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Identifying Asymmetric Comovements of International Stock Market Returns

by Fuchun Li

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Fuchun Li

Financial Stability Department
Bank of Canada
Ottawa, Ontario, Canada K1A 0G9
fuchunli@bankofcanada.ca

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Abstract

Based on a new approach for measuring the comovements between stock market returns, we provide a nonparametric test for asymmetric comovements in the sense that stock market downturns will lead to stronger comovements than market upturns. The test is used to detect whether asymmetric comovements exist in international stock markets. We find the following empirical facts. First, asymmetric comovements exist between the United States (U.S.) stock market and the stock markets for Canada, France, Germany, and the United Kingdom (U.K.), but the data are unable to reject the null hypothesis of the symmetric comovements between the U.S. and Japanese stock markets. Second, either a larger negative drop or a positive increase in stock prices leads to stronger comovements of stock market returns, indicating that comovements in the data are different from comovements implied by a bivariate symmetric distribution, which implies that comovements tend to zero as the market returns become more positive or more negative.

JEL classification: G150, G19, F210, C490
Bank classification: Financial stability; Financial system regulation and policies; International topics; Econometric and statistical methods

Résumé

Proposant une nouvelle approche en matière de mesure des covariations entre les rendements boursiers, l'auteur applique un test non paramétrique en vue d'établir si les covariations entre les marchés boursiers internationaux sont asymétriques, en l'occurrence, plus prononcées en phase de baisse qu'en phase de hausse des marchés. Il fait les constats empiriques suivants. D'abord, il existe des covariations asymétriques entre le marché boursier des États-Unis et ceux du Canada, de la France, de l'Allemagne et du Royaume-Uni, mais les données ne permettent pas de rejeter l'hypothèse nulle de symétrie des covariations entre le marché boursier des États-Unis et celui du Japon. Ensuite, plus la diminution ou la montée des cours boursiers est importante, plus les covariations entre les rendements boursiers sont fortes. Ce résultat est contraire à celui qu'implique une distribution symétrique bivariée, à savoir que les covariations tendent vers zéro lorsque les rendements boursiers deviennent très positifs ou très négatifs.

Classification JEL : G150, G19, F210, C490
Classification de la Banque : Stabilité financière; Réglementation et politiques relatives au système financier; Questions internationales; Méthodes économétriques et statistiques

1 Introduction

The study of the comovements between asset markets is a central issue in finance as it has important practical implications in asset allocation and risk management. Since the seminal work of Grubel (1968) on the benefits of international portfolio diversification (see also, Levy and Sarnat (1979) and Agmon (1972)), this topic has received special attention in international finance. In fact, a growing body of literature has emerged more recently on studying comovements of international stock market returns (see, King et al. (1994), Lin et al. (1994), Longin and Solnik (2001), Karolyi and Stulz (1996), Forbes and Rigobon (2002), Brooks and Del Negro (2006), Okimoto (2008), among others). In particular, many of these studies present empirical evidence suggesting an asymmetric pattern in comovements of international stock market returns, in the sense that stock returns show stronger comovements when they go down than when they go up.

All the empirical evidence mentioned above in favor of asymmetric comovements is based on the Pearson correlation coefficient as the measure of comovements.[1] It is well known that the validity of the Pearson correlation coefficient as the measure of comovements crucially depends on the assumptions that the relationship between two variables is linear and that the two variables are jointly normally distributed. However, a number of empirical studies have documented that a linear relationship based on the normal distribution assumption clearly fails to explain the stylized facts observed in data and that it is highly undesirable to perform various policy evaluations, risk management and financial forecasts (Granger (2002), Rodriguez (2007), and papers therein).

The necessity of searching for reliable measures of comovements has prompted researchers to use alternative approaches to measure comovements in international stock markets. For example,

[1] Pearson correlation coefficient between two random variables x and y with expected values μ_x and μ_y and standard deviations σ_x and σ_y is defined as $\rho_{x,y} = \frac{E[(x-\mu_x)(y-\mu_y)]}{\sigma_x \sigma_y}$.

using Kendall's tau coefficient as a measure of comovements, Rodrigue (2007) studies financial contagion of international stock markets, and Manner (2010) tests for asymmetric comovements. Even though Kendall's tau coefficient can capture nonlinear relationships that are not possible to measure with Pearson correlation coefficient, it depends on a basic assumption that the bivariate observations are mutually independent, which violates the visible stylized facts in empirical finance that the autocorrelation of stock returns exists at fixed intervals. Recently, many researchers have used a copula approach to model the behaviors of comovements in stock returns. The copula approach is a useful tool in modeling the possibly nonlinear comovements, but it cannot be used directly to test whether there exist asymmetric comovements in the data.

This paper proposes a new nonparametric approach to measure the comovements between two stock returns. The approach is based on the difference between the probability that the two returns simultaneously move up or down and the probability that they move up or down in opposite directions. Since the approach does not impose either a linear relationship or a restriction of the function form of the joint distribution of the two returns, it allows for maximal flexibility in fitting the data. It also does not suffer from the heteroscedasticity associated with Pearson correlation (Forbes and Rigobon, 2002). Given the nonparametric measure of comovements, we propose a nonparametric test for asymmetric comovements in the sense that market downturns lead to stronger comovements of stock returns than do market upturns. The test statistic is shown to follow an asymptotic chi-square distribution under the null hypothesis that the strength of the downside comovements is not significantly different from the strength of the upside comovements, where the downside comovements are defined as the comovements resulting from market turndowns, and similarly the upside comovements occur after market upturns.

Monte Carlo simulations show that our test performs rather well in finite samples. The test is applied to detect whether asymmetric comovements exist between the U.S. stock market and some main international stock markets. We find the following empirical facts. First, asymmetric comovements exist between the U.S. stock market and the stock markets of Canada, France, Germany, and U.K., but the test is unable to reject the null hypothesis of symmetric comovements of stock markets between U.S. and Japan. Second, either a larger negative drop or a positive increase in stock prices tends to lead to stronger comovements, which indicate that comovements in the data differ from comovements implied by a symmetric distribution suggesting that comovements tend to zero as returns become positive or negative larger.

The remainder of this paper is organized as follows. In section 2, we propose a new measure of the comovements between two stock market returns. Based on the measure of comovements, we propose a nonparametric test for asymmetric comovements of stock market returns. The size and power performances of the test are examined by Monte Carlo study in section 3. In section 4, the test is applied to investigate whether asymmetric comovements exist between the U.S. stock market and main international stock markets. Section 5 offers some conclusions. The proofs are provided in the Appendix.

2 A Nonparametric Test for Asymmetric Comovements

In this section, we propose a nonparametric approach to measure comovements between two stock market returns. Based on this approach, a nonparametric test is then proposed to detect if market turndowns lead to stronger comovements than market upturns.

2.1 A Nonparametric Measure of Comovements

Let R_t^1 and R_t^2 denote the logarithmic returns on two stock markets in period of time t. Suppose R_t^1 and R_t^2 are standardized to have zero mean and unit variance in order to simplify both the computation and statistical analysis. We begin by formally defining the measure of comovements between R_t^1 and R_t^2 as,

$$cm(R^1,R^2) \equiv Pr[(R_{t+1}^1 - R_t^1)(R_{t+1}^2 - R_t^2) > 0] - Pr[(R_{t+1}^1 - R_t^1)(R_{t+1}^2 - R_t^2) < 0], \tag{1}$$

where $Pr[(R_{t+1}^1 - R_t^1)(R_{t+1}^2 - R_t^2) > 0]$ is the probability that R_t^1 moves up or down to R_{t+1}^1 whenever R_t^2 moves up or down to R_{t+1}^2, while $Pr[(R_{t+1}^1 - R_t^1)(R_{t+1}^2 - R_t^2) < 0]$ is the probability that R_t^1 moves up or down to R_{t+1}^1 in the opposite direction as R_t^2 moves to R_{t+1}^2. If $cm(R^1,R^2) > 0$, we say that the movement from (R_t^1,R_t^2) to (R_{t+1}^1,R_{t+1}^2) is a concordant comovement, and that otherwise it is discordant. Intuitively, the concordant comovement indicates that the two returns have more opportunities to move up or down simultaneously. Since the measure of comovements does not impose either a linear relationship or any assumption on the joint distribution of the two returns, it allows for maximal flexibility in fitting the data. The measure of comovements satisfies the following important properties:

(i) it is defined for every pair of R_t^1 and R_t^2;

(ii) it is bounded between -1 and 1, i.e., $-1 \leq cm(R^1,R^2) \leq 1$;

(iii) it is symmetric, i.e., $cm(R^1,R^2) = cm(R^2,R^1)$;

(iv) if R_t^1 and R_t^2 are independent, then $cm(R^1,R^2) = 0$;

(v) $cm(-R^1,R^2) = cm(R^1,-R^2) = -cm(R^1,R^2)$;

(vi) for any monotonic transformation $T(\cdot)$, $cm(T(R^1),T(R^2)) = cm(R^1,R^2)$.

