system.

will be identified as zero if no approximation or noise errors are introduced. When these errors are present, the estimated value of the coefficient is very small under the restriction that the problem is well-conditioned. The identification of a partial differential equation with an extraneous term and no measurement noise is considered in the following example. The effects of adding noise to the system are discussed in example 3.4.3.3.

Let the dynamics of a system be characterized by the diffusion equation given by (2.12). The numerical data of the state variable u is obtained once again from equation (2.23). Given this data the problem is to find the mathematical model of the system. Due to a lack of a priori knowledge, it is assumed that the system dynamics are characterized by equation (2.30) instead of (2.12). Thus, the assumed mathematical model has an extraneous term, $(c_2) \frac{\partial u}{\partial x}$.

The analysis proceeds in exactly the same manner as that presented for the identification of c_1 and c_2 in example 2.3.3.1. The values for Δx and Δt are also unchanged. However, in this example, equation (2.12) is evaluated at points separated by Δx units in space rather than four Δt units in time. The calculated values for \tilde{c}_1 and \tilde{c}_2 appear in Figure 2.10.

2.4 SUMMARY

The identification of the mathematical model describing a distributed parameter system was considered in this chapter. It was assumed that exact measurements of the state variable were available to identify

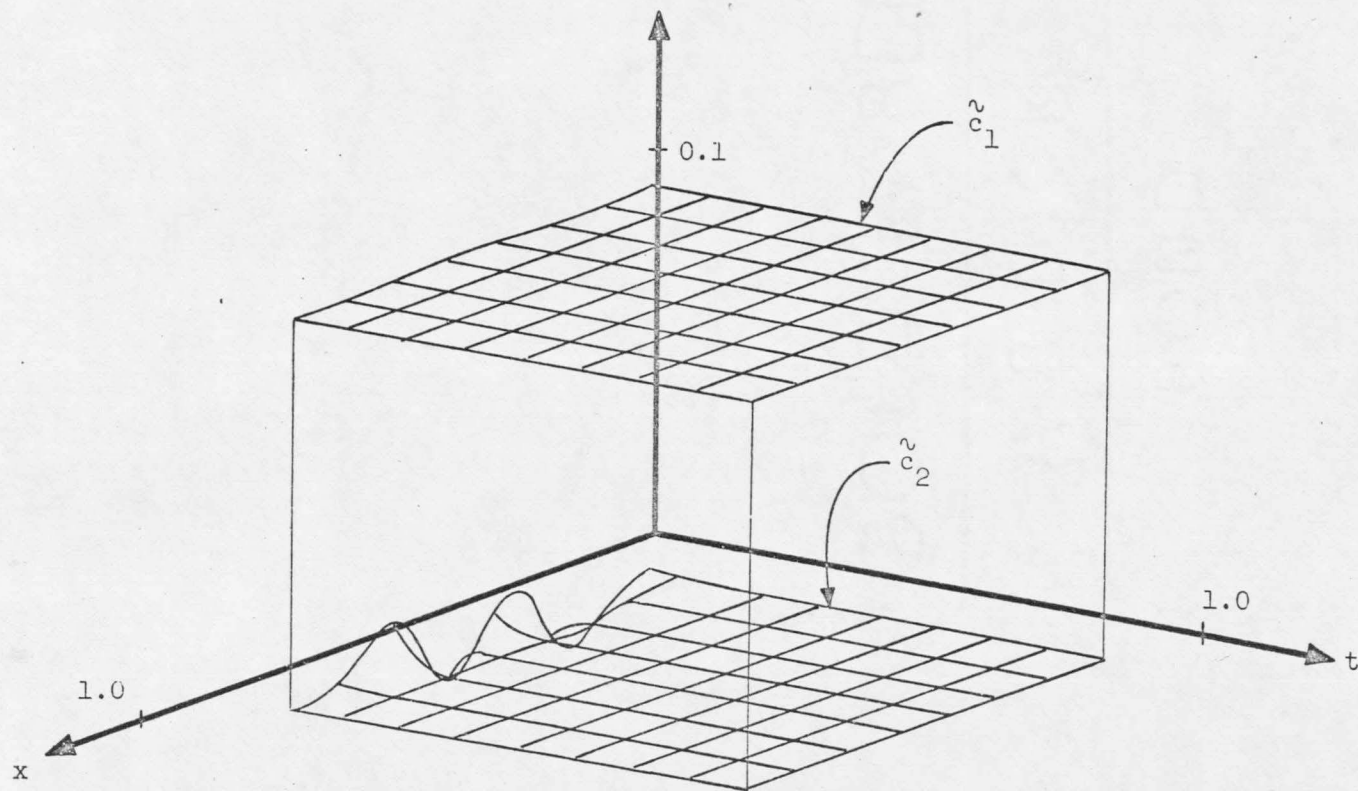


Figure 2.10 Identification of c_1 and c_2 when c_2 is the coefficient of an extraneous term.

unknown constants in the system partial differential equation and/or boundary conditions. Derivatives of the state variable were approximated by finite differences. Methods for reducing the errors resulting from these approximations were presented. Results of digital computer simulations were given to illustrate the ideas presented in the chapter. The identification of the unknown constants when the state variable measurements are not exact is considered in the next chapter.

CHAPTER 3

IDENTIFICATION OF DISTRIBUTED PARAMETER

SYSTEMS WITH MEASUREMENT NOISE

3.1 INTRODUCTION

This chapter is devoted to the identification of unknown constants in partial differential equations when measurements of the state variables of a distributed system are corrupted by noise. The introduction of noise requires that the unknown parameters must be estimated. The least squares estimator is found to be well suited for the problem, because its application requires no knowledge of the statistics of the noise. However, for linear systems, knowing the mean and variance of the transducer noise makes it possible to modify the least squares estimator to obtain better accuracy. The convergence of the least squares estimator and the modified least squares estimator is discussed. Results of digital computer simulations are given to illustrate the accuracy of the two estimators.

The errors due to finite difference approximation and measurement noise are functions of the increments in time and space used in the finite difference formulae. Increasing the increments reduces errors resulting from measurement noise but increases finite difference approximation errors. A procedure is presented which leads to the determination of the spatial and temporal increments which minimize the combined effect of the two errors. The technique is applied to examples.

Finally, the problem of ill-conditioning introduced in Chapter 2 is discussed in greater detail. The origins of ill-conditioning, the methods for testing for ill-conditioning and a procedure to reduce its effect are examined.

3.1.1 Measurement Devices

In Chapter 2 finite differences are used to approximate derivatives of the state variables. It is seen that the resulting finite difference expressions are functions of the state variables, and the analysis is presented under the assumption that the state variables are known exactly. However, in a real life situation, the state variables must be measured with transducers. This introduces additional error due to imperfections in the transducers, external noise, and interactions between the system and the transducer.

In a distributed system transducers may be required to measure such state variables as temperature, pressure, velocity, chemical concentration, electric field intensity and magnetic flux. A detailed discussion of individual transducers would necessarily be very involved and several books, such as the one by Doebelin [20], have been written on the subject. However some statistical characteristics of the noise introduced by the transducers must be considered here.

Let the noise generated by a transducer at point x_i and associated instruments be denoted by $w_i(t)$. Frequently, this noise is independent of the noise produced by transducers located at other points. Furthermore, the statistics of $w_i(t)$ may vary from transducer to transducer. Since the finite difference equations require measurements at discrete points in time, $w_i(t)$ is sampled every Δt seconds. If the samples of $w_i(t)$ are taken at sufficiently large intervals of time, the samples $w_{i,j}$ ($j = 1, 2, \dots$), where j denotes the j^{th} sample of $w_i(t)$, may be

may be assumed independent in the probability sense.

As a way of formulating the above discussion and establishing a framework from which to proceed, the transducers will be assumed to have the following characteristics.

- (3a) The noise generated in each transducer is statistically independent from the noise produced by any other transducer.
- (3b) The random samples taken from a given transducer are statistically independent.
- (3c) The noise generated by each transducer satisfies the ergodic theorem [21].
- (3d) The statistics may vary from transducer to transducer, and may or may not be known a priori.

The above assumptions, of course, eliminate the case where the noise generated by the different transducers are correlated. Furthermore, the sampling rate may be very high with the result that the sequence of samples taken from a particular transducer are not statistically independent. Identification of a distributed system with correlated measurement noise is not investigated in this thesis.

3.2 RECURSIVE ESTIMATOR

3.2.1 Problem Formulation

In Chapter 2 the matrix equation (2.7), rewritten here

$$H \underline{c} = \underline{g}$$

is generated for the system equation (2.3) so that a solution for the unknown vector \underline{c} can be obtained.

Since the elements of H and \underline{g} are now considered to be corrupted with noise it is necessary to evaluate H and \underline{g} either at different points in space or at different instants of time so as to generate enough data for estimation of \underline{c} . Let k ($k = 1, 2, \dots$) denote a specific spatial and/or time point at which equation (2.7) is evaluated; thus

$$H_k \underline{c} = \underline{g}_k \quad k = 1, 2, \dots \quad (3.1)$$

The problem is to identify the unknown vector \underline{c} under the following conditions:

- (3e) No spatial or time derivatives can be measured directly.
- (3f) The measurements are corrupted with noise whose statistics may or may not be known.
- (3g) The identification must be carried out using normal operating data; that is, no test signals that perturb the system operation are allowed.
- (3h) The boundary conditions may or may not be known.

3.2.2 Introductory Example

Before proceeding to the general development for the solution of the estimation problem, a simple example is presented to demonstrate the effects of measurement noise.

Consider the diffusion equation which is rewritten here for convenience

$$\alpha \frac{\partial^2 u}{\partial x^2}(x_i, t_j)_k = \frac{\partial u}{\partial t}(x_i, t_j)_k \quad (3.2)$$

Equation (3.1) in this example reduces to a scalar expression with each

k corresponding to a unique combination of i, j .

Since the derivatives cannot be measured directly, they are evaluated approximately by measuring $u_{i,j}$ and using the finite differences

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{\hat{u}_{i,j+1} - \hat{u}_{i,j-1}}{2\Delta t} \quad (3.3)$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{\hat{u}_{i+1,j} - 2\hat{u}_{i,j} + \hat{u}_{i-1,j}}{\Delta x^2} \quad (3.4)$$

where $\hat{u}_{i,j}$ denotes the measured value of $u_{i,j}$; that is,

$$\hat{u}_{i,j} = u_{i,j} + w_{i,j} \quad i, j = 1, 2, \dots, S \quad (3.5)$$

Thus, in addition to the discretization errors, errors due to measurement noise have been introduced.

Substituting for $\hat{u}_{i,j}$ from equation (3.5) in equations (3.3) and (3.4) and comparing with equations (2.14) and (2.17) it is seen that the errors due to measurement noise are

$$W_1 = \frac{w_{i,j+1} - w_{i,j-1}}{2\Delta t} \quad (3.6)$$

and

$$W_2 = \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{\Delta x^2} \quad (3.7)$$

where W_1 and W_2 denote errors related to $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ respectively.

It may be noted that in the evaluation of $\frac{\partial u}{\partial t}$, the dominant discretization error term is proportional to Δt^2 (see subsection 2.3.1) and the error due to measurement noise is proportional to $1/\Delta t$. Thus, if a very small value of Δt is chosen to minimize the discretization error, the error due to measurement noise becomes excessive. If, on the other hand, a very large value of Δt is chosen to minimize the measurement

noise error, the discretization error becomes excessive. A similar argument applies for approximating $\frac{\partial^2 u}{\partial x^2}$. This leads to the problem, considered in section 3.3, for the optimal selection of Δx and Δt such that the error in approximating the derivatives is minimum.

An equivalent expression for equation (3.2) in the presence of approximation error and measurement noise can be formulated by defining $\hat{\alpha}_k$ to be

$$\hat{\alpha}_k = \left(\frac{\hat{u}_{i,j+1} - \hat{u}_{i,j-1}}{2\Delta t} \right) \left(\frac{\Delta x^2}{\hat{u}_{i+1,j} - 2\hat{u}_{i,j} + \hat{u}_{i-1,j}} \right)_k \quad k = 1, 2, \dots, S \quad (3.8)$$

Notice that $\hat{\alpha}_k$ is a random variable since it is a function of random variables. The aim is to estimate the unknown parameter α given the sequence of the random variable $\hat{\alpha}_k$ ($k = 1, 2, \dots, S$).

3.2.3 General Mathematical Model

The ideas presented in the introductory example can be made more general. Thus, in anticipation of the requirements of the following subsections, a general mathematical model is derived in this subsection. The development is based on the assumption that equation (3.1) satisfies the following additional requirement.

(3i) The elements of H_k and g_k are linear in $u(x_i, t_j)_k$ and its partial derivatives.

A discussion of the complications arising in the nonlinear case is given in subsection (3.2.9).

When the pertinent elements of H_k and g_k are approximated by finite differences, equation (3.1) suggests the relationship

$$\hat{H}_k \hat{c}_k = \hat{g}_k \quad (3.9)$$

where \hat{H}_k and \hat{g}_k denote matrices whose elements are finite difference approximations of H_k and g_k as shown in equation (2.32) for a specific system. \hat{c}_k is a vector in lieu of c to ensure the equality in (3.9).

Let $o(H_k)$ and $o(g_k)$ denote the error matrices due to finite difference approximation; these error matrices are given by

$$\hat{H}_k = H_k + o(H_k) \quad (3.10)$$

$$\hat{g}_k = g_k + o(g_k) \quad (3.11)$$

Also let $\Delta \hat{c}_k$ denote the error vector in \hat{c}_k defined by

$$\hat{c}_k = c + \Delta \hat{c}_k \quad (3.12)$$

When the measurements of the state variables are corrupted by noise, equation (3.9) becomes,

$$\hat{H}_k \hat{c}_k = \hat{g}_k \quad (3.13)$$

where \hat{H}_k and \hat{g}_k denote the values of \hat{H}_k and \hat{g}_k when their elements are evaluated using noise corrupted data. \hat{c}_k is a vector in lieu of \hat{c}_k to ensure the equality in (3.13). Let $n(H_k)$ and $n(g_k)$ denote the error matrices due to noise, then

$$\hat{H}_k = \hat{H}_k + n(H_k) \quad (3.14)$$

$$= H_k + o(H_k) + n(H_k) \quad (3.15)$$

$$\hat{g}_k = \hat{g}_k + n(g_k) \quad (3.16)$$

$$= g_k + o(g_k) + n(g_k) \quad (3.17)$$

Define $\Delta \hat{c}_k$ to be the error vector in \hat{c}_k due to noise, thus:

$$\begin{aligned}\hat{\underline{c}}_k &= \underline{\tilde{c}}_k + \Delta \hat{\underline{c}}_k \\ &= \underline{c} + \Delta \underline{\tilde{c}}_k + \Delta \hat{\underline{c}}_k\end{aligned}\quad (3.18)$$

Thus the total error, $\Delta \hat{\underline{c}}_k$, in $\hat{\underline{c}}_k$ due to finite differences and noise is given by

$$\Delta \hat{\underline{c}}_k = \Delta \underline{\tilde{c}}_k + \Delta \hat{\underline{c}}_k \quad (3.19)$$

Substitution of (3.19) into (3.18) yields

$$\hat{\underline{c}}_k = \underline{c} + \Delta \hat{\underline{c}}_k \quad (3.20)$$

In anticipation of the requirements for the derivation of the least squares estimator, an expression for $\Delta \underline{\tilde{c}}_k$ is obtained here. Substituting for \hat{H}_k , \hat{g}_k and $\hat{\underline{c}}_k$ in (3.13) from equations (3.15), (3.17) and (3.20), yields

$$[H_k + o(H_k) + n(H_k)] [\underline{c} + \Delta \hat{\underline{c}}_k] = \underline{g}_k + o(\underline{g}_k) + n(\underline{g}_k) \quad (3.21)$$

In view of equation (3.1), equation (3.21) simplifies to

$$\Delta \hat{\underline{c}}_k = H_k^{-1} \{ [o(\underline{g}_k) + n(\underline{g}_k)] - [o(H_k) + n(H_k)] \underline{c} \} \quad (3.22)$$

provided \hat{H}_k is nonsingular.

3.2.4 Selection of the Estimator

There are several estimation schemes from which the most suitable must be selected. The classical techniques include least squares, least mean squares, maximum likelihood and Bayesian estimators. In addition to the classical methods, averaging is worthy of consideration due to its computational simplicity. Of all these approaches, the least squares estimator was found to be the most appropriate for the following reasons.

The least squares estimator [22] is a linear estimator which requires no knowledge of the statistics of the noise. However, when the moments of the noise generated by each transducer are known, it is shown in this thesis how the least squares estimator can frequently be modified to improve the accuracy of the approximation.

All of the other classical estimators require an a priori knowledge of the statistics of the noise vector. The noise vector is defined by returning to the general mathematical model. When equations (3.14) and (3.16) are substituted into (3.13) the results may be expressed by

$$\hat{H}_k \hat{c}_k = \hat{g}_k + [n(\hat{g}_k) + n(H_k) \hat{c}_k] \quad (3.23)$$

The noise vector is the $N \times 1$ column vector contained within the brackets on the right hand side of equation (3.23). A knowledge of the statistics of this vector requires an a priori knowledge of the transducer statistics. As was pointed out in the problem formulation this information may or may not be available; but even when the transducer statistics are known serious problems remain.

The elements of \hat{g}_k and H_k are linear partial derivatives by assumption (3i). When these elements are approximated by finite differences, the resulting equations are linear functions of the state variable. The noise is introduced when measurements of the state variables are taken. Since the finite difference expressions are linear, the elements of $n(H_k)$ and $n(\hat{g}_k)$ are linear functions of the random samples. Thus, the calculation of the density function of the elements of $n(\hat{g}_k)$ and $n(H_k)$ requires the calculation of a density function of a linear combination of random

samples taken from distributions with known density functions. This is usually not a simple task. To make the job even more difficult, once the density function of $n(H_k)$ is known, the density function of $n(H_k)\hat{c}_k$ must be determined. This requires some sort of initial estimate of \hat{c}_k be made before the density function of linear combinations of random variables with known density functions is calculated.

When the noise vector is known to have a white gaussian distribution, some interesting relationships exist between the estimators. Aoki [22] shows that the maximum likelihood estimator and the least squares estimator are identical, under the restriction that the V_k matrix (defined in Appendix A.1) is chosen to be the inverse of the covariance matrix. In this special case, the least squares estimator is also closely related to the least mean square estimator used in Kalman filtering. Sorenson [23] has shown that when the state variable in the Kalman filter is a constant, as is the \underline{c} vector in this thesis, the Kalman estimator is identical to the least squares estimator. Here again, the matrix V_k in the least squares estimator must be the inverse of the covariance matrix.

The Bayesian estimator requires the density function of the noise vector be known to estimate a random variable with a known density. Since \underline{c} is not a random variable this estimator appears to be an unreasonable choice for this problem.

In summary all of the classical estimators except the least squares estimator are poorly suited for estimating \underline{c} . In the case of the least mean square estimator and maximum likelihood estimator this is due to the

difficulty in determining the distribution of the noise vector. The Bayesian estimator, on the other hand, is intended to be used when \underline{c} is a random variable with a known density function. Finally, the averaging technique has been applied during the course of this investigation, but it rarely estimates the parameters as well as the least squares estimator. A comparison of the results using averages and the least squares estimator is given in section (3.4).

3.2.5 Least Squares Estimator

The least squares estimator is presented in this section for the scalar case. The matrix formulation for the least squares estimator appears in Appendix A.1. Examination of the scalar case is conducted first to allow introduction of basic ideas without the excessive algebra that necessarily accompanies manipulations with matrices.

The general mathematical model is utilized in the derivation of the least squares estimator. The notation presented in subsection 3.2.3 is altered to emphasize the fact that the development is being performed for the scalar case. In the scalar notation capital letters denoting matrices are replaced by small letters, and the underlining used to denote column vectors is omitted. Thus the scalar notation for the matrix H_k is h_k and g_k is the scalar form of the column vector \underline{g}_k .

The least squares estimate of c is by definition the value of ξ which minimizes the performance index

$$J_S(\xi) = \sum_{k=1}^S (\hat{h}_k \xi - \hat{g}_k)^2 v_k \quad (3.24)$$

where the v_k 's are positive real numbers.

