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Identification of source term for the ill-posed Rayleigh–Stokes problem by Tikhonov regularization method

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Abstract

In this paper, we study an inverse source problem for the Rayleigh–Stokes problem for a generalized second-grade fluid with a fractional derivative model. The problem is severely ill-posed in the sense of Hadamard. To regularize the unstable solution, we apply the Tikhonov method regularization solution and obtain an a priori error estimate between the exact solution and regularized solutions. We also propose methods for both a priori and a posteriori parameter choice rules. In addition, we verify the proposed regularized methods by numerical experiments to estimate the errors between the regularized and exact solutions.

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Keywords: Rayleigh–Stokes problem; Fractional derivative; Ill-posed problem; Tikhonov regularization method

1 Introduction

In this paper, we consider the Rayleigh–Stokes problem for a generalized second-grade fluid model with fractional derivative

$$\begin{cases} \partial_t u - (1 + \gamma \partial_t^\alpha)\Delta u = f(x)\chi(t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,T) = g(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ is a smooth domain with boundary $\partial\Omega$, and $T > 0$ is a given time. Here $\gamma > 0$ is a constant, u_0 is the initial data in $L^2(\Omega)$, $\partial_t = \partial/\partial t$, and ∂_t^α is the Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$ defined by [1, 2]

$$\partial_t^\alpha f(t) = \frac{d}{dt}\int_0^t \omega_{1-\alpha}(t-s)f(s)\,ds, \quad \omega_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

Based on our search results, recently, the Rayleigh–Stokes problem is studied by many authors with many different approaches such as

The Rayleigh–Stokes problem (1.1) plays an important role in describing the behavior of some non-Newtonian fluids [3]. The direct problems, i.e., initial and boundary value problems for the Rayleigh–Stokes problem, have been studied in [3]. Numerical solutions of Rayleigh–Stokes problem for a heated generalized second-grade fluid with fractional derivatives have been considered and developed in some previous papers by Dehghan et al. [4–7]. Many various numerical methods, such as the finite element method, have been applied for solving the forward problem for Rayleigh–Stokes equation, for example, in [3, 8–10]. In [3] the authors considered a fractional derivative anomalous diffusion model.

In practical problems, most of fluid flows and transport processes are distributed parameters, where the parameters used in the modeling equations, such as physical parameters, sink/source terms, initial and boundary conditions, and so on, are not easily obtained from the observations. To deal with this matter, the inverse problem of parameter identification has been applied.

The inverse source problem for fractional diffusion have many important applications in physical practice. Some works on well-posedness of the inverse source problem have been studied by Kirane et al. [11] and Tatar et al. [12]. Triet et al. [13] study the inverse source problem for the Rayleigh–Stokes problem with a fractional derivative model. To regularize the unstable solution, the authors apply a general filter method for constructing regularized solution, and the convergence rate of this method also has been investigated. The fractional derivative model is also studied by Dumitru et al. (see [14–18]).

The latter observation has been considered in many previous studies on linear inverse problems, such as [19–21]. Consequently, the spectral method studied in [19–21] is a special result obtained by choosing a specific filter.

To the best of our knowledge, the research results on inverse problems of the Rayleigh–Stokes problem are still limited. The research works do not deal much with regularization of ill-posed problems. Especially, the evaluation of a priori and a posteriori parameters have not been considered. Problem (1.1) is the forward problem when the source function $F = F(x,t)$ is appropriately given whereas an inverse source problem based on problem (1.1) is determining the source term F at a previous time from its value $u(x,T) = g(x)$ given at the final time T, where $g \in H^2(\Omega) \cap H_0^1(\Omega)$.

In this work, we give another way for approaching the ill-posedness of an inverse source problem. We deliver a Tikhonov regularization method to consider the above Gaussian random model. The right-hand side is a function represented in the form of variable separation. To determine the source term $f(x)$, we require the following assumptions: The functions (g,F) are approximated by the noisy observation data $(g^\varepsilon, F^\varepsilon)$ such that

$$\|g - g^\epsilon\|_{L^2(\Omega)} \le \epsilon, \qquad \|\chi - \chi^\epsilon\|_{C[0,T]} \le \epsilon, \tag{1.2}$$

$$\chi_0 \le \chi(t), \chi^\epsilon(t) \le \chi_1, \quad \forall t \in [0,T]. \tag{1.3}$$

This paper is organized as follows. In Sect. 2, we introduce some notations on Gaussian random models. The main results are given in Sect. 3, including the Tikhonov regularization method and its stability estimates under a priori and a posteriori parameters.

2 Regularization of the inverse source problem by the Tikhonov method

2.1 Preliminaries

In this section, we introduce some useful definitions and preliminary results.

Definition 2.1 Let $\{\lambda_p, \phi_p\}$ be the eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in Ω. The family of eigenvalues $\{\lambda_p\}_{p=1}^{\infty}$ satisfy $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_p \le \cdots$, where $\lambda_p \to \infty$ as $p \to \infty$:

$$\begin{cases} \Delta\phi_p(x) = -\lambda\phi_p(x), & x \in \Omega, \\ \phi_p(x) = 0, & x \in \partial\Omega. \end{cases}$$

Definition 2.2 For $k > 0$, we define

$$H^k(\Omega) := \left\{ v \in L^2(\Omega); \sum_{p=1}^{\infty} \lambda_p^k \big|\langle v, \phi_p\rangle\big|^2 < +\infty \right\} \tag{2.1}$$

equipped with the norm

$$\|v\|_{H^k(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_p^k \big|\langle v, \phi_p\rangle\big|^2 \right)^{\frac{1}{2}}.$$

Applying an eigenfunction expansion, the solution of the Rayleigh–Stokes problem is obtain in the from

$$u(x,t) = \sum_{p=1}^{+\infty} \mathcal{H}_p(\alpha,t)\big\langle u_0(x), \phi_p(x)\big\rangle + \sum_{p=1}^{\infty} \left(\int_0^t \mathcal{H}_p(\alpha, t-s)\chi(s)\, ds f_p(x) \right) \phi_p(x), \tag{2.2}$$

where $F_p(s) = \chi(s)\langle f(x), \phi_p(x)\rangle$, and $\mathcal{H}_p(\alpha,t)$ satisfies the equation

$$\begin{cases} \frac{d}{dt}\mathcal{H}_p(\alpha,t) + \lambda_p(1 + \gamma\partial_t^{\alpha})\mathcal{H}_p(\alpha,t) = 0, & t \in (0,T), \\ \mathcal{H}_p(\alpha,0) = 1. \end{cases} \tag{2.3}$$

Taking $t = T$ and $u_0 = 0$, we get

$$\begin{aligned} g(x) &= \sum_{p=1}^{\infty} \left[\int_0^T \mathcal{H}_p(\alpha, T-s) F_p(s)\, ds \right] \phi_p(x) \\ &= \sum_{p=1}^{\infty} \left[\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\, ds \right] f_p \phi_p(x), \end{aligned} \tag{2.4}$$

where $F_p(s) = \chi(s) f_p$. Hence the source function f is given by the Fourier series

$$f(x) = \sum_{p=1}^{\infty} f_p \phi_p(x) = \sum_{p=1}^{\infty} \frac{\langle g(x), \phi_p(x)\rangle}{\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\, ds} \phi_p(x). \tag{2.5}$$

Using [22], we obtain

$$\mathcal{L}\big(\mathcal{H}_p(\alpha,t)\big) = \frac{1}{t + \gamma\lambda_p t^{\alpha} + \lambda_p}. \tag{2.6}$$

Lemma 2.3 *The functions* $\mathcal{H}_p(\alpha,t)$, $p=1,2,\dots$, *are equal to*

$$\mathcal{H}_p(\alpha,t)=\int_0^{\infty} e^{-rt}K_p(\alpha,r)\,dr,$$

where

$$K_p(\alpha,r)=\frac{\gamma}{\pi}\frac{\lambda_p r^{\alpha}\sin\alpha\pi}{(-r+\lambda_p\gamma r^{\alpha}\cos\alpha\pi+\lambda_p)^2+(\lambda_p\gamma r^{\alpha}\sin\alpha\pi)^2}.$$

Proof See [22]. □

In the following lemma, we present a useful estimate.

