Identifying Second-Order Models of Mechanical Structures in Physical Coordinates: An Orthogonal Complement Approach

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Abstract— The problem of identifying the mass, damping, and stiffness matrices of a mechanical structure is a well known constrained system identification problem in the literature. The constraints come from the symmetry of the mass, damping, and stiffness matrices, as well as the number of sensors and actuators placed on the structure. Here we present two solutions to this problem, one based on a structured system identification approach and the other based on a similarity transformation approach. The latter approach takes advantage of the non-uniqueness of the problem to force the solution to a particular basis. Examples of both approaches show the feasibility of the methods, and it is expected to shed light on solving the most restrictive of the structural identification class of problems.

I. INTRODUCTION

System identification theory, in its most general form, consists of finding a mathematical model of a dynamical system based on a set of recorded inputs and outputs from some experiment. The type and choice of models depends on the type of application at hand. When identifying finite element model formulations, the identification of the mass, damping, and stiffness matrices is of primary concern. Thus, the model must be formulated in second order matrix differential equations form or in what is known as physical coordinates. The parameter matrices in physical coordinates can be identified using experimental dynamic data (see, for example, the works of [1], [29]). As with any physical system identification procedure, there is always an identifiability condition on the physical parameter matrices of the system. In the structural identification problem this identifiability condition depends on the number of sensors and/or actuators in the structure. Whether the mass, damping, and stiffness matrices can be uniquely identified from input/output data will depend on the number of sensors and actuators placed on the degrees-of-freedom (DOF) of the structure. Nevertheless, the identification of the system in modal coordinates, followed by updating of a pre-existing finite element model, seems to be the most widely employed approach. Some noteworthy research efforts in this direction are those of [6], [7], [8], [9], [11], [25].

The conversion from a second order form to first order differential equation form has also received considerable attention as shown by the works of [12], [13], [16], [17], [18], [20], [21], [22], [34]. However, in this approach, issues of non-uniqueness of the parameters arise if a state space model is used to identify the parameters of the second order model. In addition to the system parameters, a similarity transformation has to be found.

When updating structural models in second order form the modal parameters used are the undamped (normal) modal parameters, whereas in the first order form, the identified modal parameters are complex and equal to the damped modal parameters of the second order formulation. Identifying the undamped modal parameters from the identified complex modes constitutes an important problem, and the study by [28] presents a well documented discussion. One assumption quite often used in the literature is that the vibrational modes of the second order model are uncoupled (modal damping). Arguably the most commonly employed method to retrieve the undamped modal parameters is the so-called standard method (see [2], [12], [15]). One limitation of this method is that it loses its validity when the system is highly coupled. An alternative approach used by many authors focuses on how to obtain the undamped modal parameters from the complex modal parameters for the case of general damping. Some of the most noteworthy discussions include the works of [3], [4], [5], [10], [12], [30], [31], [35], [36].

Looking closely at this inverse problem, one might be interested in obtaining the parameters of the second order model directly from the input/output data. This constitutes a structured system identification problem as discussed in [23]. On the other hand, if one tries to obtain the second order parameters from the identified state space model (first order form), the various approaches impose different limitations on the number of sensors and actuators employed, when all the modes of the structure have been identified. For instance, the case of having as many actuators and sensors as the number of identified modes has been discussed by [35]. Alternatively, in [3] this requirement was lessened to only the number of sensors being equal to the number of identified modes, with a single DOF containing an actuator-sensor pair (also known as a co-located sensor-actuator pair). Later on [30], [31] improved the requirement to the case where the number of actuators is equal to the number of second order modes, with at least a co-located sensor-actuator pair. In [21] it is shown that the physical parameters of the second order model can be obtained by solving a symmetric complex eigenvalue problem. The requirements are that all DOFs should contain either a sensor, an actuator, or both, with at least one co-located sensor-actuator pair.

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In this paper we present an orthogonal complement approach to the identification of the physical parameters of a second order model from input/output data. Here we present two solutions to the problem, one based on a structured system identification approach as in [23], and the other based on a similarity transformation approach [26], [21], [27]. Both approaches require that the number of sensors equal the number of DOFs of the structure, with at least one co-located sensor-actuator pair. The rest of the paper is as follows: In Section 2 we present the structured system identification approach. In Section 3 we present the similarity transformation approach. Section 4 is devoted to a common example. Section 5 presents the conclusions.

II. IDENTIFICATION OF MECHANICAL STRUCTURES: A DIRECT INPUT-OUTPUT APPROACH

Consider a mechanical structure with N DOFs, whose motion is described by the following system of second-order differential equations

$$
\begin{aligned}
\mathcal{M}\ddot{q}(t) + \mathcal{C}\dot{q}(t) + \mathcal{K}q(t) &= \mathcal{B}d(t) \quad (1) \\
y(t) &= \mathcal{H}q(t), \quad (2)
\end{aligned}
$$

where $q(t) \in \mathbb{R}^N$, $d(t) \in \mathbb{R}^r$, $y(t) \in \mathbb{R}^m$, $\mathcal{M} \in \mathbb{R}^{N \times N}$, $\mathcal{C} \in \mathbb{R}^{N \times N}$, $\mathcal{K} \in \mathbb{R}^{N \times N}$, $\mathcal{B} \in \mathbb{R}^{N \times r}$, and $\mathcal{H} \in \mathbb{R}^{m \times N}$. Furthermore, the matrices $\mathcal{B}$ and $\mathcal{H}$ are known matrices with binary elements $\{0, 1\}$ that account for where the sensors and actuators are placed with respect to the DOF of the structure. We assume the matrices $\{\mathcal{M}, \mathcal{C}, \mathcal{K}\}$ are unknown but symmetric and positive definite. In what follows we will assume that $r \leq N$ and $m \leq N$. To put the above problem in the right perspective, we measure the position of m DOFs and excite r of the N DOFs. With this information we would like to find a symmetric triplet $\{\mathcal{M}, \mathcal{C}, \mathcal{K}\}$.