The extremum of $J_S(\xi)$ is found by setting its derivative equal to zero. Thus

$$\frac{\partial J_S(\xi)}{\partial \xi} = \sum_{k=1}^S 2(\hat{h}_k \xi - \hat{g}_k) v_k \hat{h}_k = 0 \quad (3.25)$$

Let c_S^* denote the value of ξ which satisfies equation (3.25). Then

$$c_S^* = \frac{\sum_{k=1}^S \hat{h}_k v_k \hat{g}_k}{\sum_{k=1}^S \hat{h}_k^2 v_k} \quad (3.26)$$

That J_S is in fact a minimum at $\xi = c_S^*$, and not a maximum is apparent from the second derivative of J_S

$$\frac{\partial^2 J_S(\xi)}{\partial \xi^2} = 2 \sum_{k=1}^S \hat{h}_k^2 v_k \quad (3.27)$$

since \hat{h}_k^2 and v_k are nonnegative and if \hat{h}_k is nonzero for at least one k , $\partial^2 J_S(\xi) / \partial \xi^2$ is positive. Therefore, c_S^* , the extremum point, is a local minimum for $J_S(\xi)$.

In order to obtain a recursive relationship for the estimate c_S^*

define

$$p_S^{-1} = \sum_{k=1}^S \hat{h}_k^2 v_k \quad (3.28)$$

$$= p_{S-1}^{-1} + \hat{h}_S^2 v_S \quad (3.29)$$

$$\tilde{p}_S^{-1} = \sum_{k=1}^S \tilde{h}_k^2 v_k \quad (3.30)$$

$$= \tilde{p}_{S-1}^{-1} + \tilde{h}_S^2 v_S \quad (3.31)$$

and

$$\hat{p}_S^{-1} = \sum_{k=1}^S \hat{h}_k^2 v_k \quad (3.32)$$

$$= \hat{p}_{S-1}^{-1} + \hat{h}_S^2 v_S \quad (3.33)$$

In view of (3.32), equation (3.26) reads

$$\begin{aligned} c_S^* &= \hat{p}_S \sum_{k=1}^S \hat{h}_k v_k \hat{g}_k \quad (3.34) \\ &= \hat{p}_S \left(\sum_{k=1}^{S-1} \hat{h}_k v_k \hat{g}_k + \hat{h}_S v_S \hat{g}_S \right) \end{aligned}$$

Multiplying the first term in the brackets by $\hat{p}_{S-1}^{-1} \hat{p}_{S-1}$ and substituting for \hat{p}_{S-1}^{-1} from equation (3.33) yields

$$c_S^* = \hat{p}_S \left[(\hat{p}_S^{-1} - \hat{h}_S^2 v_S) \hat{p}_{S-1} \sum_{k=1}^{S-1} \hat{h}_k v_k \hat{g}_k + \hat{h}_S v_S \hat{g}_S \right]$$

But from equation (3.34)

$$\hat{p}_{S-1} \sum_{k=1}^{S-1} \hat{h}_k v_k \hat{g}_k = c_{S-1}^*$$

Hence

$$\begin{aligned} c_S^* &= \hat{p}_S (\hat{p}_S^{-1} c_{S-1}^* - \hat{h}_S^2 v_S c_{S-1}^* + \hat{h}_S v_S \hat{g}_S) \\ &= c_{S-1}^* + \Delta c_S^* \quad (3.35) \end{aligned}$$

where

$$\Delta c_S^* = \hat{p}_S \hat{h}_S v_S (\hat{g}_S - \hat{h}_S c_{S-1}^*) \quad (3.36)$$

The recursive relationship for estimating c given in equation (3.35) may be derived in other ways. Aoki [22] redefines $J_S(\xi)$ in terms of Δc_S^* and then minimizes the performance measure with respect to this quantity.

A similar approach was presented earlier by Ho [24].

A close examination of equation (3.36) reveals that as S becomes very large Δc_S^* approaches zero, since \hat{p}_S approaches zero and the remaining quantities on the right hand side of the equation are bounded. Also note that as long as \hat{h}_k and \hat{g}_k are finite for all k , c_S^* which minimizes J is also finite.

Though the recursive estimator provides insight into how the estimator is related to the estimate of the previous stage, application of this estimator requires more computation than does an alternate scheme presented in Appendix A.2. The computational savings is particularly marked when c is an N -vector and matrix operations must be performed.

3.2.6 Convergence of the Least Squares Estimator

In this section, the convergence of the least squares estimator for the scalar case is discussed. The results of the convergence study are compared with the true value of the parameter.

Substituting equation (3.20) into equation (3.13) and substituting the resulting expression for \hat{g}_k into equation (3.34) yields

$$c_S^* = \hat{p}_S \sum_{k=1}^S \hat{h}_k v_k \hat{h}_k (c + \Delta c_k) \quad (3.37)$$

Recalling the definition of \hat{p}_{S-1} in equation (3.32) the above expression simplifies to

$$c_S^* = c + \hat{p}_S \sum_{k=1}^S \hat{h}_k^2 v_k \Delta c_k \quad (3.38)$$

The second term on the right hand side of equation (3.38) is the error term resulting in the estimation of c at stage S . Let e_S denote this error.

In general, when the measurements are taken under the same circumstances, one measurement is no more accurate than any other measurement. Consequently, there is no reason to weight the measurements of any stage heavier by selecting a large v_k at that stage. From hereon it is assumed that for all k , v_k equals v , a constant. Using this assumption in conjunction with equations (3.22) and (3.15) e_S can be expressed as

$$e_S = \frac{\sum_{k=1}^S \{h_k + [o(h_k) + n(h_k)]\} \{[o(g_k) + n(g_k)] - [o(h_k) + n(h_k)]c\}}{\sum_{k=1}^S [h_k + o(h_k) + n(h_k)]^2} \quad (3.39)$$

Under normal circumstances the error e_S is not zero unless the noise and approximation errors are zero, in which case the numerator of equation (3.39) vanishes. However, the error is bounded when c_S^* is bounded because c is finite. In the next subsection e_S is approximated and a modified least squares estimator is presented which tends to compensate for e_S .

3.2.7 A Modified Least Squares Estimator

The least squares estimate is modified by adding a correction term which is an estimate of the error e_S . Since the denominator in equation (3.39) (the expansion of $[\hat{v}_S]^{-1}$) can be obtained by measurements the problem of estimating the denominator does not arise. An estimate for

the numerator which cannot be measured is now obtained. No claim to mathematical rigour is made in obtaining this estimate. The philosophy has been to get an estimate of e_s even if it is approximate in order to improve upon the estimate of c . The validity of the approximation is borne out by results of digital computer simulations.

In Chapter 2 results of the digital computer simulations show that the error in the estimate of a parameter due to discretization is small for the system dynamics considered. For these same systems, operating under the same conditions and using the same estimator, when measurement noise is added the estimation error increases dramatically. In the case of the diffusion equation, for example, when the noise to signal ratio is less than or equal to 0.01, the estimation error goes up to as high as sixty-eight percent. The introductory example in subsection (3.2.2) provides an explanation for this large change in percentage error. W_1 and W_2 , the noise errors in evaluating the derivatives, are at least of the order $1/\Delta t$ and $1/\Delta x$ whereas the discretization errors are, at most, of the order Δt^2 and Δx^2 . Since Δt and Δx are usually much smaller than one and in any case always less than one, the values of W_1 and W_2 can be large even if the noise, $w_{i,j}$, is small. For these reasons the discretization error terms in the numerator of equation (3.39) are assumed negligible. The following additional assumptions are also made.

(3j) The mean of the noise generated by each transducer is zero, and the variance known.

(3k) $n(h_k)$ and $n(g_k)$ have no common terms. Note that $n(h_k)$ and

$n(g_k)$ in the introductory example satisfy this assumption because the $w_{i,j+1}$ and $w_{i,j-1}$ in W_1 do not appear in W_2 .

(31) x is the only spatial variable in the partial differential equations.

(3m) The measurement noise and the state variables are uncorrelated.

Under the assumption that the discretization errors $o(h_k)$ and $o(g_k)$ are negligible, equation (3.39) becomes

$$e_S = \frac{\sum_{k=1}^S [h_k + n(h_k)] [n(g_k) - n(h_k)] c}{\sum_{k=1}^S [h_k + n(h_k)]^2} \quad (3.40)$$

If S is finite, the numerator and denominator can be divided by it to give the approximation

$$\begin{aligned} e_S &\approx \frac{\frac{1}{S} \sum_{k=1}^S [h_k + n(h_k)] [n(g_k) - n(h_k)]}{\frac{1}{S} \sum_{k=1}^S [h_k + n(h_k)]^2} \\ &= \frac{\frac{1}{S} \sum_{k=1}^S n(h_k) n(g_k) - \frac{c}{S} \sum_{k=1}^S n(h_k)^2 + \frac{1}{S} \sum_{k=1}^S h_k n(g_k) - \frac{c}{S} \sum_{k=1}^S h_k n(h_k)}{\frac{1}{S} \sum_{k=1}^S h_k^2 + \frac{2}{S} \sum_{k=1}^S h_k n(h_k) + \frac{1}{S} \sum_{k=1}^S n(h_k)^2} \end{aligned} \quad (3.41)$$

Consider the sequence of samples of noise generated by a transducer located at point x_i . When several samples have been taken (in other words S is finite but large) an unbiased estimator for the mean is given by

$$\bar{w}_i = \frac{1}{S} \sum_{j=1}^S w_{i,j} \quad (3.42)$$

$$= 0 \quad (3.43)$$

An unbiased estimator for the variance σ_i^2 of $w_i(t)$ is given by the approximation

$$\sigma_i^2 \approx \frac{1}{S} \sum_{j=1}^S w_{i,j}^2 \quad (3.44)$$

By assumption (3a) in section (3.1) $w_i(t)$ and $w_k(t)$ are statistically independent for $i \neq k$. Since $w_i(t)$ and $w_k(t)$ have zero mean, an unbiased estimator for the covariance of $w_i(t)$ and $w_k(t)$ is

$$\lambda = \frac{1}{S} \sum_{j=1}^S w_{i,j} w_{k,j} \quad (3.45)$$

But the covariance of two independent random variables is zero. Therefore

$$\frac{1}{S} \sum_{j=1}^S w_{i,j} w_{k,j} = 0 \quad (3.46)$$

The above approximations are useful in evaluating expressions containing $n(h_k)$ and $n(g_k)$ in (3.41) which are linear functions of $w_{i,j}$; that is

$$n(h_k) = \left(\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} a_{r,s} w_{r,s} \right)_k \quad (3.47)$$

$$n(g_k) = \left(\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} b_{r,s} w_{r,s} \right)_k \quad (3.48)$$

where the $a_{i,j}$'s and $b_{i,j}$'s are constants determined by the finite differences. $2Q+1$ and $2R+1$ are the number of transducers and samples specified by the difference equations. For illustration consider the introductory example in subsection 3.2.2 where the errors due to noise are given by equations (3.6) and (3.7). In this case $n(h_k)$ and $n(g_k)$

are w_1 and w_2 respectively and are rewritten here for convenience.

$$n(h_k) = \left[\left(\frac{1}{\Delta x^2} \right) w_{i+1,j} + \left(\frac{-2}{\Delta x^2} \right) w_{i,j} + \left(\frac{1}{\Delta x^2} \right) w_{i-1,j} \right]_k \quad (3.49)$$

$$n(g_k) = \left[\left(\frac{1}{2\Delta t} \right) w_{i,j+1} + \left(\frac{-1}{2\Delta t} \right) w_{i,j-1} \right]_k \quad (3.50)$$

A comparison of equation (3.49) with (3.47) shows $a_{i+1,j} = 1/\Delta x^2$, $a_{i,j} = -2/\Delta x^2$, $a_{i-1,j} = 1/\Delta x^2$ and $a_{r,s} = 0$ for other values of r and s . Similarly for equation (3.50) and (3.48) $b_{r,s}$ is zero with the exception of $b_{i,j+1} = 1/2\Delta t$ and $b_{i,j-1} = -1/2\Delta t$. As before, k denotes the k th set of i 's and j 's. Notice that even though the $w_{i,j}$'s are different for each k ($k = 1, 2, 3, \dots$) the $a_{r,s}$'s and $b_{r,s}$'s in equations (3.47) and (3.48) do not change.

Approximation (3.41) for e_S contains terms with $n(h_k)$ and $n(g_k)$ summed from 1 to S . These sums can be approximated in terms of the mean and variance of the individual transducer noise w_i .

When equation (3.47) is summed over the k 's and both sides are divided by S there results

$$\begin{aligned} \frac{1}{S} \sum_{k=1}^S n(h_k) &= \frac{1}{S} \sum_{k=1}^S \left(\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} a_{r,s} w_{r,s} \right)_k \\ &= \frac{1}{S} \sum_{k=1}^S \left(\sum_{s=j-R}^{j+R} a_{i-Q,s} w_{i-Q,s} \right)_k \\ &\quad + \frac{1}{S} \sum_{k=1}^S \left(\sum_{s=j-R}^{j+R} a_{i-Q+1,s} w_{i-Q+1,s} \right)_k + \dots \\ &\quad + \frac{1}{S} \sum_{k=1}^S \left(\sum_{s=j-R}^{j+R} a_{i+Q,s} w_{i+Q,s} \right)_k \end{aligned} \quad (3.51)$$

This expression may be approximated in terms of the statistics of the transducer noise. Let the points i, j corresponding to each k ($k = 1, 2, \dots, S$) be taken at the same spatial points, but separated in time by some multiple of the sampling interval, say $m\Delta t$. The reason for selecting the points in this manner is based upon practical considerations. The points may be chosen in any fashion as long as they differ in time and/or space. However, if points are selected so that they differ in space, additional transducers are required. On the other hand, displacing the points in time presents no problem. Each set of points uses measurements taken from a minimum number of transducers. Picking the points so that they are separated by several sampling intervals is advisable to insure that each set of measurements is unrelated to the previous set. However, when this is not practical, the sets of points can be separated by a single sampling interval.

When the k 's are selected in the above fashion, each term on the right hand side of (3.51) is nearly zero. For instance, consider the first term. Since the $a_{i,j}$'s are constants specified by the finite difference formulae and the $w_{i,j}$'s are random samples, the first term can be expressed as

$$\frac{1}{S} \sum_{k=1}^S \left(\sum_{s=j-R}^{j+R} a_{i-Q,s} w_{i-Q,s} \right)_k = \sum_{s=j-R}^{j+R} a_{i-Q,s} \frac{1}{S} \sum_{k=1}^S (w_{i-Q,s})_k$$

where the order of summation has been interchanged. In this case, the interchange of order of summation is valid provided the sum is a finite number. Since consecutive values of k denote a separation of $m\Delta t$ in time,

the summation with respect to k can be replaced by a summation with respect to s as follows

$$\begin{aligned}
 \frac{1}{S} \sum_{k=1}^S \left(\sum_{s=j-R}^{j+R} a_{i-Q,s} w_{i-Q,s} \right)_k &= \sum_{s=j-R}^{j+R} a_{i-Q,s} \frac{1}{S} \sum_{s=0}^{S-1} w_{i-Q,s+mS} \\
 &= a_{i-Q,j-R} \frac{1}{S} \sum_{s=0}^{S-1} w_{i-Q,j-R+mS} \\
 &\quad + a_{i-Q,j-R+1} \frac{1}{S} \sum_{s=0}^{S-1} w_{i-Q,j-R+1+mS} + \dots \\
 &\quad + a_{i-Q,j+R} \frac{1}{S} \sum_{s=0}^{S-1} w_{i-Q,j+R+mS}
 \end{aligned} \tag{3.52}$$

When $m\Delta t$ is large enough for the sequence of samples to be independent, each of the terms on the right hand side of equation (3.52) is approximately zero by expression (3.43). Thus, the first term of equation (3.51) is approximately zero. A similar argument may be used to show the remaining terms on the right hand side of equation (3.51) are approximately zero. Therefore,

$$\frac{1}{S} \sum_{k=1}^S n(h_k) \approx 0 \tag{3.53}$$

Similarly

$$\frac{1}{S} \sum_{k=1}^S n(g_k) \approx 0 \tag{3.54}$$

When terms of the form

$$\frac{1}{S} \sum_{k=1}^S n(h_k)^2 = \frac{1}{S} \sum_{k=1}^S \left(\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} a_{r,s} w_{r,s} \right)_k^2$$

are expanded, the results contain terms with $w_{r,s}^2$ as well as cross product terms in $w_{r,s}$. When these terms are summed over k , the cross product terms are approximately zero due to assumptions (3a) and (3b) leaving the approximation

$$\frac{1}{S} \sum_{k=1}^S n(h_k)^2 = \frac{1}{S} \sum_{k=1}^S \left(\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} a_{r,s}^2 w_{r,s}^2 \right)_k \quad (3.56)$$

Consider the expansion of the right hand side of (3.56) when $r = i-Q$. By applying the same arguments used in the paragraph above equation (3.52) it is possible to obtain the expression

$$\begin{aligned} \frac{1}{S} \sum_{k=1}^S \left(\sum_{s=j-R}^{j+R} a_{i-Q,s}^2 w_{i-Q,s}^2 \right)_k &= a_{i-Q,j-R}^2 \frac{1}{S} \sum_{s=0}^{S-1} w_{i-Q,j-R+m s}^2 \\ &+ a_{i-Q,j-R+1}^2 \frac{1}{S} \sum_{s=0}^{S-1} w_{i-Q,j-R+1+m s}^2 + \dots \\ &+ a_{i-Q,j+R}^2 \frac{1}{S} \sum_{s=0}^{S-1} w_{i-Q,j+R+m s}^2 \end{aligned} \quad (3.57)$$

But by (3.44) this expression may be approximated by

$$\begin{aligned} \frac{1}{S} \sum_{k=1}^S \left(\sum_{s=j-R}^{j+R} a_{i-Q,s}^2 w_{i-Q,s}^2 \right)_k &= a_{i-Q,j-R}^2 \sigma_{i-Q}^2 \\ &+ a_{i-Q,j-R+1}^2 \sigma_{i-Q}^2 + \dots + a_{i-Q,j+R}^2 \sigma_{i-Q}^2 \\ &= \sigma_{i-Q}^2 (a_{i-Q,j-R}^2 + a_{i-Q,j-R+1}^2 + \dots + a_{i-Q,j+R}^2) \end{aligned} \quad (3.58)$$

Therefore, equation (3.55) is given by the approximation

$$\frac{1}{S} \sum_{k=1}^S n(h_k)^2 = \sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} a_{r,s}^2 \sigma_r^2 \quad (3.59)$$

$$= \sigma^2 \quad (3.60)$$

where

$$\sigma^2 \triangleq \sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} a_{r,s}^2 \sigma_r^2 \quad (3.61)$$

Finally, in view of assumption (3k) and expression (3.46) the following approximation can be made

$$\frac{1}{S} \sum_{k=1}^S n(h_k) n(g_k) = \frac{1}{S} \sum_{k=1}^S \left(\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} a_{r,s} w_{r,s} \right)_k \left(\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} b_{r,s} w_{r,s} \right)_k \quad (3.62)$$

$$\approx 0 \quad (3.63)$$

Expressions (3.60) and (3.63) may be substituted into (3.41) with the result

$$e_S = \frac{-c\sigma^2 + \frac{1}{S} \sum_{k=1}^S h_k n(g_k) - \frac{c}{S} \sum_{k=1}^S h_k n(h_k)}{\frac{1}{S} \sum_{k=1}^S h_k^2} \quad (3.64)$$

The error term can be simplified further by considering terms of the type $\frac{1}{S} \sum_{k=1}^S h_k n(h_k)$. Since the state variable and the noise are assumed statistically independent, h_k and $n(h_k)$ are also statistically independent. Furthermore, the expected value of $n(h_k)$ is zero. This is shown by first taking the expected value of equation (3.53).

$$E[n(h_k)] = E \left[\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} a_{r,s} w_{r,s} \right]_k$$

$$= \left(\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} a_{r,s} E[w_{r,s}] \right)_k \quad (3.65)$$

where E denotes the expected value. By assumption (3j) the transducer means are zero. Therefore

$$E[h_k n(h_k)] = h_k E[n(h_k)]$$

$$= 0 \quad (3.66)$$

and

$$\frac{1}{S} \sum_{k=1}^S h_k n(h_k) \approx 0 \quad (3.67)$$

Approximation (3.67) follows from the fact that the expected value of each term in the expansion of the left hand side of approximation (3.67) is zero. Since S is a large number, the summation of terms in the left hand side of (3.67) is approximately zero. A similar argument may be made to show

$$\frac{1}{S} \sum_{k=1}^S h_k n(g_k) \approx 0 \quad (3.68)$$

Application of the results of approximation (3.67) and (3.68) to (3.64) gives

$$e_S = - \frac{c\sigma^2}{\frac{1}{S} \sum_{k=1}^S \hat{h}_k^2} \quad (3.69)$$

$$= - S\sigma^2 \hat{p}_S c \quad (3.70)$$

Expression (3.70) provides an estimate of e_S in terms of \hat{p}_S , the "variance" of the noise σ^2 , and the unknown parameter c . Since c is unknown, the best that can be done at stage S is to replace c by its modified estimate at stage $S-1$, thus

$$e_S \approx - S\sigma^2 \hat{p}_S c_{S-1}^{**} \quad (3.71)$$

where c_{S-1}^{**} is the modified least square estimate of c at the stage $S-1$.