These properties indicate that our nonparametric measure possesses the same properties as the Pearson correlation coefficient, while also having some properties that the Person correlation coefficient does not have. For example, the Pearson correlation coefficient of R_t^1 and R_t^2 equals 1, if and only if a linear relationship exists between R_t^1 and R_t^2 almost surely, i.e., two constants a and b exist such that $R_t^1 = aR_t^2 + b$, almost surely. However, $cm(R^1,R^2) = 1$ does not need to have a linear relationship between R_t^1 and R_t^2. For example, let x_t be a real time series, $R_t^1 = x_t^2$, and $R_t^2 = x_t^4$, then $cm(R^1, R^2) = 1$, but it is obvious that the relationship between R_t^1 and R_t^2 is a nonlinear relationship.

Given the nonparametric measure of comovements, we measure the downside comovements, which are defined as comovements resulting from market turndowns, as follows,

$$\begin{aligned} cm^-(c) &= Pr[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) > 0 \mid R_t^1 < -c, R_t^2 < -c] \\ &\quad - Pr[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) < 0 \mid R_t^1 < -c, R_t^2 < -c], \end{aligned} \tag{2}$$

where c is a nonnegative constant. Similarly, the upside comovements, which occur after market upturns, are measured by,

$$\begin{aligned} cm^+(c) &= Pr[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) > 0 \mid R_t^1 > c, R_t^2 > c] \\ &\quad - Pr[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) < 0 \mid R_t^1 > c, R_t^2 > c]. \end{aligned} \tag{3}$$

$cm^-(c)$ measures the possibilities that the two returns R_{t+1}^1 and R_{t+1}^2 move up or down together at time $t+1$ after stock market downturns at time t, while $cm^+(c)$ measures the possibility that R_{t+1}^1 and R_{t+1}^2 move in the same directions at time $t+1$ after stock market upturns at time t. The following theorem 1 gives the closed solutions for $cm^-(c)$ and $cm^+(c)$.

Theorem 1 *Let R_t^1 and R_t^2 be two stationary processes with joint probability and distribution functions $f(x,y)$ and $F(x,y)$, respectively, with margin distribution functions H(x)(of R_t^1) and $G(x)$ (of R_t^2). Then we have,*

$$cm^+(c) = \frac{2\int_c^\infty \int_c^\infty [1 - G(x) - H(y) + 2F(x,y)] f(x,y) dxdy}{\int_c^\infty \int_c^\infty f(x,y) dxdy} - 1 \tag{4}$$

$$cm^-(c) = \frac{2\int_{-\infty}^{-c} \int_{-\infty}^{-c} [1 - G(x) - H(y) + 2F(x,y)] f(x,y) dxdy}{\int_{-\infty}^{-c} \int_{-\infty}^{-c} f(x,y) dxdy} - 1. \tag{5}$$

If $F(x,y)$ is a symmetric distribution, then

$$cm^{+}(c) = cm^{-}(c) = \frac{2[\int_{c}^{\infty}\int_{c}^{\infty} F(x,y)f(x,y)dxdy + \int_{-\infty}^{-c}\int_{-\infty}^{-c} F(x,y)f(x,y)dxdy]}{\int_{-\infty}^{-c}\int_{-\infty}^{-c} f(x,y)dxdy} - 1. \quad (6)$$

Proof: see the Appendix.

2.2 A Nonparametric Test for Asymmetric Comovements

Given the measures of the downside and upside comovements, we are interested in testing whether the strength of the downside comovements is not significantly different from the strength of the upside comovements. Thus, the null hypothesis is,

$$H_0 : cm^{+}(c) = cm^{-}(c), \text{ for all } c \geq 0. \quad (7)$$

If the null hypothesis is rejected, asymmetric comovements must exist. The alternative hypothesis is,

$$H_a : cm^{+}(c) \neq cm^{-}(c), \text{ for some } c \geq 0. \quad (8)$$

Hong et al. (2007) propose a test for asymmetric correlations of stock returns. Their test is based on the comparison between downside correlations and upside correlations. We construct our test in a similar way to Hong et.al (2007). However, our test is based on the comparison between the downside comovements and the upside comovements, which are defined in equations (2) and (3).

Let $c_1, c_2, ..., c_m$ be m chosen nonnegative positive numbers. If the null hypothesis is true, the following $m \times 1$ difference vector

$$cm^{+} - cm^{-} = [cm^{+}(c_1) - cm^{-}(c_1), ..., cm^{+}(c_m) - cm^{-}(c_m)]' \quad (9)$$

must be equal to zero.

To construct a feasible test statistic, we need to estimate both $cm^-(c_i)$ and $cm^+(c_i)$ for $1 \le i \le m$.

Let $\{R_t^1, R_t^2\}_{t=1}^T$ be a set of sample with size T. For a given c_i, $Pr[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) > 0 \mid R_t^1 < -c_i, R_t^2 < -c_i]$ and $Pr[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) < 0 \mid R_t^1 < -c, R_t^2 < -c_i]$ can respectively be estimated by,

$$\frac{\sum_{t=1}^{n-1} I[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) > 0] I[R_t^1 < -c_i, R_t^2 < -c_i]}{\sum_{t=1}^{n-1} I[R_t^1 < -c_i, R_t^2 < -c_i]} \tag{10}$$

and

$$\frac{\sum_{t=1}^{n-1} I[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) < 0] I[R_t^1 < -c_i, R_t^2 < -c_i]}{\sum_{t=1}^{n-1} I[R_t^1 < -c_i, R_t^2 < -c_i]}, \tag{11}$$

where $I[\cdot]$ is the indicator function and $S(R_t^1, R_{t+1}^1, R_t^2, R_{t+1}^2) = I[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) > 0] - I[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) < 0]$.

Putting (10) and (11) into (2) yields the estimator of $cm^-(c_i)$,

$$\hat{cm}^-(c_i) = \frac{\sum_{t=1}^{n-1} \{S(R_t^1, R_{t+1}^1, R_t^2, R_{t+1}^2)\} I[R_t^1 < -c_i, R_t^2 < -c_i]}{\sum_{t=1}^{n-1} I[R_t^1 < -c_i, R_t^2 < -c_i]}, \tag{12}$$

and we obtain the estimator of $cm^+(c_i)$ by putting (10) and (11) into (3),

$$\hat{cm}^+(c) = \frac{\sum_{t=1}^{n-1} \{S(R_t^1, R_{t+1}^1, R_t^2, R_{t+1}^2)\} I[R_{t_i}^1 > c, R_t^2 > c_i]}{\sum_{t=1}^{n-1} I[R_t^1 > c_i, R_t^2 > c_i]}. \tag{13}$$

From (9), (12), and (13), we obtain the estimator of $cm^+ - cm^-$,

$$\hat{cm}^+ - \hat{cm}^- = [\hat{cm}^+(c_1) - \hat{cm}^-(c_1), \ldots, \hat{cm}^+(c_m) - \hat{cm}^-(c_m)]'. \tag{14}$$

It can be proven (see Appendix) that under null hypothesis and regular conditions, $\sqrt{n}(\hat{cm}^+ - \hat{cm}^-)$ converges to $N[0, \Omega]$ in distribution, where Ω is a positive definite variance-covariance matrix. Ω can consistently be estimated by the following almost surely positive definite matrix,

$$\hat{\Omega} = \sum_{l=-n+1}^{n-1} k(l/p) \hat{\delta}_l, \tag{15}$$

where $\hat{\delta}_l$ is an $m \times m$ matrix with $(i,j)-th$ element,

$$\hat{\delta}_l(i,j) = \frac{1}{n}\sum_{t=|l|+1}^{n} \hat{\eta}_t(c_i)\hat{\eta}_{t-|l|}(c_j), \tag{16}$$

$$\hat{\eta}_t(c_i) = \frac{\{S(R_t^1, R_{t+1}^1, R_t^2, R_{t+1}^2) - \hat{cm}^+(c_i)\} I[R_t^1 > c, R_t^2 > c_i]}{\sum_{t=1}^{n} I[R_t^1 > c, R_t^2 > c_i]/n} - \frac{\{S(R_t^1, R_{t+1}^1, R_t^2, R_{t+1}^2) - \hat{cm}^-(c)\} I[R_t^1 < -c, R_t^2 < -c_i]}{\sum_{t=1}^{n} I[R_t^1 < -c, R_t^2 < -c_i]/n}, \tag{17}$$

$k(\cdot)$ is a kernel function that assigns a suitable weight to each lag of order l, and p is the smoothing parameter or lag truncation order.