In this section we introduce a new orthogonal complement approach to solve the above problem from a given discrete data sequence $\{d_k, y_k\}_{k=0}^{K-1}$ obtained from some input/output experiment. We assume the data is measured at equidistant time intervals, $t_k = t_i + k\Delta t$, where t_i is an initial time (usually taken as $t_i = 0$) and Δt is the sampling period. Thus, the input and output equations are of the form $u_k = \mathcal{B}d_k$ and $y_k = \mathcal{H}q_k$, respectively, where $u_k = u(t_k)$, $y_k = y(t_k)$, $d_k = d(t_k)$, and $q_k = q(t_k)$, for $k = 0, 1, \ldots, K-1$, are the sampled values. If we now approximate the first and second derivatives of $q(t)$, using a forward difference scheme, we respectively obtain

$$
\begin{aligned}
\dot{q}(t_k) &= \frac{q(t_{k+1}) - q(t_k)}{\Delta t} = \frac{q_{k+1} - q_k}{\Delta t} \\
\ddot{q}(t_k) &= \frac{\dot{q}(t_{k+1}) - \dot{q}(t_k)}{\Delta t} \\
&= \frac{\frac{q(t_{k+2}) - q(t_{k+1})}{\Delta t} - \frac{q(t_{k+1}) - q(t_k)}{\Delta t}}{\Delta t} \\
&= \frac{q_{k+2} - 2q_{k+1} + q_k}{\Delta t^2}.
\end{aligned}
$$

Let us now discretize equations (1) – (2), using the above derivative approximations, i.e.,

$$
\mathcal{M}\left(\frac{q_{k+2} - 2q_{k+1} + q_k}{\Delta t^2}\right) + \mathcal{C}\left(\frac{q_{k+1} - q_k}{\Delta t}\right) + \mathcal{K}q_k = u_k.
$$

If we now rearrange terms with same indices, we obtain a 2nd-order matrix difference equation of the form

$$
\begin{aligned}
\left(\frac{\mathcal{M}}{\Delta t^2} - \frac{\mathcal{C}}{\Delta t} + \mathcal{K}\right)q_k &+ \left(\frac{-2\mathcal{M}}{\Delta t^2} + \frac{\mathcal{C}}{\Delta t}\right)q_{k+1} \\
&+ \left(\frac{\mathcal{M}}{\Delta t^2}\right)q_{k+2} = u_k.
\end{aligned}
$$

Let us now rename the coefficients of this last equation as

$$
\begin{aligned}
\mathcal{X}_0 &= \frac{\mathcal{M}}{\Delta t^2} - \frac{\mathcal{C}}{\Delta t} + \mathcal{K} \quad (3) \\
\mathcal{X}_1 &= \frac{-2\mathcal{M}}{\Delta t^2} + \frac{\mathcal{C}}{\Delta t} \quad (4) \\
\mathcal{X}_2 &= \frac{\mathcal{M}}{\Delta t^2}. \quad (5)
\end{aligned}
$$

Then we have the standard 2nd-order matrix difference equation

$$
\begin{aligned}
\mathcal{X}_0 q_k + \mathcal{X}_1 q_{k+1} + \mathcal{X}_2 q_{k+2} &= u_k \quad (6) \\
y_k &= \mathcal{H}q_k. \quad (7)
\end{aligned}
$$

In order to solve (6) – (7), we need to make the following assumptions:
  1) The number of actuators is equal to the number of DOFs of the structure, i.e., $r = N$.
  2) The number of sensors is equal to the number of DOFs of the structure, i.e., $m = N$.

This translates to the following properties $\mathcal{B} = I_N$ and $\mathcal{H} = I_N$. These two assumptions are the most restrictive but are not uncommon in the literature. The 2nd-order matrix difference equation now becomes

$$
\mathcal{X}_0 y_k + \mathcal{X}_1 y_{k+1} + \mathcal{X}_2 y_{k+2} = u_k. \quad (8)
$$

Since we have excitations and measurements $\{u_k, y_k\}_{k=0}^{K-1}$, we can use these to write the 2nd-order matrix difference equation (8) as an overdetermined linear system of equations. That is, let $\mathcal{X}^T = \begin{bmatrix} \mathcal{X}_0 & \mathcal{X}_1 & \mathcal{X}_2 & -I_N \end{bmatrix}$ and

$$
\mathcal{A} = \begin{bmatrix}
y_0 & y_1 & y_2 & \cdots & y_{K-3} & y_{K-2} & y_{K-1} \\
y_1 & y_2 & y_3 & \cdots & y_{K-2} & y_{K-1} & 0_{N \times 1} \\
y_2 & y_3 & y_4 & \cdots & y_{K-1} & 0_{N \times 1} & 0_{N \times 1} \\
u_0 & u_1 & u_2 & \cdots & u_{K-3} & u_{K-2} & u_{K-1}
\end{bmatrix},
$$

where $0_{n_1 \times n_2}$ denotes a zero matrix of size $n_1 \times n_2$ and I_{n_1} denotes an $n_1 \times n_1$ identity matrix. Then we have

$$
\mathcal{X}^T \mathcal{A} = 0_{N \times K}, \quad (9)
$$

where $\mathcal{X} \in \mathbb{R}^{4N \times N}$ and $\mathcal{A} \in \mathbb{R}^{4N \times K}$. When the data is noise-free, $\text{rank}\{\mathcal{A}\} = 3N$ and equation (9) can be solved using a singular value decomposition (SVD) of $\mathcal{A}$ as follows.