As stated earlier, the modified least squares estimate is the least squared estimate minus the error, thus

$$\begin{aligned} c_S^{**} &= c_S^* - e_S \\ &= c_S^* + S\sigma^2 \hat{p}_S c_{S-1}^{**} \end{aligned} \quad (3.72)$$

with

$$c_1^{**} \triangleq 0 \quad (3.73)$$

It is shown in the next subsection that the accuracy of the modified least squares estimator is generally better than that of the least squares estimator. Furthermore, this scheme has been applied to several problems with significant success.

3.2.8: Convergence of the Least Squares Estimator and Modified Least Squares Estimator.

In this section, an approximate expression for the error term in the modified least squares estimator is derived. This result is then compared and contrasted with the error term developed previously for the least squares estimator.

When the error term, e_S , in the least squares estimator is approximated by the expression

$$e_S \approx -S\sigma^2 \hat{p}_S c \quad (3.74)$$

then equation (3.38) can be approximated by

$$c_S^* \approx c - S\sigma^2 \hat{p}_S c \quad (3.75)$$

Now applying (3.75) repeatedly to equations (3.72) and (3.73) gives the following result.

When $S = 2$

$$\begin{aligned} c_2^{**} &= c_2^* + 0 \\ &= c - 2v\sigma^2 \hat{p}_2 c \end{aligned}$$

and when $S = 3$

$$\begin{aligned} c_3^{**} &= c_3^* + 3v\sigma^2 \hat{p}_3 c_2^{**} \\ &= (c - 3v\sigma^2 \hat{p}_3 c) + 3v\sigma^2 \hat{p}_3 (c - 2v\sigma^2 \hat{p}_2 c) \\ &= c - (2)(3)(v\sigma^2)^2 \hat{p}_2 \hat{p}_3 c \end{aligned}$$

and in general

$$c_S^{**} = c \left[1 - S!(v\sigma^2)^{S-1} \prod_{k=2}^S \hat{p}_k \right] \quad (3.76)$$

The second term within the brackets of expression (3.76) can be approximated by assuming $n(h_k) \gg o(h_k)$ and by applying expressions (3.60) and (3.67) with the result

$$\begin{aligned} S! (v\sigma^2)^{S-1} \prod_{k=2}^S \hat{p}_k &= \frac{S! (v\sigma^2)^{S-1}}{\prod_{k=2}^S \left[v \sum_{\ell=1}^k h_\ell^2 \right]} \\ &= \frac{S! v^{S-1} (\sigma^2)^{S-1}}{v^{S-1} \prod_{k=2}^S \left[\sum_{\ell=1}^k h_\ell^2 + k\sigma^2 \right]} \\ &= \frac{S! (\sigma^2)^{S-1}}{\prod_{k=2}^S \left[\sum_{\ell=1}^k h_\ell^2 + k\sigma^2 \right]} \end{aligned} \quad (3.77)$$

For convenience of notation define A_2 to be the quantity on the right hand side of approximation (3.77). Then expression (3.76) is given

approximately by

$$c_S^{**} \approx c [1 - A_2] \quad (3.78)$$

This expression for the approximate error of c_S^{**} can be compared with that derived for c_S^* . First, however, e_S in expression (3.69) must be further approximated by once again assuming $n(h_k) \gg o(h_k)$ and applying approximations (3.60) and (3.67) to obtain

$$\begin{aligned} e_S &= - \frac{c\sigma^2}{\frac{1}{S} \sum_{k=1}^S [h_k + n(h_k) + o(h_k)]^2} \\ &= - \frac{\sigma^2 c}{\frac{1}{S} \sum_{k=1}^S h_k^2 + \sigma^2} \end{aligned} \quad (3.79)$$

Therefore, c_S^* can be approximated by the expression

$$c_S^* \approx c[1 - A_1] \quad (3.80)$$

where A_1 is defined by

$$A_1 \triangleq \frac{\sigma^2}{\frac{1}{S} \sum_{k=1}^S h_k^2 + \sigma^2} \quad (3.81)$$

The comparison of the convergence of c_S^{**} and c_S^* may now be made by examining A_1 and A_2 . To begin with, A_1 and A_2 are functions of the same parameters and these parameters are nonnegative. Both estimates are approximately equal to c when A_1 and A_2 are zero. This is reasonable for only σ^2 appears in the numerators of these expressions and as the variance of the noise becomes smaller the error decreases. When σ^2 increases without bound, A_1 and A_2 approach unity. Therefore, c_S^{**} and c_S^* can be

expected to lie somewhere between 0 and c.

Additional comparisons can be made by establishing bounds on h_k^2 . Assume the system is operating under transient conditions. Then h_k^2 can be expected to be bounded above and below by the positive real numbers U^2 and ϵ^2 . Therefore

$$\epsilon^2 \leq h_k^2 \leq U^2 \quad k = 1, 2, \dots, S \quad (3.82)$$

It is true, of course, that in a dynamic system h_k can be identically zero at a finite number of points in time and space. However, it may be assumed that this happens rather infrequently.

Bounds on A_1 and A_2 can now be found in terms of those on h_k^2 . Since the quantities in A_1 are all nonnegative, application of the inequality of (3.82) yields

$$\frac{\sigma^2}{\frac{1}{S} \sum_{k=1}^S U^2 + \sigma^2} \leq A_1 \leq \frac{\sigma^2}{\frac{1}{S} \sum_{k=1}^S \epsilon^2 + \sigma^2} \quad (3.83)$$

This inequality can be simplified further as follows

$$\frac{\sigma^2}{U^2 + \sigma^2} \leq A_1 \leq \frac{\sigma^2}{\epsilon^2 + \sigma^2}$$

$$\frac{1}{\frac{U^2}{\sigma^2} + 1} \leq A_1 \leq \frac{1}{\frac{\epsilon^2}{\sigma^2} + 1} \quad (3.84)$$

Similarly A_2 is bounded by

$$\frac{S!(\sigma^2)^{S-1}}{\prod_{k=2}^S \prod_{\ell=1}^k [U^2 + k\sigma^2]} \leq A_2 \leq \frac{S!(\sigma^2)^{S-1}}{\prod_{k=2}^S \prod_{\ell=1}^k [\epsilon^2 + k\sigma^2]} \quad (3.85)$$

which can also be further simplified. Therefore,

$$\frac{S!(\sigma^2)^{S-1}}{\prod_{k=2}^S k[U^2+\sigma^2]} \leq A_2 \leq \frac{S!(\sigma^2)^{S-1}}{\prod_{k=2}^S k[\varepsilon^2+\sigma^2]}$$

$$\frac{S!(\sigma^2)^{S-1}}{S!(U^2+\sigma^2)^{S-1}} \leq A_2 \leq \frac{S!(\sigma^2)^{S-1}}{S!(\varepsilon^2+\sigma^2)^{S-1}}$$

$$\frac{1}{(\frac{U^2}{\sigma^2} + 1)^{S-1}} \leq A_2 \leq \frac{1}{(\frac{\varepsilon^2}{\sigma^2} + 1)^{S-1}} \quad (3.86)$$

Reference to the inequalities of (3.84) and (3.86) shows that A_1 and A_2 lie between zero and unity as expected. The bounds on A_1 in (3.84) emphasize the fact that the error in the least squares estimator is, in general, nonzero when σ^2 is finite. From the upper bound it is seen that A_1 is small only when ε^2 is much larger than σ^2 . In contrast to this result, for a given set of bounds on h_k^2 and a specified variance, the bounds on A_2 are smaller than those on A_1 when S is greater than two. In fact, when ε^2 is nonzero, the upper bound on A_2 approaches zero as S becomes infinitely large. In general then, the modified least squares estimate can be expected to be more accurate than the least squares estimate.

3.2.9 Nonlinear Problems

The least squares estimator developed in subsection 3.2.5 may also be applied to the identification of systems described by nonlinear partial differential equations, provided that the equations are linear in c . In the nonlinear case, however, an examination of the convergence of the estimator is quite involved. The fact that the least squares

estimator in the linear case contains a nonzero error suggests that the error in the nonlinear problem is also nonzero. However, until an analytic expression for this error is found, there is no way to develop a modified least squares estimator.

The nature of the difficulties encountered with the identification of a nonlinear partial differential equation is illustrated by the equation

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 c = \frac{\partial u}{\partial t} \quad (3.87)$$

Notice equation (3.87) appears similar in structure to the diffusion equation already considered.

The formulation given in equation (3.1) is still applicable, as are equations (3.9) and (3.13) when noise and approximation errors are included. However, equations (3.10) and (3.14) are no longer valid. This is demonstrated by approximating $\partial^2 u / \partial x^2$ by finite differences and introducing noise to get

$$\begin{aligned} \hat{h}_k &= [(u_{xx} + o(u_{xx}) + n(u_{xx}))_k]^2 \\ &= [u_{xx}^2 + o(u_{xx})^2 + n(u_{xx})^2 \\ &\quad + 2\{u_{xx} o(u_{xx}) + u_{xx} n(u_{xx}) + o(u_{xx})n(u_{xx})\}]_k \end{aligned} \quad (3.88)$$

where $u_{xx} = \partial^2 u / \partial x^2$, $o(u_{xx})$ is the error in approximating u_{xx} and $n(u_{xx})$ is the noise term. The right hand side of equation (3.88) cannot be separated into terms containing only noise and approximation errors as was done in equation (3.15), since it has cross product terms containing u_{xx} , discretization error and the noise.

In order to apply the least squares estimate, equation (3.38), an expression for Δc_k is found by following the procedure used previously for the linear case. Thus, for the scalar case, the substitution of equation (3.20) into (3.13) yields

$$\hat{h}_k(c + \Delta c_k) = \hat{g}_k \quad (3.89)$$

Therefore, Δc_k is given by the expression

$$\Delta c_k = \frac{1}{\hat{h}_k} (\hat{g}_k - \hat{h}_k c) \quad (3.90)$$

When (3.17), (3.87) and (3.88) are substituted into (3.90) there results

$$\begin{aligned} \Delta c_k = \frac{1}{\hat{h}_k} \{ [o(g_k) + n(g_k)] - [o(u_{xx})^2 + n(u_{xx})^2 + 2u_{xx}o(u_{xx}) \\ + 2u_{xx}n(u_{xx}) + 2o(u_{xx})n(u_{xx})] c \}_k \end{aligned} \quad (3.91)$$

Substitution of equation (3.91) into (3.38) yields the least squares estimate

$$\begin{aligned} c_S^* = c + \hat{p}_S v. \sum_{k=1}^S \hat{h}_k \{ [o(g_k) + n(g_k)] - [o(u_{xx})^2 + n(u_{xx})^2 \\ + 2u_{xx}o(u_{xx}) + 2u_{xx}n(u_{xx}) + 2o(u_{xx})n(u_{xx})] c \}_k \end{aligned} \quad (3.92)$$

The estimation error could be written in terms of the statistics of the transducer noise by making use of approximations similar to those of subsection 3.2.6. The analysis is quite involved and is not carried out in this thesis. However, an examination of equation (3.92) reveals the following information about the approximate value of the identification error.

Since \hat{h}_k contains $n(u_{xx})^2$, terms with $n(u_{xx})^4$ appear in both the denominator and numerator. When $n(u_{xx})^4$ is evaluated using transducer

statistics, see equation (3.55), it will be approximated by terms which are a function of the fourth moment of the transducer noise. Furthermore, terms containing $n(u_{xx})^2$ and $n(u_{xx})$ will also be present. Therefore, a modified estimator would require a knowledge of the fourth moment about the mean in addition to the mean and variance of the transducer noise.

The error will also contain terms which are products of $o(u_{xx})$ and $n(u_{xx})^2$. These terms must be included in the expression for the approximate error. As a result, a modified scheme would require an estimate of $o(u_{xx})$. This problem is discussed briefly in subsection 3.3.1.

3.3 SELECTION OF OPTIMAL INCREMENTS IN TIME AND SPACE

In section 3.2.2 a dilemma is presented. The error due to approximating derivatives with finite differences is decreased by making the sampling interval and spacing between transducers as small as possible. However, the error resulting from measurement noise is reduced by increasing the sampling interval and transducer spacing. Hence, there is a tradeoff between errors due to noise and approximations. This suggests that there exists an optimal sample interval and transducer spacing which minimizes the combined effect of these two errors. The expressions for these optimal values turn out to be functions of the variance of the transducer noise and the parameter c . The transducer statistics may be determined in a straight forward manner by performing tests on the devices. However, c is the quantity which must be estimated and an a priori knowledge of this quantity negates the need to perform the identification.

Despite the need to make an initial estimate of c , the procedure is very useful. Selecting a combination of Δx and Δt , with no criteria other than physical constraints, can lead to inferior results. By using all information available to make an initial estimate of c and accepting rather large errors, an estimate of the optimal Δx and Δt can be obtained. Though this combination is not precisely the optimum, it will generally lead to a more accurate identification than can be obtained by simply guessing. For algebraic simplicity the scalar case is considered in this section and the matrix case is considered in Appendix A.3.

3.3.1 Index of Performance

For the problem considered in section 3.2 the aim is to find Δx and Δt such that $|\Delta c_k|$ is minimum. Since Δc_k is a random variable it is reasonable to minimize the index of performance

$$I_1 = E[\Delta c_k^2] \quad (3.93)$$

with respect to Δx and Δt . However, this is not a very convenient index of performance for the following reason.

When equation (3.15) is substituted into the scalar form of equation (3.22), Δc_k becomes

$$\Delta c_k = \frac{[o(g_k) + n(g_k)] - [o(h_k) + n(h_k)]c}{h_k + n(h_k) + o(h_k)} \quad (3.94)$$

Substituting equation (3.94) into equation (3.93) results in a very complicated expression to minimize. An examination of Δc_k reveals that the numerator of equation (3.93) has far greater effect on the

minimization of I_1 than does the denominator. This conclusion results from the observation that $n(h_k)$ and $o(h_k)$ are usually small with respect to h_k . Now if the minimization were performed with respect to the numerator of I_1 only, the problem would be made much more tractable.

Therefore, let the new performance index be given by

$$\begin{aligned}
 I_2 &= E\{[o(g_k) + n(g_k)] - [o(h_k) + n(h_k)]c\}^2 \\
 &= E\{[n(h_k)c - n(g_k)] + [o(h_k)c - o(g_k)]\}^2 \\
 &= E\{[n(h_k)c - n(g_k)]^2\} + [o(h_k)c - o(g_k)]^2 \\
 &\quad + 2[o(h_k)c - o(g_k)] E[n(h_k)c - n(g_k)] \quad (3.95)
 \end{aligned}$$

I_2 may be simplified further by making use of equations (3.47) and (3.48). When the expected value of $n(h_k)$ is taken there results

$$\begin{aligned}
 E[n(h_k)] &= E\left[\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} a_{r,s} w_{r,s}\right]_k \\
 &= \left[\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} a_{r,s} E[w_{r,s}]\right]_k
 \end{aligned}$$

But by assumption (3j) the mean of the noise generated by each transducer is zero. Therefore

$$E[n(h_k)] = 0 \quad (3.96)$$

Similarly for $n(g_k)$ taking the expectation results in

$$\begin{aligned}
 E[n(g_k)] &= E\left[\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} b_{r,s} w_{r,s}\right]_k \\
 &= \left[\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} b_{r,s} E[w_{r,s}]\right]_k \\
 &= 0 \quad (3.97)
 \end{aligned}$$

Substituting equations (3.96) and (3.97) into (3.95) reduces I_2 to the expression

$$I_2 = E[\{n(h_k)c - n(g_k)\}^2] + [o(h_k)c - o(g_k)]^2 \quad (3.98)$$

When equations (3.47) and (3.48) are substituted in the first term on the right hand side of equation (3.98)

$$E[\{n(h_k)c - n(g_k)\}^2] = E[\{\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} (a_{r,s}c - b_{r,s})w_{r,s}\}_k^2] \quad (3.99)$$

Since assumptions (3a) and (3b) state the $w_{i,j}$'s are statistically independent and the means are zero by assumption (3j)

$$E[\{n(h_k)c - n(g_k)\}^2] = [\sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} (a_{r,s}c - b_{r,s})^2 E(w_{r,s}^2)]_k \quad (3.100)$$

Equation (3.100) may be expressed in terms of the variance of the transducer noise. Assume the points i,j corresponding to each k , are separated in time. The justification for not changing the spatial points is presented in subsection 3.2.7. $E[w_{r,s}^2]$ at the k th point is known to be the variance of the noise generated by the transducer located at x_r . Therefore

$$E[\{n(h_k)c - n(g_k)\}^2] = \sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} (a_{r,s}c - b_{r,s})^2 \sigma_r^2 \quad (3.101)$$

Notice that the subscript k has been dropped because the quantities in the equation do not change as k takes on new values. Substituting equation (3.101) into (3.98) results in the following expression for I_2 .

$$I_2 = \sum_{s=j-R}^{j+R} \sum_{r=i-Q}^{i+Q} (a_{r,s}c - b_{r,s})^2 \sigma_r^2 + [o(h_k)c - o(g_k)]^2 \quad (3.102)$$

As was stated at the beginning of this section, σ_r^2 can be calculated and an initial estimate of c can be made. The problem remains to evaluate $o(h_k)$ and $o(g_k)$. The nature of the difficulties encountered in obtaining the finite difference error may be illustrated by returning once again to the introductory example.

The error terms for the central difference approximations of equations (2.17) and (2.14) are given respectively by the expressions

$$o(h_k) = \left[\left(\frac{\Delta x^2}{12} \right) \frac{\partial^4 u}{\partial x^4} \right]_k + \text{higher order terms in } \Delta x^2 \quad (3.103)$$

$$o(g_k) = \left[\left(\frac{\Delta t^2}{6} \right) \frac{\partial^3 u}{\partial t^3} \right]_k + \text{higher order terms in } \Delta t^2 \quad (3.104)$$

As long as Δt and Δx are less than one, the higher order terms in Δx^2 and Δt^2 are negligible with respect to the first term on the right hand side of (3.103) and (3.104) and may be neglected. Since Δx and Δt are known this leaves $[\partial^4 u / \partial x^4]_k$ and $[\partial^3 u / \partial t^3]_k$ to be evaluated for the points in time and space corresponding to $k = 1, 2, \dots, S$. This could be done by approximating the derivatives with finite difference equations and then performing tests on the system. However, in order to obtain $\partial^4 u / \partial x^4$ with accuracy to within an error of order Δx^2 , at least five transducers are necessary. Furthermore, the selection of Δx and Δt is an off-line process and the values of the derivatives for $k = 1, 2, \dots, S$ are not known a priori. Hence, in order to proceed further with the selection of Δx and Δt , it is necessary to replace $o(h_k)$ and $o(g_k)$ by some representative terms.

A simple technique has been applied with considerable success which by-passes the difficulty in determining the derivatives occurring in the

finite difference errors. The approach is to take only the functions of Δx and Δt which appear in the dominant error term. Returning to the introductory example, the errors $o(h_k)$ and $o(g_k)$ would be replaced by Δx^2 and Δt^2 , respectively.

When the performance index has been formulated for a particular system, I_2 is nonlinear in Δx and Δt . Furthermore, Δx and Δt must be constrained from above and below. The lower constraint on Δx is the physical size of the transducers while the maximum sampling rate determines the lower limit on Δt . However, these lower constraints are not as critical as the upper constraints.

