BULL. AUSTRAL. MATH. SOC. MOS 4650, 4655
VOL. 2 (1970), 253-260.

Identities in tensor products of Banach algebras

R. J. Loy

Let A_1, A_2 be Banach algebras, $A_1 \otimes A_2$ their algebraic tensor product over the complex field. If $\|\cdot\|_\alpha$ is an algebra norm on $A_1 \otimes A_2$ we write $A_1 \otimes_\alpha A_2$ for the $\|\cdot\|_\alpha$-completion of $A_1 \otimes A_2$. In this note we study the existence of identities and approximate identities in $A_1 \otimes_\alpha A_2$ versus their existence in A_1 and A_2. Some of the results obtained are already known, but our method of proof appears new, though it is quite elementary.

1. Preliminaries

The four results collected here are probably already known in one form or another; the proofs are included for completeness.

PROPOSITION 1. *Let A be a Banach algebra, $\{e_\lambda\}$, $\{f_\mu\}$ nets in A such that $\|e_\lambda x-x\| \to 0$, $\|xf_\mu-x\| \to 0$ uniformly on the unit ball of A. Then A has an identity.*

Proof. Take $s \in \{e_\lambda\}$ such that $\|sx-x\| \le \frac{1}{2}$ for $\|x\| \le 1$. Then s is not a left topological divisor of zero, for otherwise there would be $\{x_n\} \subseteq A$, $\|x_n\| = 1$, with $sx_n \to 0$, contradicting $\|sx_n-x_n\| \le \frac{1}{2}$ for each n. Similarly there is $t \in A$ which is not a right topological divisor of zero.

Received 10 January 1970.

https://doi.org/10.1017/S0004972700041873 Published online by Cambridge University Press

Now for each positive integer n take $e_n \in \{e_\lambda\}$, $f_n \in \{f_\mu\}$ such that $\|e_n x - x\| \le \frac{1}{n}$, $\|x f_n - x\| \le \frac{1}{n}$ for $\|x\| \le 1$. Then certainly $\|(e_n - e_m)t\| \le \left(\frac{1}{n} + \frac{1}{m}\right)\|t\|$, $\|s(f_n - f_m)\| \le \left(\frac{1}{n} + \frac{1}{m}\right)\|s\|$ and so, by the choice of s and t, $\{e_n\}$, $\{f_n\}$ are Cauchy, and so converge to elements $e, f \in A$. But then e and f are respectively left and right identities for A, and so $e = f$ is an identity.

We will also require the following modification of Proposition 1. As usual ν, $\hat{}$ denote spectral radius and Gelfand transform respectively.

PROPOSITION 2. *Let A be a commutative Banach algebra, $\{e_\lambda\}$ a net in A such that $\nu(e_\lambda x - x) \to 0$ uniformly for $\nu(x) \le 1$. Then there is an idempotent $e \in A$ with $\hat{e} \equiv 1$.*

Proof. By the same argument as in Proposition 1 there is a sequence $\{e_n\} \subseteq A$ with $\nu(e_n x - x) \to 0$ uniformly for $\nu(x) \le 1$, and $\nu(e_n - e_m) \to 0$. It follows that $\{\hat{e}_n\}$ converges uniformly to the constant function 1 on the carrier space of A, which is thus compact. Also, for n sufficiently large $\hat{e}_n$ is bounded away from zero, and the elementary argument of [5], pp. 171-2 now furnishes the desired idempotent e.

PROPOSITION 3. *Let A be a Banach algebra which does not consist entirely of right (left) topological divisors of zero. If A has a left (right) approximate identity $\{d_\rho\}$ then it has a bounded left (right) approximate identity. Indeed, if A is commutative and $\{d_\rho\}$ is countable, then $\{d_\rho\}$ is bounded.*