With (14) and (15), we are ready to define the test statistic for testing the null hypothesis of symmetric comovements as,

$$T_{cm} = n(\hat{cm}^+ - \hat{cm}^-)'\hat{\Omega}^{-1}(\hat{cm}^+ - \hat{cm}^-). \tag{18}$$

The following theorem provides the asymptotic null distribution of the test statistic.

Theorem 2 *Suppose Assumptions A.1-A.4 hold and $p = p(n) \to \infty, p/n \to 0$ as $n \to \infty$. Under the null hypothesis, we have,*

$$T_{cm} = n(\hat{cm}^+ - \hat{cm}^-)'\hat{\Omega}^{-1}(\hat{cm}^+ - \hat{cm}^-) \xrightarrow{d} \chi^2_m, \text{ as } n \to \infty, \tag{19}$$

where χ^2_m is a chi-square distribution with degrees of freedom m.

Proof: see the Appendix.

Theorem 2 indicates that our test statistic has an asymptotic chi-square distribution with degrees of freedom m. Consequently, the critical values for our test are available, making its applications easy to carry out. Since the test statistic does not rely on the assumption that the data are drawn from a given model, it is a model free test, which can directly be used to test whether the asymmetric comovements exist in data.

The asymptotic bias of kernel estimators depends on the smoothness of the kernel at zero and on the smoothness of the spectral density matrix $f(\lambda)$ of $\{\delta_l\}$ at zero, where δ_l is an $m \times m$ matrix with $(i,j)-th$ element $\delta_l(i,j) = \frac{1}{n}\sum_{t=|l|+1}^{n} \eta_t(c_i)\eta_{t-|l|}(c_j), \eta_t(c) = lim_{n\to\infty}\hat{\eta}_t(c)$. For $q \in [0,\infty)$, following Andrews (1991), we define

$$k_q = lim_{x\to 0}\frac{1-k(x)}{|\,x\,|^q}. \tag{20}$$

The smoother the kernel is at zero, the larger is the value of q for which k_q is finite. Andrews (1991) defines the optimal smoothing parameters $\{p_n^*\}$ as follows,

$$p_n^* = (qk_q^2\alpha(q)n/\int k^2(x)dx))^{1/(2q+1)}, \tag{21}$$

where $\alpha(q)$ is a function of the unknown spectral density matrix $f(\lambda)$ (equation 5.2 in Andrews, 1991).

3 Finite Sample Performance

In this section, we use Monte Carlo simulations to examine the finite-sample performance of the test. The data are generated from two AR(1)-GARCH(1, 1) models,

$$R_t^1 = \mu + \phi R_{t-1}^1 + \eta_t^1, \tag{22}$$

$$R_t^2 = \mu + \phi R_{t-1}^2 + \eta_t^2, \tag{23}$$

where $\eta_t^i = \sqrt{h_t^i}\varepsilon_t^i$, and $h_t^i = \alpha + \beta h_{t-1}^i + \gamma(R_t^i - \mu - \phi R_{t-1}^i)^2$, for $i = 1,2$. The parameter values are set as $(\mu,\phi,\alpha,\beta,\gamma) = (0.01, 0.05, 0.05, 0.85, 0.1)$, and ε_t^1 and ε_t^2 are supposed to follow a joint distribution $H(x,y)$. According to copula theory, a unique distribution function $C(\cdot,\cdot)$ exists, which is called a copula, such that for all $(x,y) \in R^2, H(x,y) = C(F_{\varepsilon_t^1}(x), F_{\varepsilon_t^2}(y))$, where $F_{\varepsilon_t^1}(x)$ and $F_{\varepsilon_t^2}(y)$ are corresponding marginal distributions of ε_t^1 and ε_t^2.

By using different specifications of the copula function $C(F_{\varepsilon_t^1}(x), F_{\varepsilon_t^2}(y))$, equations (22) and (23) can come up with different data-generating processes. A normal copula is $C_{nor}(u,v,\rho) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$, where Φ_ρ is the standard bivariate distribution function with correlation ρ. A bivariate Clayton copula is $C_{clay}(u,v,\tau) = (u^{-\tau} + v^{-\tau} - 1)^{-1/\tau}$, where parameter $\tau > 0$. From equations (4) and (5) in theorem 1, we know that a normal copula generates symmetric comovements, while a bivariate Clayton copula produces asymmetric comovements. The first copula used to generate data is the mixture copula that combines a normal copula with a Clayton copula,

$$C_{mix}(u,v;\rho,\tau,k) = kC_{nor}(u,v,\rho) + (1-k)C_{clay}(u,v,\tau), \tag{24}$$

where k is the mixture parameter, which determines the strength of asymmetric comovements in the data generated from the mixture copula. The lower k is, the more the mixture copula generates asymmetries in the data. The second copula is the Generalized Joe-Clayton (GJC) copula, which is proposed by Patton (2006) and is specified as follows,

$$C^{GJC}(u,v;\tau^u,\tau^l) = 0.5(C^{JC}(u,v;\tau^u,\tau^l) + C^{JC}(1-u,1-v;\tau^u,\tau^l + u + v - 1), \tag{25}$$

where C^{JC} is the Joe-Clayton copula, which is given by,

$$C^{JC}(u,v;\tau^u,\tau^l) = 1 - (1 - \{[1-(1-u)^\kappa]^{-\sigma} - 1\}^{-1/\sigma})^{1/\kappa}, \tag{26}$$

where $\kappa = 1/log_2(2-\tau^u), \sigma = -1/log_2(\tau^l)$ and $\tau^u \in (0,1], \tau^l \in (0,1]$. The GJC copula has two tail comovement parameters, τ^u and τ^l. It can produce symmetric comovements when $\tau^u = \tau^l$, while it generates asymmetric comovements if $\tau^u \neq \tau^l$. The larger the distance $|\tau^u - \tau^l|$ is, the more the GJC produces asymmetries in the data. Given the relationship between the distribution function and its copula function, we generate data by following the way in Okimoto (2008). In this simulation, we choose four sets of exceedance levels: $c_1 = 0; (c_1, c_2) = (0, 0.5); (c_1, c_2, c_3) =$

$(0, 0.5, 1); (c_1, c_2, c_3, c_4) = (0, 0.5, 1, 1.5)$. The Bartlett kernel, $k(z) = (1 - |z|)I(|z| < 1)$, is used to estimate the variance-covariance matrix, and the soothing parameter p is determined by the data with the procedure in Andrew (1991).

Table 1 reports the estimated size and power of our test. The second column of table 1 reports the estimated sizes when the symmetric comovements are generated from either the mixture copula ($k = 0$) or the GJC copula ($\tau^u = \tau^l = 0.1$), which indicates that our test has satisfactory size performance at all four sets of exceedance levels for sample size as small as 250. Columns 3-5 of table 1 report the estimated power of our test when the null hypothesis is that asymmetric comovements do not exist but in fact the data are generated from the mixture copula ($k \neq 0$) or the GJC copula ($\tau^u \neq \tau^l$). When the data are generated from the mixture copula, the test's power always increases rapidly with respect to the decrease in k, which indicates that our test becomes more powerful when the data are more asymmetric. For example, for $T = 1000$ and $(c_1, c_2, c_3, c_4) = (0, 0.5, 1, 1.5)$, the power increases to 96.6% when k decreases to 0.25. When the data are generated from the GJC copula, our test power increases with respect to the increase in the distance $|\tau_u - \tau_l|$, which indicates that more asymmetric the data, more powerful the test is. Overall, the results of our Monte Carlo simulation indicate that our test has satisfactory size performance, and good power in detecting asymmetric comovements implied by the data considered.

4 Do Asymmetric Comovements Exist in International Stock Markets?

We apply our test to examine whether market downturns will lead to stronger comovements than market upturns between the U.S. stock market and the stock markets of Canada, France, German, Japan, and U.K.. Following Longin and Solnik (2001) and Okimoto (2008), our empirical study is based on monthly total stock market index for six countries: Canada, France, Germany, Japan, U.K., and U.S.. The data are obtained from Datastream with the sample period from 1973:1 to 2009:3. We define stock returns for each country as 100 times the change in the natural logarithm of each country's stock index.