$$
\begin{aligned}
\mathcal{A} &= U\Sigma V^T \\
&= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0_{3N \times (K-3N)} \\ 0_{N \times 3N} & 0_{N \times (K-3N)} \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \\
&= U_1 \Sigma_1 V_1^T,
\end{aligned}
$$

where $U_1 \in \mathbb{R}^{4N \times 3N}$, $U_2 \in \mathbb{R}^{4N \times N}$, $\Sigma_1 \in \mathbb{R}^{3N \times 3N}$, $V_1 \in \mathbb{R}^{K \times 3N}$, $V_2 \in \mathbb{R}^{K \times (K-3N)}$. In the above SVD we have that $U^T U = U U^T = I_{4N}$ and $V^T V = V V^T = I_K$. Thus we obtain the following orthogonal complement problem

$$U_2^T \mathscr{A} = 0_{N \times K}. \tag{10}$$

If we now let $U_2^T = \begin{bmatrix} U_{21}^T & U_{22}^T & U_{23}^T & U_{24}^T \end{bmatrix}$. Then a solution to (9) of the form $\mathscr{X}^T = -U_{24}^{-T} U_2^T$, does not consider the symmetry of $\mathscr{X}_0$, $\mathscr{X}_1$, and $\mathscr{X}_2$. In order to do so, we need to enforce the symmetry of (3) – (5). This leads to the following symmetry constrained problem, which in turn makes use of the orthogonal complement data, i.e.,

$$U_{24}^T \left(\frac{\mathscr{M}}{\Delta T^2} - \frac{\mathscr{C}}{\Delta T} + \mathscr{K} \right) = -U_{21}^T \tag{11}$$

$$U_{24}^T \left(-\frac{2\mathscr{M}}{\Delta T^2} + \frac{\mathscr{C}}{\Delta T} \right) = -U_{22}^T \tag{12}$$

$$U_{24}^T \left(\frac{\mathscr{M}}{\Delta T^2} \right) = -U_{23}^T \tag{13}$$

$$U_{24}^T \left(\frac{\mathscr{M}^T}{\Delta T^2} - \frac{\mathscr{C}^T}{\Delta T} + \mathscr{K}^T \right) = -U_{21}^T \tag{14}$$

$$U_{24}^T \left(-\frac{2\mathscr{M}^T}{\Delta T^2} + \frac{\mathscr{C}^T}{\Delta T} \right) = -U_{22}^T \tag{15}$$

$$U_{24}^T \left(\frac{\mathscr{M}^T}{\Delta T^2} \right) = -U_{23}^T. \tag{16}$$

Let $Q = Q^T$ with $Q \in \mathbb{R}^{N \times N}$ be an arbitrary symmetric matrix, then one can apply a "shuffle matrix" operator [24] to obtain $\text{vec}\{Q^T\} = \mathscr{F} \cdot \text{vec}\{Q\}$, where $\mathscr{F}$ is a given by

$$\mathscr{F} = \begin{bmatrix} I_{N^2}(1:N:N^2, 1:N^2) \\ I_{N^2}(2:N:N^2, 1:N^2) \\ \vdots \\ I_{N^2}(N:N:N^2, 1:N^2) \end{bmatrix}, \tag{17}$$

and for an arbitrary square matrix Q, $\text{vec}\{Q\} \in \mathbb{R}^{N^2 \times 1}$ is a column vector that stacks all columns of Q from left to right into a long vector. $\mathscr{F}$ is the matrix obtained from the rows of the $N^2 \times N^2$ identity matrix, I_{N^2}, by taking every Nth row starting with the first, then every Nth row starting with the second row, and so on, until the last block obtained by taking every Nth row starting with the Nth row. Equation (17) uses MATLAB[1] notation.

Let us now define

$$E_1 = I_N \otimes U_{24}^T \tag{18}$$

$$E_2 = (I_N \otimes U_{24}^T) \cdot \mathscr{F} \tag{19}$$

$$L_1 = -\text{vec}\{U_{21}^T\} \tag{20}$$

$$L_2 = -\text{vec}\{U_{22}^T\} \tag{21}$$

$$L_3 = -\text{vec}\{U_{23}^T\}. \tag{22}$$

[1]MATLAB is a registered trademark of the Math Works, Inc.

