Identification of Image Noise Sources in Digital Scanner Evaluation

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ABSTRACT

For digital image acquisition systems, analysis of image noise often focuses on random sources, such as those associated with quantum signal detection and signal-independent fluctuations. Other important noise sources result in pixel-to-pixel sensitivity variations that introduce repeatable patterns into the image data. In addition, because most analyses use a nominally uniform target area to estimate image noise statistics, target noise can often masquerade as noise introduced by the device under test. We describe a method for distilling various fixed-pattern and temporal noise sources. The method uses several replicate digital images acquired in register. In some cases, however, evaluation of several digital scanners reveals scan-to-scan variation in the image registration to the input test target. To overcome this limitation, a modified noise estimation method is described. This includes a step to correct this scan-to-scan misregistration. We also show that, in some cases, measurement of temporal and fixed-pattern noise sources can be achieved via the noise color covariance from a single test image.

Keywords: image noise, digital scanner, image quality, fixed pattern noise, noise cracking

1. INTRODUCTION

In the design, selection, and testing of image acquisition devices it is often useful to understand the magnitude and nature of unwanted image fluctuations. For several sources of this image noise, a statistical description is appropriate.[1-3] For example, the photon, or shot noise related to quantum detection is often modeled as a Poisson distributed random process. Other sources have both stochastic and predictable components. Pixel-to-pixel detector sensitivity variations that introduce repeatable patterns into the image data may appear to be random from unit to unit, but somewhat repeatable from scan-to-scan for a given scanner. When evaluating digital scanners, an additional source of image noise is the test target used in the measurement. This can be due to, e.g., surface texture, scratches, or the microstructure of the printing technology used. Because most analyses call for a nominally uniform image region over which image noise statistics are to be obtained, target noise can often masquerade as noise introduced by the device under test.

In a previous paper,[4] we described a method for distilling various fixed-pattern and temporal noise sources. The method uses several replicate digital images, acquired in register. A similar sampling is included in a recent ISO standard.[5] Practical evaluation of several digital scanners, however, can reveal scan-to-scan variation in the image registration to the input test target. While this is usually not a problem for normal product operation, it makes the above method less accurate. Here we extend our previous method to accommodate this characteristic and report on the use of these techniques, sometimes referred to as *noise cracking*. While the second-order statistics of the image noise (variance, rms) per pixel are our primary interest, the proposed methods can be extended to include the autocovariance or noise-power spectrum.[1-3,6] We start with a summary of the previous method.

2. MEASUREMENT OF NOISE COMPONENTS

We can state our objective as a variance component analysis[7] of several noise sources from sets of observations (image data). A simple additive noise model, which separates effective temporal and fixed pattern noise (FPN) contributions,[8] is adopted. For a uniform image area, the total variance is taken as the sum of its components,

$$\sigma_{total}^2 = \sigma_{random}^2 + \sigma_{fp}^2 , \tag{1}$$

where σ^2_{random} is for the temporally uncorrelated (from image to image) noise sources, and σ^2_{fp} is due to fixed pattern components. We do not require all noise sources to be independent and additive; we are merely interested in the effective components as in Eq. (1). It should be understood that observed image noise will usually vary with (mean) signal level and color record. For digital cameras and scanners, the fixed pattern noise can result from several sources. When testing a print scanner, image fluctuations can be introduced by, e.g., the platen (glass), input target, and imager. This leads to a breakdown of fixed pattern noise variance, e.g.,

$$\sigma^2_{fp} = \sigma^2_{target} + \sigma^2_{platen} + \sigma^2_{imager} . \tag{2}$$

The procedure and analysis that follows are based on the capturing and processing of image data, which allow the suppression of one or more of these sources, so that the remaining sources can be estimated. The following notation is observed whenever possible,

{x: V} is a set of replicate image arrays, gathered while varying parameter **V**. For example,
{x: } is a set acquired by simple repeated scanning varying only in time,
{x: target} is a set acquired by moving the target location between each sample image acquisition.

An image data set is denoted by x_{pqr}, p = 1, …, P pixels, q = 1, …, Q lines, r = 1, …, R replicates.

2.1 Data Collection

In order to estimate the random temporal and fixed pattern noise variance components of Eq. 1, a set of digital images is need, as shown in Fig. 1. Each digital image is a simple replicate with no change in hardware or software settings, or placement of the test target with uniform areas. This image set, **{x: }**, is spatially registered for a fixed detector.