Descriptive statistics on these time series are presented in Table 2. France and the U.K. markets exhibit positive skewness, while the stock markets of Canada, Germany, Japan, and U.S. exhibit negative skewness, which indicates that the distributions of these individual stock returns display asymmetries.[2] The Jarque-Bera statistics strongly reject normality. The results of a formal augmented Dickey-Fuller non-stationary test suggest that the null hypothesis of non-stationary is rejected at the 5 % significance level. Since the test is known to have low power, which is the probability of rejecting the null hypothesis when it is not true, even a slight rejection means that stationary of the series is very likely.

We examine the comovements of the U.S. stock market with the other five stock markets separately. Hence, we have five country pairs: U.S. and Canada, U.S. and Frence, U.S. and Germany, U.S. and U.K., and U.S. and Japan. We calculate the test statistic T_{cm} by setting the exceedance levels $C = [0, 0.5, 1, 1.5]$. In the empirical work, we use the Bartlett kernel, and the smoothing parameter is determined by the data with the procedure in Andrews (1991). Table 3 provides the results for testing symmetric comovements at a 5% significant level for the five country pairs. We find statistically significant evidence of the asymmetric comovements between the U.S. stock market and the stock markets of Canada, Frence, Germany, and U.K., but the data cannot reject the null hypothesis of symmetric comovements between the U.S. and Japanese stock markets.

To get a visual impression of asymmetric comovements, in figures 1-5 we plot the estimations of the probabilities of downside and upside comovements for the five country pairs. The figures for

[2] The skewness or asymmetry in the distribution of individual stock returns has been reported by numerous authors (Harvey and Siddique (2000), Ait-Sahalia and Brandt (2001), Patton (2004), among others.) over the last three decades.

U.S. and Canada, U.S. and Frence, U.S. and Germany, and U.S. and U.K., provide clear pictorial representations of the asymmetric comovements. There are two main features of the plots. First, there is a sharp break evident at $c = 0$, where the conditional event changes from $(R_t^1 < 0, R_t^2 < 0)$ to $(R_t^1 > 0, R_t^2 > 0)$, which suggests that negative returns will lead to stronger comovements than positive returns. Obviously, we observe that far from being symmetric, the downside comovements (for negative levels) are always greater than the upside comovements (for positive levels). Second, instead of tapering off to zero, as in the case of a bivariate normal distribution, the downside comovements tend to increase as returns become more negative. Unlike the empirical results obtained by Longin and Solnic (2001), our empirical results indicate that instead of decreasing to zero, the upside comovements tend to increase as returns become more positive. The comovements in the data differ from the comovements generated by a bivariate symmetric distribution, such as a bivariate normal distribution, which predicts symmetric comovements and both upside and downside comovements decrease to zero. Figure 4 shows that downside comovements are not always higher than their respective upside comovements, graphically suggesting that we cannot reject the null hypothesis of symmetric comovements of stock returns between the U.S and Japanese stock markets, which is in line with the result in table 3.

5 Conclusion

In this paper, we propose a new approach to measure comovements between two stock returns. The measure of comovements reflects the direction of the movement between the two returns and allows for the nonlinear relationship between the two returns. Based on the measure of comovements, we develop a nonparametric test to detect whether asymmetric comovements exist in the sense that stock market downturns will lead to stronger comovements than market upturns. Monte

Carlo simulations show that our test has satisfactory size performance and good power in detecting asymmetric comovements in the data considered.

Applying the test to international stock markets, we find the following empirical facts: first, asymmetric comovements exist between the U.S. stock market and the stock markets of Canada, France, Germany, and U.K., but the test is unable to reject the null hypothesis of symmetric comovements of stock markets between U.S. and Japan. Second, unlike the empirical results from Longin and Solnic (2001), our empirical results indicate that the upside comovements and the downside comovements tend to increase as returns become more positive and more negative, respectively, which indicates that the comovements in the data are different from the comovements implied by a bivariate symmetric distribution, which suggests that comovements tend to zero when returns become more positive or more negative. The empirical results prompt us to use more flexible models to model the comovements of international stock markets.

Table 1: **Percentage Rejections of the H_0**

$(\varepsilon_t^1, \varepsilon_t^2)$ **Generated from the Mixture Copula**

	k=1.00	k=0.75	k=0.5	k=0.25
T=250				
{0}	0.043	0.175	0.432	0.801
{0, 0.5 }	0.051	0.141	0.331	0.722
{0, 0.5, 1}	0.049	0.121	0.275	0.713
{0, 0.5, 1, 1.5}	0.063	0.135	0.208	0.699
T=500				
{0}	0.035	0.327	0.778	0.963
{0, 0.5 }	0.046	0.201	0.521	0.951
{0, 0.5, 1}	0.050	0.204	0.407	0.899
{0, 0.5, 1, 1.5}	0.042	0.175	0.399	0.894
T=1000				
{0}	0.051	0.518	0.861	0.995
{0, 0.5 }	0.040	0.401	0.800	0.991
{0, 0.5,1}	0.060	0.358	0.715	0.987
{0, 0.5, 1, 1.5}	0.054	0.302	0.663	0.966

$(\varepsilon_t^1, \varepsilon_t^2)$ **Generated from the GJC Copula**

	$(\tau_u = \tau_l)$	$(\tau_u = 0.1, \tau_l = 0.3)$	$(\tau_u = 0.1, \tau_l = 0.4)$	$(\tau_u = 0.1, \tau_l = 0.5)$
T=250				
{0}	0.049	0.231	0.395	0.539
{0, 0.5 }	0.061	0.234	0.318	0.440
{0, 0.5,1}	0.054	0.182	0.290	0.369
{0, 0.5, 1, 1.5}	0.036	0.107	0.144	0.169
T=500				
{0}	0.046	0.447	0.608	0.788
{0, 0.5 }	0.069	0.387	0.517	0.733
{0, 0.5, 1}	0.062	0.346	0.500	0.671
{0, 0.5, 1, 1.5}	0.047	0.180	0.299	0.321
T=1000				
{0}	0.051	0.733	0.893	0.971
{0, 0.5 }	0.060	0.638	0.850	0.957
{0, 0.5, 1}	0.053	0.587	0.796	0.938
{0, 0.5, 1, 1.5}	0.039	0.434	0.599	0.697

The table reports the size and power of T_{cm}. The nominal size of the test is set at 5%. The data-generating processes are: $R_t^i = 0.01 + 0.05R_{t-1}^1 + \eta_t^i$, where $\eta_t^i = \sqrt{h^i}\varepsilon_t^i$, and $h_t^i = 0.05 + 0.85h_{t-1}^i + 0.1(R_t^i - 0.01 - 0.05R_{t-1}^i)^2, i = 1, 2$. The results are based on 1000 simulations drawn from the mixture copula and Generalized Joe-Clayton copula, which are used to generate $(\varepsilon_t^1, \varepsilon_t^2)$. $H_0 : cm^+(c) = cm^-(c)$ for all $c \geq 0$, $H_a : cm^+(c) \neq cm^-(c)$, for some $c \geq 0$.

Table 2: **Summary Statistics of the Data**

	Canada	France	Germany	Japan	U.K.	U.S.
Mean	0.4985	0.5617	0.5616	0.1820	0.5523	0.4354
Std.Dev	3.9062	5.1000	4.4068	4.4323	4.8145	3.8428
Skewness	-1.0205	0.7233	-1.1552	-0.5133	0.1729	-1.0740
Kurtosis	7.2196	4.6244	7.0015	5.3253	13.0876	7.4678
Min	-24.3267	-20.1684	-25.1595	-24.8746	-25.2891	-23.8226
Max	11.3938	14.0825	10.7125	13.7299	37.0598	11.2983
Jarque-Bera	397.2957	85.5647	386.0702	116.8302	1842	444.4021
5% Critical value	5.9915	5.9915	5.9915	5.9915	5.9915	5.9915
Augmented D.F. test statistic	-14.749	-14.561	-14.513	-14.635	-14.962	-15.018
5% Critical value	-2.873	-2.873	-2.873	-2.873	-2.873	-2.873

This table reports descriptive statistics of stock market returns of Canada, France, Germany, Japan U.K., and U.S.. The frequency of the data is monthly. The sample period is from 1973:1 to 2009:3. Under the null of normality, the Jarque-Bera test statistic follows a chi-squared distribution with two degrees of freedom.