Then by vectorizing (11) – (16), we get

$$\mathbf{A} = \begin{bmatrix} E_1 & -E_1 & E_1 \\ -2E_1 & E_1 & 0_{N^2 \times N^2} \\ E_1 & 0_{N^2 \times N^2} & 0_{N^2 \times N^2} \\ E_2 & -E_2 & E_2 \\ -2E_2 & E_2 & 0_{N^2 \times N^2} \\ E_2 & 0_{N^2 \times N^2} & 0_{N^2 \times N^2} \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} \text{vec}\{\frac{\mathscr{M}}{\Delta t^2}\} \\ \text{vec}\{\frac{\mathscr{C}}{\Delta t}\} \\ \text{vec}\{\mathscr{K}\} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix}.$$

Finally, solving the overdetermined system of equations

$$\mathbf{AX} = \mathbf{B}, \tag{23}$$

we obtain $\{\frac{\mathscr{M}}{\Delta t^2}, \frac{\mathscr{C}}{\Delta t}, \mathscr{K}\}$, from which $\{\mathscr{M}, \mathscr{C}, \mathscr{K}\}$ can be retrieved. However, if there is noise in the data, $\text{rank}\{\mathscr{A}\} \neq 3N$ in (9), which implies that we can only solve the problem approximately. Any algorithm will face the same challenges under severe noise conditions. Thus, we have to resort to an approximation. One way to solve the above problem is by using an errors-in-variable approach, where we accept noise in both the inputs and outputs, i.e., $y_k = \hat{y}_k + \Delta y_k$ and $u_k = \hat{u}_k + \Delta u_k$, then solve the following structured optimization problem:

Minimize $\quad \sum_{k=0}^{K-1} \left[\mathbf{tr}\{\Delta y_k^T \Delta y_k\} + \mathbf{tr}\{\Delta u_k^T \Delta u_k\} \right]$

Subject to: $\quad \mathscr{X}^T \hat{\mathscr{A}} = 0_{N \times K}$

$$\text{rank}\{\hat{\mathscr{A}}\} = 3N$$

$\hat{\mathscr{A}}$ is a block Hankel matrix,

where

$$\hat{\mathscr{A}} = \begin{bmatrix} \hat{y}_0 & \hat{y}_1 & \hat{y}_2 & \cdots & \hat{y}_{K-3} & \hat{y}_{K-2} & \hat{y}_{K-1} \\ \hat{y}_1 & \hat{y}_2 & \hat{y}_3 & \cdots & \hat{y}_{K-2} & \hat{y}_{K-1} & 0_{N \times 1} \\ \hat{y}_2 & \hat{y}_3 & \hat{y}_4 & \cdots & \hat{y}_{K-1} & 0_{N \times 1} & 0_{N \times 1} \\ \hat{u}_0 & \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_{K-3} & \hat{u}_{K-2} & \hat{u}_{K-1} \end{bmatrix}.$$

The solution to the above problem can be obtained using some recently proposed algorithms in [23].

III. IDENTIFICATION OF MECHANICAL STRUCTURES: A SIMILARITY TRANSFORMATION APPROACH

Let us convert the second order model (1) – (2) to first order or state space form. Let the states and parameter matrices be defined as

$$\dot{x}_c(t) = \begin{bmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{bmatrix} \in \mathbb{R}^{2N \times 1}, x_c(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \in \mathbb{R}^{2N \times 1}$$

$$A_c = \begin{bmatrix} 0_{N \times N} & I_N \\ -\mathscr{M}^{-1}\mathscr{K} & -\mathscr{M}^{-1}\mathscr{C} \end{bmatrix} \in \mathbb{R}^{2N \times 2N}$$

$$B_c = \begin{bmatrix} 0_{N \times r} \\ \mathscr{M}^{-1}\mathscr{B} \end{bmatrix} \in \mathbb{R}^{2N \times r}$$

$$C_c = \begin{bmatrix} \mathscr{H} & 0_{m \times N} \end{bmatrix} \in \mathbb{R}^{m \times 2N}$$

$$D_c = \begin{bmatrix} 0_{m \times r} \end{bmatrix} \in \mathbb{R}^{m \times r},$$

where the subscript c is used to denote a continuous time model, i.e.,

$$\dot{x}_c(t) = A_c x_c(t) + B_c d(t) \tag{24}$$
$$y(t) = C_c x_c(t) + D_c d(t). \tag{25}$$

Given a set of discrete input/output measurements $\{d_k, y_k\}_{k=0}^{K-1}$, a discrete model can be identified of the form

$$x_{k+1}^d = A_d x_k^d + B_d d_k \tag{26}$$
$$y_k = C_d x_k^d + B_d d_k, \tag{27}$$

where the order of the system is $n = 2N$, $x_k^d \in \mathbb{R}^{n \times 1}$, $A_d \in \mathbb{R}^{n \times n}$, $B_d \in \mathbb{R}^{n \times r}$, $C_d \in \mathbb{R}^{m \times n}$, and $D_d \in \mathbb{R}^{m \times r}$. This model can be identified with any state space identification technique from the system identification toolbox of MATLAB [32], [33]. Once the discrete model is obtained, a continuous time version can be obtained using any discrete-to-continuous transformation. The resulting model has the form

$$\dot{x}(t) = A x(t) + B d(t) \tag{28}$$
$$y(t) = C x(t) + D d(t), \tag{29}$$

where $n = 2N$ is the system order, $x(t) \in \mathbb{R}^{n \times 1}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, $C \in \mathbb{R}^{m \times n}$, and $D \in \mathbb{R}^{m \times r}$. The partitioned matrices are as follows:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

The above model (28) – (29) is a black-box model, whereas (24) – (25) is a physical model of the mechanical structure. It is well known in system identification that two equi-dimensional models of the same system are related by a non-singular similarity transformation matrix $T \in \mathbb{R}^{n \times n}$ of the form [26], [27]

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

such that

$$TA = A_c T \tag{30}$$
$$TB = B_c \tag{31}$$
$$C = C_c T \tag{32}$$
$$D = D_c. \tag{33}$$