Proof. Let $\mathcal{F}$ be the family of all finite subsets of A, and define a directed set $\Lambda = \{(F, \delta) : F \in \mathcal{F}, 1 > \delta > 0\}$ where $(F_1, \delta_1) \le (F_2, \delta_2)$ if $F_1 \subseteq F_2$ and $\delta_2 \le \delta_1$. Take $z \in A$ not a right topological divisor of zero. Then for $\lambda = (F, \delta) \in \Lambda$, take $e_\lambda \in \{d_\rho\}$ such that $\|e_\lambda y - y\| < \delta$ for $y \in F \cup \{z\}$, so that if $x \in A$, $1 > \varepsilon > 0$, $\|e_\lambda x - x\| < \varepsilon$ provided $\lambda \ge (\{x\}, \varepsilon)$. It follows that $\{e_\lambda\}$ is a left approximate identity in A. Also, $\|e_\lambda z\| < 1 + \|z\|$ for all λ, so

https://doi.org/10.1017/S0004972700041873 Published online by Cambridge University Press

that, by the choice of z , $\{e_\lambda\}$ is bounded.

The last statement is proved in [6], p. 279.

REMARK. The converse is false: $L^1(0, 1)$ is a radical algebra under convolution, so that all elements are topological divisors of zero, but has a (countable) bounded approximate identity.

Now let U_1, U_2 be seminormed spaces under p_1, p_2 respectively. Corresponding to the normed case define seminorms p_γ, p_λ on $U_1 \otimes U_2$ by

$$p_\gamma(x) = \inf\left\{\sum p_1(u_i)p_2(v_i) : x = \sum u_i \otimes v_i\right\} ,$$

$$p_\lambda(x) = \sup\left\{\left|\sum \varphi_1(u_i)\varphi_2(v_i)\right| : x = \sum u_i \otimes v_i,\ \varphi_j \in U_j^*,\ \|\varphi_j\|_{p_j} = 1,\ j = 1, 2\right\} .$$

As in the normed case p_γ is the greatest seminorm p on $U_1 \otimes U_2$ such that $p(u \otimes v) = p_1(u)p_2(v)$, $u \in U_1$, $v \in U_2$; and so in particular $p_\lambda \le p_\gamma$. Using the terminology of [1] a seminorm p on $U_1 \otimes U_2$ will be called admissible if there are positive constants m, M such that $mp_\lambda \le p \le Mp_\gamma$.

PROPOSITION 4. *Let* U_j, p_j , $j = 1, 2$ *be as above, and* p *an admissible seminorm on* $U_1 \otimes U_2$, *with* m *as above. If* $x = \sum_{i=1}^{n} u_i \otimes v_i \in U_1 \otimes U_2$ *with* $\{v_i\}$ (*or* $\{u_i\}$) *linearly independent, then* $p_1(u_i)p_2(v_i) \le \frac{1}{m}p(x)$, $i = 1, 2, \ldots, n$.

Proof. Take $1 \le i \le n$. If $p_1(u_i)p_2(v_i) = 0$ the result is immediate for this i , otherwise the Hahn-Banach theorem furnishes $\varphi_j \in U_j^*$, $j = 1, 2$ such that $\|\varphi_j\|_{p_j} = 1$, $\varphi_2(v_k) = \delta_{ik}p_2(v_k)$, $\varphi_1(u_i) = p_1(u_i)$. But then for $w = \sum s_k \otimes t_k \in U_1 \otimes U_2$,

$$\varphi_1 \otimes \varphi_2(w) = \sum \varphi_1(s_k)\varphi_2(t_k) \le p_\lambda(w) \le \frac{1}{m}p(w) ,$$ and so

$$\varphi_1 \otimes \varphi_2(x) = p_1(u_i)p_2(v_i) \le \frac{1}{m}p(x) .$$

https://doi.org/10.1017/S0004972700041873 Published online by Cambridge University Press

2. The general (non-commutative) case

For the remainder of this paper A_j will denote a Banach algebra with norm $\|\cdot\|_j$, spectral radius ν_j, $j = 1, 2$; $\|\cdot\|_\alpha$ will be an algebra norm on $A_1 \otimes A_2$ with spectral radius ν_α. If A is commutative its carrier space, with the Gelfand topology, will be denoted Φ_A.

THEOREM 1. *Let* $\|\cdot\|_\alpha$ *be an admissible algebra norm on* $A_1 \otimes A_2$. *Then* $A_1 \otimes_\alpha A_2$ *has an identity* ι *if and only if* A_1, A_2 *have identities* e, f, *and* $\iota = e \otimes f$.