As indicated in equations (3.103) and (3.104), the finite difference error terms contain higher order terms in Δx^2 and Δt^2 . If Δx and Δt are equal to or greater than one the errors are unbounded and the finite differences are no longer valid. Therefore, the upper constraint on Δx and Δt is one. It should be noted that physical considerations may override this upper constraint. For example, if the length of the material is one unit and three transducers are required, Δx can be no larger than 1/2 unit.

The application of the selection of optimal Δx and Δt to nonlinear problems suffers from the same type of difficulties encountered in formulating the modified least squares estimator in subsection 3.2.9. To demonstrate this consider the illustration used in that subsection.

Once again, the aim is to minimize $E[\Delta c_k^2]$ and the important minimization occurs in the numerator of this performance index. The

expression for the numerator of Δc_k is available in equation (3.91). Since the numerator contains u_{xx} in addition to the error terms resulting from finite differences and measurement noise, it too must be approximated. Thus, a point is reached where estimates of most of the parameters in the performance index must be made, and the accuracy of the optimization technique becomes very questionable.

Since the independent variables are constrained, the problem fits into the general category of a nonlinear programming problem. Since several texts, such as the one by Hadley [25], have been written on this subject, it will not be considered here. In the examples which follow, the upper constraints prove to be very critical in selecting the optimal Δx and Δt .

3.4 EXAMPLES AND RESULTS OF DIGITAL COMPUTER SIMULATIONS

Examples illustrating the identification of systems in the absence of measurement noise are presented in section 2.3. This portion of the thesis is devoted to examples demonstrating the estimation of system parameters in the presence of measurement noise. The least squares estimator and modified least squares estimators are used in all the examples. The accuracy of these estimators is compared with the results obtained by taking averages. The technique for selecting the optimal Δx and Δt is applied.

Before proceeding to the examples, some comments concerning the digital computer simulations used for this section are in order. The partial differential equations and associated boundary conditions in the

examples all have analytic solutions. The solutions were calculated on the digital computer to generate the values of the state variables at the desired points in space and time. The resulting data was accurate to the least four significant figures. Noise was then added to the state variable providing the data from which the estimation of the unknown constants in the original partial differential equation could be made.

The noise was obtained by using the output of a random number generator. The generator was a subroutine written for the digital computer. The numbers were random samples from a density function with a uniform distribution. The mean and variance of the distribution were known to be 0.5 and 1/12 respectively. However, these parameters were changed as desired by shifting the origin and scaling the abscissa. In all the examples which follow no effort was made to change the statistics from transducer to transducer. Hence, the noise added to the state variables in each example are random samples taken from the same distribution with the same probability density function.

3.4.1 Estimation of a Single Constant

The identification of α in the diffusion equation with boundary conditions given in subsections 2.3.1 and 2.3.2 is now considered. The system in subsection 2.3.1 starts from an initial state and proceeds to a specified steady-state condition. This system is referred to as the unforced example. The forced example in subsection 2.3.2 has boundary conditions which keep the system in a dynamic state. Before proceeding

to a discussion of the individual examples, some relationships are developed which apply to both.

The modified least squares estimator is given by equation (3.72).

All the parameters in the equation are known except σ_c^2 . Since the statistics of all the transducers are assumed to be the same,

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \dots = \sigma_c^2 \quad (3.105)$$

The utilization of (3.105) in equation (3.61) gives the results

$$\sigma^2 = \sigma_c^2 \frac{\sum_{s=j-1}^{j+1} \sum_{r=i-1}^{i+1} a_{r,s}^2}{\quad} \quad (3.106)$$

The $a_{r,s}$'s and $b_{r,s}$'s were determined in the paragraph following equation (3.50). When this information is substituted into equation

(3.106), σ^2 becomes

$$\begin{aligned} \sigma^2 &= \sigma_c^2 \left[\left(\frac{1}{\Delta x^2}\right)^2 + \left(\frac{-2}{\Delta x^2}\right)^2 + \left(\frac{1}{\Delta x^2}\right)^2 \right] \\ &= \frac{6\sigma_c^2}{\Delta x^4} \end{aligned} \quad (3.107)$$

Substituting equation (3.107) into equation (3.72) yields

$$c_S^{***} = c_S^* + \frac{6Sv\sigma_c^2 p_S}{\Delta x^4} c_S^{***} \quad (3.108)$$

where $c_1^{***} = 0$ in accordance with equation (3.73).

The optimal Δx and Δt minimize I_2 in equation (3.102) with Δx^2 and Δt^2 replacing $o(h_k)$ and $o(g_k)$. Substituting for the $a_{r,s}$'s and $b_{r,s}$'s in equation (3.102), the index of performance simplifies to

$$\begin{aligned} I_2 &= \sigma_c^2 \left[\left(\frac{c}{\Delta x^2}\right)^2 + \left(\frac{-2c}{\Delta x^2}\right)^2 + \left(\frac{c}{\Delta x^2}\right)^2 + \left(\frac{1}{2\Delta t}\right)^2 + \left(\frac{-1}{2\Delta t}\right)^2 \right] \\ &\quad + (\Delta x^2 c - \Delta t^2)^2 \end{aligned}$$

Hence

$$I_2 = \frac{6\sigma_c^2 c^2}{\Delta x^4} + \frac{\sigma_c^2}{2\Delta t^2} + (\Delta x^2 c - \Delta t^2)^2 \quad (3.109)$$

Consider now the unforced example. In order to find Δx and Δt that minimize I_2 it is necessary to know σ_c^2 and c . For this example, σ_c^2 was chosen to be 0.01/12. The value of c used in the generation of the data of the state variable was $1/\pi^2$. Under normal circumstances the initial estimate of c may be very poor. However, the actual value of c will be used in this section to illustrate the best possible improvement attainable with the proposed optimization technique. Thus, knowing σ_c^2 and c , I_2 becomes

$$I_2 = \frac{5 \times 10^{-3}}{(\pi \Delta x)^4} + \frac{.01}{24\Delta t^2} + \left[\left(\frac{\Delta x}{\pi} \right)^2 - \Delta t^2 \right]^2 \quad (3.110)$$

The solution for u in equation (2.23) was based upon a length of material of unity. Since the number of spatial measurements required by the finite difference formulae is three, the spacing between transducers can be no larger than 1/2 unit. When I_2 is minimized with respect to Δx and Δt with an upper constraint of 0.5 for Δx , the resulting optimal combination is $\Delta x = 0.5$ and $\Delta t = 0.27$. The fact that the optimal combination of Δx and Δt requires that Δx be as large as the bound permits is not unexpected. Arguments leading to the modified least squares estimator were based upon the assumption that errors due to noise are much larger than those resulting from finite difference approximations, when Δx and Δt are less than unity. Since the performance index was formulated to minimize the combined effect of the errors due to noise and finite difference approximations, the minimization will lead to a rather large value for Δx .

In the digital computer simulations the upper bound on Δx was reduced from 0.5 to 0.45. In an actual physical system of length unity, selecting $\Delta x = 0.5$ would require the placement of two transducers at each end of the material. If the transducers were thermocouples this would be impractical. The resulting optimal value for Δt with $\Delta x = 0.45$ was 0.27.

In this example the weighting function v in the least squares estimator was assumed to be unity. Since σ_c is known, the modified least squares estimator can be written

$$c_S^{**} = c_S^* + \frac{5 \times 10^{-3} \hat{S}_p}{\Delta x^4} c_{S-1}^{**} \quad (3.111)$$

The least squares estimate and the modified least squares estimate using the optimal Δx and Δt are given in Figure 3.1. The data was plotted by an IBM 1627 Model II and is accurate to within 0.01 inches. Also included in this figure is the identification of c by taking the mean of \hat{c}_k that is,

$$c_{AVE} = \frac{1}{S} \sum_{k=1}^S \hat{c}_k \quad (3.112)$$

An examination of the results of Figure 3.1 shows that the least squares estimator estimated α more accurately than did the modified least squares estimator. This appears to contradict the conclusions reached in subsection 3.2.8. However, these results are reasonable in light of the results of the digital computer calculations made in the example of subsection 2.3.1.

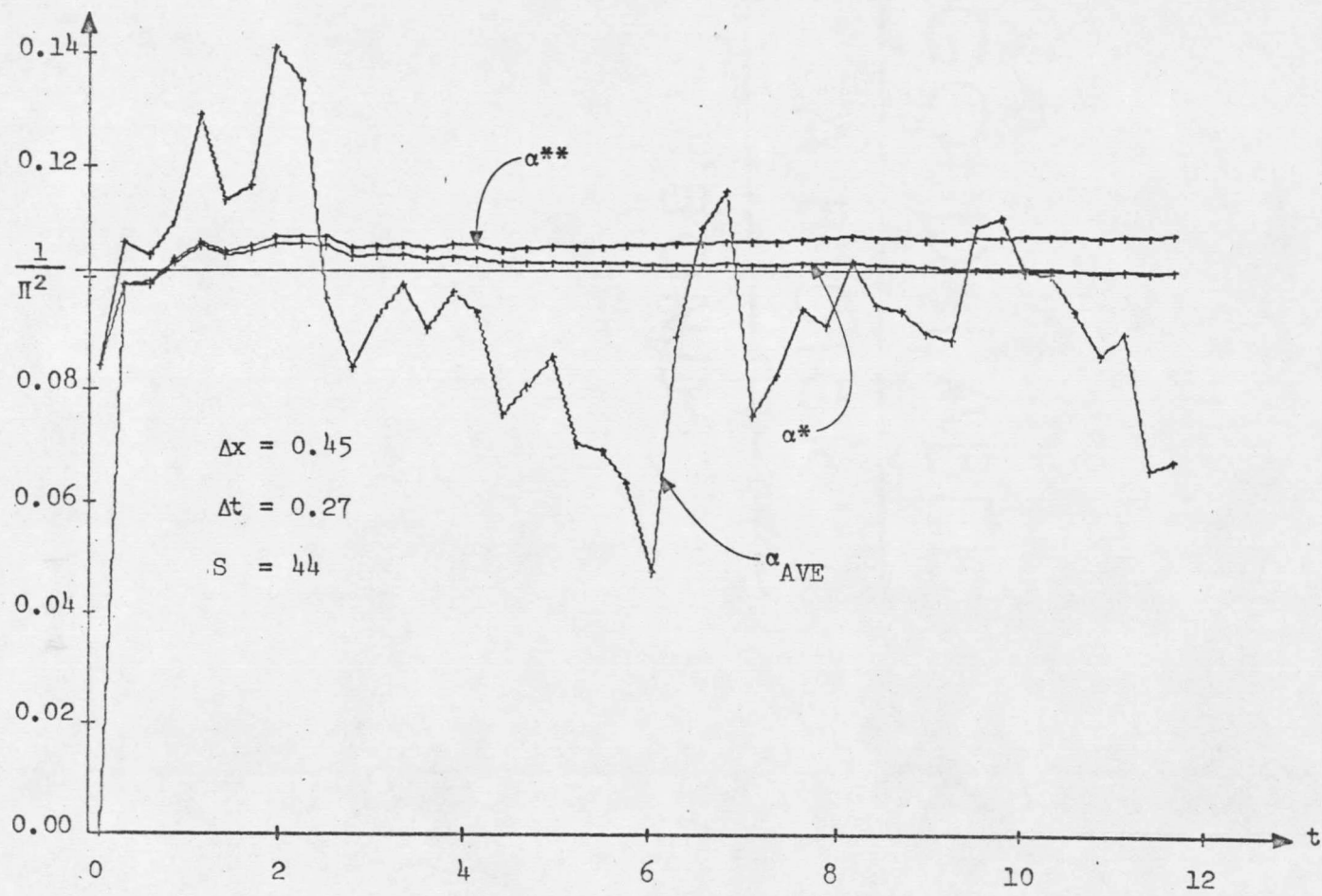


Figure 3.1 Estimation of α in the unforced example.

When the central difference formula of equation (2.19) is used to calculate $\hat{\alpha}$ the array of Figure 2.2 is generated. Notice the finite difference error of the calculations centered about $\Delta x = 0.5$ is always positive. Now even though the finite difference error at each stage is very small with respect to the error due to measurement noise, the cumulative effect of the finite difference error in this example was significant.

The dynamics in the unforced example are of a special type. Figure 2.1 shows that in the absence of measurement noise the dynamics are decaying monotonically with respect to time. The forced system, on the other hand has dynamics which more closely typify those encountered in distributed parameter systems. As a result, the finite difference error is at times positive and at other times negative.

The boundary conditions for the forced system are given in equations (2.25), (2.26) and (2.27). The analytic solution of the diffusion equation is found by the method of finite cosine transforms and appears in equation (2.29). The parameters selected for this example are $\alpha = 1$, $\omega = 0.6$, $\beta = 2/117$, $L = 1.0$ and $k = 8$. After equation (2.29) was calculated on the digital computer, measurement noise with a variance of $10^{-4}/12$ and zero mean was added.

As in the unforced example, the weighting function v is assumed to be unity in all calculations. Since σ_c^2 is known, the modified least squares estimator of equation (3.108) can be written

$$c_S^{**} = c_S^* + \frac{5 \times 10^{-5} \hat{S}_P S}{\Delta x^4} c_{S-1}^{**} \quad (3.113)$$

The initial estimate of α was chosen to be unity for the reason given earlier in this section. With the c and σ_c^2 selected, equation (3.109) becomes

$$I_2 = \frac{5 \times 10^{-5}}{\Delta x^4} + \frac{10^{-4}}{24 \Delta t^2} + (\Delta x^2 - \Delta t^2)^2 \quad (3.114)$$

Once again, the upper bound on Δx is 1/2 unit because $L = 1.0$.

However, due to the physical restrictions encountered in placing transducers at the boundary of a system, the upper bound on Δx was set at 0.45. For this upper bound the optimal Δx is 0.45 and the optimal Δt is 0.45.

A comparison of the accuracy using the least squares estimator and modified least squares estimator can be made by referring to the data plotted in Figure 3.2. The estimation error of the modified least squares estimator is less than 1/2 percent for 123 stages while the least squares error for the same number of stages was over 15 percent.

Digital computer simulations performed during the course of the investigations revealed a significant characteristic of the estimation scheme presented in this chapter. It was found that using finite difference formulae which decreased the approximation error resulted in increased estimation error when measurement noise was present. Thus the use of approximations of (2.16) and (2.18) rather than (2.14) and (2.17) in the estimation of α in the diffusion equation produces inferior results. The reason for this is apparent.

The use of additional measurements of the state variable in space and time reduces the error due to finite difference approximations.

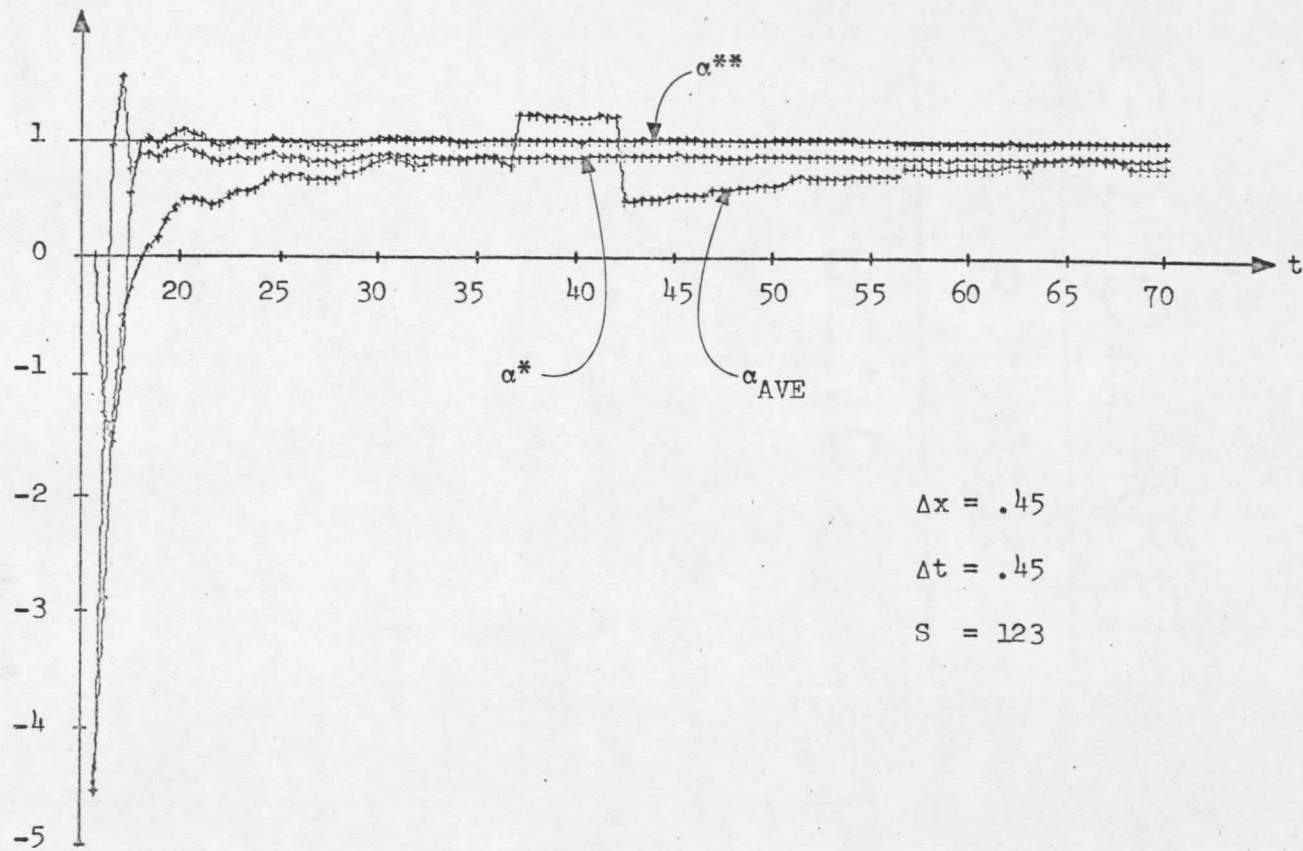


Figure 3.2 Estimation of α in the forced example.

However, these additional measurements introduce extra terms in the error resulting from measurement noise. This error is, therefore, made even larger than that resulting from finite differences.

3.4.2 Estimation of Boundary Conditions

This section is devoted to the identification of β in the boundary condition, equation (2.26), when measurement noise is added to the example of subsection 2.3.2. The estimation is performed with the least squares estimator and a modified least squares estimator.

The problem can be formulated in terms of the general mathematical model by letting

$$h_k = u(L, t)_k \quad (3.115)$$

and

$$g_k = \frac{\partial u}{\partial x}(L, t)_k \quad (3.116)$$

In this case x is fixed ($x = L$). Therefore, the subscript k denotes the k^{th} point in time.

Backward differences are used to approximate g_k in equation (3.116). To reduce the effect of measurement noise a finite difference formula utilizing three points in space is used instead of the five point approximation of subsection 2.3.2. The following finite difference approximation has an error of order Δx^2 .

$$g_k = \frac{1}{2\Delta x} (3u_{L,j} - 4u_{L-1,j} + u_{L-2,j})_k \quad (3.117)$$

Since h_k in equation (3.115) can be measured directly

$$o(h_k) = 0 \quad (3.118)$$

The addition of noise to the system results in the following expressions for $n(g_k)$ and $n(h_k)$.

$$n(g_k) = \frac{1}{2\Delta x} (3w_{L,j} - 4w_{L-1,j} + w_{L-2,j})_k \quad (3.119)$$

$$n(h_k) = (w_{L,j})_k \quad (3.120)$$

Consider the selection of the optimal Δx . Since $o(g_k)$, $n(g_k)$ and $n(h_k)$ are independent of Δt , it does not enter into the minimization of I_2 in equation (3.102), thus

$$I_2 = \sum_{r=L-3}^L (a_r c - b_r)^2 \sigma_r^2 + [o(h_k)c - o(g_k)]^2 \quad (3.121)$$

Since σ_r 's are identical, substituting the result of equation (3.118) into (3.121) yields

$$I_2 = \sigma_c^2 \sum_{r=L-3}^L (a_r c - b_r)^2 + o(g_k)^2 \quad (3.122)$$

In view of equations (3.47) and (3.48), the coefficients a_r and b_r are obtained from equations (3.119) and (3.120) as

$$a_L = 1, a_{L-1} = a_{L-2} = 0, b_L = \frac{3}{2\Delta x}, b_{L-1} = \frac{-4}{2\Delta x} \text{ and } b_{L-2} = \frac{1}{2\Delta x}$$

Hence, equation (3.122) can also be expressed as

$$\begin{aligned} I_2 &= \sigma_c^2 \left[\left(c - \frac{3}{2\Delta x}\right)^2 + \left(\frac{4}{2\Delta x}\right)^2 + \left(\frac{-1}{2\Delta x}\right)^2 \right] + o(g_k)^2 \\ &= \frac{\sigma_c^2}{4\Delta x^2} [(2c\Delta x - 3)^2 + 17] + o(g_k)^2 \end{aligned} \quad (3.123)$$

When $o(g_k)$ is approximated by Δx^2 in the dominant error term, the performance index is given approximately by the expression

$$I_2 \approx \frac{\sigma_c^2}{4\Delta x^2} [(2c\Delta x - 3)^2 + 17] + \Delta x^4 \quad (3.124)$$

The values of the parameters selected for this example are: $\alpha = \omega = 1$, $-\beta = -2/117$, $L = 1/2$, $v = 1$ and $k = 117/40$. The additive noise (again taken from the uniform distribution) had a variance of $0.01^2/12$ and a mean of zero. With the parameters selected, I_2 in expression (3.124) becomes

$$I_2 = \frac{(0.01)^2}{48\Delta x^2} \left[\left(\frac{-4\Delta x}{117} - 3 \right)^2 + 17 \right] + \Delta x^4 \quad (3.125)$$

When the minimization of I_2 was performed, the extremum occurred at $\Delta x = 0.18$. Thus, the measurements of the state variable were taken at $x_{L-2} = 0.14$, $x_{L-1} = 0.32$ and $x_L = 0.50$.