Using partitioned matrices, (30) – (32) become

$$T_1 A_1 + T_2 A_3 = T_3 \tag{34}$$
$$T_1 A_2 + T_2 A_4 = T_4 \tag{35}$$
$$T_3 A_1 + T_4 A_3 = -\mathcal{M}^{-1}\mathcal{K}T_1 - \mathcal{M}^{-1}\mathcal{C}T_3 \tag{36}$$
$$T_3 A_2 + T_4 A_4 = -\mathcal{M}^{-1}\mathcal{K}T_2 - \mathcal{M}^{-1}\mathcal{C}T_4 \tag{37}$$
$$T_1 B_1 + T_2 B_2 = 0_{N \times r} \tag{38}$$
$$T_3 B_1 + T_4 B_2 = \mathcal{M}^{-1}\mathcal{B} \tag{39}$$
$$C_1 = \mathcal{H}T_1 \tag{40}$$
$$C_2 = \mathcal{H}T_2. \tag{41}$$

If we now let $\mathscr{X}_s^T$ and $\mathscr{A}_s$ be defined as

$$\mathscr{X}_s^T = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 & \mathscr{M}^{-1}\mathscr{B} & I_N \end{bmatrix}$$

$$\mathscr{A}_s = \begin{bmatrix} A_1 & A_2 & B_1 & 0_{N \times r} \\ A_3 & A_4 & B_2 & 0_{N \times r} \\ -I_N & 0_{N \times N} & 0_{N \times r} & B_1 \\ 0_{N \times N} & -I_N & 0_{N \times r} & B_2 \\ 0_{r \times N} & 0_{r \times N} & 0_{r \times r} & -I_r \\ 0_{m \times N} & 0_{m \times N} & 0_{m \times r} & 0_{m \times r} \end{bmatrix}$$

$$\begin{bmatrix} I_N & 0_{N \times N} \\ 0_{N \times N} & I_N \\ 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & 0_{N \times N} \\ 0_{r \times N} & 0_{r \times N} \\ -\mathscr{H}^\dagger C_1 & -\mathscr{H}^\dagger C_2 \end{bmatrix},$$

where $\mathscr{H}^\dagger$ denotes the pseudo-inverse of $\mathscr{H}$. Then (34) – (35) and (38) – (41) can be written as an orthogonal complement problem of the form

$$\mathscr{X}_s^T \mathscr{A}_s = 0_{N \times (4N + 2r)}. \tag{42}$$

Given that $\mathscr{A}_s \in \mathbb{R}^{(4N + r + m) \times (4N + 2r)}$, it is easy to verify that $\text{rank}\{\mathscr{A}_s\} = 4N + r$. Since $CB = 0_{m \times r}$, if we take (col 4 of $\mathscr{A}_s$) $\times B_1$ + (col 5 of $\mathscr{A}_s$) $\times B_2$, we see that it is equal to (col 3 of $\mathscr{A}_s$), thus the above rank property. However, in order to be able to solve (42), we must enforce assumption (2) $(m = N)$, as well as the condition $r \le N$. This is a less restrictive set of conditions than assumptions (1) – (2) combined. In this case $\mathscr{A}_s \in \mathbb{R}^{(5N + r) \times (4N + 2r)}$ and it is guaranteed that there exists a matrix $\mathscr{X}_s \in \mathbb{R}^{(5N + r) \times N}$ such that (42) is satisfied.

If we compute the SVD of $\mathscr{A}_s$, we obtain

$$\mathscr{A}_s = \begin{bmatrix} \mathscr{U}_1 & \mathscr{U}_2 \end{bmatrix} \begin{bmatrix} \mathscr{S}_1 & 0_{(4N+r) \times r} \\ 0_{N \times (4N+r)} & 0_{N \times r} \end{bmatrix} \begin{bmatrix} \mathscr{V}_1^T \\ \mathscr{V}_2^T \end{bmatrix},$$

where $\mathscr{U}_1 \in \mathbb{R}^{(5N+r) \times (4N+r)}$, $\mathscr{U}_2 \in \mathbb{R}^{(5N+r) \times N}$, $\mathscr{V}_1 \in \mathbb{R}^{(4N+r) \times (4N+2r)}$, $\mathscr{V}_2 \in \mathbb{R}^{(4N+r) \times r}$, $\mathscr{S}_1 \in \mathbb{R}^{(4N+r) \times (4N+r)}$, from which we obtain $\mathscr{U}_2^T \mathscr{A}_s = 0_{N \times (4N+2r)}$. If we now make the partition $\mathscr{U}_2^T = \begin{bmatrix} \mathscr{U}_{21}^T & \mathscr{U}_{22}^T & \mathscr{U}_{23}^T & \mathscr{U}_{24}^T & \mathscr{U}_{25}^T & \mathscr{U}_{26}^T \end{bmatrix}$, we obtain the solution

$$\mathscr{X}_s^T = (\mathscr{U}_{26}^T)^{-1} \mathscr{U}_2^T. \tag{43}$$

Then T and $\mathscr{H}_1 = \mathscr{M}^{-1}\mathscr{B}$ can be obtained from

$$T = \begin{bmatrix} (\mathscr{U}_{26}^T)^{-1}\mathscr{U}_{21}^T & (\mathscr{U}_{26}^T)^{-1}\mathscr{U}_{22}^T \\ (\mathscr{U}_{26}^T)^{-1}\mathscr{U}_{23}^T & (\mathscr{U}_{26}^T)^{-1}\mathscr{U}_{24}^T \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$
$$\mathscr{H}_1 = (\mathscr{U}_{26}^T)^{-1}\mathscr{U}_{25}^T.$$