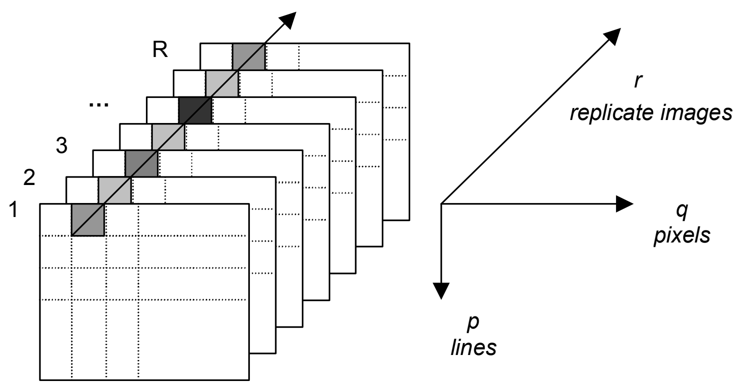

Figure 1. Set of image data used to estimate fixed pattern and random noise components

This data set can be used to estimate the components of Eq. 1, however, a second set is needed to estimate the statistics of the fixed pattern noise components. The second set, **{y: target}**, is acquired while moving (translating) the test target a short distance between repeated image captures.

2.2 Fixed Pattern and Random Noise Estimation

The first step in the isolation of the random (frame-to-frame temporal) image noise is to compute the overall variance for all pixel values in the set of R registered replicate digital images, **{x }**. This is done in two steps. After calculating the grand sample mean in Eq. 3, the total sample variance is found

$$\bar{x} = \frac{1}{PQR} \sum_{p=1}^{P} \sum_{q=1}^{Q} \sum_{r=1}^{R} x_{pqr} \tag{3}$$

$$s_{total}^2 = \frac{1}{PQR-1}\sum_{p=1}^{P}\sum_{q=1}^{Q}\sum_{r=1}^{R}\left(x_{pqr} - \bar{x}\right)^2 \quad . \tag{4}$$

Note that in Eq. 4 each pixel (observation) is treated as independent. The fixed pattern noise, however, has been observed R times for each image pixel location. This suggests that an inter-image averaging may be useful in identifying this noise source. This is accomplished by,

$$\bar{x}_{pq} = \frac{1}{R}\sum_{r=1}^{R} x_{pqr} \;\; , \text{ for all } p, q \tag{5}$$

where $\bar{x}_{pq}$ is an array of mean values. This array can provide an estimate of the fixed pattern noise variance, however, a direct estimate of the random temporal noise variance is first obtained via the inter-image sample variance array. This is given by,

$$s_{pq}^2 = \frac{1}{R-1}\sum_{r=1}^{R}\left(x_{pqr} - \bar{x}_{pq}\right)^2 \; , \;\; \text{for all } p, q \, . \tag{6}$$

This array is then used in a pooled estimate of the random temporal noise variance,

$$\hat{\sigma}_{random}^2 = \bar{s}_{pq}^2 = \frac{1}{PQ}\sum_{p=1}^{P}\sum_{q=1}^{Q} s_{pq}^2 \; , \tag{7}$$

where ^ indicates an estimate.

The fixed pattern variance estimate is computed from the arrays of Eqs. 3 and 7

$$\hat{\sigma}_{fp}^2 = \frac{1}{PQ-1}\sum_{p=1}^{P}\sum_{q=1}^{Q}\left(\bar{x}_{pq} - \bar{x}\right)^2 - \frac{\bar{s}_{pq}^2}{R} \, , \tag{8}$$

where the double summation term is the mean-squared error of the array of inter-image means, and the last term of the RHS ensures that an unbiased estimate is computed, as shown in the Appendix. The two estimated noise variances can then be combined to see to what extent Eq. 2 holds for the imaging system under study,

$$\hat{\sigma}_{total}^2 \overset{?}{=} \hat{\sigma}_{random}^2 + \hat{\sigma}_{fp}^2 \, , \quad \text{where} \quad \hat{\sigma}_{total}^2 = s_{total}^2 \, .$$