Table 3: **Testing Symmetric Comovement between U.S. and International Stock Markets**

	Canada	France	Germany	Japan	U.K.
Test Statistic	19.5571	20.6290	15.0320	5.2044	11.9383
5% Critical value	9.4877	9.4877	9.4877	9.4877	9.4877
Results	Reject	Reject	Reject	Reject	Reject

This table reports the testing results of symmetric comovement between U.S. stock market and stock markets of Canada, France, Germany, Japan and U.K.. The null hypothesis is symmetric comovement. $C = \{0, 0.5, 1, 1.5\}$ is used, and 5% critical value is 9.4877. The sample period is from 1973 : 1 to 2009 : 3.

Appendix

All convergencies are taken as $n \to \infty$. To derive the null asymptotic distribution, we need the following regularity conditions.

Assumption A.1. *$\{R_t^1, R_t^2\}$ is a mean zero, and bivariate stationary $\alpha-$mixing process with $\alpha-$mixing coefficient satisfying $\sum_{j=1}^{\infty} \alpha(j)^{(\kappa-1)/\kappa} < \infty$, and $Sup_{t \le 1}(E|R_t^1|^{4\kappa} + E|R_t^2|^{4\kappa}) < \infty$ for some $\kappa > 1$.*

Assumption A.2. *The kernel function $k(\cdot) : R \to [-1, 1]$, is symmetric about zero and is continuous at all points except a finite number of them on R, with $k(0) = 1$ and $\int_{-\infty}^{\infty} |k(z)|dz < \infty$.*

Assumption A.3. *The bandwidth $p = p(T) \to \infty, p/T \to 0$ as the sample size $T \to \infty$.*

Assumption A.4. *The kernel function $k(x)$ satisfies: (i)$|k(x)| \le c|x|^{-b}$ for some $b > 1 + 1/q$, where $q \in (0, \infty)$ is such that $k_q \in (0, \infty)$, and (ii) $|k(x) - k(y)| \le c|x - y|$ $\forall x, y \in R$.*

Assumption A.1 allows for the existence of volatility clustering, which is a well-known stylized fact for most financial time series. The mixing condition is commonly used for a nonlinear time series analysis. This condition characterizes temporal dependence in return series and rules out long memory processes. It is well known that returns of equity returns have weak serial correlation. Therefore, the mixing condition is quite reasonable in the present context. Assumptions A.2 and A.3 are standard conditions on the kernel function $k(\cdot)$ and bandwidth p. These conditions are sufficient when we use nonstochastic bandwidths. Assumption 4 imposes some extra conditions on the kernel function, which is needed when we use data-dependent bandwidth $\hat{p}$. Many commonly used kernels, such as the Bartlett, Parzen, and quadratic-spectral kernels, are included. But, Assumption 4 rules out the truncated and Daniell kernels.

Proof of Theorem 1:

Because of $Pr[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) < 0 \mid R_t^1 > c, R_t^2 > c] = 1 - Pr[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) >$

$0 \mid R_t^1 > c, R_t^2 > c]$, and $Pr[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) > 0 \mid R_t^1 > c, R_t^2 > c] = Pr[R_{t+1}^2 \geq R_t^2, R_{t+1}^1 \geq R_t^1 \mid R_t^1 > c, R_t^2 > c] + Pr[R_{t+1}^2 \leq R_t^2, R_{t+1}^1 \leq R_t^1 \mid R_t^1 > c, R_t^2 > c]$, we have,

$$
\begin{aligned}
cm^+(c) &= 2Pr[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) > 0 \mid R_t^1 > c, R_t^2 > c] - 1 \\
&= \frac{2Pr[R_{t+1}^2 \geq R_t^2, R_{t+1}^1 \geq R_t^1, R_t^1 \geq c, R_t^2 \geq c]}{Pr[R_t^1 \geq c, R_t^2 \geq c]} \\
&\quad + \frac{2Pr[R_{t+1}^2 \leq R_t^2, R_{t+1}^1 \leq R_t^1, R_t^1 \geq c, R_t^2 \geq c]}{Pr[R_t^1 \geq c, R_t^2 \geq c]} - 1 \\
&= \frac{2\int_c^\infty \int_c^\infty Pr[R_{t+1}^1 \geq y, R_{t+1}^2 \geq x] f(x,y)dxdy}{\int_c^\infty \int_c^\infty f(x,y)dxdy} \\
&\quad + \frac{2\int_c^\infty \int_c^\infty Pr[R_{t+1}^1 \leq y, R_{t+1}^2 \leq x] f(x,y)dxdy}{\int_c^\infty \int_c^\infty f(x,y)dxdy} - 1 \\
&= \frac{2\int_c^\infty \int_c^\infty [1 - G(x) - H(y) + 2F(x,y)] f(x,y)dxdy}{\int_c^\infty \int_c^\infty f(x,y)dxdy} - 1. \qquad \text{(A.1)}
\end{aligned}
$$

To obtain (A.1), we have used the fact that $\int_c^\infty \int_c^\infty Pr[R_{t+1}^1 \geq y, R_{t+1}^2 \geq x] f(x,y)dxdy = \int_c^\infty \int_c^\infty [1 - H(x) - G(y) + F(x,y)] f(x,y)dxdy$. Similarly, we can prove that,

$$
\begin{aligned}
cm^-(c) &= 2Pr[(R_{t+1}^2 - R_t^2)(R_{t+1}^1 - R_t^1) > 0 \mid R_t^1 < -c, R_t^2 < -c] - 1 \\
&= \frac{2\int_{-\infty}^{-c} \int_{-\infty}^{-c} [1 - G(x) - H(y) + 2F(x,y)] f(x,y)dxdy}{\int_{-\infty}^{-c} \int_{-\infty}^{-c} f(x,y)dxdy} - 1. \qquad \text{(A.2)}
\end{aligned}
$$

From (A.1) and (A.2), we obtain (4) and (5) in Theorem 1, respectively. If $F(x,y)$ is a symmetric distribution ($Pr[R_t^1 \geq x, R_t^2 \geq y] = Pr[R_t^1 \leq -x, R_t^2 \leq -y], and f(x,y) = f(-x,-y)$, almost for all (x,y)), then we have,

$$
\begin{aligned}
cm^+(c) &= \frac{Pr[R_{t+1}^1 \leq R_t^1, R_{t+1}^2 \leq R_t^2, R_t^1 > c, R_t^2 > c]}{Pr[R_t^1 > c, R_t^2 > c]} \\
&\quad + \frac{Pr[R_{t+1}^1 \leq -R_t^1, R_{t+1}^2 \leq -R_t^2, R_t^1 > c, R_t^2 > c]}{Pr[R_t^1 > c, R_t^2 > c]} \\
&= \frac{\int_c^\infty \int_c^\infty Pr[R_{t+1}^1 \leq x, R_{t+1}^2 \leq y] f(x,y)dxdy}{\int_c^\infty \int_c^\infty f(x,y)dxdy} \\
&\quad + \frac{\int_c^\infty \int_c^\infty Pr[R_{t+1}^1 \leq -x, R_{t+1}^2 \leq -y] f(x,y)dxdy}{\int_c^\infty \int_c^\infty f(x,y)dxdy} \\
&= \frac{\int_c^\infty \int_c^\infty F(x,y) f(x,y)dxdy + \int_{-\infty}^c \int_{-\infty}^c F(x,y) f(x,y)dxdy}{\int_c^\infty \int_c^\infty f(x,y)dxdy}, \qquad \text{(A.3)}
\end{aligned}
$$

and

$$
\begin{aligned}
cm^-(c) &= \frac{Pr[R^1_{t+1} \le R^1_t, R^2_{t+1} \le R^2_t, R^1_t < -c, R^2_t < -c]}{Pr[R^1_t < -c, R^2_t < -c]} \\
&\quad + \frac{Pr[R^1_{t+1} \le -R^1_t, R^2_{t+1} \le -R^2_t, R^1_t < -c, R^2_t < -c]}{Pr[R^1_t < -c, R^2_t < -c]} \\
&= \frac{\int_{-\infty}^{-c}\int_{-\infty}^{-c} Pr[R^1_{t+1} \le x, R^2_{t+1} \le y] f(x,y)dxdy}{\int_{-\infty}^{-c}\int_{-\infty}^{-c} f(x,y)dxdy} \\
&\quad + \frac{\int_{-\infty}^{-c}\int_{-\infty}^{-c} Pr[R^1_{t+1} \le -x, R^2_{t+1} \le -y] f(x,y)dxdy}{\int_{-\infty}^{-c}\int_{-\infty}^{-c} f(x,y)dxdy} \\
&= \frac{\int_{-\infty}^{-c}\int_{-\infty}^{-c} F(x,y)f(x,y)dxdy + \int_{c}^{\infty}\int_{c}^{\infty} F(x,y)f(x,y)dxdy}{\int_{-\infty}^{-c}\int_{-\infty}^{-c} f(x,y)dxdy}. \qquad (A.4)
\end{aligned}
$$

Form $(A.3)$ and $(A.4)$ and $\int_{-\infty}^{-c}\int_{-\infty}^{-c} f(x,y)dxdy = \int_{c}^{\infty}\int_{c}^{\infty} f(x,y)dxdy$, we have $cm^+(c) = cm^-(c)$.