If we now define $\mathscr{H}_2 = \mathscr{M}^{-1}\mathscr{K}$ and $\mathscr{H}_3 = \mathscr{M}^{-1}\mathscr{C}$, then (36) – (37) can be solved from

$$\begin{bmatrix} \mathscr{H}_2 & \mathscr{H}_3 \end{bmatrix} = -\begin{bmatrix} T_3 & T_4 \end{bmatrix} A T^{-1}. \tag{44}$$

We now need to find $\{\mathscr{M}, \mathscr{C}, \mathscr{K}\}$, taking into account the symmetry conditions on these. From the symmetry

constraints and $\mathscr{H}_i$, for $i = 1, 2, 3$, we can write the system of equations as

$$\mathscr{B} = I_N \mathscr{M} \mathscr{H}_1 \tag{45}$$

$$\mathscr{K} = I_N \mathscr{M} \mathscr{H}_2 \tag{46}$$

$$\mathscr{C} = I_N \mathscr{M} \mathscr{H}_3 \tag{47}$$

$$I_N \mathscr{M} \mathscr{H}_2 = \mathscr{H}_2^T \mathscr{M} I_N \tag{48}$$

$$I_N \mathscr{M} \mathscr{H}_3 = \mathscr{H}_3^T \mathscr{M} I_N. \tag{49}$$

Now vectorizing (45) – (49) and using the property $\text{vec}\{XYZ\} = (Z^T \otimes X)\text{vec}\{Y\}$, we obtain

$$\mathscr{Q} = \mathscr{P}^\dagger \mathscr{R}, \tag{50}$$

where

$$\mathscr{P} = \begin{bmatrix} (\mathscr{H}_1^T \otimes I_N) & 0_{Nr \times N^2} & 0_{Nr \times N^2} \\ (\mathscr{H}_2^T \otimes I_N) & -I_{N^2} & 0_{N^2 \times N^2} \\ (\mathscr{H}_3^T \otimes I_N) & 0_{N^2 \times N^2} & -I_{N^2} \\ (\mathscr{H}_2^T \otimes I_N) - (I_N \otimes \mathscr{H}_2^T) & 0_{N^2 \times N^2} & 0_{N^2 \times N^2} \\ (\mathscr{H}_3^T \otimes I_N) - (I_N \otimes \mathscr{H}_3^T) & 0_{N^2 \times N^2} & 0_{N^2 \times N^2} \end{bmatrix}$$

$$\mathscr{Q} = \begin{bmatrix} \text{vec}\{\mathscr{M}\} \\ \text{vec}\{\mathscr{K}\} \\ \text{vec}\{\mathscr{C}\} \end{bmatrix}, \quad \mathscr{R} = \begin{bmatrix} \text{vec}\{\mathscr{B}\} \\ 0_{N^2 \times 1} \\ 0_{N^2 \times 1} \\ 0_{N^2 \times 1} \\ 0_{N^2 \times 1} \end{bmatrix}.$$

IV. EXAMPLES

We now present an example of a mechanical structure with $N = 3$ DOFs and a true set of parameters $\{\mathscr{M}, \mathscr{C}, \mathscr{K}\}$ given by

$$\mathscr{M} = \begin{bmatrix} 0.8 & 0.0 & 0.0 \\ 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 1.2 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} 0.4 & -0.1 & -0.1 \\ -0.1 & 0.4 & -0.1 \\ -0.1 & -0.1 & 0.4 \end{bmatrix}$$

$$\mathscr{K} = \begin{bmatrix} 4.0 & -1.0 & -1.0 \\ -1.0 & 4.0 & -1.0 \\ -1.0 & -1.0 & 4.0 \end{bmatrix}.$$

Direct Input-Output Approach:

In this method we assume that $r = N$ and $m = N$ so that all DOFs contain both a sensor and an actuator. Thus, $\mathscr{B} = I_N$ and $\mathscr{H} = I_N$. The system was simulated with a continuous time state space model with parameter matrices

$$A_c = \begin{bmatrix} 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \\ -5.00 & 1.25 & 1.25 & -0.50 & 0.13 & 0.13 \\ 0.50 & -2.00 & 0.50 & 0.05 & -0.20 & 0.05 \\ 0.83 & 0.83 & -3.33 & 0.08 & 0.08 & -0.33 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0_{3 \times 3} \\ I_3 \end{bmatrix}, \quad C_c = \begin{bmatrix} I_3 & 0_{3 \times 3} \end{bmatrix}, \quad D_c = \begin{bmatrix} 0_{3 \times 3} \end{bmatrix}.$$

The sampling time used in the simulation was $\Delta t = 0.0001$ and $K = 100,000$ data points were used. The resulting

physical parameter matrices were

$$\mathscr{M} = \begin{bmatrix} 0.8000 & -0.0002 & -0.0001 \\ -0.0002 & 2.0001 & 0.0000 \\ -0.0001 & 0.0000 & 1.2000 \end{bmatrix}$$

$$\mathscr{C} = \begin{bmatrix} 0.4001 & -0.1000 & -0.1001 \\ -0.1000 & 0.4004 & -0.1001 \\ -0.1001 & -0.1001 & 0.4001 \end{bmatrix}$$

$$\mathscr{K} = \begin{bmatrix} 4.0001 & -1.0003 & -1.0002 \\ -1.0003 & 4.0001 & -1.0001 \\ -1.0002 & -1.0001 & 4.0001 \end{bmatrix}.$$

As can be seen, the computed physical parameters agree closely with the true physical parameters.