Lemma 2.4 *Let* $\alpha\in(\frac{1}{2},1)$. *We have the following estimate for all* $t\in[0,T]$:

$$\mathcal{H}_p(\alpha,t)\geq\frac{C(\gamma,\alpha,\lambda_1)}{\lambda_p}, \tag{2.7}$$

and there exists $\mathcal{D}$ *such that*

$$\int_0^T\left|\mathcal{H}_p(\alpha,t)\right|^2dt\leq\frac{\mathcal{D}^2}{\lambda_p^2}\frac{T^{2\alpha-1}}{2\alpha-1}, \tag{2.8}$$

where

$$C(\gamma,\alpha,\lambda_1)=\gamma\sin(\alpha\pi)\int_0^{+\infty}\frac{e^{-rT}r^{\alpha}\,dr}{\gamma^2r^{2\alpha}+\frac{r^2}{\lambda_1^2}+1}. \tag{2.9}$$

Proof See [23]. □

Lemma 2.5 *From* (2.7) *of Lemma* 2.4 *we get*

$$\int_0^T\mathcal{H}_p(\alpha,T-s)\,ds\geq\int_0^T\frac{C(\gamma,\alpha,\lambda_1)}{\lambda_p}\,ds=\frac{TC(\gamma,\alpha,\lambda_1)}{\lambda_p}. \tag{2.10}$$

Next, from (2.10) *by putting* $\inf_{t\in[0,T]}|\chi^{\epsilon}(t)|=\chi_0$ *we have*

$$\frac{1}{\int_0^T\mathcal{H}_p(\alpha,T-s)\chi^{\epsilon}(s)\,ds}\leq\frac{1}{\chi_0}\frac{1}{\int_0^T\mathcal{H}_p(\alpha,T-s))\,ds}\leq\frac{\lambda_p}{\chi_0TC(\gamma,\alpha,\lambda_1)}. \tag{2.11}$$

2.2 The ill-posedness of the inverse source problem

Theorem 2.6 *The inverse source problem is ill-posed.*

Proof Define the linear operator $\mathcal{K}:L^2(\Omega)\to L^2(\Omega)$ by

$$\begin{aligned}\mathcal{K}f(x)&=\sum_{p=1}^{\infty}\left[\int_0^T\mathcal{H}_p(\alpha,s)\chi(s)\,ds\right]\langle f(x),\phi_p(x)\rangle\phi_p(x)\\&=\int_{\Omega}k(x,\omega)f(\omega)\,d\omega,\end{aligned} \tag{2.12}$$

where

$$k(x,\omega) = \sum_{p=1}^{\infty}\left[\int_0^T \mathcal{H}_p(\alpha,s)\chi(s)\,ds\right]\phi_p(x)\phi_p(\omega).$$

Since $k(x,\omega) = k(\omega,x)$, $\mathcal{K}$ is a self-adjoint operator. Next, we will prove its compactness. Define the finite rank operators $\mathcal{K}_N$ by

$$\mathcal{K}_N f(x) = \sum_{p=1}^{N}\left[\int_0^T \mathcal{H}_p(\alpha,s)\chi(s)\,ds\right]\langle f(x),\phi_p(x)\rangle\phi_p(x). \tag{2.13}$$

Then from (2.12) and (2.13) we have

$$\begin{aligned}\|\mathcal{K}_N f - \mathcal{K}f\|^2_{L^2(\Omega)} &= \sum_{p=N+1}^{\infty}\left[\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds\right]^2\left|\langle f(x),\phi_p(x)\rangle\right|^2\\ &\le \|\chi\|^2_{C([0,T])}\sum_{p=N+1}^{\infty}\frac{\mathcal{D}^2}{\lambda_p^2}\frac{T^{2\alpha-1}}{2\alpha-1}\left|\langle f(x),\phi_p(x)\rangle\right|^2\\ &\le \|\chi\|^2_{C([0,T])}\frac{\mathcal{D}^2}{\lambda_N^2}\frac{T^{2\alpha-1}}{2\alpha-1}\sum_{p=N+1}^{\infty}\left|\langle f(x),\phi_p(x)\rangle\right|^2.\end{aligned}$$

This implies that

$$\|\mathcal{K}_N f - \mathcal{K}f\|_{L^2(\Omega)} \le \|\chi\|_{C[0,T]}\frac{DT^{\alpha-\frac{1}{2}}}{\lambda_N\sqrt{2\alpha-1}}\|f\|_{L^2(\Omega)}. \tag{2.14}$$

Therefore $\|\mathcal{K}_N - \mathcal{K}\| \to 0$ in the sense of operator norm in $L(L^2(\Omega);L^2(\Omega))$ as $N \to \infty$. Also, $\mathcal{K}$ is a compact operator. Next, the singular values for the linear self-adjoint compact operator $\mathcal{K}$ are

$$\psi_p = \int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds, \tag{2.15}$$

and the corresponding eigenvectors ϕ_p form an orthonormal basis in $L^2(\Omega)$. From (2.12), the inverse source problem we introduced can be formulated as the operator equation

$$\mathcal{K}f(x) = g(x), \tag{2.16}$$

and by Kirsch [24] we conclude that it is ill-posed. To illustrate ill-posed problems, we present an example. Let us choose the input final data $g^k(x) = \frac{\phi_k(x)}{\sqrt{\lambda_k}}$. By (2.5) the source term corresponding to g^k is

$$\begin{aligned}f^k(x) &= \sum_{p=1}^{\infty}\frac{\langle g^k(x),\phi_p(x)\rangle}{\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds}\phi_p(x) = \sum_{p=1}^{\infty}\frac{\langle\frac{\phi_k(x)}{\sqrt{\lambda_k}},\phi_p(x)\rangle}{\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds}\phi_p(x)\\ &= \frac{\phi_k(x)}{\sqrt{\lambda_k}\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds}.\end{aligned} \tag{2.17}$$

Let us choose the other input final data $g = 0$. By (2.5) the source term corresponding to g is $f = 0$. The error in L^2-norm between two input final data is

$$\|g^k - g\|_{L^2(\Omega)} = \frac{1}{\sqrt{\lambda_k}}. \tag{2.18}$$

Therefore

$$\lim_{k\to+\infty} \|g^k - g\|_{L^2(\Omega)} = \lim_{k\to+\infty} \frac{1}{\sqrt{\lambda_k}} = 0, \tag{2.19}$$

and the error in L^2 norm between the corresponding source terms is

$$\|f^k - f\|^2_{L^2(\Omega)} = \frac{1}{\lambda_k(\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds)^2}. \tag{2.20}$$

Hence

$$\|f^k - f\|_{L^2(\Omega)} = \frac{1}{\sqrt{\lambda_k}\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds}. \tag{2.21}$$

From (2.21), combined with Lemma 2.4, we have

$$\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds \le \|\chi\|_{C[0,T]} \frac{\mathcal{D}}{\lambda_N} \frac{T^{\alpha-\frac{1}{2}}}{\sqrt{2\alpha-1}}. \tag{2.22}$$

We obtain

$$\|f^k - f\|_{L^2(\Omega)} \ge \frac{\sqrt{\lambda_k}}{\mathcal{D}\|\chi\|_{C[0,T]}} \left(\frac{\sqrt{2\alpha-1}}{T^{\alpha-\frac{1}{2}}}\right). \tag{2.23}$$

Since $\alpha > \frac{1}{2}$, this leads to

$$\lim_{k\to+\infty} \|f^k - f\|_{L^2(\Omega)} > \lim_{k\to+\infty} \frac{\sqrt{\lambda_k}}{\mathcal{D}\|\chi\|_{C[0,T]}} \left(\frac{\sqrt{2\alpha-1}}{T^{\alpha-\frac{1}{2}}}\right) = +\infty. \tag{2.24}$$