2.4 Fixed Pattern Noise Component Estimation

If a separation of scanner and target-induced fixed pattern noise variance is desired, the second set of digital images can be used. Recall that each digital image in **{y: target}** was captured over the same scanner platen area and with the same region of the scanning detector array, but with shifted target. The inter-record average of these images, therefore, will provide a measure with reduced random noise and FPN due to the target. Thus the fixed pattern noise due to the imager is estimated using a procedure similar to that of Eqs. 7 and 8. First the grand mean, $\bar{y}$, and mean and variance arrays are computed

$$s_{y\,pq}^2 = \frac{1}{R-1}\sum_{r=1}^{R}\left(y_{pqr} - \bar{y}_{pq}\right)^2 \, ; \quad \bar{y}_{pq} = \frac{1}{R}\sum_{r=1}^{R} y_{pqr} \, . \tag{9}$$

A pooled variance estimate of the sum of target FPN and random components is

$$\bar{s}_{y\,pq}^2 = \frac{1}{PQ}\sum_{p=1}^{P}\sum_{q=1}^{Q} s_{y\,pq}^2 \, .$$

The fixed pattern noise variance for the imager (detector, optics and platen) is found from the mean-squared fluctuations across the mean array, corrected as in Eq. 8 for the bias due to the random and target noise sources

$$\hat{\sigma}^2_{imager} = \frac{1}{PQ-1}\sum_{p=1}^{P}\sum_{p=1}^{Q}\left(\bar{y}_{pq} - \bar{y}\right)^2 - \frac{\bar{s}^2_{y\,pq}}{R}. \tag{10}$$

The FPN variance due to the target is

$$\hat{\sigma}^2_{target} = \hat{\sigma}^2_{fp} - \hat{\sigma}^2_{imager}. \tag{11}$$

The following characterize the imaging system noise components

s^2_{total}	total noise
$\hat{\sigma}^2_{fp}$	total fixed pattern
$\hat{\sigma}^2_{imager}$	fixed pattern from imager, optics and platen
$\hat{\sigma}^2_{random}$	random temporal.

2.5 Example

Several gray Munsel matte paper samples were used as uniform areas on a test target. Several replicate digital images were collected with software driver parameters set consistent with general print scanning operations (e.g., 24 bit color, gamma=2.2, 300 pixels per inch). Five digital images (R=5) were chosen as set **{x: }** in the above noise source estimation procedure. Figure 2 shows the results for thirty patches for the green color record, each with a different mean signal level. Note that particularly for high mean values, the observed image noise is dominated by fixed pattern image fluctuations. As a test of whether Eq. 2 holds for this scanner, the fixed pattern random temporal variance values were added and the resulting RMS values compared with that for the total noise. The results, shown in Fig. 3, were in agreement, indicating than Eq. 2 approximately holds for this device.

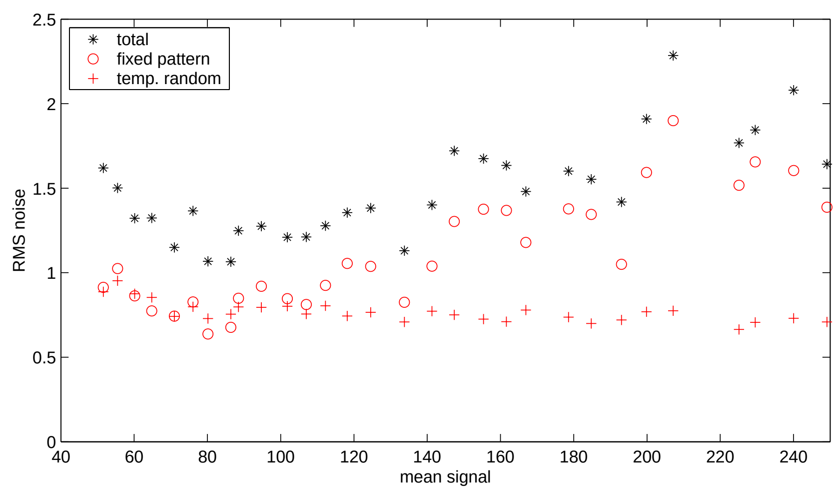

Figure 2. Measured total, fixed pattern and random RMS noise for a desk top scanner using temporal method (units are digital signal values, 0-255)