Proof. Suppose that $A_1 \otimes_\alpha A_2$ has an identity ι, the converse being immediate. Let $\varepsilon > 0$, and take $x = x_\varepsilon \in A_1 \otimes A_2$ with $\|x-\iota\|_\alpha < \varepsilon$. Then if $x = \sum u_i \otimes v_i$ and $\|s \otimes t\|_\alpha \leq 1$,

$$\left\|\sum u_i s \otimes v_i t - s \otimes t\right\|_\alpha < \varepsilon , \quad \left\|\sum su_i \otimes tv_i - s \otimes t\right\|_\alpha < \varepsilon .$$

Now let $\sum u_j' s \otimes v_j'$, $\sum su_k'' \otimes v_k''$ be alternative expressions for $\sum u_i s \otimes v_i t$, $\sum su_i \otimes tv_i$ respectively, where $v_1' = v_1'' = t$ and $\{v_j'\}$, $\{v_k''\}$ are linearly independent sets. Then

$$\left\|(u_1's-s) \otimes t + \sum_{j\geq 2} u_j's \otimes v_j'\right\|_\alpha < \varepsilon , \quad \left\|(su_1''-s) \otimes t + \sum_{k\geq 2} su_k'' \otimes v_k''\right\|_\alpha < \varepsilon ,$$

and so by Proposition 4 $\|u_1's-s\|_1\|t\|_2 < \frac{\varepsilon}{m}$, $\|su_1''-s\|_1\|t\|_2 < \frac{\varepsilon}{m}$, where $m\|\cdot\|_\lambda \leq \|\cdot\|_\alpha \leq M\|\cdot\|_\gamma$. Now u_1', u_1'' depend on t only, not upon s, and so, noting that if $\|s\|_1 \leq 1$ then $\|s \otimes t\|_\alpha \leq M\|t\|_2$, it follows that $\|u_1's-s\|_1 \leq \frac{M}{m}\varepsilon$, $\|su_1''-s\|_1 \leq \frac{M}{m}\varepsilon$, for $\|s\|_1 \leq 1$. Taking $\varepsilon = 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots$ we thus obtain sequences $\{e_n\}$, $\{e_n'\} \subseteq A_1$ such such that $\|e_ns-s\|_1 \to 0$, $\|se_n'-s\|_1 \to 0$ uniformly on the unit ball of A_1, so that by Proposition 1 A_1 has an identity e.

Similarly A_2 has an identity f, whence $e \otimes f$ is an identity on

https://doi.org/10.1017/S0004972700041873 Published online by Cambridge University Press

$A_1 \otimes_\alpha A_2$ and so must equal ι.

By an equally simple argument we have the following.

THEOREM 2. *Let* $\|\cdot\|_\alpha$ *be an admissible algebra norm on* $A_1 \otimes A_2$. *If* A_1 *and* A_2 *each possess a bounded left (right) approximate identity then so does* $A_1 \otimes_\alpha A_2$. *Conversely, if* $A_1 \otimes_\alpha A_2$ *has a left (right) approximate identity then so do* A_1 *and* A_2.

Proof. Let $m\|\cdot\|_\lambda \le \|\cdot\|_\alpha \le M\|\cdot\|_\gamma$, and take $\{e_\lambda\}$, $\{f_\mu\}$ bounded left approximate identities in A_1 and A_2 respectively, with $\sup\|e_\lambda\|_1 \le C$, $\sup\|f_\mu\|_2 \le C$ for some C. Then the set $\{e_\lambda \otimes f_\mu\} \subseteq A_1 \otimes A_2$ is bounded, $\sup\|e_\lambda \otimes f_\mu\|_\alpha \le MC^2$, and with the product direction is a left approximate identity in $A_1 \otimes A_2$ (under $\|\cdot\|_\alpha$). Since $A_1 \otimes A_2$ is dense in $A_1 \otimes_\alpha A_2$ it follows easily that $\{e_\lambda \otimes f_\mu\}$ is a bounded left approximate identity in $A_1 \otimes_\alpha A_2$.