Proof of Theorem 2:

We first use the Cramer-Wold device to show that $\sqrt{n}(\hat{cm}^+ - \hat{cm}^-)$ converges to $N(0,\Omega)$ in distribution. Let $\lambda = (\lambda_1, ..., \lambda_m)'$ be an $m \times 1$ vector such that $\lambda'\lambda = 1$. We have

$$
\begin{aligned}
&\lambda'(\hat{cm}^+ - \hat{cm}^-) \\
= \;& \sum_{i=1}^{m} \lambda_i [\hat{cm}^+(c_i) - \hat{cm}^-(c_i)] \\
= \;& \frac{1}{n}\sum_{t=1}^{n}\sum_{i=1}^{m} \lambda_i [S(R^1_t, R^1_{t+1}, R^2_t, R^2_{t+1}) - cm^+(c_i)] I[R^1_t > c_i, R^2_t > c_i] / \frac{1}{n}\sum_{t=1}^{n} I[R^2_t > c_i, R^1_t > c_i] \\
& - \frac{1}{n}\sum_{t=1}^{n}\sum_{i=1}^{m} \lambda_i [S(R^1_t, R^1_{t+1}, R^2_t, R^2_{t+1}) - cm^-(c_i)] I[R^1_t < -c_i, R^2_t < -c_i] / \frac{1}{n}\sum_{t=1}^{n} I[R^2_t < -c_i, R^1_t < -c_i] \\
& + \sum_{i=1}^{m} \lambda_i (cm^+(c_i) - cm^-(c_i)) \\
= \;& \frac{1}{n}\sum_{t=1}^{n}\sum_{i=1}^{m} \lambda_i (\eta^+_t(c_i) - \eta^-_t(c_i)) + \frac{1}{n}\sum_{t=1}^{n}\sum_{i=1}^{m} \lambda_i \eta^+_t(c_i) \Big(\frac{Pr[R^1_t > c_i, R^2_t > c_i]}{\sum_{t=1}^{n} I[R^1_t > c_i, R^2_t > c_i]/n} - 1\Big) \\
& - \frac{1}{n}\sum_{t=1}^{n}\sum_{i=1}^{m} \lambda_i \eta^-_t(c_i) \Big(\frac{Pr[R^1_t < -c_i, R^2_t < -c_i]}{\sum_{t=1}^{n} I[R^1_t < -c_i, R^2_t < -c_i]/n} - 1\Big) + \sum_{i=1}^{m} \lambda_i (cm^+(c_i) - cm^-(c_i)) \\
= \;& \frac{1}{n}\sum_{t=1}^{n}\sum_{i=1}^{m} \lambda_i (\eta^+_t(c_i) - \eta^-_t(c_i)) + o_p\Big(\frac{1}{n}\sum_{t=1}^{n}\sum_{i=1}^{m} \lambda_i (\eta^+_t(c_i) - \eta^-_t(c_i))\Big) \\
= \;& \frac{1}{n}\sum_{t=1}^{n} \eta_t + o_p\Big(\frac{1}{n}\sum_{t=1}^{n} \eta_t\Big) \qquad (A.5)
\end{aligned}
$$

where $\eta_t = \sum_{t=1}^{m} \lambda_i(\eta_t^+(c_i) - \eta_t^-(c_i))$ and

$$\eta_t^+(c_i) = \frac{\{S(R_t^1, R_{t+1}^1, R_t^2, R_{t+1}^2) - cm^+(c_i)\}I[R_t^1 > c_i, R_t^2 > c_i]}{Pr[R_t^1 > c_i, R_t^2 > c_i]},$$
$$\eta_t^-(c_i) = \frac{\{S(R_t^1, R_{t+1}^1, R_t^2, R_{t+1}^2) - cm^-(c_i)\}I[R_t^1 < -c_i, R_t^2 < -c_i]}{Pr[R_t^2 < -c_i, R_t^1 < -c_i]}.$$

In addition, in$(A.5)$ we use the fact that under null hypothesis, $cm^+(c_i) = cm^-(c_i)$ for $i = 1, ...m$. From $A.5$, we know that to prove $\sqrt{n}(\hat{cm}^+ - \hat{cm}^-)$ converges to $N(0, \Omega)$, we consider

$$\sqrt{n}\lambda'(\hat{cm}^+ - \hat{cm}^-) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \eta_t + o_p(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \eta_t). \tag{A.6}$$

We have

$$V = lim_{n\to\infty} var[n^{-1/2} \sum_{t=1}^{n} \eta_t] = \sum_{l=-\infty}^{\infty} cov(\eta_t, \eta_{t-l})$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_i \lambda_j \Omega_{ij} = \lambda' \Omega \lambda. \tag{A.7}$$

Because Ω is a positive definite matrix, $0 < V < \infty$ for all λ such that $\lambda'\lambda = 1$. By the central limit theorem for mixing processes (White 1984, Theorem 5.19), we have $\lambda\sqrt{n}(\hat{cm}^+ - \hat{cm}^-)/\sqrt{V}$ converges to $N(0, 1)$ in distribution. It follows from the Cramer-Wold device that $\sqrt{n}(\hat{cm}^+ - \hat{cm}^-)$ converges to $N(0, \Omega)$ in distribution. Therefore, we have,

$$n(\hat{cm}^+ - \hat{cm}^-)'\Omega^{-1}(\hat{cm}^+ - \hat{cm}^-) \xrightarrow{d} \chi_m^2. \tag{A.8}$$

Next, we show that $\hat{\Omega} \xrightarrow{p} \Omega$. We only need to prove that $\hat{\Omega}_{ij} \xrightarrow{p} \Omega_{ij}$, for $1 \leq i, j \leq m$. Let $\tilde{\Omega}_{ij} = \sum_{l=-n+1}^{n-1} k(l/p)\delta_l(i, j)$, and $\Omega_{ij}^n = \sum_{l=1-n}^{n-1} E\delta_l(i, j)$ then we have

$$\hat{\Omega}_{ij} - \Omega_{ij} = (\hat{\Omega}_{ij} - \tilde{\Omega}_{ij}) + (\tilde{\Omega}_{ij} - E\tilde{\Omega}_{ij}) + (E\tilde{\Omega}_{ij} - \Omega_{ij}^n) + (\Omega_{ij}^n - \Omega_{ij}) \tag{A.9}$$

We have

$$\frac{\sqrt{n}}{p} \mid \hat{\Omega}_{ij} - \tilde{\Omega}_{ij} \mid \leq \frac{1}{p} \sum_{l=-n+1}^{n-1} \mid k(l/p) \mid\mid \sqrt{n}(\hat{\delta}_l(ij) - \delta_l(i, j)) \mid . \tag{A.10}$$

It is straightforward to prove that $|\sqrt{n}(\hat{\delta}_l(ij)-\delta_l(i,j))| = O_p(1)$ uniformly for $-n+1 \leq l \leq n+1$ and a given i,j. This result, and the fact that $\frac{1}{p}\sum_{l=-n+1}^{n-1} | k(l/p) | \rightarrow \int_{-\infty}^{\infty} | k(x) | \, dx$ imply that $\hat{\Omega}_{ij} - \tilde{\Omega}_{ij} \xrightarrow{p} 0$.

It follows from Proposition 1(a) in Andrews (1991) that $var(\tilde{\Omega}_{ij}) = O(p/n)$. Consequently, by Chebyshev's inequality, we have

$$\tilde{\Omega}_{ij} - E\tilde{\Omega}_{ij} = O_p(\sqrt{p/n}) = o_p(1). \tag{A.11}$$

By Proposition 1(b) in Andrews (1991), $E\tilde{\Omega}_{ij} - \Omega^n_{ij} = o(1)$, $q = 0$. Obviously Ω^n_{ij} converges to Ω_{ij}.