Combining (2.19) and (2.24), we conclude that the inverse source problem is ill-posed. □

2.3 Conditional stability of source term *f*

In this section, we introduce a conditional stability estimate of this inverse source problem. We impose the following a priori bound on the exact solution $f(x)$:

$$\|f\|_{H^k(\Omega)} \le E, \tag{2.25}$$

where E and k are positive constants. We have the following:

Theorem 2.7 *Let $f \in H^k(\Omega)$ be such that $\|f\|_{H^k(\Omega)} \le E$ for some $E > 0$. Then we have the estimate*

$$\|f\|_{L^2(\Omega)} \le \mathcal{C}(k,T)E^{\frac{1}{k+1}}\|g\|^{\frac{k}{k+1}}_{L^2(\Omega)},$$

where

$$\mathcal{C}(k,T) = \frac{1}{\chi_0^{\frac{k}{k+1}} T^{\frac{k}{k+1}} C^{\frac{k}{k+1}}(\gamma,\alpha,\lambda_1)}. \tag{2.26}$$

Proof From (2.5), using the Hölder inequality, we have

$$\begin{aligned}
\|f\|^2_{L^2(\Omega)} &= \sum_{p=1}^{\infty} \left| \frac{\langle g(x),\phi_p(x)\rangle}{\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds} \right|^2 \\
&= \sum_{p=1}^{\infty} \frac{|\langle g(x),\phi_p(x)\rangle|^{\frac{2}{k+1}} |\langle g(x),\phi_p(x)\rangle|^{\frac{2k}{k+1}}}{|\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds|^2} \\
&\le \left[\sum_{p=1}^{\infty} \frac{|\langle g(x),\phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds|^{2k+2}} \right]^{\frac{1}{k+1}} \left[\sum_{p=1}^{\infty} \left|\langle g(x),\phi_p(x)\rangle\right|^2 \right]^{\frac{k}{k+1}} \\
&\le \left[\sum_{p=1}^{\infty} \frac{|\langle f(x),\phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds|^{2k}} \right]^{\frac{1}{k+1}} \|g\|^{\frac{2k}{k+1}}_{L^2(\Omega)}.
\end{aligned} \tag{2.27}$$

Using Lemma 2.5, we have

$$\begin{aligned}
\sum_{p=1}^{\infty} \frac{|\langle f(x),\phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds|^{2k}} &\le \sum_{p=1}^{\infty} \frac{\lambda_p^{2k}|\langle f(x),\phi_p(x)\rangle|^2}{\chi_0^{2k} T^{2k} C^{2k}(\gamma,\alpha,\lambda_1)} \\
&= \frac{\|f\|^2_{H^k(\Omega)}}{\chi_0^{2k} T^{2k} C^{2k}(\gamma,\alpha,\lambda_1)}.
\end{aligned} \tag{2.28}$$

Combining (2.27) and (2.28), we get

$$\begin{aligned}
\|f\|^2_{L^2(\Omega)} &\le \frac{\|f\|^{\frac{2}{k+1}}_{H^k(\Omega)}}{\chi_0^{\frac{2k}{k+1}} T^{\frac{2k}{k+1}} C^{\frac{2k}{k+1}}(\gamma,\alpha,\lambda_1)} \|g\|^{\frac{2k}{k+1}}_{L^2(\Omega)} \\
&\le \mathcal{C}^2(k,T) E^{\frac{2}{k+1}} \|g\|^{\frac{2k}{k+1}}_{L^2(\Omega)},
\end{aligned} \tag{2.29}$$

where $\mathcal{C}(k,T) = \frac{1}{\chi_0^{\frac{k}{k+1}} T^{\frac{k}{k+1}} C^{\frac{k}{k+1}}(\gamma,\alpha,\lambda_1)}$ is a constant depending on $C(\gamma,\alpha,\lambda_1)$. □

2.4 The Tikhonov regularization method

Applying the Tikhonov regularization method, we solve the inverse source problem, which minimizes the function f in the following quantity in $L^2(\Omega)$:

$$\|\mathcal{K}f - g\|^2_{L^2(\Omega)} + \beta^2 \|f\|^2_{L^2(\Omega)}, \tag{2.30}$$

and its minimized value f_β satisfies

$$\mathcal{K}^*\mathcal{K}f_\beta(x) + \beta^2 f_\beta(x) = \mathcal{K}^* g(x). \tag{2.31}$$

Due to singular value decomposition for compact self-adjoint operator $\mathcal{K}$ as in (2.15), we have

$$f_\beta(x) = \sum_{p=1}^{\infty} \frac{\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} \langle g(x), \phi_p(x)\rangle \phi_p(x). \tag{2.32}$$

If the observed data $(\chi^\epsilon(t), g^\epsilon(x))$ of $(\chi(t), g(x))$ are with noise level ϵ, that is,

$$\|g - g^\epsilon\|_{L^2(\Omega)} \le \epsilon, \qquad \|\chi - \chi^\epsilon\|_{C[0,T]} \le \epsilon, \tag{2.33}$$

then we can present a regularized solution as

$$f_\beta^\epsilon(x) = \sum_{p=1}^{\infty} \frac{\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \langle g^\epsilon(x), \phi_p(x)\rangle \phi_p(x). \tag{2.34}$$

3 The choices of regularization parameter β and convergence results

In this section, we consider an a priori strategy and a posteriori choice rule to find the regularization parameter. Under each choice of the regularization parameter, we can obtain a convergence estimate.

3.1 An a priori choice rule

Choose the regularization parameter β. The next theorem shows that the choice β is valid under suitable assumptions.

Theorem 3.1 *Let f be as in (2.5), and let the noise assumption (2.33) and the a priori condition (2.25) hold. Then the error estimate between the exact solution and its regularized solution is as follows*:

(a) *If $0 < k \le 1$, then by choosing $\beta = (\frac{\epsilon}{E})^{\frac{1}{k+1}}$ we have the convergence estimate*

$$\begin{aligned} &\|f(x) - f_\beta^\epsilon(x)\|_{L^2(\Omega)} \\ &\quad \le \left[\frac{5E^{\frac{1}{k+1}}}{4|\chi_0|\lambda_1^k} + \frac{1}{2} + \sqrt{\left(\frac{1}{2|\chi_0|TC(\gamma,\alpha,\lambda_1)}\right)^2 + 1}\right] E^{\frac{1}{k+1}} \epsilon^{\frac{k}{k+1}}. \end{aligned} \tag{3.1}$$

(b) *If $k > 1$, then by choosing $\beta = (\frac{\epsilon}{E})^{\frac{1}{2}}$ we have the convergence estimate*

$$\|f(x) - f_\beta^\epsilon(x)\|_{L^2(\Omega)} \le \left[\frac{5E^{\frac{1}{2}}}{4|\chi_0|\lambda_1^k} + \frac{1}{2} + \frac{\lambda_1^{1-k}}{2|\chi_0|TC(\gamma,\alpha,\lambda_1)}\right] E^{\frac{1}{2}} \epsilon^{\frac{1}{2}}. \tag{3.2}$$

We first give two lemmas.