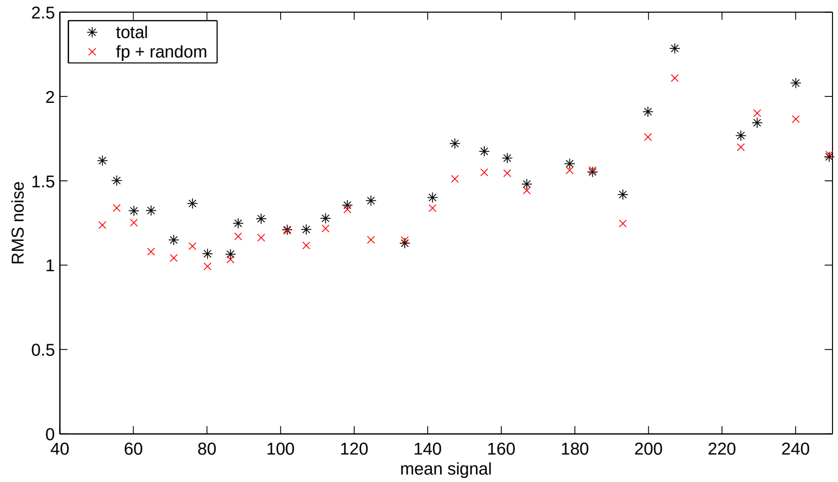

Figure 3. Comparison of total noise and the combined (in quadrature) fixed pattern and temporal random sources for a desk top scanner using temporal method

3. MODIFIED METHOD

The above method for distilling the various image noise sources for image acquisition systems has been used successfully. For several systems which include a moveable imager assembly, such as desktop scanners with a linear detector array, minor translation errors from scan-to-scan make the acquisition of the registered data set **{x: }** difficult. The same situation occurs in high speed document and film scanners. One solution is to screen the digital images for minimal misregistration and select those most compatible with the analysis. In some cases this is not practical. Note that most minor, usually one-dimensional, image translation is normally of no importance during scanning operations, since replicate scans are rarely compared in this way.

3.1 Data and Analysis

Since the intent of most device measurement is to conducted measurements *in situ*, rather than in an unrealistic (stationary) fixture, the challenge was to modify the above method to accommodate this situation. A solution is to first detect the magnitude of the scan-to-scan translation, and then to re-register the replicate image arrays with respect to the input target features. Figure 4 shows an outline of the modified procedure, which we will now describe in detail.

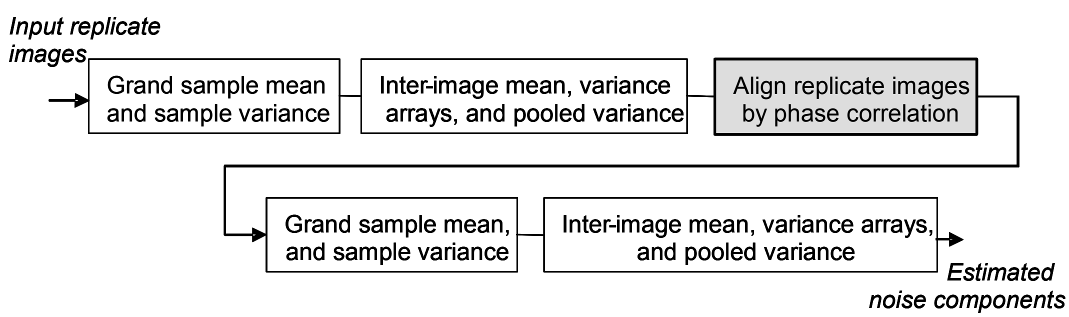

Figure 4. Steps in the modified method

Consider the set of R replicate image arrays, where each is now subject to a translation error with respect to the input test target. In the notation used above, this set is **{x: target}**. We will use many of the same steps, starting with computing the grand sample mean, $\bar{x}$, and variance, s^2_{total}, using Eqs. 3 and 4. As before, we then compute the inter-image mean, $\bar{x}_{pq}$, and variance, $s^2_{x\,pq}$ arrays by Eqs. 5 and 6. From the variance array, the pooled variance is computed for all pixels as in Eq. 7, however; since the input target is not registered with this data set, this inter-image pooled variance includes both random temporal and target variation. The pooled variance is

$$\bar{s}^2_{x\,pq} = \frac{1}{PQ}\sum_{p=1}^{P}\sum_{q=1}^{Q} s^2_{pq} \, . \tag{12}$$