In order to derive a modified least squares estimator for this example, it is necessary to return to expression (3.41). Approximations (3.60), (3.67) and (3.68) are valid, but (3.63) is not. The reason approximation (3.63) does not apply is apparent from equations (3.119) and (3.120). Since both expressions contain $w_{L,j}$, $n(g_k)$ and $n(h_k)$ are correlated. Hence

$$\frac{1}{S} \sum_{k=1}^S n(h_k)n(g_k) = \frac{1}{S2\Delta x} \sum_{k=1}^S [(3w_{L,j} - 4w_{L-1,j} + w_{L-2,j})w_{L,j}]_k \quad (3.126)$$

By approximation (3.46) the summation of cross product terms in $w_{i,j}$ are nearly zero. As a result, expression (3.126) simplifies to

$$\frac{1}{S} \sum_{k=1}^S n(h_k)n(g_k) = \frac{3}{2\Delta x} \left[\frac{1}{S} \sum_{k=1}^S (w_{L,j})_k \right] \quad (3.127)$$

Since the variance of the noise generated by each transducer is σ_c^2 , approximation (3.44) may be substituted into (3.127) to give

$$\frac{1}{S} \sum_{k=1}^S n(h_k) n(g_k) = \frac{3\sigma_c^2}{2\Delta x} \quad (3.128)$$

Therefore equation (3.41) can be written

$$e_s = \hat{S}p_{S^2} v(-c\sigma^2 + \frac{3\sigma_c^2}{2\Delta x}) \quad (3.129)$$

where σ^2 is defined by equation (3.61). The modified least squares estimator is given by

$$c_s^{**} = c_s^* + \hat{S}p_{S^2} v(\sigma^2 c_{s-1}^{**} - \frac{3\sigma_c^2}{2\Delta x}) \quad (3.130)$$

Before the modified least squares estimator can be applied, σ^2 in equation (3.130) must be written in terms of σ_c^2 . In this example equation (3.61) is given by

$$\begin{aligned} \sigma^2 &= \sigma_c^2 \sum_{r=L-3}^L a_r^2 \\ &= \sigma_c^2 \end{aligned} \quad (3.131)$$

Substitution of equation (3.131) into (3.130) yields

$$c_s^{**} = c_s^* + \hat{S}p_{S^2} \sigma_c^2 (c_{s-1}^{**} - \frac{3}{2\Delta x}) \quad (3.132)$$

The result of estimating the boundary condition in the presence of noise is shown in Figure 3.3. In this example the modified least squares estimator and least squares estimator lie between 0 and β as predicted in subsection 3.2.8. The value of the estimators were nearly the same though the error in the estimation of β using the modified least squares estimator was 10.1 percent while the least squares estimate has an error of 12.4 percent.

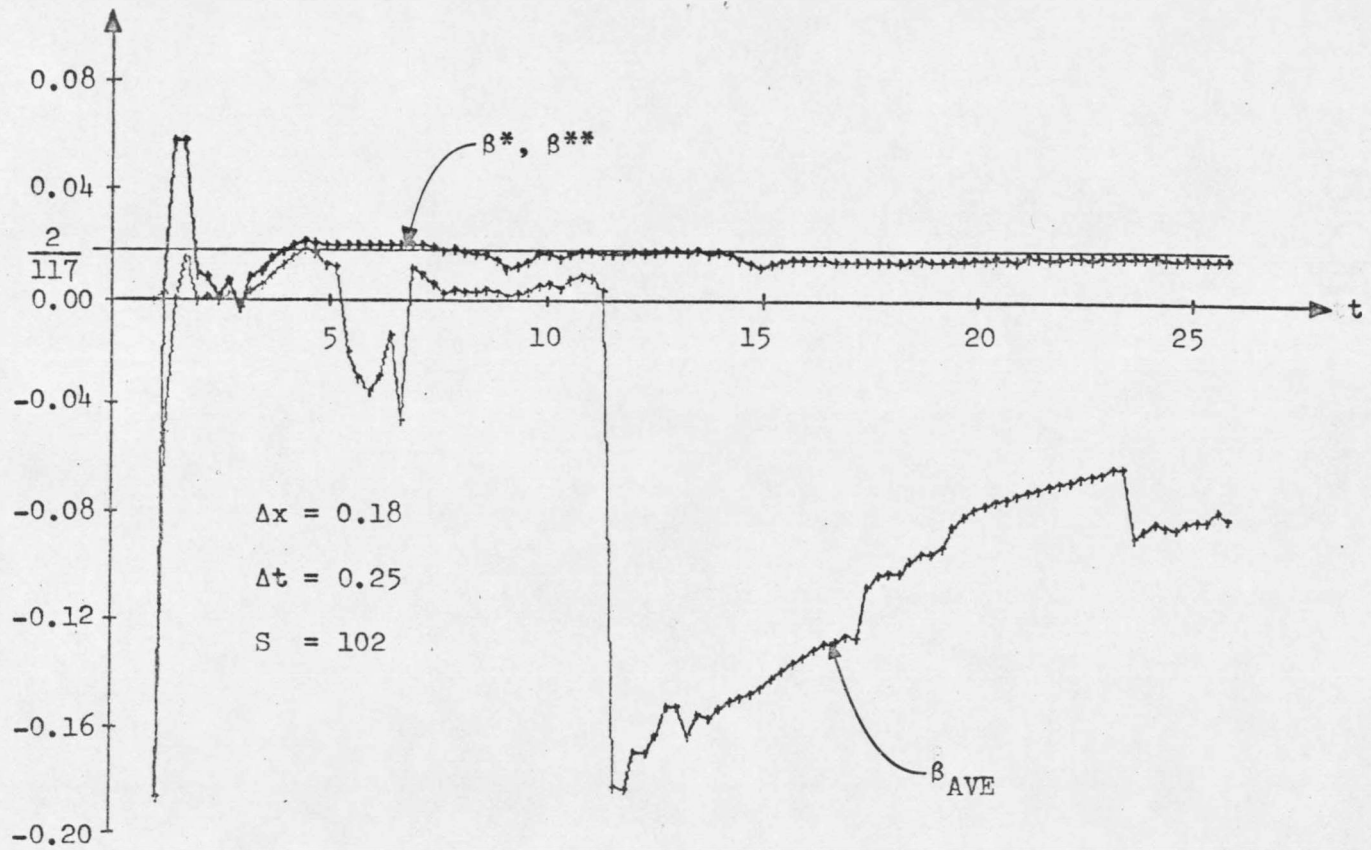


Figure 3.3 Estimation of β in the boundary conditions.

3.4.3 Estimation of Two Constants

3.4.3.1 Linear System

Consider once again the linear partial differential equation with two unknown constants given by equation (2.30). The identification of the unknown parameters was considered in example 2.3.3.1 when the measurements were known to be exact. In this section the parameters are estimated in the presence of measurement noise, using a least squares estimator and a modified least squares estimator.

A general development for the matrix modified least squares estimator is presented in Appendix A.1. The modified least squares estimator is given by equation (A.17). In the derivation the following assumptions are made: (1) The elements of H_k and g_k are linear partial derivatives; (2) $n(H_k)$ and $n(g_k)$ are independent in the probability sense; and (3) the mean of the transducer noise is zero.

Since the example under consideration satisfies all three of these conditions, the modified least squares estimator is established as soon as \hat{Q}_S in equation (A.17) has been determined.

Q_S is defined by equation (A.16) to be

$$Q_S = \sum_{k=1}^S n(H_k)^T V_k n(H_k) \quad (3.133)$$

Since there is no reason to weight one of the stages heavier than any other stage, it is assumed the matrix V_k ($k=1, 2, \dots, S$) is a constant. In this example V_k is taken to be the identity matrix; thus, equation (3.133) can be written

$$Q_S = \sum_{k=1}^S n(H_k) \hat{n}(H_k) \quad (3.134)$$

The first order derivatives in equation (2.30) were approximated by the central difference of (2.14) while (2.17) was used to approximate the second order derivatives. For this example, equation (2.30) was evaluated at points separated by Δt units in time. The addition of measurement noise to the finite difference equations results in the following expressions for the noise.

$$n(H_k) = \begin{bmatrix} \frac{1}{\Delta x^2} (w_{i+1,j} - 2w_{i,j} + w_{i-1,j}) & \frac{1}{2\Delta x} (w_{i+1,j} - w_{i-1,j}) \\ \frac{1}{\Delta x^2} (w_{i+1,j+1} - 2w_{i,j+1} + w_{i-1,j+1}) & \frac{1}{2\Delta x} (w_{i+1,j+1} - w_{i-1,j+1}) \end{bmatrix}_k \quad (3.135)$$

and

$$n(g_k) = \begin{bmatrix} \frac{1}{2\Delta t} (w_{i,j+1} - w_{i,j-1}) \\ \frac{1}{2\Delta t} (w_{i,j+2} - w_{i,j}) \end{bmatrix}_k \quad (3.136)$$

When the product $n(H_k) \hat{n}(H_k)$ is formed for $k = 1, 2, \dots, S$ and the results summed according to equation (3.134), the contribution of the cross product terms is negligible. This claim is based upon the relationship given in expression (3.46). However, consider the remaining terms contained in the elements of Q_S

$$q_{S11} = \left(\frac{1}{\Delta x^2} \right)^2 \sum_{k=1}^S [w_{i+1,j}^2 + (-2w_{i,j})^2 + w_{i-1,j}^2 + w_{i+1,j+1}^2 + (-2w_{i,j+1})^2 + w_{i-1,j+1}^2]_k \quad (3.137)$$

$$q_{S12} = q_{S21} = \frac{1}{2\Delta x^3} \sum_{k=1}^S [w_{i+1,j}^2 - w_{i-1,j}^2 + w_{i+1,j+1}^2 - w_{i-1,j+1}^2]_k \quad (3.138)$$

$$q_{S22} = \left(\frac{1}{2\Delta x}\right)^2 \sum_{k=1}^S [w_{i+1,j}^2 + (-w_{i-1,j})^2 + w_{i+1,j+1}^2 + (w_{i-1,j+1})^2]_k \quad (3.139)$$

where

$$Q_S = \begin{bmatrix} q_{S11} & q_{S12} \\ q_{S21} & q_{S22} \end{bmatrix} \quad (3.140)$$

If the transducer variance is again denoted by σ_c , expression (3.44) can be used to approximate the elements of Q_S with the result

$$\begin{aligned} q_{S11} &= \frac{2S\sigma_c^2}{\Delta x^4} [1 + 4 + 1] \\ &= \frac{12S\sigma_c^2}{\Delta x^4} \end{aligned} \quad (3.141)$$

$$\begin{aligned} q_{S12} &= q_{S21} = \frac{S\sigma_c^2}{2\Delta x^3} [1 - 1 + 1 - 1] \\ &= 0 \end{aligned} \quad (3.142)$$

$$\begin{aligned} q_{S22} &= \frac{2S\sigma_c^2}{4\Delta x^2} [1 + 1] \\ &= \frac{S\sigma_c^2}{\Delta x^2} \end{aligned} \quad (3.143)$$

In this example \hat{Q}_S , the approximation for Q_S , is given by

$$\hat{Q}_S = \begin{bmatrix} \frac{12S\sigma_c^2}{\Delta x^4} & 0 \\ 0 & \frac{S\sigma_c^2}{\Delta x^2} \end{bmatrix} \quad (3.144)$$

The optimal selection of Δx and Δt is based upon equation (A.30).

Since the samples are independent and are taken from a uniform distribution with variance σ_c^2 and zero mean

$$E[n(H_k)' n(H_k)] = \begin{bmatrix} \frac{12 \sigma_c^2}{\Delta x^4} & 0 \\ 0 & \frac{\sigma_c^2}{\Delta x^2} \end{bmatrix} \quad (3.145)$$

and

$$\begin{aligned} E[n(\underline{g}_k)' n(\underline{g}_k)] &= \left(\frac{1}{2\Delta t} \right)^2 E [w_{i,j+1}^2 + (-w_{i,j-1})^2 \\ &\quad + (w_{i,j+2})^2 + (-w_{i,j})^2]_k \\ &= \frac{\sigma_c^2}{\Delta t^2} \end{aligned} \quad (3.146)$$

Premultiplying equation (3.145) by \underline{c}' and postmultiplying by \underline{c} yields

$$\underline{c}' E[n(H_k)' n(H_k)] \underline{c} = \frac{12 \sigma_c^2 c_1^2}{\Delta x^4} + \frac{\sigma_c^2 c_2^2}{\Delta x^2} \quad (3.147)$$

Thus the first two terms on the right hand side of equation (A.30) have been determined leaving only the last term to be found. If the elements of $o(H_k)$ and $o(\underline{g}_k)$ are approximated by the dominant error term there results

$$o(H_k) \approx \begin{bmatrix} \Delta x^2 & \Delta x^2 \\ \Delta x^2 & \Delta x^2 \end{bmatrix} \quad (3.148)$$

and

$$o(\underline{g}_k) \approx \begin{bmatrix} \Delta t^2 \\ \Delta t^2 \end{bmatrix} \quad (3.149)$$

Then

$$[o(H_k)_c - o(g_k)] = \begin{bmatrix} (c_1 + c_2)\Delta x^2 - \Delta t^2 \\ (c_1 + c_2)\Delta x^2 - \Delta t^2 \end{bmatrix} \quad (3.150)$$

and

$$[o(H_k)_c - o(g_k)]' [o(H_k)_c - o(g_k)] = 2[(c_1 + c_2)\Delta x^2 - \Delta t^2]^2 \quad (3.151)$$

Substitution of equations (3.146) and (3.147) into (A.30) and application of approximation (3.151) yields the expression

$$I_A = \frac{12\sigma_c^2 c_1^2}{\Delta x^4} + \frac{\sigma_c^2 c_2^2}{\Delta x^2} + \frac{\sigma_c^2}{\Delta t^2} + 2[(c_1 + c_2)\Delta x^2 - \Delta t^2]^2 \quad (3.152)$$

The boundary conditions selected for this example were given earlier in equation (2.33). The data for the state variable was obtained by calculating equation (2.34) with the parameters $c_1 = 1/\pi^2$ and $c_2 = 2/\pi^2$. The noise added to this data had a variance $\sigma_c^2 = (0.05)^2/12$. These numbers were also substituted into expression (3.152) and the minimization of I_A was performed with respect to Δx and Δt . The optimum combination was found to be $\Delta x = 0.50$ and $\Delta t = 0.27$. However, for reasons given earlier, an upper limit of 0.30 was placed on Δx . The optimal Δt with this constraint on Δx is $\Delta t = 0.22$.

The results of the estimation of c_1 and c_2 are displayed in Figures 3.4 and 3.5 respectively. The data obtained using the least squares estimator and modified least squares estimator can be compared with the average, which is calculated from the expression

$$\bar{c}_{AVE} = \frac{1}{S} \sum_{k=1}^S \hat{H}_k^{-1} \hat{g}_k \quad (3.153)$$

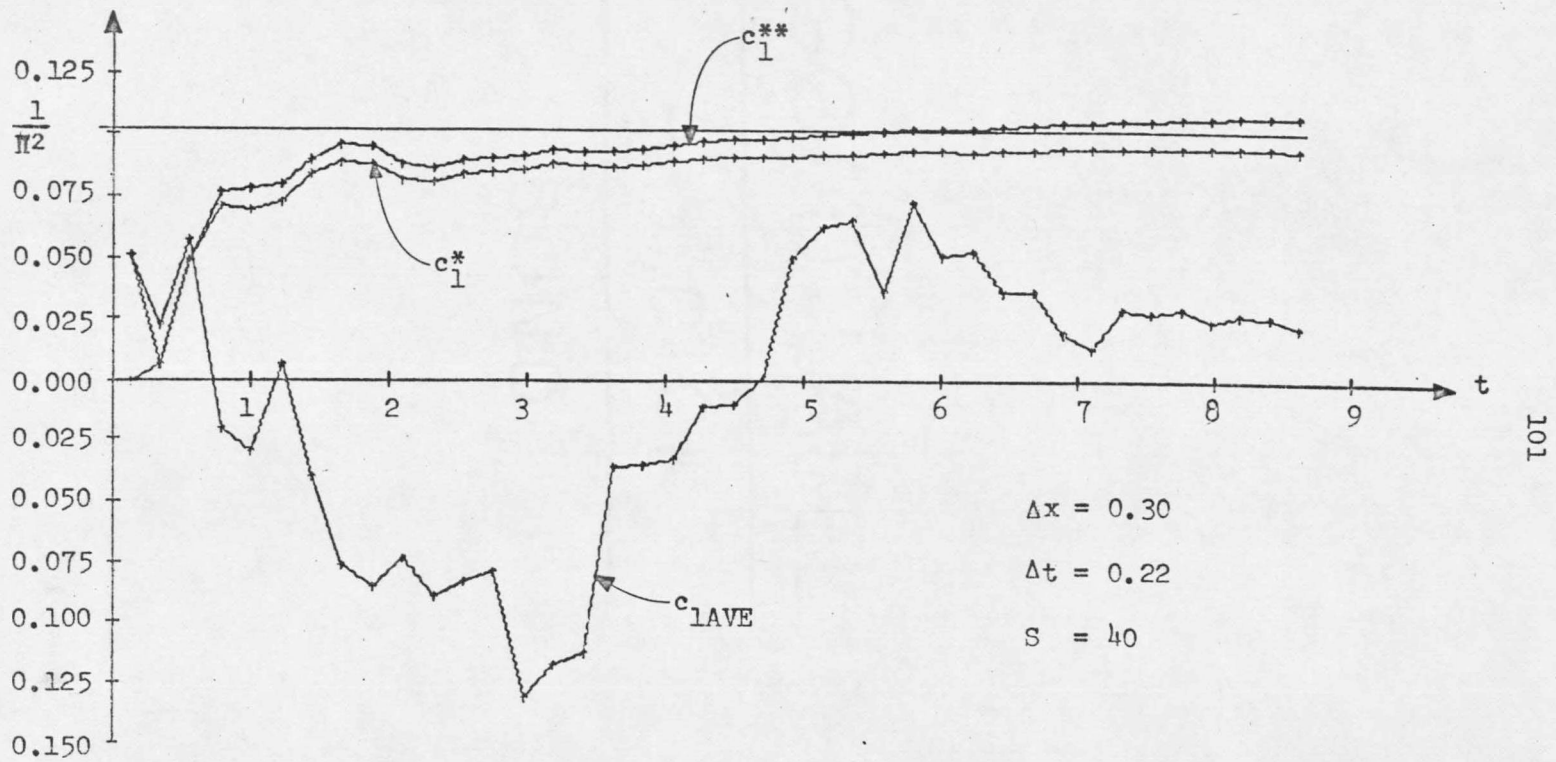


Figure 3.4 Estimation of c_1 in the linear system with two unknown coefficients.