Lemma 3.2 *Assume that (2.33) holds. Then we have the estimate*

$$\|f_\beta(x) - f_\beta^\epsilon(x)\|_{L^2(\Omega)} \le \frac{5\epsilon\|f\|_{L^2(\Omega)}}{4|\chi_0|} + \frac{\epsilon}{2\beta}. \tag{3.3}$$

Proof From (2.32) and (2.34) we have

$$\begin{aligned} f_\beta(x) - f_\beta^\epsilon(x) &= \sum_{p=1}^\infty \biggl(\frac{\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} \\ &\quad - \frac{\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \biggr) \bigl\langle g(x), \phi_p(x) \bigr\rangle \phi_p(x) \\ &\quad + \sum_{p=1}^\infty \frac{\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \bigl\langle g(x) - g^\epsilon(x), \phi_p(x) \bigr\rangle \phi_p(x) \\ &= \sum_{p=1}^\infty \frac{\beta^2 \int_0^T \mathcal{H}_p(\alpha, T-s)(\chi(s) - \chi^\epsilon(s))\,ds}{(\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2)(\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2)} \\ &\quad \times \bigl\langle g(x), \phi_p(x) \bigr\rangle \phi_p(x) \\ &\quad + \sum_{p=1}^\infty \frac{\int_0^T \mathcal{H}_p(\alpha, T-s)(\chi(s) - \chi^\epsilon(s))\,ds | \int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} \\ &\quad \times \frac{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \bigl\langle g(x), \phi_p(x) \bigr\rangle \phi_p(x) \\ &\quad + \sum_{p=1}^\infty \frac{\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \bigl\langle g(x) - g^\epsilon(x), \phi_p(x) \bigr\rangle \phi_p(x) \\ &\le \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \mathcal{Q}_1 &= \sum_{p=1}^\infty \frac{\beta^2 \int_0^T \mathcal{H}_p(\alpha, T-s)(\chi(s) - \chi^\epsilon(s))\,ds}{(\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2)(\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2)} \\ &\quad \times \bigl\langle g(x), \phi_p(x) \bigr\rangle \phi_p(x), \\ \mathcal{Q}_2 &= \sum_{p=1}^\infty \frac{\int_0^T \mathcal{H}_p(\alpha, T-s)(\chi(s) - \chi^\epsilon(s))\,ds | \int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} \\ &\quad \times \frac{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \bigl\langle g(x), \phi_p(x) \bigr\rangle \phi_p(x), \\ \mathcal{Q}_3 &= \sum_{p=1}^\infty \frac{\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \bigl\langle g(x) - g^\epsilon(x), \phi_p(x) \bigr\rangle \phi_p(x). \end{aligned}$$

We continue estimating the error in three steps.

Step 1. Estimate of $\|\mathcal{Q}_1\|_{L^2(\Omega)}$. Using the inequality $a^2 + b^2 \ge 2ab$, $a, b \ge 0$, we get

$$\begin{aligned} &\|\mathcal{Q}_1\|_{L^2(\Omega)}^2 \\ &\quad \le \sum_{p=1}^\infty \biggl[\frac{\beta^2 \int_0^T \mathcal{H}_p(\alpha, T-s)(\chi(s) - \chi^\epsilon(s))\,ds}{(\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2)(\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2)} \biggr]^2 \\ &\quad \times \bigl| \bigl\langle g(x), \phi_p(x) \bigr\rangle \bigr|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{p=1}^{\infty} \frac{|\beta^2 \int_0^T \mathcal{H}_p(\alpha, T-s)(\chi(s)-\chi^\epsilon(s))\,ds|^2}{16\beta^4 |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2 |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \big|\langle g(x), \phi_p(x)\rangle\big|^2 \\
&\le \sum_{p=1}^{\infty} \frac{\|\chi-\chi^\epsilon\|_{C[0,T]}^2 |\int_0^T \mathcal{H}_p(\alpha, T-s)\,ds|^2}{16|\chi_0|^2 |\int_0^T \mathcal{H}_p(\alpha, T-s)\,ds|^2} \frac{|\langle g(x), \phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} \\
&\le \frac{\|\chi-\chi^\epsilon\|_{C[0,T]}^2}{16|\chi_0|^2} \sum_{p=1}^{\infty} \big|\langle f(x), \phi_p(x)\rangle\big|^2 = \frac{\|\chi-\chi_\epsilon\|_{C[0,T]}^2}{16|\chi_0|^2} \|f\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{3.5}
$$

Hence

$$
\|\mathcal{Q}_1\|_{L^2(\Omega)} \le \frac{\|\chi-\chi^\epsilon\|_{C[0,T]} \|f\|_{L^2(\Omega)}}{4|\chi_0|} \le \frac{\epsilon \|f\|_{L^2(\Omega)}}{4|\chi_0|}. \tag{3.6}
$$

Step 2. Estimate of $\|\mathcal{Q}_2\|_{L^2(\Omega)}$. We have

$$
\begin{aligned}
&\|\mathcal{Q}_2\|_{L^2(\Omega)}^2 \\
&\quad\le \sum_{p=1}^{\infty} \bigg[\frac{\int_0^T \mathcal{H}_p(\alpha, T-s)(\chi(s)-\chi^\epsilon(s))\,ds| \int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} \\
&\qquad\times \frac{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \bigg]^2 \big|\langle g(x), \phi_p(x)\rangle\big|^2 \\
&\quad\le \sum_{p=1}^{\infty} \frac{(\int_0^T \mathcal{H}_p(\alpha, T-s)(\chi(s)-\chi^\epsilon(s))\,ds)^2}{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2 |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \big|\langle g(x), \phi_p(x)\rangle\big|^2 \\
&\quad\le \sum_{p=1}^{\infty} \frac{\|\chi-\chi^\epsilon\|_{C[0,T]}^2 (\int_0^T \mathcal{H}_p(\alpha, T-s)\,ds)^2}{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \frac{|\langle g(x), \phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} \\
&\quad\le \sum_{p=1}^{\infty} \frac{\|\chi-\chi^\epsilon\|_{C[0,T]}^2}{|\chi_0|^2} \frac{|\langle g(x), \phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} \\
&\quad\le \frac{\|\chi-\chi^\epsilon\|_{C[0,T]}^2}{|\chi_0|^2} \sum_{p=1}^{\infty} \big|\langle f(x), \phi_p(x)\rangle\big|^2 = \frac{\|\chi-\chi^\epsilon\|_{C[0,T]}^2}{|\chi_0|^2} \|f\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Hence

$$
\|\mathcal{Q}_2\|_{L^2(\Omega)} \le \frac{\|\chi-\chi^\epsilon\|_{C[0,T]}}{|\chi_0|} \|f\|_{L^2(\Omega)} \le \frac{\epsilon \|f\|_{L^2(\Omega)}}{|\chi_0|}. \tag{3.7}
$$

Step 3. Estimate of $\|\mathcal{Q}_3\|_{L^2(\Omega)}$.

$$
\begin{aligned}
\|\mathcal{Q}_3\|_{L^2(\Omega)}^2 &\le \sum_{p=1}^{\infty} \bigg(\frac{\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds}{\beta^2 + (\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds)^2} \big\langle g(x)-g^\epsilon(x), \phi_p(x)\big\rangle \bigg)^2 \\
&\le \frac{1}{4\beta^2} \sum_{p=1}^{\infty} \big|\langle g(x)-g^\epsilon(x), \phi_p(x)\rangle\big|^2 \\
&= \frac{1}{4\beta^2} \|g-g^\epsilon\|_{L^2(\Omega)}^2 \le \frac{\epsilon^2}{4\beta^2}.
\end{aligned}
\tag{3.8}
$$

Hence

$$\|\mathcal{Q}_3\|_{L^2(\Omega)} \le \frac{\epsilon}{2\beta}. \tag{3.9}$$

Combining (3.6), (3.7), and (3.9), we get

$$\begin{aligned}\|f_\beta(x) - f_\beta^\epsilon(x)\|_{L^2(\Omega)} &\le \|\mathcal{Q}_1\|_{L^2(\Omega)} + \|\mathcal{Q}_2\|_{L^2(\Omega)} + \|\mathcal{Q}_3\|_{L^2(\Omega)} \\ &\le \frac{\epsilon\|f\|_{L^2(\Omega)}}{4|\chi_0|} + \frac{\epsilon\|f\|_{L^2(\Omega)}}{|\chi_0|} + \frac{\epsilon}{2\beta} \\ &= \frac{5\epsilon\|f\|_{L^2(\Omega)}}{4|\chi_0|} + \frac{\epsilon}{2\beta}. \end{aligned} \tag{3.10}$$

The proof of Lemma 3.2 is completed. □

To obtain the boundedness of bias, we usually need some a priori condition. By Tikhonov's theorem we can restrict L^{-1} to the continuous image of a compact set M. Thus we assume that f is in a compact subset of $L^2(\Omega)$. From now on, we assume that $\|f\|_{H^2(\Omega)} \le E$ for $k > 0$.