Conversely, let $\{d_\rho\}$ be a left approximate identity in $A_1 \otimes_\alpha A_2$. Let F be a finite subset of A_1, $K = \max\{\|s\|_1 : s \in F\} + 1$, $\delta > 0$, and take $t \in A_2$, $\|t\|_2 = 1$. Choose $x \in \{d_\rho\}$ such that $\|x(s \otimes t) - s \otimes t\|_\alpha < \frac{\delta m}{2M}$, $s \in F$, and then take $\sum u_i \otimes v_i \in A_1 \otimes A_2$ with $\left\|x - \sum u_i \otimes v_i\right\|_\alpha < \frac{\delta m}{2KM^2}$. But then $\left\|\sum u_i s \otimes v_i t - s \otimes t\right\|_\alpha < \frac{\delta m}{M}$ for all $s \in F$. Proceeding as in Theorem 1 it follows that there is $u \in A_1$ with $\|us-s\|_1 < \delta$ for $s \in F$. Now proceed as in Proposition 3, but without the element z, to obtain a net $\{e_\lambda\}$, consisting of such u, which is a left approximate identity in A_1.

Similarly A_2 has a left approximate identity.

REMARK. The first half of this result appears known, it is used implicitly in [4], Theorem 2.2. The present author has been unable to determine whether addition of the hypothesis of boundedness of $\{d_\rho\}$ in the converse half would ensure boundedness of the resulting nets $\{e_\lambda\}$, $\{f_\mu\}$ in A_1, A_2 respectively. However, if A_1 and A_2 are commutative and $\{d_\rho\}$ is countable and unbounded then not both $\{e_\lambda\}$, $\{f_\mu\}$ are bounded, for otherwise $\{e_\lambda \otimes f_\mu\}$ is a bounded approximate identity in $A_1 \otimes_\alpha A_2$, contradicting [6], p. 279. In the general case, if A_1 and A_2 do not consist entirely of right (left) topological divisors of zero then Proposition 3 shows that they have bounded left (right) approximate identities, and hence so does $A_1 \otimes_\alpha A_2$. Thus if A_1 and A_2 are commutative and $\{d_\rho\}$ is countable, then $\{d_\rho\}$ is bounded.

3. The commutative case

The first result concerning identities in $A_1 \otimes_\alpha A_2$ was that of Gelbaum [2], Theorem 4, who considered the case A_1, A_2 commutative semisimple, and $\|\cdot\|_\alpha = \|\cdot\|_\gamma$. This case is included in Theorem 1 above. Recently Lardy and Lindberg [3] have defined an algebra norm $\|\cdot\|_\alpha$ on $A_1 \otimes A_2$ to be a spectral tensor norm in the case $\nu_\alpha(u \otimes v) = \nu_1(u)\nu_2(v)$, $u \in A_1$, $v \in A_2$. They showed that the natural map of $\Phi_{A_1 \otimes_\alpha A_2}$ into $\Phi_{A_1} \times \Phi_{A_2}$ is surjective if and only if $\|\cdot\|_\alpha$ is spectral, and in this case $A_1 \otimes_\alpha A_2$ has an identity if and only if A_1 and A_2 have identities. In this section we obtain an elementary proof of this result.

LEMMA. *Let A_1, A_2 be commutative Banach algebras, $\|\cdot\|_\alpha$ an algebra norm on $A_1 \otimes A_2$. Then $\|\cdot\|_\alpha$ is a spectral tensor norm if and only if ν_α is an admissible seminorm on $A_1 \otimes A_2$, taking the seminorms ν_1, ν_2 on A_1, A_2 . Indeed, $\|\cdot\|_\alpha$ is spectral if and only if $\nu_\alpha = \nu_\lambda$.*

Proof. Suppose $\|\cdot\|_\alpha$ is spectral. Then by [3], Theorem 1, every multiplicative linear functional on $A_1 \otimes A_2$ is $\|\cdot\|_\alpha$-continuous, and so if $x \in A_1 \otimes A_2$,

$$\begin{aligned}\nu_\alpha(x) &= \sup\{|\phi(x)| : \phi \in (A_1 \otimes A_2)^* ,\ \phi \text{ multiplicative}\}\\ &= \sup\{|\varphi \otimes \psi(x)| : (\varphi, \psi) \in \Phi_{A_1} \times \Phi_{A_2}\}\\ &\le \nu_\lambda(x) .\end{aligned}$$

https://doi.org/10.1017/S0004972700041873 Published online by Cambridge University Press