Similarity Transformation Approach

In this method all previous conditions are the same, except for $r = 1$, which is a less restrictive condition. Thus, $\mathscr{B} = e_1$ and $\mathscr{H} = I_N$, where e_1 is the first column of I_N. The data was generated from the same model as before but with only one input and is shown in Fig. 1. First a discrete time model was identified using a subspace system identification technique from MATLAB. Then the discrete time model was converted to continuous time form using the MATLAB function d2c. The parameter matrices of the identified continuous time model were then used to solve (42) and eventually, equation (50). The physical parameters obtained were
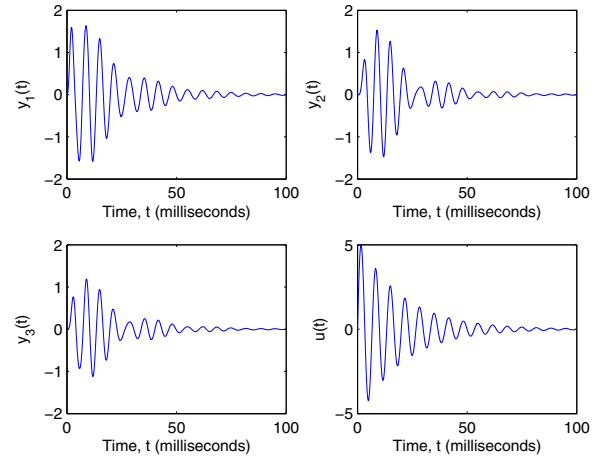

Fig. 1. Input and output data used in the example.

$$\mathscr{M} = \begin{bmatrix} 0.8000 & 0.0000 & 0.0000 \\ 0.0000 & 2.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.2000 \end{bmatrix}$$

$$\mathscr{C} = \begin{bmatrix} 0.4000 & -0.1000 & -0.1000 \\ -0.1000 & 0.4000 & -0.1000 \\ -0.1000 & -0.1000 & 0.4000 \end{bmatrix}$$

$$\mathscr{K} = \begin{bmatrix} 4.0000 & -1.0000 & -1.0000 \\ -1.0000 & 4.0000 & -1.0000 \\ -1.0000 & -1.0000 & 4.0000 \end{bmatrix}.$$

As can be seen from the results, there are no discrepancies between the true and computed models.

V. Conclusions

We have introduced two algorithms for identifying the physical parameters of mechanical structures. As an inverse problem this is a challenging one due to the fact that the data is discrete and the model is continuous in time. Furthermore, the matrices to be identified must be symmetric. There are other challenges such as the number of sensors and actuators needed to monitor the structure. In reality, the lower the number of sensors and actuators, the better the model. The first method assumes that each degree of freedom has both an actuator and a sensor. This presents some limitations compared to the works of [21]. In the similarity transformation approach, we assumed that the structure contains a full set of sensors and at least one actuator. Thus the condition that $r + m = N + 1$ as proposed by [21], although this is not the general case. More work needs to be done along these lines. The similarity transformation approach is an extension of the works of [27] to mechanical structures and a special case of the works of [26], representing an exact rank case. Finally, we point out that the problem presented herein is normally solved with nonlinear methods. Here we presented two linear methods for its solution.

References

[1] M. S. Agbabian, S. F. Masri, R. K. Miller, and T. K. Caughey, "System Identification Approach to Detection of Structural Changes", ASCE Journal of Engineering Mechanics, Vol. 117, No. 2, pp. 370-390, 1991.

[2] K. F. Alvin, "Second-Order Structural Identification via State Space Based System Realizations", Ph.D. Thesis, University of Colorado, 1993.

[3] K. F. Alvin and K. C. Park, "Second-Order Structural Identification Procedure via State - Space - Based System Identification", AIAA Journal, Vol. 32, No. 2, pp. 397-406, 1994.

[4] K. F. Alvin, L. D. Peterson, and K. C. Park, "Method for Determining Minimum - Order Mass and Stiffness Matrices from Modal Test Data", AIAA Journal, Vol. 33, No. 1, pp. 128-135, 1995.

[5] E. Balmès, "New Results on the Identification of Normal Modes from Experimental Complex Modes", Mechanical Systems and Signal Processing, Vol. 11, No. 2, pp. 229-243, 1997.

[6] M. Baruch, "Optimal Correction of Mass and Stiffness Matrices Using Measured Modes", AIAA Journal, Vol. 20, No. 11, pp. 1623-1626, 1982.

[7] M. Baruch, "Modal Data are Insufficient for Identification of Both Mass and Stiffness Matrices", AIAA Journal, Vol. 35, No. 11, pp. 1797-1798, 1997.

[8] J. L. Beck and L. S. Katafygiotis, "Updating Models and Their Uncertainties. I: Bayesian Statistical Framework", Journal of Engineering Mechanics, Vol. 124, No. 4, pp. 455-461, 1998.

[9] A. Berman, "Mass Matrix Correction Using an Incomplete Set of Measured Modes", AIAA Journal, Vol. 17, No. 10, pp. 1147-1148, 1979.

[10] S. Y. Chen, M. S. Ju, and Y. G. Tsuei, "Extraction of Normal Modes for Highly Coupled Incomplete Systems with General Damping", Mechanical Systems and Signal Processing, Vol. 10, No. 1, pp. 93-106, 1996.