Lemma 3.3 *Let $f \in H^k(\Omega)$ and suppose that $\|f\|_{H^k(\Omega)} \le E$ for some $E > 0$. Then we have the estimate*

$$\|f(x) - f_\beta(x)\|_{L^2(\Omega)} \le \begin{cases} E\beta^k\sqrt{(\frac{1}{2|\chi_0|TC(\gamma,\alpha,\lambda_1)})^2 + 1}, & 0 < k < 1, \\ \frac{E\lambda_1^{1-k}}{2|\chi_0|TC(\gamma,\alpha,\lambda_1)}\beta, & k \ge 1. \end{cases} \tag{3.11}$$

Proof From (2.32) and (2.5), using the Parseval identity, we get

$$\begin{aligned}&\|f(x) - f_\beta(x)\|^2_{L^2(\Omega)} \\ &= \sum_{p=1}^{+\infty} \frac{\beta^4|\langle g(x), \phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2[\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2]^2} \\ &= \sum_{p=1}^{+\infty} \frac{\beta^4\lambda_p^{-2k}\lambda_p^{2k}|\langle g(x), \phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2[\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2]^2} \\ &\le \sup_{p\in\mathbb{N}}|\mathcal{M}(p)|^2 \sum_{p=1}^{+\infty} \frac{\lambda_p^{2k}|\langle g(x), \phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} \\ &= \sup_{p\in\mathbb{N}}|\mathcal{M}(p)|^2\|f\|^2_{H^k(\Omega)}. \end{aligned} \tag{3.12}$$

Hence

$$\mathcal{M}(p) = \frac{\beta^2\lambda_p^{-k}}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2}. \tag{3.13}$$

Next, we estimate $\mathcal{M}(p)$. Applying the Cauchy inequality and Lemma 2.5, for $\chi \geq \chi_0$, we get

$$\mathcal{M}(p) \leq \frac{\beta^2 \lambda_p^{-k}}{2\beta \int_0^T \mathcal{H}(p, T-s)\chi(s)\,ds} \leq \frac{\beta \lambda_p^{1-k}}{2|\chi_0| TC(\gamma, \alpha, \lambda_1)}. \tag{3.14}$$

We consider two cases.

Case 1. If $k \geq 1$, then

$$\lambda_p^{1-k} = \frac{1}{\lambda_p^{k-1}} \leq \frac{1}{\lambda_1^{k-1}} = \lambda_1^{1-k}. \tag{3.15}$$

Combining (3.12), (3.13), and (3.14), we obtain

$$\left\| f(x) - f_\beta(x) \right\|_{L^2(\Omega)} \leq \frac{\beta \lambda_1^{1-k}}{2|\chi_0| TC(\gamma, \alpha, \lambda_1)} \|f\|_{H^k(\Omega)} \leq \frac{E\lambda_1^{1-k}}{2|\chi_0| TC(\gamma, \alpha, \lambda_1)} \beta. \tag{3.16}$$

Case 2. $0 < k < 1$. Choose any $q \in (0, 1)$. We $\mathbb{N} = \mathcal{L}_1 \cup \mathcal{L}_2$, where

$$\mathcal{L}_1 = \left\{ p \in \mathbb{N}, \lambda_p^{1-k} \leq \beta^{-q} \right\}, \qquad \mathcal{L}_2 = \left\{ p \in \mathbb{N}, \lambda_p^{1-k} > \beta^{-q} \right\}. \tag{3.17}$$

From (3.12) and (3.17) we have:

$$\begin{aligned}
&\left\| f(x) - f_\beta(x) \right\|_{L^2(\Omega)}^2 \\
&\quad = \sup_{p \in \mathcal{L}_1} \left[\frac{\beta \lambda_p^{1-k}}{2|\chi_0| TC(\gamma, \alpha, \lambda_1)} \right]^2 \sum_{p \in \mathcal{L}_1} \lambda_p^{2k} \left| \langle f(x), \phi_p(x) \rangle \right|^2 \\
&\qquad + \sum_{p \in \mathcal{L}_2} \left[\frac{\beta^2 \lambda_p^{-k}}{\beta^2 + |\int_0^T \mathcal{H}(p, T-s)\chi(s)\,ds|^2} \right]^2 \lambda_p^{2k} \left| \langle f(x), \phi_p(x) \rangle \right|^2 \\
&\quad \leq \left[\frac{1}{2|\chi_0| TC(\gamma, \alpha, \lambda_1)} \right]^2 \beta^{2-2q} \|f\|_{H^k(\Omega)}^2 + \sup_{p \in \mathcal{L}_2} \lambda_p^{-2k} \sum_{p=1}^{\mathcal{L}_2} \lambda_p^{2k} \left| \langle f(x), \phi_p(x) \rangle \right|^2 \\
&\quad \leq \left[\frac{1}{2|\chi_0| TC(\gamma, \alpha, \lambda_1)} \right]^2 \beta^{2-2q} \|f\|_{H^k(\Omega)}^2 + \beta^{\frac{2qk}{1-k}} \|f\|_{H^k(\Omega)}^2.
\end{aligned} \tag{3.18}$$

Choosing $q = 1 - k$, from $\|f(x)\|_{H^k(\Omega)} \leq E$ we have

$$\begin{aligned}
\left\| f(x) - f_\beta(x) \right\|_{L^2(\Omega)}^2 &\leq \left(\frac{1}{2|\chi_0| TC(\gamma, \alpha, \lambda_1)} \right)^2 \beta^{2k} E^2 + \beta^{2k} E^2 \\
&= \beta^{2k} E^2 \left(\left(\frac{1}{2|\chi_0| TC(\gamma, \alpha, \lambda_1)} \right)^2 + 1 \right).
\end{aligned} \tag{3.19}$$

This implies that

$$\left\| f(x) - f_\beta(x) \right\|_{L^2(\Omega)} \leq \beta^k E \sqrt{\left(\frac{1}{2|\chi_0| TC(\gamma, \alpha, \lambda_1)} \right)^2 + 1}. \tag{3.20}$$

□

Now we continue to prove Theorem 3.1.

If $0 \le k \le 1$, then from Lemmas 3.2 and 3.3 we have

$$\begin{aligned} &\left\| f(x) - f_\beta^\epsilon(x) \right\|_{L^2(\Omega)} \\ &\quad \le \left\| f(x) - f_\beta(x) \right\|_{L^2(\Omega)} + \left\| f_\beta(x) - f_\beta^\epsilon(x) \right\|_{L^2(\Omega)} \\ &\quad \le \frac{5\epsilon \|f\|_{L^2(\Omega)}}{4|\chi_0|} + \frac{\epsilon}{2\beta} + \beta^k E \sqrt{\left(\frac{1}{2|\chi_0| TC(\gamma,\alpha,\lambda_1)} \right)^2 + 1}. \end{aligned} \tag{3.21}$$

By choosing the parameter regularization

$$\beta = \left(\frac{\epsilon}{E} \right)^{\frac{1}{k+1}} \quad \text{and} \quad \|f\|_{L^2(\Omega)} \le \frac{1}{\lambda_1^k} \|f\|_{H^k(\Omega)} \le \frac{E}{\lambda_1^k}$$

we obtain

$$\begin{aligned} &\left\| f(x) - f_\beta^\epsilon(x) \right\|_{L^2(\Omega)} \\ &\quad \le \frac{5E}{4|\chi_0|\lambda_1^k}\epsilon + \frac{1}{2} E^{\frac{1}{k+1}} \epsilon^{\frac{k}{k+1}} + E^{\frac{1}{k+1}} \epsilon^{\frac{k}{k+1}} \sqrt{\left(\frac{1}{2|\chi_0| TC(\gamma,\alpha,\lambda_1)} \right)^2 + 1} \\ &\quad \le \left[\frac{5E^{\frac{k}{k+1}} \epsilon^{\frac{1}{k+1}}}{4|\chi_0|\lambda_1^k} + \frac{1}{2} + \sqrt{\left(\frac{1}{2|\chi_0| TC(\gamma,\alpha,\lambda_1)} \right)^2 + 1} \right] E^{\frac{1}{k+1}} \epsilon^{\frac{k}{k+1}}. \end{aligned} \tag{3.22}$$