$$= \hat{\sigma}^2_{random} + \hat{\sigma}^2_{target} \, .$$

The mean-squared fluctuations across the inter-image mean array is computed, this is taken as equal to the sum of three terms,

$$s^2_{\bar{x}_{pq}} = \frac{1}{PQ-1}\sum_{p=1}^{P}\sum_{q=1}^{Q}\left(\bar{x}_{pq} - \left(\frac{1}{PQ}\sum_{p=1}^{P}\sum_{q=1}^{Q}\bar{x}_{pq}\right)\right)^2 , \tag{13}$$

$$= \hat{\sigma}^2_{imager} + \frac{\hat{\sigma}^2_{random}}{R} + \frac{\hat{\sigma}^2_{target}}{R} \, ,$$

using the results of the Appendix.

As shown in Fig. 4, the next step is to translate the image arrays in **{x, target}** so that each is in register with respect to the input target. This can be done using a phase correlation[9] approach in either the spatial domain or by discrete Fourier transforms. Details of this approach will not be discussed here, except to note that the method can be used to detect and report image misregistration prior to translating the arrays. The aligned version of the image data set will now not be aligned with respect to the imager in the scanner under test, and we will use this in the analysis.

The 'alligned' data set, we denote as **{y: imager}**. As before, we compute the inter-image mean, $\bar{y}_{pq}$, and variance, $s^2_{y\,pq}$ arrays by Eqs. 5 and 6. From the variance array, the pooled variance is computed for all pixels as in Eq. 7, however, since the imager is now not registered with this data set, this inter-image pooled variance includes both random temporal and imager variation, so the mean pooled variance, as in Eq. 12, is

$$\bar{s}^2_{y\,pq} = \hat{\sigma}^2_{random} + \hat{\sigma}^2_{imager} \, . \tag{14}$$

As before, we compute the sample variance across the inter-image mean array, as in Eq. 13 is the sum of three terms

$$s^2_{\bar{y}_{pq}} = \hat{\sigma}^2_{target} + \frac{\hat{\sigma}^2_{random}}{R} + \frac{\hat{\sigma}^2_{imager}}{R} \, . \tag{15}$$

For the above modified method, the following noise variance estimates are taken as characterizing the system under test,

$$s^2_{total}$$

$$\hat{\sigma}^2_{imager} = s^2_{\bar{x}_{pq}} - \frac{\bar{s}^2_{x\,pq}}{R}$$

$$\hat{\sigma}^2_{target} = s^2_{\bar{y}_{pq}} - \frac{\bar{s}^2_{y\,pq}}{R}$$

$$\hat{\sigma}^2_{random} = s^2_{total} - \hat{\sigma}^2_{imager} - \hat{\sigma}^2_{target} \, .$$

4. SIMPLIFIED COLOR CORRELATION METHOD

4.1 Data and Estimation Procedure

While the original noise distillation method is intended to be most general, as we have seen the requirements for the input image data may not always be met for digital scanners. The modified method, with the addition of the automated spatial registration step can accommodate the evaluation of most digital scanners. For field use, however, it is not always convenient to collect a series of replicate images. The search for a simplified method based on single test images led to an investigation into the use of the inter-record noise covariance. Consider the case where a neutral gray target is scanned for the purposes of noise evaluation. If the imager or target were subject to fixed pattern noise sources, we would expect a correlation between fluctuations in the red, green and blue color records. We observe this when scanning photographic film, where the structure of the film grain introduces both spatial and color correlation into the image fluctuations. Inkjet prints also can impart such noise correlation.

If we have a three-color digital image, the (3x3) color covariance matrix, in common notation given by

$$\Sigma_{rgb} = \begin{bmatrix} \sigma_{rr} & \sigma_{rg} & \sigma_{rb} \\ \sigma_{rg} & \sigma_{gg} & \sigma_{gb} \\ \sigma_{rb} & \sigma_{gbr} & \sigma_{bb} \end{bmatrix},$$

which has six unique elements. The diagonal elements are the variances values, $\sigma_{rr} = \sigma_r^2$, etc. As for the RMS noise (variance), the covariance matrix for image noise will often vary with mean signal level. For nominally uniform image data, the covariance matrix can be estimated element-by-element by the sample covariance, e.g.,

$$\hat{\sigma}_{rg} = \frac{1}{PQ}\sum_{p=1}^{P}\sum_{q=1}^{Q}\left(r_{pq} - \bar{r}\right)\left(g_{pq} - \bar{g}\right); \quad \bar{r} = \frac{1}{PQ}\sum_{p=1}^{P}\sum_{q=1}^{Q} r_{pq}\ ,\ \bar{g} = \frac{1}{PQ}\sum_{p=1}^{P}\sum_{q=1}^{Q} g_{pq} \tag{16}$$

where the red, green and blue image arrays are r_{pq}, g_{pq}, b_{pq}.