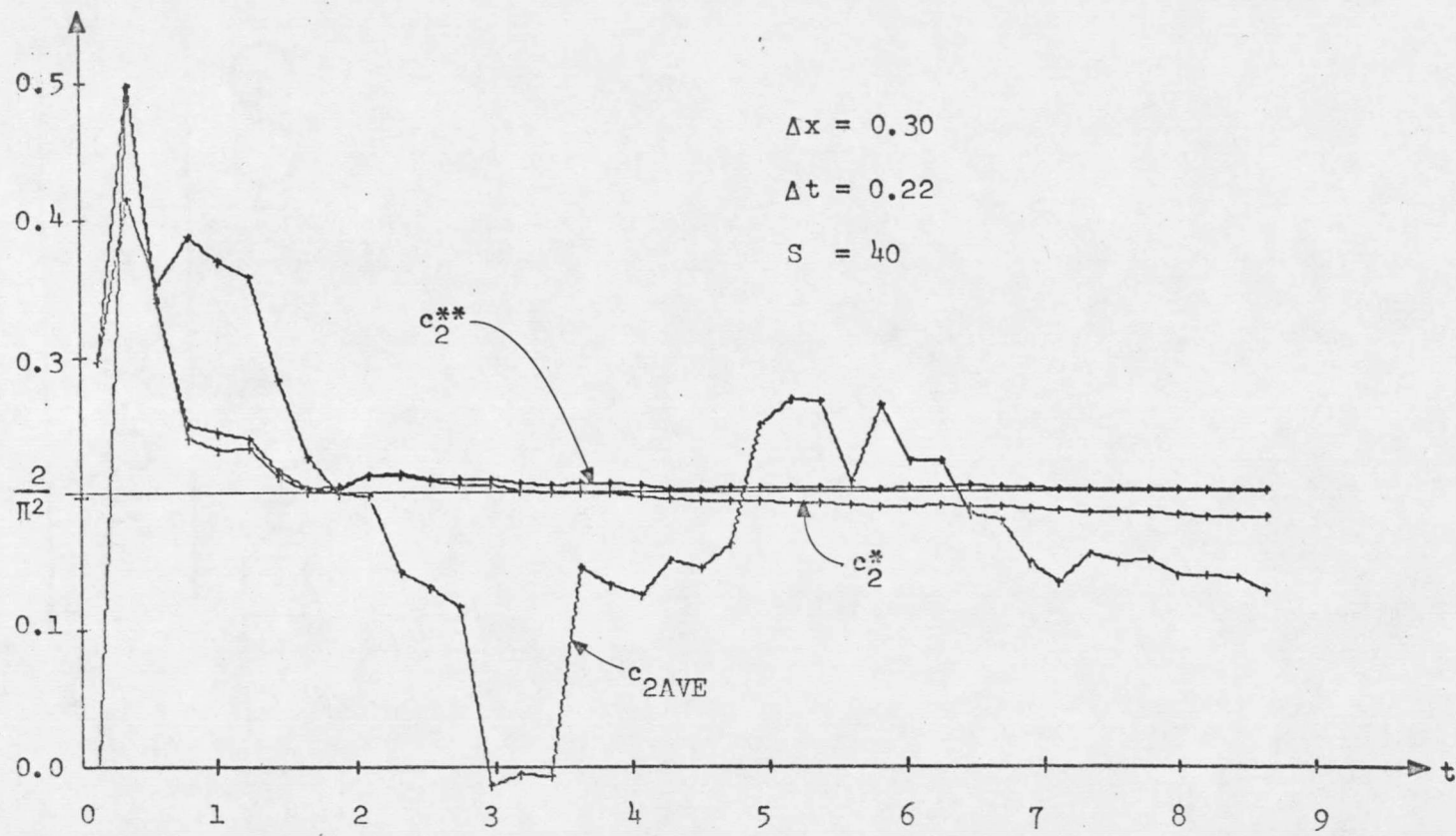


Figure 3.5 Estimation of c_2 in the linear system with two unknown coefficients.

3.4.3.2 Mathematical Model Containing an Extraneous Term

The identification of a system containing an extraneous term, when no measurement noise was present, was given in example 2.3.3.3. Now consider the estimation of the coefficient of the extraneous term of the same system when the measurements are corrupted by measurement noise.

The discussion of example 3.4.3.1 provides the relationships necessary to perform the estimation in this example, even though equation (2.30) is evaluated at points separated by Δx units in space rather than Δt units in time. This interchangeability is possible because the statistics of the measurement noise are assumed to be unchanged for the different transducers. Thus, the modified least squares estimator in this example is obtained by substituting \hat{Q}_S in equation (3.144) into equation (A.17). The selection of the optimal Δx and Δt may be performed by minimizing I_A in expression (3.152) with respect to these quantities. Notice that I_A in this problem reduces to the correct expression for the determination of the optimal Δx and Δt in the diffusion equation when c_2 is set equal to zero. The fact that (3.152) with $c_2 = 0$ is larger than (3.109) by a factor of two has no influence on the values of Δx and Δt which minimize the performance index.

The data was obtained by calculating equation (2.23) and adding noise with a variance of $(0.05)^2/12$. The optimal combination of Δx and Δt for this value of variance was calculated and found to be 0.3 and 0.21 respectively. When the estimation was performed the data

shown in Figure 3.6 resulted. The curves plotted for the modified least squares estimator and least squares estimator for c_2 were almost coincident. The average values for c_1 and c_2 were not shown because c_2 exceeded the range of the plot. c_{1AVE} ranged from 0.111 to -0.025 with the final estimate being 0.021. c_{2AVE} varied from 0.539 to -0.212 with 0.073 being the estimate at the last stage.

3.5 ILL-CONDITIONING

The concept of ill-conditioning was introduced in subsection 2.2.2. In this thesis ill-conditioning is understood to mean ill-conditioning with respect to inversion. When the matrix H_k in equation (3.1) is well-conditioned with respect to inversion, the introduction of error resulting from finite difference approximation and measurement noise has little effect on the inverse of H_k . If H_k is ill-conditioned, however, these errors are greatly exaggerated when the inverse is calculated. This section is devoted to demonstrating how ill-conditioning of H_k arises, what measures can be taken to minimize the incidence of ill-conditioning, what techniques are available to test H_k for ill-conditioning, and the significance of the ill-conditioning in the estimation problem.

The difficulties arising from ill-conditioning may be illustrated by the system

$$x_1 + x_2 = 2 \quad (3.154)$$

$$x_1 + 1.01x_2 = 2.01 \quad (3.155)$$

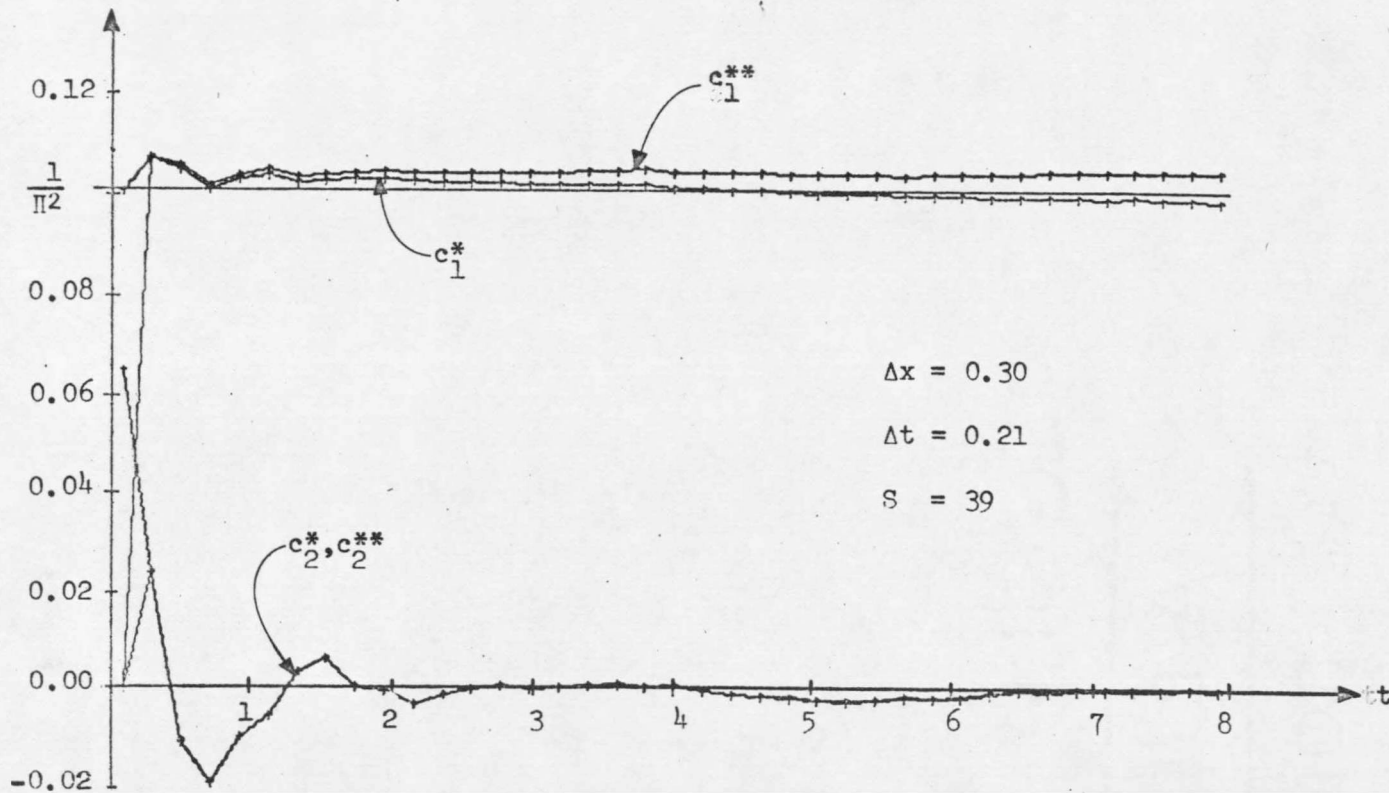


Figure 3.6 Estimation of c_1 and c_2 when c_2 is the coefficient of an extraneous term.

The solution of this pair of equations is $x_1 = x_2 = 1.0$. Now consider the change in the solution resulting from a very small change in the coefficients of equation (3.155). One such equation is given by

$$1.001x_1 + x_2 = 2.01 \quad (3.156)$$

The x_1 and x_2 satisfying equations (3.154) and (3.156) are 10 and -8 respectively. Thus, a change in the coefficients of equation (3.155) amounting to less than one percent resulted in at least a 900 percent change in the solution.

The circumstances illustrated in equations (3.154), (3.155) and (3.156) could occur in equation (3.1). This is easily demonstrated by returning to the system in example (2.3.3.1). When the matrix equation is written in terms of its elements

$$\begin{bmatrix} u_{xx1} & u_{x1} \\ u_{xx2} & u_{x2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} u_{t1} \\ u_{t2} \end{bmatrix} \quad (3.157)$$

where

$$u_{xxi} = \left(\frac{\partial^2 u}{\partial x^2} \right)_i \quad i = 1, 2 \quad (3.158)$$

$$u_{xi} = \left(\frac{\partial u}{\partial x} \right)_i \quad i = 1, 2 \quad (3.159)$$

$$u_{ti} = \left(\frac{\partial u}{\partial t} \right)_i \quad i = 1, 2 \quad (3.160)$$

If points 1 and 2 are selected in such a way that the dynamics change very little between them, the following relationships apply

$$u_{xx2} = u_{xx1} + \Delta u_{xx1} \quad (3.161)$$

$$u_{x2} = u_{x1} + \Delta u_{x1} \quad (3.162)$$

$$u_{t2} = u_{t1} + \Delta u_{t1} \quad (3.163)$$

where

$$|\Delta u_{xx1}| \ll |u_{xx1}|, \text{ etc.}$$

When equations (3.161), (3.162) and (3.163) are substituted into equation (3.157) the following pair of equations result

$$u_{xx1}c_1 + u_{x1}c_2 = u_{t1} \quad (3.164)$$

$$(u_{xx1} + \Delta u_{xx1})c_1 + (u_{x1} + \Delta u_{x1})c_2 = u_{t1} + \Delta u_{t1} \quad (3.165)$$

These equations are of the same type as equations (3.154) and (3.155) and are liable to be ill-conditioned.

Ill-conditioning in the more general case arises in exactly the same way as that demonstrated above. If points are selected in time and space very close together the dynamics change very little. Since the elements in each of the N columns of H_k contain the same partial derivatives evaluated at N points, the rows are very nearly the same. The only way this undesirable situation can be avoided is by spacing the points far enough apart that the dynamics change significantly. This, of course, can be done by selecting the points with large spatial separation and allowing several samples between points. However, separating the points in space requires additional transducers. This is not always possible for physical or economical reasons. Under these circumstances it will be necessary to make the separation in time only.

There are several ways of detecting ill-conditioning of H_k in

equation (3.1). Some of the more important indicators are [26]:

1. Small changes in the elements of H_k result in large changes in \underline{c} .
2. The determinant of H is very small.
3. Elements of H_k^{-1} are very large compared to elements of H_k .
4. If $H_k \underline{c} = \underline{g}_k$ is solved by placing the augmented matrix $\begin{bmatrix} H_k & \underline{g}_k \\ \hline & \end{bmatrix}$ in diagonal form, some elements along the main diagonal are very small compared to other elements along the diagonal.
5. When the eigenvalues of H_k are calculated, one or more of them is very small.

When a method is selected to detect ill-conditioning, one is usually interested in finding a procedure which can be performed readily on a digital computer. This is particularly true when the conditioning of H_k is in question, for the identification scheme presented in this thesis requires a digital computer to perform on-line computations.

Most of the tests for conditioning require a comparison of several quantities. After these comparisons have been completed, it is necessary to establish a rigid criteria as to whether or not the matrix is ill-conditioned. For example, in item 4 above, the relative size of all the main diagonal elements must be compared. The problem is to establish what discrepancy in size must exist before the matrix is declared to be ill-conditioned.

Tests involving the determinant of H_k circumvent this dilemma, for a single number results from the calculation. However, using the value of the determinant as a criteria for conditioning can be misleading.

This may be demonstrated by the system described in equations (3.154) and (3.155). The determinant of the coefficients on the left hand side of this set is 0.01. When equations (3.154) and (3.155) are multiplied by 100 the determinant becomes 100. Now, if the value of the determinant were the only criteria, the system having the determinant of 0.01 would appear to be ill-conditioned and that of 100 well-conditioned. This contradicts the fact that multiplication of a set of equations by a constant does not effect the solution for the unknown.

This problem of scaling can be eliminated by normalizing the system in some fashion. Conte [26] defines the normalized determinant of H_K by

$$\text{Norm}|H| = \frac{|H|}{\kappa_1 \kappa_2 \dots \kappa_N} \quad (3.166)$$

where

$$\kappa_i = [h_{i1}^2 + h_{i2}^2 + \dots + h_{iN}^2]^{1/2} \quad i = 1, 2, \dots, N \quad (3.167)$$

He states the criteria for ill-conditioning for the matrix H_K is

$$\text{Norm}|H| \ll 1 \quad (3.168)$$

When this test is applied to the system in equations (3.154) and (3.155).

$$\kappa_1 = \sqrt{2} \quad \text{and} \quad \kappa_2 = \sqrt{2.0201} \quad (3.169)$$

Therefore

$$\text{Norm}|H| = \frac{0.01}{\kappa_1 \kappa_2} = 0.005 \quad (3.169)$$

Hence the set is ill-conditioned.

This criteria is very easily mechanized on the digital computer. The test appears to be quite reliable, though the calculation of $\text{Norm}|H_K|$ is rather lengthy when the dimensions of the matrix are large.

Ill-conditioning of H_k is of most concern when \underline{c} is estimated directly from equation (3.13). Since

$$\hat{\underline{c}}_k = \hat{H}_k^{-1} \hat{\underline{g}}_k \quad (3.170)$$

errors in H_k resulting from finite difference approximations and measurement noise cause large errors in $\hat{\underline{c}}_k$. However, when the recursive schemes of section 3.2 are used, the errors resulting from ill-conditioning are not serious as long as H_k is well-conditioned for almost all k .

This claim is supported by recalling that all the recursive schemes require a matrix inverse to be calculated in \hat{P}_S^{-1} only. But \hat{P}_S^{-1} is defined by equation (A.6) to be

$$\hat{P}_S^{-1} = \sum_{k=1}^S \hat{H}_k^{-1} \hat{V}_k \hat{H}_k^{-1} \quad (3.171)$$

$$= \hat{P}_{S-1}^{-1} + \hat{H}_S^{-1} \hat{V}_S \hat{H}_S^{-1} \quad (3.172)$$

Now if \hat{H}_S is ill-conditioned and \hat{P}_{S-1}^{-1} is well-conditioned the sum of equation (3.172) will be well-conditioned. Since some of the \hat{H}_k 's in equation (3.171) are assumed to be well-conditioned, \hat{P}_{S-1}^{-1} can be expected to be well-conditioned.

When the H_k 's are all ill-conditioned, the above argument is no longer valid. However, the material on ill-conditioning presented by Bellman and Kalaba [27] suggests the introduction of a type of Lagrange multiplier to improve the conditioning of P_S . This prompted this author to consider the minimization of the performance index

$$J_S''(\underline{\xi}) = \sum_{k=1}^S [\|\hat{H}_k \underline{\xi} - \hat{\underline{g}}_k\|_{V_k}^2 + \lambda_k \|\underline{\xi} - \underline{c}_{k-1}^*\|^2] \quad (3.173)$$

where λ_k is a positive number and the remaining notation is described in Appendix A.1. The minimization of equation (3.173) is performed in exactly the same manner as the minimization of equation (A.1) with the result

$$\underline{c}_S^* = \hat{P}_S^{-1} \sum_{k=1}^S \hat{H}_k^T \hat{V}_k^{-1} \hat{g}_k \quad (3.174)$$

where \hat{P}_S^{-1} is defined by the expression

$$\hat{P}_S^{-1} = \sum_{k=1}^S (\hat{H}_k^T \hat{V}_k^{-1} \hat{H}_k + \lambda_k I) \quad (3.175)$$

$$= \hat{P}_{S-1}^{-1} + \hat{H}_S^T \hat{V}_S^{-1} \hat{H}_S + \lambda_S I \quad (3.176)$$

Notice that equations (3.174) and (A.5) are identical in form but \hat{P}_S^{-1} is defined in a slightly different fashion. The recursive relationships for \underline{c}_S^* in terms of \underline{c}_{S-1}^* are likewise identical in form and are given in equation (A.8).

The difference in the definitions of \hat{P}_S is very significant. Bellman and Kalaba state that even though $\sum_{k=1}^S \hat{H}_k^T \hat{V}_k^{-1} \hat{H}_k$ is ill-conditioned the addition of the terms $\sum_{k=1}^S \lambda_k I$ reduces the ill-conditioning significantly. Unfortunately, in the systems simulated by the author, the addition of λ_k increased the error in the identification. The reason for this result is believed to stem from the fact that only some of the matrices H_k , $k = 1, 2, \dots, S$ were found to be ill-conditioned. If a system could be simulated where all the H_k 's were ill-conditioned the estimator of equation (3.174) might very well provide better results than those of equation (A.5).

3.6 SUMMARY

The identification of unknown constants in partial differential equations when measurements of the state variables are corrupted by noise is considered in Chapter 3. The classical least squares estimator is presented. The least squares estimator requires no a priori knowledge of the statistics of the measurement noise. When the statistics are known and the differential equation is linear, a modified least squares estimator may be used. It is shown that the estimation error of the modified least squares estimator is generally less than the least squares estimator estimation error. This claim is supported by the results of digital computer simulations.

The magnitude of errors resulting from measurement noise and finite difference approximation are functions of the increments in time and space used in the finite difference formulae. These increments must be less than unity to insure that the finite difference errors are bounded. Because of this upper bound, the errors resulting from measurement noise are of far greater significance than errors arising from finite differences. The use of more accurate finite difference formulae requiring additional measurements of the state variable increases the estimation error. An index of performance is presented which is minimized with respect to the temporal and spatial increments. The minimization reduces the combined effects of the two types of errors.