If $k > 1$, then from Lemmas 3.2 and 3.3 we have the estimate

$$\begin{aligned} \left\| f(x) - f_\beta^\epsilon(x) \right\|_{L^2(\Omega)} &\le \left\| f(x) - f_\beta(x) \right\|_{L^2(\Omega)} + \left\| f_\beta(x) - f_\beta^\epsilon(x) \right\|_{L^2(\Omega)} \\ &\le \frac{5\epsilon \|f\|_{L^2(\Omega)}}{4|\chi_0|} + \frac{1}{2}\frac{\epsilon}{\beta} + \frac{\beta \lambda_1^{1-k}}{2|\chi_0| TC(\gamma,\alpha,\lambda_1)} E. \end{aligned} \tag{3.23}$$

Since $\beta = (\frac{\epsilon}{E})^{\frac{1}{2}}$ and $\|f\|_{L^2(\Omega)} \le \frac{1}{\lambda_1^k} \|f\|_{H^k(\Omega)} \le \frac{E}{\lambda_1^k}$, we have

$$\begin{aligned} \left\| f(x) - f_\beta^\epsilon(x) \right\|_{L^2(\Omega)} &\le \frac{5E}{4|\chi_0|\lambda_1^k}\epsilon + \frac{1}{2} E^{\frac{1}{2}} \epsilon^{\frac{1}{2}} + \frac{\lambda_1^{1-k}}{2|\chi_0| TC(\gamma,\alpha,\lambda_1)} E^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \\ &\le \left[\frac{5E^{\frac{1}{2}} \epsilon^{\frac{1}{2}}}{4|\chi_0|\lambda_1^\gamma} + \frac{1}{2} + \frac{\lambda_1^{1-k}}{2|\chi_0| TC(\gamma,\alpha,\lambda_1)} \right] E^{\frac{1}{2}} \epsilon^{\frac{1}{2}}. \end{aligned} \tag{3.24}$$

3.2 An a posteriori parameter choice

In this section, we consider the choice of the a posteriori regularization parameter in Morozov's discrepancy principle (see in [1]). Suppose $\tau > 1$ is a given fixed constant.

Choose the regularization parameter $\beta = \beta(\epsilon)$ as the solution of the equation

$$\left\| \mathcal{K} f_\beta^\epsilon - g^\epsilon \right\|_{L^2(\Omega)} = \tau \epsilon, \tag{3.25}$$

where $0 < \tau\epsilon \le \|g^\epsilon\|_{L^2(\Omega)}$.

Lemma 3.4 *Set $\mathcal{P}(\beta) = \|\mathcal{K} f_\beta^\epsilon - g^\epsilon\|_{L^2(\Omega)}$. Then we have:*

(a) *$\mathcal{P}(\beta)$ is a continuous function.*

(b) $\mathcal{P}(\beta) \to 0$ *as* $\beta \to 0$.
(c) $\mathcal{P}(\beta) \to \|g^\epsilon\|_{L^2(\Omega)}$ *as* $\beta \to \infty$.
(d) $\mathcal{P}(\beta)$ *is a strictly decreasing function for any* $m \in (0, +\infty)$.

Lemma 3.4 shows that there exists a unique solution of equation (3.25).

Lemma 3.5 *If* (3.25) *holds, then the regularization parameter* β *satisfies*

$$\beta \geq \frac{2\lambda_1^k |\chi_0|(\tau - 1)\epsilon}{|\chi_1| E}. \tag{3.26}$$

Proof By (2.34), for every $g^\epsilon \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\left\| \mathcal{K} f_\beta^\epsilon - g^\epsilon \right\|_{L^2(\Omega)}^2 = \sum_{p=1}^{\infty} \left(\frac{\beta^2}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \right)^2 \left\| g^\epsilon \right\|_{L^2(\Omega)}^2. \tag{3.27}$$

Using (3.25), we obtain

$$\tau\epsilon = \sum_{p=1}^{\infty} \left(\frac{\beta^2}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \right) \left\| g^\epsilon \right\|_{L^2(\Omega)}. \tag{3.28}$$

We have

$$\left\| g^\epsilon \right\|_{L^2(\Omega)} \leq \left\| g^\epsilon - g \right\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}. \tag{3.29}$$

Combining (3.28), (3.29), and $\frac{\beta^2}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} < 1$, we obtain

$$\tau\epsilon \leq \left\| g^\epsilon - g \right\|_{L^2(\Omega)} + \sum_{p=1}^{\infty} \left(\frac{\beta^2}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \right) \|g\|_{L^2(\Omega)}. \tag{3.30}$$

Because $\tau > 1$, by (1.2) we get

$$(\tau - 1)\epsilon \leq \sum_{p=1}^{\infty} \left(\frac{\beta^2}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \right) \|g\|_{L^2(\Omega)}. \tag{3.31}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{p=1}^{\infty} \left(\frac{\beta^2}{\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2} \right)^2 \|g\|_{L^2(\Omega)}^2 \\
&\quad \leq \sum_{p=1}^{\infty} \left(\frac{\beta^2 \int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds}{[\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2]\lambda_p^k} \right)^2 \frac{\lambda_p^{2k} |\langle g(x), \phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} \\
&\quad \leq \sum_{p=1}^{\infty} \left(\frac{\beta^2 \int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds}{[\beta^2 + |\int_0^T \mathcal{H}_p(\alpha, T-s)\chi^\epsilon(s)\,ds|^2]\lambda_p^k} \right)^2 \frac{\lambda_p^{2k} |\langle g(x), \phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2} \\
&\quad \leq \sum_{p=1}^{\infty} \mathcal{E}(p)^2 \frac{\lambda_p^{2k} |\langle g(x), \phi_p(x)\rangle|^2}{|\int_0^T \mathcal{H}_p(\alpha, T-s)\chi(s)\,ds|^2},
\end{aligned} \tag{3.32}$$

where $\mathcal{E}(p) = \frac{\beta^2 \int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds}{(\beta^2+|\int_0^T \mathcal{H}_p(\alpha,T-s)\chi^\epsilon(s)\,ds|^2)\lambda_p^k}$. We estimate $\mathcal{E}(p)$ as follows:

$$\begin{aligned}\mathcal{E}(p) &= \frac{\beta^2 \int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds}{(\beta^2+|\int_0^T \mathcal{H}_p(\alpha,T-s)\chi^\epsilon(s)\,ds|^2)\lambda_p^k} \le \frac{\beta^2 \int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds}{2\lambda_p^k\beta \int_0^T \mathcal{H}_p(\alpha,T-s)\chi^\epsilon(s)\,ds} \\ &\le \frac{\beta|\chi_1|}{2\lambda_1^k|\chi_0|}. \end{aligned} \tag{3.33}$$

Therefore, combining (3.31), (3.32), and (3.33), we conclude that

$$(\tau-1)\epsilon \le \frac{\beta|\chi_1|}{2\lambda_1^k|\chi_0|}\|f\|_{H^k(\Omega)} \le \frac{\beta|\chi_1|E}{2\lambda_1^k|\chi_0|}. \tag{3.34}$$