Suppose that the bandwidth parameter p is a function of the data, which is expressed as $\hat{p}$, and $\hat{\Omega}^*$ is the kernel estimator of Ω.

$$\begin{aligned}
\hat{\Omega}^*_{ij} - \hat{\Omega}_{ij} &= 2\sum_{l=1}^{r(n)}(k(l/\hat{p}) - k(l/p))\hat{\delta}_l(c_i,c_j) \\
&\quad +2\sum_{j=r(n)+1}^{n-1} k(l/\hat{p})\hat{\delta}_l(c_i,c_j) \\
&\quad -2\sum_{l=r(n)+1}^{n-1} k(l/p)\hat{\delta}_l(c_i,c_j) \\
&= 2I_{n1}(i,j) + 2I_{n2}(i,j) - 2I_{n3}(i,j),
\end{aligned} \tag{A.12}$$

where $r(n)$ is defined as in Andrews (equation 7.2 in Andrews, 1991). We need to show that $I_{nm}(i,j) \rightarrow 0$ for $m = 1,2,3$. For I_{n1}, we have

$$\begin{aligned}
|I_{n1}(i,j)| &\leq c\sum_{l=1}^{r(n)} l|\frac{1}{\hat{p}} - \frac{1}{p}||\hat{\delta}_l(c_i,c_j)| \\
&\leq c\sum_{l=1}^{r(n)} l|\frac{1}{\hat{p}} - \frac{1}{p}||\hat{\delta}_l(c_i,c_j) - \delta_l(c_i,c_j)| + c\sum_{l=1}^{r(n)} l|\frac{1}{\hat{p}} - \frac{1}{p}||\delta_l(c_i,c_j)|.
\end{aligned} \tag{A.13}$$

To show $I_{n1}(i,j) \rightarrow 0$,we only need to show that $\sum_{l=1}^{r(n)} l|\frac{1}{\hat{p}} - \frac{1}{p}||\hat{\delta}_l(c_i,c_j) - \delta_l(c_i,c_j)| \rightarrow 0$ and $\sum_{l=1}^{r(n)} l|\frac{1}{\hat{p}} - \frac{1}{p}||\delta_l(c_i,c_j)| \rightarrow 0$.

Since $\hat{\delta}_l(c_i,c_j)-\delta_l(c_i,c_j)=O_p(n^{-1/2})$ uniformly for l, we have

$$\begin{aligned}\sum_{l=1}^{r(n)} l\left|\frac{1}{\hat{p}}-\frac{1}{p}\right|\left|\hat{\delta}_l(c_i,c_j)-\delta_l(c_i,c_j)\right| &\leq c\left|\frac{1}{\hat{p}}-\frac{1}{p}\right|r(n)\sum_{l=1}^{r(n)}\left|\hat{\delta}_l(c_i,c_j)-\delta_l(c_i,c_j)\right| \\ &= O_p(n^{\frac{-1+2v-q-1/2}{2q+1}})=o_p(1), \qquad \text{(A.14)}\end{aligned}$$

because of assumption A.4 and $q>1/2$.

$$\sum_{l=1}^{r(n)} l\left|\frac{1}{\hat{p}}-\frac{1}{p}\right|\left|\delta_l(c_i,c_j)\right| \leq \left|\frac{1}{\hat{p}}-\frac{1}{p}\right|r(n)\sum_{l=1}^{\infty}\left|\delta_l(c_i,c_j)\right| = O(n^{\frac{v-1}{2q+1}})=o_p(1), \qquad \text{(A.15)}$$

because of $v<1$ and $\sum_{l=1}^{\infty}|\delta_l(c_i,c_j)|<\infty$. (A.13), (A.14), and (A.15) indicate that $I_{n1}(i,j)\to 0$.

$$\begin{aligned}I_{n2}(i,j) &\leq \sum_{l=r(n)+1}^{n-1} k(l/\hat{p})|\hat{\delta}_l(i,j)-\delta_l(i,j)| + \sum_{l=r(n)+1}^{n-1} k(l/\hat{p})|\delta_l(i,j)| \\ &= I_{n2}^1(i,j)+I_{n2}^2(i,j). \qquad \text{(A.16)}\end{aligned}$$

For $I_{n2}^1(i,j)$, we have

$$\begin{aligned}I_{n2}^1(i,j) &\leq c\sum_{l=r(n)+1}^{n-1}(l/\hat{p})^{-b}|\hat{\delta}_l(i,j)-\delta_l(i,j)| \\ &= O_p(n^{\frac{b}{2q+1}-1/2})\sum_{l=r(n)}^{\infty} l^{-b} \\ &= O_p(n^{b/(2q+1)-1/2-v(b-1)/(2q+1)})=o_p(1). \qquad \text{(A.17)}\end{aligned}$$

For $I_{n2}^2(i,j)$, we have

$$I_{n2}^2(i,j)\leq \sum_{l=r(n)}^{\infty}|\delta_l(i,j)|=o_p(1). \qquad \text{(A.18)}$$

A.13 and A.14 imply that $I_{n2}(i,j)=o_p(1)$. Using a similar way as we show $I_{n2}(i,j)=o_p(1)$, we can show that $I_{n3}(i,j)=o_p(1)$.

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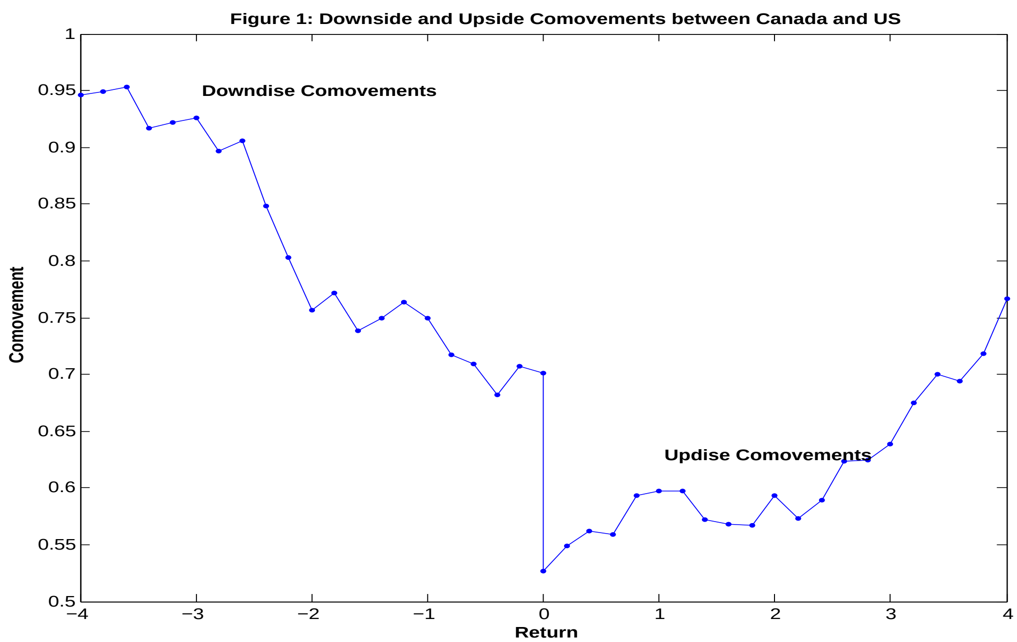


Figure 1: Downside and Upside Comovements between Canada and US
Downdise Comovements
Updise Comovements
Comovement
Return
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
−4
−3
−2
−1
0
1
2
3
4

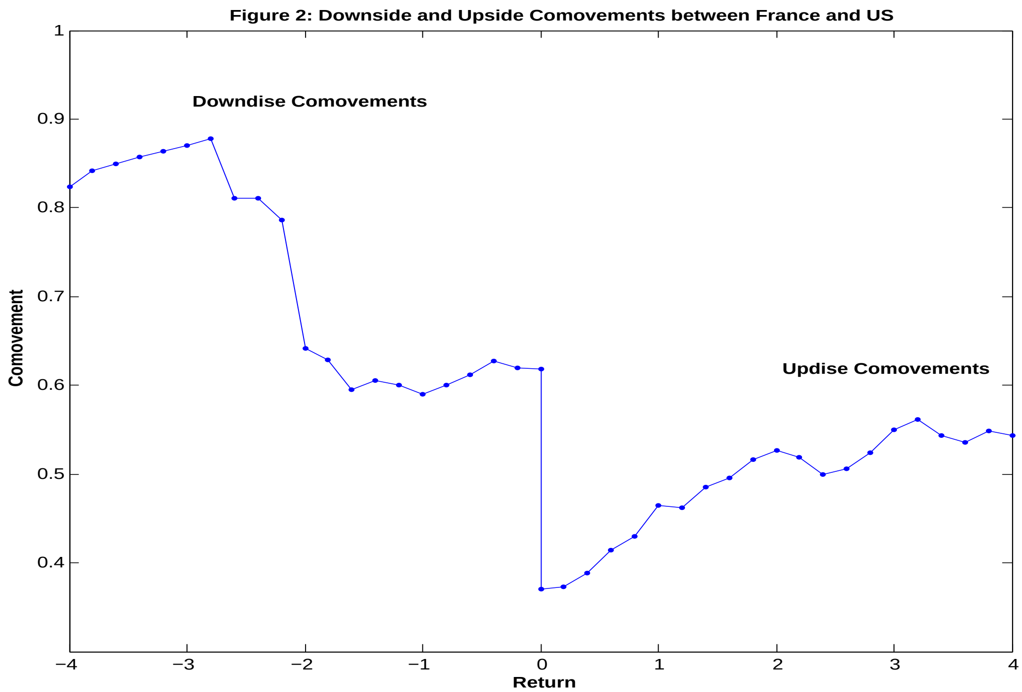


Figure 2: Downside and Upside Comovements between France and US
Downdise Comovements
Updise Comovements
Comovement
Return
1
0.9
0.8
0.7
0.6
0.5
0.4
−4
−3
−2
−1
0
1
2
3
4