Several aspects of an image acquisition system can influence the noise color correlation. Unwanted mixing of the three color-records, crosstalk, at the detector or during signal readout can introduce or modify the off-diagonal elements of the matrix. In addition, signal mixing via a matrix or 3-D look-up table operation, as part of normal image processing will modify the matrix.[10] The presence of inter-color fixed pattern noise will also influence the color variance matrix. For digital scanners with little to no color signal mixing or color channel misregistration, it was suggested that the estimated color covariance matrix could provide a measure of fixed pattern noise due to the input target

Since the amplitude of image noise observed in digital images is usually different in each of the color records, we chose to use the normalized form of the covariance matrix, to estimate the relative amount of variation (fraction of the variance). The elements of the correlation matrix are scaled as

$$c_{rg} = \frac{\sigma_{rg}}{\sqrt{\sigma_{rr}\sigma_{gg}}},$$

and the diagonal elements are unity. The fraction of the variance component in the red image record due to the green and blue record was taken as, c_{rg} and c_{rb}, respectively. One estimate of the random temporal variance in the red record, therefore, is

$$\hat{\sigma}^2_{r\,random1} = \hat{\sigma}^2_{r\,total}\left(1 - c_{rg}\right). \tag{17}$$

However, a second estimate is provided by the red-blue correlation,

$$\hat{\sigma}^2_{r\,random2} = \hat{\sigma}^2_{r\,total}\left(1 - c_{rb}\right) \tag{18}$$

Combining these two values in a simple average provided an estimate of the random temporal noise variance in the red color record,

$$\hat{\sigma}^2_{r\,random} = \sigma^2_{r\,total}\left(1 - \frac{c_{rg} + c_{rb}}{2}\right), \tag{19}$$

and the corresponding estimate of the fixed pattern noise,

$$\hat{\sigma}^2_{r\,fp} = \hat{\sigma}^2_{r\,total} - \hat{\sigma}^2_{r\,random} = \sigma^2_{r\,total}\left(\frac{c_{rg} + c_{rb}}{2}\right). \tag{20}$$

It should be noted that, if needed, we could also correct this estimate for the correlation between green and blue color records in the estimate of the red fixed pattern noise.

4.2 Example

Figure 5 compares the color correlation and original temporal methods for RMS random noise estimation. Image processing and registration assumptions for the two techniques were verified. The test target contained a series of gray uniform patches, and was scanned using a desktop scanner at 600 pixels per inch sampling. The results from the correlation method and a single test image were consistent with those from the temporal noise cracking method with $R = 5$. As in this example, we observe that the correlated noise estimation results have more variability than those from the temporal method. The general conclusions regarding the fraction of noise due to random temporal sources, however, are consistent.

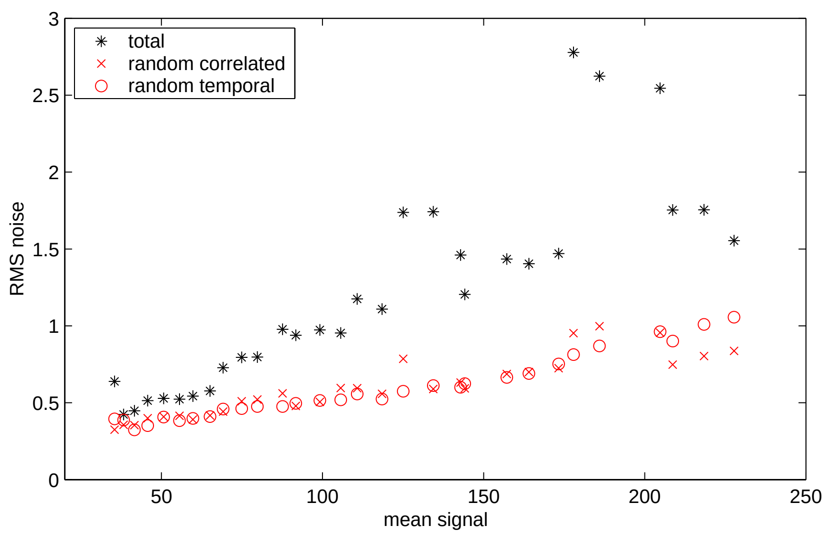

Figure 5. Comparison of simplified color correlation method and original temporal method for estimating the RMS random noise of a desk top scanner