Hence we obtain

$$\beta \ge \frac{2\lambda_1^k|\chi_0|(\tau-1)\epsilon}{|\chi_1|E} \tag{3.35}$$

which gives the required result. □

Theorem 3.6 *Suppose the a priori conditions* (1.2) *and* (2.25) *hold and the regularization parameter* β *is given by* (3.25). *Then we have the error estimate*

$$\begin{aligned}&\big\|f(x)-f_\beta^\epsilon(x)\big\|_{L^2(\Omega)} \\ &\quad\le \mathcal{C}(k,T)E^{\frac{1}{k+1}}(\tau+1)^{\frac{k}{k+1}}\epsilon^{\frac{k}{k+1}} + \frac{5\epsilon E}{4|\chi_0|\lambda_1^k} + \frac{|\chi_1|E}{4\lambda_1^k|\chi_0|(\tau-1)}.\end{aligned} \tag{3.36}$$

Proof By the triangle inequality we have

$$\big\|f(x)-f_\beta^\epsilon(x)\big\|_{L^2(\Omega)} \le \big\|f(x)-f_\beta(x)\big\|_{L^2(\Omega)} + \big\|f_\beta(x)-f_\beta^\epsilon(x)\big\|_{L^2(\Omega)}. \tag{3.37}$$

Using $\|f\|_{L^2(\Omega)} \le \frac{1}{\lambda_1^k}\|f\|_{H^k(\Omega)} \le \frac{E}{\lambda_1^k}$, from Lemmas 3.2 and 3.5 we obtain

$$\big\|f_\beta(x)-f_\beta^\epsilon(x)\big\|_{L^2(\Omega)} \le \frac{5\epsilon\|f\|_{L^2(\Omega)}}{4|\chi_0|} + \frac{\epsilon}{2\beta} \tag{3.38}$$

$$\le \frac{5\epsilon E}{4|\chi_0|\lambda_1^k} + \frac{|\chi_1|E}{4\lambda_1^k|\chi_0|(\tau-1)}. \tag{3.39}$$

For the first part of the right-hand side of (3.37), we have

$$\begin{aligned}&\mathcal{K}f(x)-\mathcal{K}f_\beta(x) \\ &\quad= \sum_{p=1}^\infty \frac{\beta^2}{\beta^2+|\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds|^2}\big\langle g(x),\phi_p(x)\big\rangle\phi_p(x) \\ &\quad= \sum_{p=1}^\infty \frac{\beta^2}{\beta^2+|\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds|^2}\big\langle g(x)-g^\epsilon(x),\phi_p(x)\big\rangle\phi_p(x) \\ &\qquad+ \sum_{p=1}^\infty \frac{\beta^2}{\beta^2+|\int_0^T \mathcal{H}_p(\alpha,T-s)\chi(s)\,ds|^2}\big\langle g^\epsilon(x),\phi_p(x)\big\rangle\phi_p(x).\end{aligned}$$

Using (1.2) and (3.25), we get

$$\left\|\mathcal{K}f(x)-\mathcal{K}f_\beta(x)\right\|_{L^2(\Omega)}\le(\tau+1)\epsilon. \tag{3.40}$$

We also have

$$\begin{aligned}
&\left\|f(x)-f_\beta(x)\right\|^2_{H^k(\Omega)}\\
&\quad=\sum_{p=1}^{+\infty}\left(\frac{\beta^2}{\beta^2+|\int_0^T\mathcal{H}_p(\alpha,T-s)\chi^\epsilon(s)\,ds|^2}\right)^2\frac{\lambda_p^{2k}|\langle g(x),\phi_p(x)\rangle|^2}{|\int_0^T\mathcal{H}_p(\alpha,T-s)\chi(s)\,ds|^2}\\
&\quad\le\sum_{p=1}^{+\infty}\frac{\lambda_p^{2k}|\langle g(x),\phi_p(x)\rangle|^2}{|\int_0^T\mathcal{H}_p(\alpha,T-s)\chi(s)\,ds|^2}=E^2.
\end{aligned}$$

From Theorem 2.7 we have

$$\left\|f(x)-f_\beta(x)\right\|_{L^2(\Omega)}\le\mathcal{C}(k,T)E^{\frac{1}{k+1}}(\tau+1)^{\frac{k}{k+1}}\epsilon^{\frac{k}{k+1}}. \tag{3.41}$$

Therefore

$$\left\|f(x)-f^\epsilon_\beta(x)\right\|_{L^2(\Omega)}\le\mathcal{C}(k,T)E^{\frac{1}{k+1}}(\tau+1)^{\frac{k}{k+1}}\epsilon^{\frac{k}{k+1}}+\frac{5\epsilon E}{4|\chi_0|\lambda_1^k}+\frac{|\chi_1|E}{4\lambda_1^k|\chi_0|(\tau-1)}.$$ □

4 Numerical experiment

In this section, we present a numerical result with $\Omega=(0,1)$. Recall that the problem is given by

$$\begin{cases}\partial_t u-(1+\gamma\partial_t^\alpha)\Delta u=f(x)\chi(t), & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & x\in\partial\Omega,\\ u(x,0)=u_0(x), & x\in\Omega,\\ u(x,T)=g(x), & x\in\Omega.\end{cases} \tag{4.1}$$

Fix the parameter $\gamma=1$. The couple $(g^\epsilon,\chi^\epsilon)$ determined below plays the role of measured data with random noise:

$$\begin{aligned}g^\epsilon(\cdot)&=g(\cdot)+\epsilon\,\text{rand}(\cdot),\\ \chi^\epsilon(\cdot)&=\chi(\cdot)+\epsilon\,\text{rand}(\cdot),\end{aligned} \tag{4.2}$$

where rand() $\in(-1,1)$ is a random number. We can easily verify the validity of the inequality

$$\left\|g-g^\epsilon\right\|_{L^2(\Omega)}\le\epsilon,\qquad\left\|\chi-\chi^\epsilon\right\|_{C[0,T]}\le\epsilon. \tag{4.3}$$

In (4.1), we have $u(x,t)=\sin(\pi x)t^{\alpha+1}$. By a simple calculation we get $f(x)=\sin(\pi x)$ and $\chi(t)=(\alpha+1)t^\alpha+\pi^2t^{\alpha+1}+\frac{\pi^2(\alpha+1)}{\Gamma(2)}t$. Combining this with (4.2), we get

$$\begin{aligned}\chi^\epsilon(s)&=(\alpha+1)s^\alpha+\pi^2s^{\alpha+1}+\frac{\pi^2(\alpha+1)}{\Gamma(2)}s+\epsilon\,\text{rand}(\cdot),\\ g^\epsilon(x)&=\sin(\pi x)+\epsilon\,\text{rand}(\cdot).\end{aligned} \tag{4.4}$$

Following (2.32), f can be rewritten as

$$f(x) = \sum_{p=1}^{\infty} \frac{\langle g(x), \phi_p(x)\rangle}{\int_0^T \mathcal{H}(p, T-s)\chi(s)\,ds}\phi_p(x). \tag{4.5}$$

Next, we can rewrite the term $\mathcal{H}(p, T-s)$ as follows:

$$\mathcal{H}(p, T-s) = \int_0^{\infty} e^{-r(T-s)}K_p(r)\,dr = \lim_{M\to\infty}\int_0^M e^{-r(T-s)}K_p(r)\,dr. \tag{4.6}$$

Combining (4.5) and (4.6), we get

$$\begin{aligned} f(x) &= \sum_{p=1}^{\infty} \frac{\langle g(x), \phi_p(x)\rangle}{\int_0^T \mathcal{H}(p, T-s)\chi(s)\,ds}\phi_p(x) \\ &= \sum_{p=1}^{\infty} \frac{\langle g(x), \phi_p(x)\rangle}{\lim_{M\to\infty}\int_0^T (\int_0^M e^{-r(T-s)}K_p(r)\,dr)\chi(s)\,ds}\phi_p(x) \end{aligned}$$

with M large enough. However, to be able to calculate, we choose $M = 300$. Using the composite Simpson rule of numerical integration in Matlab, we have the following approximates of $f \in L^2(0,1)$:

$$\int_0^1 G(x)\,dx \approx \frac{1}{3N_x}\sum_{k=1}^{N_x/2}\bigl[G(x_{2k-2}) + 4G(x_{2k-1}) + G(x_{2k}))\bigr],$$

where $x_k = \frac{k}{N_x}$, $x_0 = 0$, $x_{N_x} = 1$.