Now let $x = \sum u_i \otimes v_i$, and take $\varepsilon > 0$. Then there are $\varphi_j \in A_j^*$, $\|\varphi_j\|_{\nu_j} = 1$, $j = 1, 2$, with $\nu_\lambda(x) \le \left|\sum \varphi_1(u_i)\varphi_2(v_i)\right| + \varepsilon$. Since $|\varphi_1(u)| \le \nu_1(u)$, $u \in A_1$, there is $\varphi_1' \in \Phi_{A_1}$ with $\left|\varphi_1'\left(\sum u_i\varphi_2(v_i)\right)\right| \ge \left|\varphi_1\left(\sum u_i\varphi_2(v_i)\right)\right|$ so that $\nu_\lambda(x) \le \left|\sum \varphi_1'(u_i)\varphi_2(v_i)\right| + \varepsilon$. Similarly, there is $\varphi_2' \in \Phi_{A_2}$ with $\nu_\lambda(x) \le \left|\sum \varphi_1'(u_i)\varphi_2'(v_i)\right| + \varepsilon \le \nu_\alpha(x) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary it follows that $\nu_\lambda(x) \le \nu_\alpha(x)$. Thus $\nu_\alpha = \nu_\lambda$ and so ν_α is certainly admissible.

Suppose conversely that ν_α is admissible, with $m\nu_\lambda \le \nu_\alpha$, and that $\|\cdot\|_\alpha$ is not spectral. Since $\nu_\alpha(u \otimes v) \le \nu_1(u)\nu_2(v)$ for all $u \in A_1$, $v \in A_2$, there must be u, v with $\nu_\alpha(u \otimes v) \le k\nu_1(u)\nu_2(v)$ for some k, $0 < k < 1$. But then

$\nu_\alpha(u^n \otimes v^n) \le k^n\nu_1(u^n)\nu_2(v^n) = k^n\nu_\lambda(u^n \otimes v^n) \le \frac{k^n}{m}\nu_\alpha(u^n \otimes v^n)$, which is impossible for n sufficiently large. Thus $\|\cdot\|_\alpha$ is spectral, and so by the above $\nu_\alpha = \nu_\lambda$.

THEOREM 3 (Lardy and Lindberg). *Let A_1, A_2 be commutative Banach algebras, $\|\cdot\|_\alpha$ an algebra norm on $A_1 \otimes A_2$. If $\|\cdot\|_\alpha$ is a spectral tensor norm then $A_1 \otimes_\alpha A_2$ has an identity if and only if A_1 and A_2 have identities.*

Proof. Suppose $A_1 \otimes_\alpha A_2$ has an identity, the converse being immediate. Arguing as in Theorem 1, but using ν_1, ν_2, ν_α in place of $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\alpha$, there is $\{e_n\} \subseteq A_1$ with $\nu_1(e_ns-s) \to 0$ uniformly for $\nu_1(s) \le 1$. But then by Proposition 2 there is an idempotent $e \in A_1$ with $\hat{e} \equiv 1$. Similarly there is an idempotent $f \in A_2$ with $\hat{f} \equiv 1$. The result now follows as in [3], Theorem 4.

https://doi.org/10.1017/S0004972700041873 Published online by Cambridge University Press

References

[1] H.R. Fischer, "Über eine Klasse topologischer Tensorprodukte", *Math. Ann.* 150 (1963), 242-258.

[2] B.R. Gelbaum, "Tensor products and related questions", *Trans. Amer. Math. Soc.* 103 (1962), 525-548.

[3] L.J. Lardy and J.A. Lindberg Jr, "On maximal regular ideals and identities in the tensor product of commutative Banach algebras", *Canad. J. Math.* 21 (1969), 639-647.

[4] Kjeld B. Laursen, "Ideal structure in generalized group algebras", *Pacific J. Math.* 30 (1969), 155-174.

[5] Charles E. Rickart, *General theory of Banach algebras* (Van Nostrand, Princeton, New Jersey, 1960).

[6] C. Robert Warner and Robert Whitley, "A characterization of regular maximal ideals", *Pacific J. Math.* 30 (1969), 277-281.

Carleton University,
Ottawa, Canada.