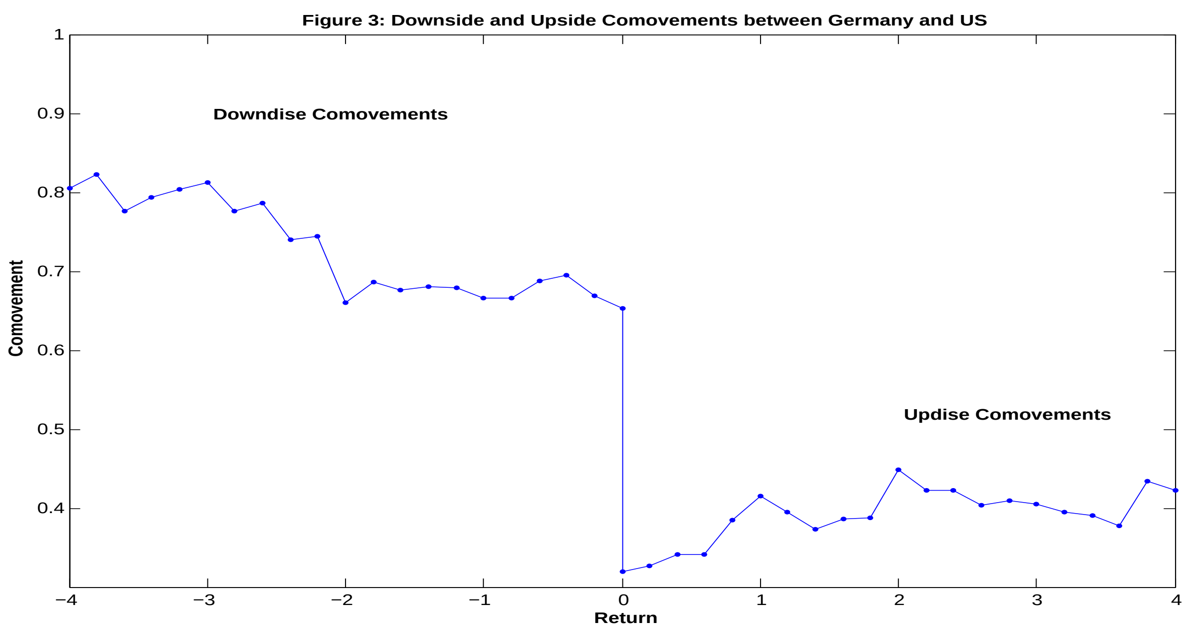
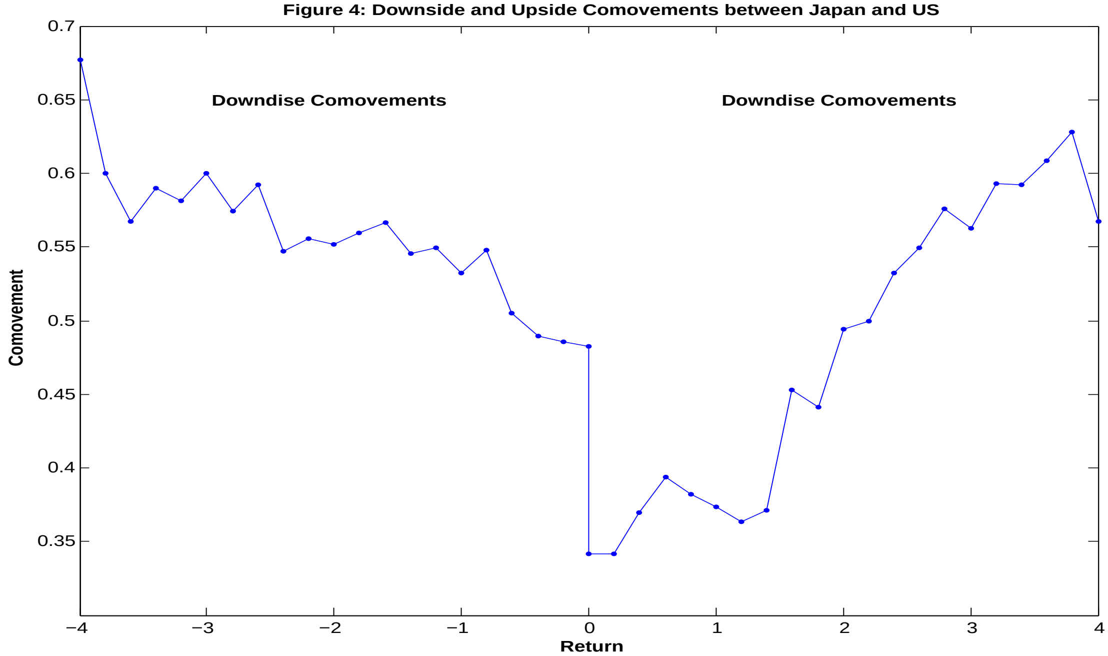


Figure 3: Downside and Upside Comovements between Germany and US
Downdise Comovements
Updise Comovements
Comovement
Return
1
0.9
0.8
0.7
0.6
0.5
0.4
−4
−3
−2
−1
0
1
2
3
4

Figure 4: Downside and Upside Comovements between Japan and US



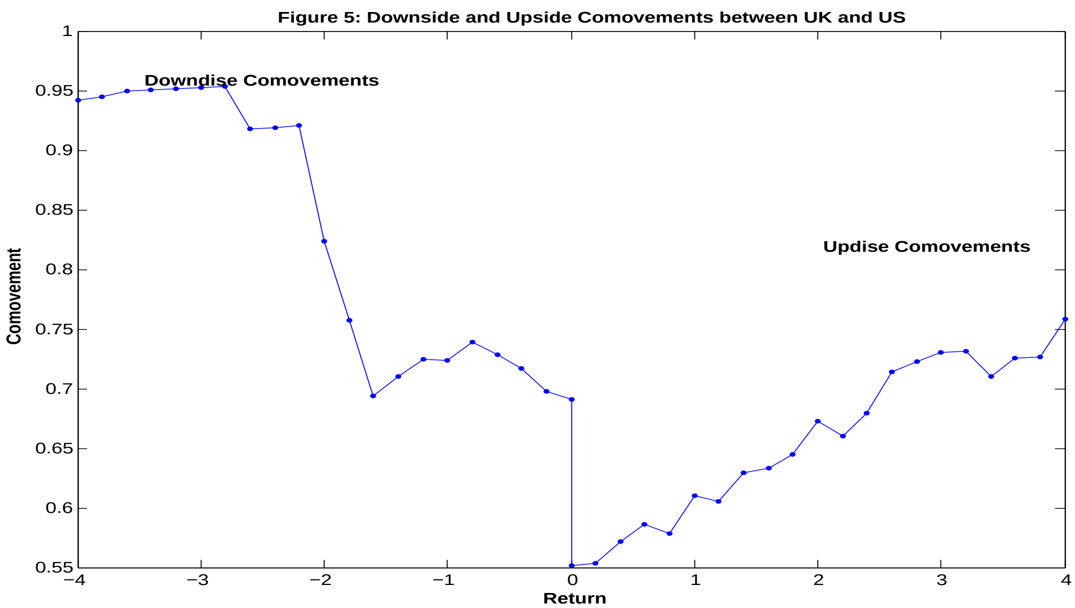


Figure 5: Downside and Upside Comovements between UK and US
Downdise Comovements
Updise Comovements
Comovement
Return
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
−4
−3
−2
−1
0
1
2
3
4