Similarly, we have the approximates of $f_\beta^\epsilon \in L^2(0,1)$. In practice, it is very difficult to obtain the value of M for the a priori parameter choice rule without having an exact solution. We thus try taking $\|f\|_{H^2(\Omega)} \le M$ with $E \approx 10$, leading to $\beta_{\text{pri}} = (\frac{\epsilon}{E})^{\frac{1}{2}}$ for the a priori parameter choice rule and $\beta_{\text{pos}} = \frac{\epsilon}{E(\tau,|\chi_0|,|\chi_1|,\alpha)}$ for the a posteriori parameter choice rule based on τ. Of course, choosing τ and α different, we have different β_{pos} with $E(\tau,|\chi_0|,|\chi_1|,\alpha) = \frac{|\chi_1|}{\sqrt{2}\lambda_1^k(\tau^2-2)|\chi_0|}\|f\|_{H^2(\Omega)}$.

In general, the whole numerical procedure is summarized in the following steps.

Step 1. As the discretization level, a uniform grid of mesh-point (x_i, t_j) is used to discretize the space and time intervals:

$$x_i = i\Delta x, \qquad \Delta x = \frac{1}{N_x}, \quad i = \overline{0, N_x}, \qquad t_j = j\Delta t, \qquad \Delta t = \frac{1}{N_t}, \quad j = \overline{0, N_t}. \tag{4.7}$$

Of course, higher values of N_x and N_t will provide more accurate and stable numerical results. In this example, we take $N_x = N_t = 512$.

Step 2. Setting $f_\beta^\epsilon(x_i) = f_{\beta,i}^\epsilon$ and $f(x_i) = f_i$, we construct two vectors containing all discrete values of f_β^ϵ and f, denoted by Λ_β^ϵ and Ψ, respectively:

$$\begin{aligned} \Lambda_\beta^\epsilon &= \begin{bmatrix} f_{\beta,0}^\epsilon & f_{\beta,1}^\epsilon & \cdots & f_{\beta,N_x}^\epsilon \end{bmatrix} \in \mathbb{R}^{N_x+1}, \\ \Psi &= \begin{bmatrix} f_0 & f_1 & \cdots & f_{N_x-1} & f_{N_x} \end{bmatrix} \in \mathbb{R}^{N_x+1}. \end{aligned} \tag{4.8}$$

Table 1 Error estimates between the exact and regularized solutions for $\tau = 1.6$, $\alpha \in \{0.65, 0.75, 0.85, 0.95\}$

	ϵ		
	0.1	0.01	0.001
	$\alpha = 0.65$		
$Err^{\beta_{\text{pri}}}$	0.067015108159255	0.047468902109316	0.041441591914833
$Err^{\beta_{\text{pos}}}$	0.098997404519191	0.044495631182970	0.040535488144887
	$\alpha = 0.75$		
$Err^{\beta_{\text{pri}}}$	0.084761583230752	0.028602688042509	0.024458932308338
$Err^{\beta_{\text{pos}}}$	0.053156432449751	0.028231628315379	0.024378712621981
	$\alpha = 0.85$		
$Err^{\beta_{\text{pri}}}$	0.130209694916768	0.021179319121018	0.015465349519243
$Err^{\beta_{\text{pos}}}$	0.122475577340357	0.024601414585993	0.015239026338959
	$\alpha = 0.95$		
$Err^{\beta_{\text{pri}}}$	0.037276991722023	0.010256764619097	0.008742004875893
$Err^{\beta_{\text{pos}}}$	0.134846446943958	0.010076794618172	0.009621545639896

Table 2 Error estimates between the exact and regularized solutions for $\tau = 1.7$, $\alpha \in \{0.65, 0.75, 0.85, 0.95\}$

	ϵ		
	0.1	0.01	0.001
	$\alpha = 0.65$		
$Err^{\beta_{\text{pri}}}$	0.111131774567399	0.047840847429863	0.040796403046666
$Err^{\beta_{\text{pos}}}$	0.104219494973211	0.047373969253811	0.040704667266564
	$\alpha = 0.75$		
$Err^{\beta_{\text{pri}}}$	0.042292184968334	0.032976631468816	0.025075858532616
$Err^{\beta_{\text{pos}}}$	0.118817648652812	0.028123860184860	0.024430674882777
	$\alpha = 0.85$		
$Err^{\beta_{\text{pri}}}$	0.130527372052525	0.022400766773086	0.014919116713563
$Err^{\beta_{\text{pos}}}$	0.082465125249822	0.020184320450563	0.014652593632991
	$\alpha = 0.95$		
$Err^{\beta_{\text{pri}}}$	0.045333767253358	0.011699213808191	0.008763429831879
$Err^{\beta_{\text{pos}}}$	0.044277712631869	0.008651113194266	0.008905712655202

Step 3. Error estimate between the exact and regularized solutions:

$$\mathbb{E} = \frac{\sqrt{\sum_{i=0}^{N_x} |f_\beta^\epsilon(x_i) - f(x_i)|^2_{L^2(0,1)}}}{\sqrt{\sum_{i=0}^{N_x} |f(x_i)|^2_{L^2(0,1)}}}. \tag{4.9}$$

The numerical results are summarized in Tables 1, 2, 3.

Table 1 show the relative error estimates between the exact solution and its regularized solution, both a priori and a posteriori, at $\tau = 1.6$ with $\alpha = 0.65$, $\alpha = 0.75$, $\alpha = 0.85$, and $\alpha = 0.95$. Table 2 show the relative error estimates between the exact solution and its regularized solution, both a priori and a posteriori at $\tau = 1.7$ with $\alpha = 0.65$, $\alpha = 0.75$, $\alpha = 0.85$, and $\alpha = 0.95$. In Tables 1 and 2, we calculate with values of $\epsilon = 10^{-1}, 10^{-2}$ and $\epsilon = 10^{-3}$. In Table 3, with two values $\epsilon = 10^{-4}$ and $\epsilon = 10^{-5}$, $\tau = 1.8$, we choose $\alpha = 0.52$, $\alpha = 0.62$, $\alpha = 0.72$, $\alpha = 0.82$, and $\alpha = 0.92$. In general, we see that the posterior parameter choice

Table 3 Error estimates between the exact and regularized solutions for $\tau = 1.8$, $\alpha \in \{0.52, 0.62, 0.72, 0.82, 0.92\}$

α	$Err^{\beta}{}^{\text{pri}}$	$Err^{\beta}{}^{\text{pos}}$
	$\epsilon = 0.0001$	
0.52	0.077602462311104	0.077544342977239
0.62	0.047271805697450	0.047192018901400
0.72	0.028296574111022	0.028307819519103
0.82	0.016920298564528	0.016962577259079
0.92	0.010151270279964	0.010110524205060
	$\epsilon = 0.00001$	
0.52	0.077472454648553	0.077480496522115
0.62	0.047159886813125	0.047166399068891
0.72	0.028302374036182	0.028294614661429
0.82	0.016904097025631	0.016899494878404
0.92	0.010096298360581	0.010097434555097

rule method converges to the exact solution faster than the prior parameter choice rule method. We also see that our proposed regularized methods have very good convergence rates to the exact solution as ϵ tends to 0.

5 Concluding remarks

In this work, we have studied the inverse source problem for the Rayleigh–Stokes equation in a second-grade generalized flow. We introduce a Tikhonov regularized method to establish an approximate solution. Then we prove an upper bound on the rate of convergence of the mean integrated squared error under some a priori condition of the sought solution. In the future work, we will try to study a numerical method for solving the ill-posedness of our inverse source problem.

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