5. CONCLUSIONS

Image capture and data processing methods for measuring the statistics of component noise sources for digital scanners have been presented. This information is useful for product comparison, performance verification, fixed pattern correction evaluation and target noise specification. The statistical analysis of variance approach can be applied to multiple registered data sets in a way that identifies temporal and repeated image fluctuation due to the digital scanner being tested, and bias introduced to the test target being used. In addition, two abbreviated methods have also been discussed. The modified method can be used for systems when registration with an input test target is impractical. When testing on a single digital image is desired, we have also shown that in some cases, the color noise correlation can also be used to separate random temporal and fixed pattern noise statistics. Application of these methods to actual scanners showed good agreement between explicit noise calculations and those inferred from the simple additive model used. In several cases, observed fixed pattern components were found to dominate system noise performance.

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APPENDIX: UNBIASED ESTIMATION OF FIXED PATTERN VARIANCE

Let the data set **{x: }** result from the sampling of the sum of independent random and fixed pattern normal random variables,

$$x_{pqr} = x_{pq} + \varepsilon_r \tag{a1}$$

where

$$\begin{aligned} \mu_{x_{pq}} &= \mu_\varepsilon = 0; \\ \sigma^2_{x_{pq}} &= \sigma^2_{fp} \\ \sigma^2_\varepsilon &= \sigma^2_{random} \end{aligned} \tag{a2}$$

The first term of the RHS Eq. a1 is due to the fixed pattern and the second term the random component. Note that for any single image pixel value x_{pq}, ε is observed once at each pixel. We form $\bar{x}_{pq}$ by computing the sample mean *at each pixel*

$$\bar{x}_{pq} = \frac{1}{R}\sum_{r=1}^{R} x_{pqr} = x_{pq} + \frac{1}{R}\sum_{r=1}^{R} \varepsilon_r\ , \quad \text{for all } p, q\ . \tag{a3}$$

This allows us to reduce the influence of random (temporal) fluctuations on the estimate of σ_{fp}^2, but not eliminate it. This can be addressed by considering the statistics of $\bar{x}_{pq}$. Using Eqs. a1

$$\begin{aligned} \mu_{\bar{x}_{pq}} &= x_{pq} \\ \sigma_{\bar{x}_{pq}}^2 &= \sigma_{x_{pq}}^2 + \frac{\sigma_\varepsilon^2}{R} \end{aligned} \tag{a4}$$

if we assume that each replicate image is an independent observation of x_{pqr}. We now estimate the fixed pattern variance by computing the mean-squared variation *across* the array $\bar{x}_{pq}$

$$s_{\bar{x}_{pq}}^2 = \frac{1}{PQ-1}\sum_{p=1}^{P}\sum_{q=1}^{Q}\left(\bar{x}_{pq} - \left(\frac{1}{PQ}\sum_{p=1}^{P}\sum_{q=1}^{Q}\bar{x}_{pq}\right)\right)^2 .$$

From Eq. a4 and a2 we see that the expected value of this estimate is

$$\mathrm{E}\left[s_{\bar{x}_{pq}}^2\right] = \sigma_{x_{pq}}^2 + \frac{\sigma_\varepsilon^2}{R} = \sigma_{fp}^2 + \frac{\sigma_{random}^2}{R}\ ,$$

where we rely on x_{pqr} being the sum of independent normal random variables. The second term represents the bias that is a decreasing function of *R*. An unbiased estimate of the fixed pattern variance is found by

$$\hat{\sigma}_{fp}^2 = s_{\bar{x}_{pq}}^2 - \frac{\hat{\sigma}_{random}^2}{R}\ ,$$

where the random temporal variance estimate is given in Eq. 7. Equations 8 and 15 use this result.