The ill-conditioning problem introduced in Chapter 2 is discussed in further detail. It turns out ill-conditioning did not present any

difficulties in the examples considered. However, a performance index is given which can be used when conditioning is a problem.

CHAPTER 4

DISCUSSION AND CONCLUSIONS

4.1 SYSTEMS WITHOUT MEASUREMENT NOISE

A method has been presented to identify the partial differential equation and the associated boundary conditions describing a distributed parameter system. The method requires that the form of the partial differential equation and the boundary conditions be known up to a set of constant parameters which are to be determined. When the precise form of the equation is not known a priori, extraneous terms may be included. It is shown in Appendix C that the coefficients of extraneous terms are identified as zeroes if there is no approximation error or measurement noise. When measurement noise and approximation error are small the identification technique yields negligible values for the coefficients of extraneous terms. An identifiability condition is also given.

The identification procedure makes use of finite differences to approximate the derivatives at specified points in time and space. In order to identify the coefficients of a partial differential equation, a knowledge of the associated boundary conditions is not necessary. Similarly, in order to identify the coefficients in the boundary conditions a knowledge of the partial differential equation is not necessary. The identification scheme can be used to identify the differential equation and the boundary conditions simultaneously. The identification scheme requires normal operating data and can be used for on-line identification.

It is necessary to have dynamics with respect to all spatial variables to make a complete identification of the partial differential equation.

If there were no dynamics with respect to one of the spatial variables, say x , all terms containing derivatives of the state variable with respect to x would vanish. Under these circumstances it would be impossible to identify unknown constants associated with these terms.

Approximation of partial derivatives by finite differences introduces limits on the accuracy of the identification. This limitation is also encountered in numerical solution of partial differential equations by finite differences. In both problems the accuracy of the finite difference formulae is improved in three ways. (1) Reduction of discretization interval, (2) use of higher order approximations for derivatives, and (3) use of high accuracy formulae. Since the numerical solution of partial differential equations are sequential or recursive in nature, stability of the formulae used for approximating the system equations is necessary. This stability requirement rules out the use of some difference formulae for numerical solution of partial differential equations. Since the identification schemes presented here is a single stage process, stability is of no concern and some of these very high accuracy formulae can be employed advantageously.

Experimental results for identification of linear and nonlinear partial differential equations and boundary conditions were obtained using digital computer simulations. The results are favorable. An examination of the experimental results presented in Chapter 2 reveals that the approximation error is increased when the rate of change of the state variable with respect to time or space is either very large

or very small. A possible method for reducing this effect is an adaptive identification. That is, sample the measurements at a high rate and perform the identification only after the change of state variable lies within determined bounds. This would tend to eliminate data taken in regions where the results are known to be poor.

4.2 SYSTEMS WITH MEASUREMENT NOISE

The identification of unknown coefficients when the measurements of the state variables are corrupted by noise has been discussed. The measurements are sampled at a predetermined rate as required for the finite difference formulae. The noise produced by the transducers is assumed to be stationary and uncorrelated. Furthermore, the noise introduced in the samples for a given transducer are also assumed to be statistically independent.

The measurement noise introduces randomness in the identification process. Consequently, an estimate of the unknown parameters is made. Of the classical estimators the least squares estimator was found to be best suited for the problem. The other schemes were eliminated because they require an a priori knowledge of the statistics of the noise vector. It is explained why the noise vector statistics are very difficult to find even when the statistics of the measurement noise are available. The least squares estimator requires no knowledge of the noise vector statistics. However, when the mean and variance of the measurement noise are available, a modified least squares estimator can be applied to linear systems. The convergence of the two estimators is compared,

and the accuracy of the modified scheme is shown to generally be superior. Simple averaging is also considered because fewer calculations are required than for the other estimators. However, the simulation results show that the accuracy obtained by averaging is generally inferior. The least squares estimator can be used to identify partial differential equations which are nonlinear in the state variable. The accuracy of the estimation can be expected to be poorer than that obtained in linear systems. Furthermore, the application of the modified least squares estimator to the nonlinear problem is of questionable value.

The errors in the identification resulting from measurement noise are much larger than those originating from finite difference approximations. The use of higher order approximations for derivatives decreases the finite difference error. However, the higher order approximations require additional measurements of the state variable and additional terms are introduced into the expressions for measurement noise. As a result, the errors due to measurement noise are increased. It should be pointed out that this characteristic of the estimation is advantageous. Finite difference equations with low order approximations require fewer measurements of the state variable and, hence, the number of measurement transducers is kept to a minimum.

A method for minimizing the combined effect of errors due to finite difference approximation and measurement noise is presented. The optimal combination of increments in time and space is determined by minimizing a performance index with respect to these quantities. The spatial and

temporal increments have an upper constraint of unity and lower bounds established by physical considerations. Hence, the performance index minimization fits into the general problem of nonlinear programming. The technique is developed for linear systems and is poorly suited for nonlinear systems. The application to linear systems is demonstrated.

The proposed identification requires the calculation of the inverse of a square matrix. It is shown that the matrix may be ill-conditioned with respect to inversion. The best way to avoid ill-conditioning is to separate the measurements in time and space when generating the rows of \hat{H} and \hat{g} . This, however, is not always possible so methods of detecting ill-conditioned matrices are presented. No problems associated with ill-conditioning were observed in the digital computer simulations. However, a technique is given to improve the accuracy of estimation when the effects of ill-conditioning are significant.

CHAPTER 5

AREAS OF FUTURE RESEARCH

5.1 ESTIMATION OF PARAMETERS WHICH CHANGE WITH TIME AND/OR SPACE

In the absence of measurement noise the vector of unknown parameters \underline{c} can be calculated directly from equation (3.9). This calculation is complete at each stage and does not require any knowledge of the results of calculations made at other stages. As a result the unknown parameters can be identified when they are changing very slowly with respect to time and/or space. In other words, no significant error is introduced when the parameter change is negligible over the time or spatial interval required for calculation of the finite difference equations at each stage.

When measurement noise is present, the least squares estimator of Chapter 3 can be used to estimate the unknown parameters. The discussion of the convergence of this estimator applies to the case where \underline{c} is a constant. However, if \underline{c} is changing very slowly, the estimation can be restarted after a specified number of stages. This approach has serious limitations in that the estimated value of the parameter represents an "average" over the estimation interval. Thus the change from stage to stage is masked by the calculation.

An approach which is more sensitive to the change in the parameters from stage to stage is much more preferable. One such scheme requires a knowledge of the nature of the change occurring in the unknown parameters. Aoki [22] discusses the case where the vector \underline{c}_k satisfies the difference equation

$$\underline{c}_k = \phi_{k-1} \underline{c}_{k-1} \quad k = 1, 2, \dots, S \quad (5.1)$$

with ϕ_k being a known $N \times N$ nonsingular matrix. Even though the relationship between \underline{c}_k and \underline{c}_{k-1} is known from stage to stage, there is no way of obtaining \underline{c}_{k-1} exactly at any given stage. Therefore, some type of estimation is required. The least squares estimator, according to Aoki, is given by the expression

$$\underline{c}_S^* = \phi_{S-1} \underline{c}_{S-1}^* + \hat{P}_S \hat{H}_S^{-1} \hat{V}_S (\underline{g}_S - \hat{H}_S \phi_{S-1} \underline{c}_{S-1}^*) \quad (5.2)$$

where

$$\hat{P}_S^{-1} = \phi_{S-1}^{-1} \hat{P}_{S-1}^{-1} \phi_{S-1}^{-1} + \hat{H}_S^{-1} \hat{V}_S \hat{H}_S \quad (5.3)$$

Notice that equation (A.8) is a special case of (5.2) resulting when ϕ_k is the identity matrix.

This still leaves the very important case where ϕ_k in equation (5.1) is not known a priori. Such a problem represents one direction of investigation which to date has not been solved.

5.2 IDENTIFICATION OF THE INTEGRAL EQUATIONS

This thesis deals with the identification of the differential equations characterizing a system, as have investigations by other authors [9, 10, 11, 13]. Frequently, however, it is mathematically simpler to work with the integral equations vice the differential equations. This is particularly true when finding the optimum controller in an adaptive system.

In order to represent a system by an integral equation, it is necessary to know its Green's function. The Green's function can be derived when the system partial differential equation and associated

boundary conditions are completely specified. Unfortunately, this is rarely an easy task. It is, therefore, much more preferable to obtain the Green's function directly using normal operating data. Much work has been done in the identification of the Green's (weighting function) function in lumped systems. However, the author is not aware of any studies of this type having been completed for distributed systems.

5.3 ESTIMATION OF PARAMETERS WHEN TRANSDUCER NOISE IS CORRELATED AND/OR NONSTATIONARY

The estimation of parameters in the presence of noise presented in this thesis is based upon the validity of assumptions (3a) through (3d). These conditions preclude the use of sets of random samples which are not independent. This dependent situation arises when the noise generated by different transducers is correlated and/or when the samples for a transducer are not separated sufficiently in time.

The derivation of the least squares estimator requires no conditions about the independence of the noise. Hence, the least squares estimator can be applied to the estimation of parameters when the samples are correlated. However, the estimation error will probably be larger than that encountered in the uncorrelated case. If sufficient information concerning the dependence of the measurement noise is available, an approach similar to that presented in Chapter 3 might be used to obtain a modified least squares estimator.

The assumptions presented in subsection 3.1.1 also eliminated the case where the statistics of noise generated by the transducers is

changing with time. Here again, the least squares estimator can be applied, but the accuracy of the estimation is in doubt. As with the uncorrelated noise much work remains to be done when the noise is nonstationary.

5.4 SENSITIVITY

Errors due to finite difference approximations and measurement noise are always present in the proposed identification scheme. This is true even though the combined effect of these errors can be reduced by proper selection of the increments in time and space. With this in mind, consider the problem of determining the errors incurred in the unknown parameter vector \underline{c} when information concerning errors due to finite difference approximations and measurement noise is available.

The sensitivity problem has been touched on briefly in this thesis. An explicit expression for the error in \underline{c} under the restriction that the elements of H and \underline{g} are linear partial derivatives is given by equation (3.22). As was stated in section (3.5), when H_k is ill-conditioned the errors present in \hat{H}_k are greatly exaggerated in the calculation of the inverse. Thus, $\Delta \underline{c}_k$ in equation (3.22) depends on the conditioning of H_k in addition to finite difference approximation and measurement noise.

Ideally, sensitivity statements can be made a priori by applying only analytic methods. This provides considerable insight into the problem without performing expensive experiments on the physical system.

In the proposed identification technique considerable work remains to be done in this area. For example, sensitivity to the transition from the continuous to the discrete mathematical model must be investigated further. According to Tomovic' [28] this problem is as yet unsolved for the general case.

When more than one constant is identified, some of these parameters are more sensitive to errors than others. An example of this may be seen by returning to the identification of c_1 and c_2 in the nonlinear example. In Figure 2.9 the finite difference errors had little effect on c_1 when t was greater than 0.4 units. c_2 , on the other hand, had detectible errors over the same range. A sensitivity study might reveal why c_2 is more sensitive to errors.

Until it is possible to conduct an analytic sensitivity analysis, an alternate study might provide worthwhile information. Consider the error in c when H_k and g_k are scalar with linear partial derivatives. The error resulting from errors in h_k and g_k were given by equation (3.90). The proposed study involves a digital computer simulation similar to those used in Chapter 3.

After the measurement noise with a known mean and variance has been added to calculations of the state variable, Δc_k for several stages is calculated using equation (3.90). The results of this calculation will have a randomness due to the measurement noise. Plot the Δc_k on a relative frequency histogram. The resulting plot has some average value and dispersion. Now, by changing the mean and variance of the

measurement noise and calculating another histogram of Δc_k , a comparison of the average value and dispersion can be made. Several such studies might provide insight into how the errors in c are related to the errors in h_k and g_k .

APPENDIX A
MATRIX FORMULATION

A.1 LEAST SQUARES ESTIMATOR AND MODIFIED LEAST SQUARES ESTIMATOR

The least squares estimator for the scalar case was presented in subsection 3.2.5. This section of the appendix is devoted to the development of the least squares estimator in the matrix case. The problem statement of subsection 3.2.1 is still applicable as is the general mathematical model presented in subsection 3.2.3. Since the development closely parallels that presented in subsection 3.2.5, the discussion is not detailed.

The least squares estimator is found by minimizing the performance index $J_S(\underline{\xi})$ with respect to $\underline{\xi}$. The performance index is given by

$$J_S(\underline{\xi}) = \sum_{k=1}^S \left\| \hat{H}_k \underline{\xi} - \hat{g}_k \right\|_{V_k}^2 \quad (\text{A.1})$$

where

$$\left\| \hat{H}_k \underline{\xi} - \hat{g}_k \right\|_{V_k}^2 \triangleq (\hat{H}_k \underline{\xi} - \hat{g}_k)^T V_k (\hat{H}_k \underline{\xi} - \hat{g}_k) \quad (\text{A.2})$$

and V_k is a positive definite symmetric matrix. Increasing the magnitude of the elements of V_k tends to make the estimator settle down with fewer estimates. However, forcing the estimator to reach a steady state value too rapidly may result in a greater error.

Taking the derivatives of equation (A.1) with respect to $\underline{\xi}$ gives

$$J_{S\xi} = 2 \left[\sum_{k=1}^S \hat{H}_k^T V_k \hat{H}_k \right] \underline{\xi} - 2 \sum_{k=1}^S \hat{H}_k^T V_k \hat{g}_k \quad (\text{A.3})$$

where $J_{S\xi}$ is defined by

$$J_{S\xi} = \left[\frac{\partial J_S}{\partial \xi_1} \quad \frac{\partial J_S}{\partial \xi_2} \quad \dots \quad \frac{\partial J_S}{\partial \xi_N} \right] \quad (\text{A.4})$$

Let \underline{c}_S^* denote the value of \underline{g} which minimizes $J_S(\underline{g})$. Since the extremum of J_S occurs when (A.3) is set equal to zero, \underline{c}_S^* is given by

$$\underline{c}_S^* = \hat{P}_S^{-1} \sum_{k=1}^S \hat{H}_k' \hat{V}_k \hat{g}_k \quad (\text{A.5})$$

where \hat{P}_S is defined by the equation

$$\hat{P}_S^{-1} = \sum_{k=1}^S \hat{H}_k' \hat{V}_k \hat{H}_k \quad (\text{A.6})$$

$$= \hat{P}_{S-1}^{-1} + \hat{H}_S' \hat{V}_S \hat{H}_S \quad (\text{A.7})$$

The estimate at stage S can be expressed in terms of the estimate of stage $S-1$. This is done by first noting equation (A.5) can be expressed

$$\underline{c}_S^* = \hat{P}_S^{-1} \left[\sum_{k=1}^{S-1} \hat{H}_k' \hat{V}_k \hat{g}_k + \hat{H}_S' \hat{V}_S \hat{g}_S \right]$$

The first term in the brackets is now multiplied by $\hat{P}_{S-1}^{-1} \hat{P}_{S-1} = I$, with I denoting the identity matrix, to get

$$\underline{c}_S^* = \hat{P}_S^{-1} \left[\hat{P}_{S-1}^{-1} \hat{P}_{S-1} \sum_{k=1}^{S-1} \hat{H}_k' \hat{V}_k \hat{g}_k + \hat{H}_S' \hat{V}_S \hat{g}_S \right]$$

Finally substituting for \hat{P}_{S-1}^{-1} from equation (A.7) and then making use of equation (A.5) yields the least squares estimator

$$\begin{aligned} \underline{c}_S^* &= \hat{P}_S^{-1} \left[(\hat{P}_S^{-1} - \hat{H}_S' \hat{V}_S \hat{H}_S) \hat{P}_{S-1}^{-1} \sum_{k=1}^{S-1} \hat{H}_k' \hat{V}_k \hat{g}_k + \hat{H}_S' \hat{V}_S \hat{g}_S \right] \\ &= \hat{P}_S^{-1} \left[\hat{P}_{S-1}^{-1} \underline{c}_{S-1}^* - \hat{H}_S' \hat{V}_S \hat{H}_S \underline{c}_{S-1}^* + \hat{H}_S' \hat{V}_S \hat{g}_S \right] \\ &= \underline{c}_{S-1}^* + \hat{P}_S^{-1} \hat{H}_S' \hat{V}_S (\hat{g}_S - \hat{H}_S \underline{c}_{S-1}^*) \end{aligned} \quad (\text{A.8})$$

The convergence of the least squares estimator is now considered. Substituting for \hat{g}_k in equation (A.5) from equation (3.13) results in the expression

$$\underline{c}_S^* = \hat{P}_S \sum_{k=1}^S \hat{H}_k^T \hat{V}_k \hat{H}_k \underline{c}_k \quad (\text{A.9})$$

Replacing \hat{c}_k in equation (A.9) with equation (3.20) and recalling the definition of \hat{P}_S one gets

$$\begin{aligned} \underline{c}_S^* &= \hat{P}_S \sum_{k=1}^S \hat{H}_k^T \hat{V}_k \hat{H}_k (\underline{c} + \Delta \underline{c}_k) \\ &= \underline{c} + \hat{P}_S \sum_{k=1}^S \hat{H}_k^T \hat{V}_k \hat{H}_k \Delta \underline{c}_k \end{aligned} \quad (\text{A.10})$$

When equation (3.22) is substituted for $\Delta \underline{c}_k$

$$\underline{c}_S^* = \underline{c} + \hat{P}_S \sum_{k=1}^S \hat{H}_k^T \hat{V}_k \{ [o(\underline{g}_k) + n(\underline{g}_k)] - [o(H_k) + n(H_k)] \underline{c} \} \quad (\text{A.11})$$

If the finite difference expressions used to approximate the elements of H_k and \underline{g}_k have no common points, $n(H_k)$ and $n(\underline{g}_k)$ are statistically independent. When the transducer noise has zero mean, the techniques of subsection 3.2.6 can be used to show

$$\sum_{k=1}^S \hat{H}_k^T \hat{V}_k [o(\underline{g}_k) + n(\underline{g}_k)] = \underline{0} \quad (\text{A.12})$$

$$\sum_{k=1}^S \hat{H}_k^T \hat{V}_k o(H_k) \underline{c} = \underline{0} \quad (\text{A.13})$$

and

$$\sum_{k=1}^S \hat{H}_k' V_k n(H_k) \underline{c} = \sum_{k=1}^S n(H_k) \hat{V}_k n(H_k) \underline{c} \quad (\text{A.14})$$

with $\underline{0}$ being defined as an $N \times 1$ null vector. Based upon the results of approximations (A.12), (A.13) and (A.14), equation (A.11) can be approximated by the expression

$$\underline{c}_S^* = \underline{c} - \hat{P}_S \hat{Q}_S \underline{c} \quad (\text{A.15})$$

where

$$\hat{Q}_S \triangleq \sum_{k=1}^S n(H_k) \hat{V}_k n(H_k) \quad (\text{A.16})$$

When \hat{Q}_S is approximated, its elements are functions of the transducer variance. This is illustrated by the example 3.4.3.1. By letting $\hat{\hat{Q}}_S$ denote the approximated value of \hat{Q}_S , the following modified least squares estimator is obtained

$$\underline{c}_S^{**} = \underline{c}_S^* + \hat{P}_S \hat{\hat{Q}}_S \underline{c}_S^{**} \quad (\text{A.17})$$

where

$$\underline{c}_1^{**} = \underline{0} \quad (\text{A.18})$$

When the transducer means are not zero an examination of equation (A.11) shows that all the terms containing $n(H_k)$ and $n(\underline{g}_k)$ contribute to the error term. However, when the means are known, the least squares estimator may be rewritten as follows to compensate for the nonzero mean.