[11] D. J. Ewins, Modal Testing: Theory and Practice, Letchworth, Research Studies Press, 1984.

[12] S. R. Ibrahim and E. C. Mikulcik, "A Method for the Direct Identification of Vibration Parameters from the free response", Shock and Vibration Bulletin, Vol. 47, No. 4, pp. 183-198, 1977.

[13] S. R. Ibrahim, "Random Decrement Technique for Modal Identification of Structures", Journal of Spacecraft and Rockets, Vol. 14, No. 11, pp. 696-700, 1977.

[14] S. R. Ibrahim, "Computation of Normal Modes from Identified Complex Modes", AIAA Journal, Vol. 21, No. 3, pp. 446-451, 1983.

[15] M. Imregun and D. J. Ewins, "Realization of Complex Mode Shapes", Proceedings of the 11th International Modal Analysis Conference, pp. 1303-1309, 1993.

[16] J. N. Juang and R. S. Pappa, "An Eigensystem Realization Algorithm for Modal Parameter Identification and Model Reduction", Journal of Guidance, Control, and Dynamics, Vol. 8, no. 5, pp. 620-627, 1985.

[17] J. N. Juang, J. E. Cooper, and J. R. Wright, "An Eigensystem Realization Algorithm Using Data Correlations (ERA/DC) for Modal Parameter Identification", Control Theory and Advanced Technology, Vol. 4, No. 1, pp. 5-14, 1988.

[18] J. N. Juang, M. Phan, L. G. Horta, and R. W. Longman, "Identification of Observer/Kalman Filter Markov Parameters: Theory and Experiments", Journal of Guidance, Control, and Dynamics, Vol. 16, 2, pp. 320-329, 1993.

[19] C. G. Koh and L. M. See, "Identification and Uncertainty Estimation of Structural Parameters", Journal of Engineering Mechanics, Vol. 120, No. 6, pp. 1219-1236, 1993.

[20] H. Luç, R. Betti, and R. W. Longman, "Identification of Linear Structural Systems Using Earthquake - Induced Vibration Data", Earthquake Engineering and Structural Dynamics, Vol. 28, pp. 1449-1467, 1999.

[21] H.Luç, "Control Theory Based System Identification", Ph.D. Thesis, Columbia University, New York, 2001.

[22] H. Luç, R. Betti, and R. W. Longman, "Obtaining Optimized Predictive Models and Modal Parameters of Structural Systems", Earthquake Engineering and Structural Dynamics, vol. 31, pp. 14131440, 2002.

[23] I. Markovsky, *Low Rank Approximation: Algorithms, Implementation, Applications*, Springer-Verlag, London, 2012.

[24] C. D. Martin, "Higher-Order Kronecker Products and Tensor Decompositions," Ph.D. dissertation, Cornell University, 2005.

[25] J. E. Mottershead and M. I. Friswell, "Model Updating in Structural Dynamics: A Survey", Journal of Sound and Vibration, Vol. 165, no. 2, pp. 347-375,1993.

[26] O. Prot, G. Mercère, and J. A. Ramos, "A null-space-based technique for estimation of linear-time invariant structured state-space representations," IFAC 16th Symposium on System Identification, pp. 191-196, 2012.

[27] J. A. Ramos and P. L. dos Santos, "Parameter estimation of discrete and continuous-time physical models: A similarity transformation approach," 49th IEEE Conference on Decision and Control, pp. 4435-4440, 2010.

[28] A. Sestieri and S. R. Ibrahim, "Analysis of Errors and Approximations in the Use of Modal Coordinates", Journal of Sound and Vibration, Vol. 177, No. 2, pp. 145-157, 1994.

[29] A. W. Smyth, S. F. Masri, T. K. Caughey, and N. F. Hunter, "Surveillance of Intricate Mechanical Systems on the Basis of Vibration Signature Analysis", ASME Journal of Applied Mechanics, Vol. 67, No. 3, pp. 540-551, 2000.

[30] D. H. Tseng, R. W. Longman and J. N. Juang, "Identification of Gyroscopic and Nongyroscopic Second Order Mechanical Systems Including Repeated Problems", Advances in Astronautical Sciences, Vol. 87, pp. 145-165, 1994.

[31] D. H. Tseng, R. W. Longman and J. N. Juang, "Identification of the Structure of the Damping Matrix in Second Order Mechanical Systems", Advances in Astronautical Sciences, Vol. 87, pp. 166-190, 1994.

[32] P. Van Overschee and B. De Moor, *Subspace identification for linear Systems*. Kluwer Academic Publisher, 1996.

[33] M. Verhaegen, "Identification of the deterministic part of MIMO state space model given in innovations form from input-output data," *Automatica*, vol. 30, no. 1, pp. 61-74, 1994.

[34] H. Vold, J. Kundrat, G. T. Rocklin, and R. Russell, "A Multiple-Input Modal Estimation Algorithm for Mini Computers." SAE Transactions, Vol. 91, No. 1, pp. 815-821, 1982.

[35] C. D. Yang and F. B. Yeh, "Identification, Reduction, and Refinement of Model Parameters by the Eigensystem Realization Algorithm", Journal of Guidance, Control, and Dynamics, Vol. 13, No. 6, pp. 1051-1059, 1990.

[36] Q. Zhang and G. Lallement, "Comparison of Normal Eigenmodes Calculation Methods Based on Identified Complex Eigenmodes", Journal of Spacecraft and Rockets, Vol. 24, pp. 69-73, 1987.