Let

$$\Gamma = \begin{bmatrix} E[n(h_{11})] & E[n(h_{12})] & \dots & E[n(h_{1N})] \\ E[n(h_{12})] & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ E[n(h_{N1})] & \dots & & E[n(h_{NN})] \end{bmatrix}$$

$$= E[n(H_k)] \quad (\text{A.19})$$

and

$$\begin{aligned} \underline{\delta} &= [E[n(g_1)] \ E[n(g_2)] \ \dots \ E[n(g_N)]] \\ &= E[n(\underline{g}_k)] \end{aligned} \quad (\text{A.20})$$

Next define the performance measure by the equation

$$J_S(\underline{\xi}) = \sum_{k=1}^S \left\| (\hat{H}_k - \Gamma)\underline{\xi} - (\hat{g}_k - \underline{\delta}) \right\|_{V_k}^2 \quad (\text{A.21})$$

This index of performance can be minimized with respect to $\underline{\xi}$ with the same steps used previously for $J_S(\underline{\xi})$. If \underline{c}_S^0 denotes the value of $\underline{\xi}$ which minimizes $J_S(\underline{\xi})$, then

$$\underline{c}_S^0 = \hat{P}_S^{-1} \sum_{k=1}^S (\hat{H}_k - \Gamma)^{-1} V_k^{-1} (\hat{g}_k - \underline{\delta}) \quad (\text{A.22})$$

with

$$\hat{P}_S^{-1} \triangleq \sum_{k=1}^S (\hat{H}_k - \Gamma)^{-1} V_k^{-1} (\hat{H}_k - \Gamma) \quad (\text{A.23})$$

The recursive estimator for \underline{c}_S^o is given by

$$\underline{c}_S^o = \underline{c}_{S-1}^o + \hat{P}_S^o (\hat{H}_S - \Gamma)^T V_S^{-1} [(\hat{g}_S - \underline{d}) - (\hat{H}_S - \Gamma) \underline{c}_{S-1}^o] \quad (\text{A.24})$$

When the error term for \underline{c}_S^o is approximated, the expression is identical to that obtained for \underline{c}_S^* when the transducers have zero mean.

A.2 ALTERNATE RECURSIVE FORM OF THE LEAST SQUARES ESTIMATOR

When the least squares estimate is actually computed, a recursive form of the estimator is normally used. This makes it possible to have an updated estimate at each stage which utilizes all the past information. Furthermore, a record of the estimate from stage to stage indicates trends in the data, such as the rate at which the estimate is reaching a steady-state value.

The recursive form of the estimator presented by most authors is given by equation (A.8). However, the author of this thesis proposes another recursive estimator which requires fewer computations than does equation (A.8).

This estimator is obtained by first defining $\hat{\underline{L}}_S$ to be

$$\hat{\underline{L}}_S = \sum_{k=1}^S \hat{H}_k^T V_k^{-1} \hat{g}_k \quad (\text{A.25})$$

$$= \hat{\underline{L}}_{S-1} + \hat{H}_S^T V_S^{-1} \hat{g}_S \quad (\text{A.26})$$

When $\hat{\underline{L}}_S$ is substituted into equation (A.5) the recursive estimator is given by

$$\underline{c}_S^* = \hat{P}_{S-S} \hat{\underline{L}}_S \quad (\text{A.27})$$

Matrix Operation	Number of Times Operation is Performed	
	Equation (A.8)	Equation (A.27)
Invert NxN	1	1
Transpose NxN	1	1
Mult. two NxN	3	2
Mult. NxN & Nx1	2	2
Add two NxN	1	1
Add two Nx1	2	1

Figure A.1 Comparison of number of operations required to calculate equation (A.8) and (A.27).

Figure A.1 provides a comparison of the number of operations required to calculate \underline{c}_S^* using equations (A.8) and (A.27). This comparison is based upon the assumption that at stage S, \hat{H}_S , V_S and \hat{g}_S are read into the computer while P_{S-1}^{-1} , \underline{c}_{S-1}^* and $\hat{\delta}_{S-1}$ are stored from the previous calculation. The disadvantage of using equation (A.27) is the requirement that the Nx1 column vector $\hat{\underline{c}}_{S-1}$ be stored for use in the next calculation. However, the number of calculations required with this estimator are less than those required for equation (A.8). Figure A.1 shows the savings is the multiplication of an NxN matrix with another NxN matrix and the addition of two Nx1 matrices.

A.3 SELECTION OF OPTIMAL TRANSDUCER SPACING AND SAMPLING RATE

A method is presented to select optimal Δx and Δt for the scalar problem in section 3.3. This result is easily extended to the determination of optimal transducer spacing and sampling rate when the system

dynamics are described by the matrix differential equation

$$\dot{H}_k \underline{c} = \underline{g}_k \quad k = 1, 2, \dots, S \quad (\text{A.28})$$

When the elements of H_k and \underline{g}_k are linear partial differentials, the addition of finite difference errors and measurement noise is described by equations (3.13) through (3.18). The arguments of section 3.3 leading to the selection of the performance index of equation (3.95) are applicable in the matrix problem. The matrix equation equivalent to the scalar index of performance is given by

$$I_A = E\{[\underline{c}'[n(H_k)\underline{c} - n(\underline{g}_k)] + [o(H_k)\underline{c} - o(\underline{g}_k)]]' \cdot \{[n(H_k)\underline{c} - n(\underline{g}_k)] + [o(H_k)\underline{c} - o(\underline{g}_k)]\}} \quad (\text{A.29})$$

The minimization of I_A is carried out with respect to Δx , Δy , Δz and Δt . In the special case where $n(H_k)$ and $n(\underline{g}_k)$ are statistically independent and have zero mean, I_A in equation (A.29) can be simplified.

Since the following relationships are true

$$E[n(H_k)] = 0$$

$$E[n(\underline{g}_k)] = \underline{0}$$

$$E[n(H_k) \hat{n}(\underline{g}_k)] = \underline{0}$$

$$E[n(\underline{g}_k) \hat{n}(H_k)] = \underline{0}'$$

equation (A.29) may be written as

$$I_A = \underline{c}' E[n(H_k) \hat{n}(H_k)] \underline{c} + E[n(\underline{g}_k) \hat{n}(\underline{g}_k)] + [o(H_k)\underline{c} - o(\underline{g}_k)]' [o(H_k)\underline{c} - o(\underline{g}_k)] \quad (\text{A.30})$$

As was necessary in the scalar problem, the variance of the transducer

noise must be known apriori and an initial estimate of c made. An application of equation (A.29) is given in example 3.4.3.1.

APPENDIX B

IDENTIFICATION OF A VECTOR PARTIAL

DIFFERENTIAL EQUATION

The problem of identification of coefficients in a partial differential equation with one state variable was considered in the body of this thesis. However, certain distributed parameter systems are best described by partial differential equations with two or more state variables. For example, the state variables in a system might be temperature, pressure, and velocity. Though systems described by several state variables are inherently more difficult to identify, the techniques used in the single variable case may readily be applied.

Let the state variables describing a distributed parameter system be denoted by u_1, u_2, \dots, u_A where the u_v 's are functions of time and the three spatial variables. The set of equations describing the system has the form

$$\begin{bmatrix} c_{11}^h h_{11} + c_{12}^h h_{12} + \dots + c_{M}^h h_{MM} \\ \cdot \\ \cdot \\ \cdot \\ c_{\Psi 1}^h h_{\Psi 1} + c_{\Psi 2}^h h_{\Psi 2} + \dots + c_{\Psi M}^h h_{\Psi M} \end{bmatrix} = \begin{bmatrix} f_{\Omega 1} \\ \cdot \\ \cdot \\ \cdot \\ f_{\Omega \Psi} \end{bmatrix} \quad (\text{B.1})$$

where the $h_{i,j}$'s in equation (B.1) may contain u 's, t , x , y , and z and derivatives of the u 's with respect to time and space.

Consider the identification of the coefficients in the i th row of equation (B.1). This equation may be written

$$c_{i1}^h h_{i1} + c_{i2}^h h_{i2} + \dots + c_{iN}^h h_{iN} = g_i \quad (\text{B.2})$$

where the $c_{i1}, c_{i2}, \dots, c_{iN}$ are the constants which must be identified and g_i contains $f_{\Omega i}$ and the terms with the known constants $c_{iN+1}, \dots,$

c_{iM} . Since equation (B.1) is of the same form as equation (2.3), the identification procedure developed for a single state variable applies.

The unknown constants may be found by evaluating equation (B.2) at N different points in time and/or space. The derivatives of the state variables in the resulting equations can be approximated by finite differences. The data required in the difference equations is obtained by making measurements of the state variable, such as temperature, pressure or velocity, at the desired points in space and time.

The requirement for measurements of different state variables at the same point in space possesses physical difficulties. The necessary transducers cannot all be placed exactly at the desired point. Some sort of cluster of transducers must be placed at the desired location. The errors introduced by this arrangement depend primarily on the number of state variables and the measurement interval selected. If only two transducer are required and the separation between the sets of transducers is large, little error will be introduced. However, several transducers with little spacing between the clusters of transducers could result in considerable error.

There is an additional source of error when two or more types of transducers are located at the same point. The measurements taken by a transducer may be adversely affected by the physical presence of other transducers. For example, let one of the state variables be the velocity of a fluid. The presence of several transducers at the point in question would alter the fluid flow and introduce error. It should be noted, however, that reduction or elimination of errors of this type might be

accomplished by clever transducer design.

The two types of errors just presented contribute to the measurement noise. There are various other sources of measurement noise such as imperfections in the transducers, external noise and interactions between the system and transducer. The least squares estimator presented in this thesis can be used to estimate the system parameters of the vector partial differential equation. This is true because \hat{H}_k and \hat{g}_k are obtained directly by performing measurements on the system with more than one type of transducer. The fact that the different types of transducers may very well have very different statistical properties is of no concern, for, unlike the other classical estimators, a knowledge of the noise vector (discussed in subsection 3.2.4) is not required.

APPENDIX C

IDENTIFICATION OF A PARTIAL DIFFERENTIAL
EQUATION MODEL WITH EXTRANEIOUS TERMS

Occasionally the exact form of the partial differential equation describing a distributed parameter system is not known a priori. Under these circumstances the assumed model of the system may include terms which are extraneous; that is, these terms are not necessary to describe the dynamics.

Consider equation (2.6) when the constant c_N is the coefficient of an extraneous term. When the inverse of H exists, c_N can be found by applying Cramer's rule.

$$c_N = \frac{\begin{vmatrix} h_{11} & \dots & h_{1N-1} & g_1 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ h_{N1} & \dots & h_{NN-1} & g_N \end{vmatrix}}{|H|} \quad (C.1)$$

where $|H|$ denotes the determinant of H . Since the term containing c_N is extraneous, the actual equation describing the system is given by

$$c_1 h_{i1} + c_2 h_{i2} + \dots + c_{N-1} h_{iN-1} = g_i \quad (C.2)$$

When (C.2) is evaluated at N different points, the following vector equation is obtained

$$\begin{bmatrix} c_1 h_{11} + c_2 h_{12} + \dots + c_{N-1} h_{1N-1} \\ \cdot \\ \cdot \\ \cdot \\ c_1 h_{N1} + c_2 h_{N2} + \dots + c_{N-1} h_{NN-1} \end{bmatrix} = \begin{bmatrix} g_1 \\ \cdot \\ \cdot \\ \cdot \\ g_N \end{bmatrix} \quad (C.3)$$

Substituting this result into equation (C.1)

$$c_N = \frac{\begin{vmatrix} h_{11} & \dots & h_{1N-1} & (c_1 h_{11} + c_2 h_{12} + \dots + c_{N-1} h_{1N-1}) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ h_{N1} & \dots & h_{NN-1} & (c_1 h_{N1} + c_2 h_{N2} + \dots + c_{N-1} h_{NN-1}) \end{vmatrix}}{|H|} \quad (C.4)$$

Since the N^{th} column in the numerator of equation (C.4) is a linear combination of the preceding $(N-1)$ columns, the determinant of the numerator is zero. Thus, the value of c_N is identified as zero.

When the elements of H are approximated with finite differences and measurement noise is added, the value of c_N will be nonzero. However, when the process is well-conditioned and the errors due to approximation and measurement noise are not too large, the value of c_N will usually turn out to be negligible compared to the coefficients of the nonextraneous terms included in the model. This is demonstrated by the results in example 2.3.3.3.

APPENDIX D
DERIVATION OF A QUADRATIC
HIGH ACCURACY FORMULA

A high accuracy formula for the identification of α in the diffusion equation was presented in subsection 2.3.1. The identification must be performed in the absence of measurement noise, but under these conditions the results are quite good. The disadvantage of the procedure is the necessity of solving a quartic equation in α and selecting the correct root from the resulting set of four possible solutions. This section is devoted to the derivation of another high accuracy formula which, though not as accurate as the quartic, requires the solution of only a quadratic in α . The derivation is suggested by Mitchell and Pearce [17] and will utilize their notation.

The high accuracy formula presented in this section is made possible by a unique recursive relationship possessed by the diffusion equation. When equation (2.13) is differentiated with respect to time there results

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t} \left[\alpha \frac{\partial^2 u}{\partial x^2} \right] \\ &= \alpha \frac{\partial^2}{\partial x^2} \left[\alpha \frac{\partial^2 u}{\partial x^2} \right] \\ &= \alpha^2 \frac{\partial^4 u}{\partial x^4} \end{aligned} \tag{D.1}$$

Successive differentiation of equation (D.1) with respect to time yields the general result

$$\frac{\partial^n u}{\partial t^n} = \alpha^n \frac{\partial^{2n} u}{\partial x^{2n}} \quad n = 1, 2, \dots \tag{D.2}$$

Now consider the Taylor's series expansion of $u_{i,j+1}$ about the point $u_{i,j}$, that is

$$u_{i,j+1} = u_{i,j} + \Delta t \frac{\partial u}{\partial t} \Big|_{i,j} + \frac{\Delta t^2}{2!} \frac{\partial^2 u}{\partial t^2} \Big|_{i,j} + \frac{\Delta t^3}{3!} \frac{\partial^3 u}{\partial t^3} \Big|_{i,j} + \dots \quad (\text{D.3})$$

When the relationship of equation (D.2) is substituted into equation (D.3)

$$u_{i,j+1} = u_{i,j} + (\alpha \Delta t) \frac{\partial^2 u}{\partial x^2} \Big|_{i,j} + \frac{(\alpha \Delta t)^2}{2!} \frac{\partial^4 u}{\partial x^4} \Big|_{i,j} + \dots \quad (\text{D.4})$$

Define p by the relationship

$$p = \frac{\alpha \Delta t}{\Delta x^2} \quad (\text{D.5})$$

Then equation (D.4) may be expressed by

$$u_{i,j+1} = u + pB + (1/2)p^2D + (1/6)p^3F + \dots \quad (\text{D.6})$$

where u, B, D, F, \dots are the values of

$$u_{i,j}, \Delta x^2 \frac{\partial^2 u}{\partial x^2} \Big|_{i,j}, \Delta x^4 \frac{\partial^4 u}{\partial x^4} \Big|_{i,j}, \Delta x^6 \frac{\partial^6 u}{\partial x^6} \Big|_{i,j}, \dots$$

By making use of substitutions similar to those used above, the following additional Taylor's series expansions are obtained.

$$u_{i,j-1} = u - pB + (1/2)p^2D - (1/6)p^3F + (1/24)p^4H - \dots \quad (\text{D.7})$$

$$u_{i+1,j} + u_{i-1,j} = 2u + B + (1/12)D + (1/360)F + (1/20,160)H + \dots \quad (\text{D.8})$$

$$\begin{aligned} u_{i+1,j\pm 1} + u_{i-1,j\pm 1} &= 2u + (1 \pm 2p)B \\ &+ (1/12 \pm p + p^2)D + [1/360 \pm (1/12)p + (1/2)p^2 \pm (1/3)p^3]I \quad (\text{D.9}) \\ &+ [1/20,160 \pm (1/360)p + (1/24)p^2 \pm (1/6)p^3 + (1/12)p^4]H + \dots \end{aligned}$$

All linear relationship of the values of u about the point i, j is selected such that the maximum possible number of u, B, D, \dots, H, \dots are eliminated. For the case where three spatial measurements and

three samples in time are used, the aim is to select the polynomials in p , denoted by a , b , c , d , e and f , to eliminate terms containing u , B , D , ... from the expression

$$\begin{aligned}
 & au_{i,j} + bu_{i,j+1} + cu_{i,j-1} + d(u_{i+1,j} + u_{i-1,j}) \\
 & + e(u_{i+1,j+1} + u_{i-1,j+1}) + f(u_{i+1,j-1} + u_{i-1,j-1}) = 0 \quad (D.10)
 \end{aligned}$$

When the coefficients of the u 's in equation (D.10) are quartic in p Mitchell and Pearce are able to eliminate terms containing u , B , D , F and H . The results are given in equations (2.20) and (2.21). By assuming the coefficients are quadratic in p it is possible to eliminate only u , B , D , and F .

This is done by first substituting equations (D.6) through (D.9) into (D.10). Since it is necessary to pick a , b , c , d , e and f in such a way that the terms containing u , B , D , F and H are identically zero, the following relationships must hold.

$$(a + b + c + 2d + 2e + 2f)u = 0 \quad (D.11)$$

$$[bp - cp + d + e(1+2p) + f(1-2p)]B = 0 \quad (D.12)$$

$$[bp^2/2 + cp^2/2 + d/12 + e(1/12 + p + p^2) + f(1/12 - p + p^2)]D = 0 \quad (D.13)$$

$$\begin{aligned}
 & [bp^3/6 - cp^3/6 + d/360 + e(1/360 + p/12 + p^2/12 + p^3/3) \\
 & + f(1/360 - p/12 + p^2/2 - p^3/3)]F = 0 \quad (D.14)
 \end{aligned}$$

In general, u , B , D and F are nonzero. Therefore, the quantities within the brackets in equations (D.11) through (D.14) must be identically zero.

The coefficients are assumed to be quadratic in p . As a result let

$$a = a_1 p^2 + a_2 p + a_3 \quad (D.15)$$

$$b = b_1 p^2 + b_2 p + b_3 \quad (\text{D.16})$$

$$c = c_1 p^2 + c_2 p + c_3 \quad (\text{D.17})$$

$$d = d_1 p^2 + d_2 p + d_3 \quad (\text{D.18})$$

$$e = e_1 p^2 + e_2 p + e_3 \quad (\text{D.19})$$

$$f = f_1 p^2 + f_2 p + f_3 \quad (\text{D.20})$$

Substituting equations (D.15) through (D.20) into equations (D.11) through (D.14) results in four equations which are quartic in p . In each equation $a_1, a_2, a_3, \dots, f_3$ must be selected such that coefficients of p vanish. Hence

$$a_1 + b_1 + c_1 + 2d_1 + 2e_1 + 2f_1 = 0$$

$$a_2 + b_2 + c_2 + 2d_2 + 2e_2 + 2f_2 = 0$$

$$a_3 + b_3 + c_3 + 2d_3 + 2e_3 + 2f_3 = 0$$

$$b_1 - c_1 + 2e_1 - 2f_1 = 0$$

$$b_2 - c_2 + d_1 + e_1 + f_1 + 2e_2 + 2f_2 = 0$$

$$b_3 - c_3 + d_2 + e_2 + 2e_3 + f_2 - 2f_3 = 0$$

$$d_3 + e_3 + f_3 = 0$$

$$b_1 + c_1 + 2e_1 + 2f_1 = 0$$

$$b_2 + c_2 + 2e_1 + 2e_2 - 2f_1 + 2f_2 = 0$$

$$6b_3 + 6c_3 + d_1 + e_1 + f_1 + 12(e_2 + e_3 - f_2 + f_3) = 0$$

$$d_2 + e_2 + f_2 + 12(e_3 - f_3) = 0$$

$$b_2 - c_2 + 3e_1 + 2e_2 + 3f_1 - 2f_2 = 0$$

$$2b_3 - 2c_3 + e_1 + 6e_2 + 4e_3 - f_1 + 6f_2 - 4f_3 = 0$$

$$d_1 + e_1 + 30e_2 + 180e_3 + f_1 - 30f_2 + 180f_3 = 0$$

$$d_2 + e_2 + 30e_3 + f_2 - 30f_3 = 0$$

The above fifteen equations contain eighteen unknown. Therefore, three unknowns may be picked arbitrarily. The values of $e_3 = 1$, $f_1 = 2$ and $f_2 = 8$ were picked to avoid fractions for the remaining coefficients.

The results are summarized by the equations

$$a = 200p^2 - 90p - 35 \quad (D.21)$$

$$b = 104p^2 + 127p + 17.5 \quad (D.22)$$

$$c = -4p^2 - 37p + 17.5 \quad (D.23)$$

$$d = -100p^2 - 9p - 2 \quad (D.24)$$

$$e = -52p^2 + p + 1 \quad (D.25)$$

$$f = 2p^2 + 8p + 1 \quad (D.26)$$

Since $p = \frac{\alpha \Delta t}{\Delta x^2}$, substituting (D.21) through (D.26) into (D.10) gives a quadratic in α .

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