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Identities involving degenerate Euler numbers and polynomials arising from non-linear differential equations

Taekyun Kim[a,b,*], Dae San Kim[c]

[a] *Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China.*
[b] *Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea.*
[c] *Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea.*

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Abstract

The purpose of this paper is to construct some new non-linear differential equations and investigate the solutions of these non-linear differential equations. In addition, we give some new identities involving degenerate Euler numbers and polynomials arising from those non-linear differential equations. ©2016 All rights reserved.

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1. Introduction

As is well known, the Euler polynomials of order $r(\in \mathbb{N})$ are defined by the generating function

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see [1–15]}). \tag{1.1}$$

When $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the higher-order Euler numbers.

*Corresponding author
Email addresses: tkkim@kw.ac.kr (Taekyun Kim), dskim@sogang.ac.kr (Dae San Kim)

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In particular, $r=1$, $E_n(x)=E_n^{(1)}(x)$ are called ordinary Euler polynomials.

In [2, 3], L. Carlitz considered the degenerate Euler polynomials which are given by the generating function

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{1.2}$$

When $x=0$, $\mathcal{E}_{n,\lambda}^{(r)}=\mathcal{E}_{n,\lambda}^{(r)}(0)$ are called the higher-order degenerate Euler numbers.

In particular, for $r=1$, $\mathcal{E}_{n,\lambda}=\mathcal{E}_{n,\lambda}^{(1)}(0)$ and $\mathcal{E}_{n,\lambda}(x)=\mathcal{E}_{n,\lambda}^{(1)}(x)$ are respectively called the degenerate Euler numbers and the degenerate Euler polynomials.

From (1.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}&=\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}} \\ &=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(x\mid\lambda)_{n-l}\,\mathcal{E}_l^{(r)}\right)\frac{t^n}{n!},\end{aligned} \tag{1.3}$$

where $(x\mid\lambda)_n=x(x-\lambda)\cdots(x-(n-1)\lambda)$.

Thus, by (1.3), we get

$$\mathcal{E}_{n,\lambda}^{(r)}(x)=\sum_{l=0}^{n}\binom{n}{l}(x\mid\lambda)_{n-l}\,\mathcal{E}_l^{(r)},\quad(n\geq 0),\quad(\text{see [3, 10, 12]}). \tag{1.4}$$

In [9, 11], Kim and Kim, and Kim developed some new methods for obtaining identities related to Bernoulli numbers of the second kind and Frobenius-Euler polynomials of higher order arising from certain non-linear differential equations.

For example,

$$\begin{aligned}&(-1)^N\sum_{j=0}^{\min\{n,N-1\}}(N-j)!\,(N-1)!H_{N-1,N-1-j}\,(n)_j\,b_{n-j}^{(N+1-j)} \\ &=\begin{cases}(-1)^N N!\prod_{l=0}^{n-1}(N-l), & \text{if } 0\leq n<N,\\ \sum_{l=0}^{n-N-1}\binom{n}{l}\frac{b_{n-l}}{n-l}\prod_{l=0}^{l+N}(n-l), & \text{if } n\geq N+1,\end{cases}\end{aligned}$$

where $H_{N,0}=1$, for all $N(\in\mathbb{N})$,

$$\begin{aligned}H_{N,1}&=H_N=1+\frac{1}{2}+\cdots+\frac{1}{N},\\ H_{N,j}&=\frac{H_{N-1,j-1}}{N}+\frac{H_{N-2,j-1}}{N-1}+\cdots+\frac{H_{0,j-1}}{1},\quad H_{0,j-1}=0\quad(2\leq j\leq N),\\ b_n^{(k)}&=\text{the } n\text{th Bernoulli numbers of the second kind with order } k\quad(\text{see [9, 11]}).\end{aligned}$$

The rising factorial sequence is defined as

$$(x)_n=x(x+1)\cdots(x+n-1)=\sum_{l=0}^{n}|S_1(n,l)|\,x^l,\quad(n\geq 0), \tag{1.5}$$

where $|S_1(n,l)|$ are called the unsigned Stirling numbers of the first kind (see [1–9, 11–13]).

The purpose of this paper is to construct some new non-linear differential equations and investigate the solutions of these non-linear differential equations. In addition, we give some new identities involving degenerate Euler numbers and polynomials arising from those non-linear differential equations.

2. Identities of degenerate Euler numbers and polynomials

Now, we construct the non-linear differential equations with the solution $F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}$.
Let

$$F = F(t) = F(t;\lambda) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}, \tag{2.1}$$

and

$$F^N = \underbrace{F \times F \times \cdots \times F}_{N-\text{times}}, \quad \text{where } N \in \mathbb{N}. \tag{2.2}$$

From (2.1), we note that

$$\begin{aligned} F^{(1)} = \frac{dF}{dt} &= \frac{-(1+\lambda t)^{\frac{1}{\lambda}}}{\left((1+\lambda t)^{\frac{1}{\lambda}}+1\right)^2(1+\lambda t)} \\ &= \frac{(-1)}{1+\lambda t}\left(F - F^2\right). \end{aligned} \tag{2.3}$$

Thus, by (2.3), we get

$$F^{(1)} = \frac{dF}{dt}(t) = \frac{(-1)}{1+\lambda t}\left(F - F^2\right). \tag{2.4}$$

From (2.4), we can derive

$$\begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} \\ &= \frac{(-1)^2\lambda}{(1+\lambda t)^2}\left(F-F^2\right) + \frac{(-1)}{(1+\lambda t)}\left(F^{(1)} - 2FF^{(1)}\right) \\ &= \frac{(-1)^2\lambda}{(1+\lambda t)^2}\left(F-F^2\right) + \frac{(-1)}{(1+\lambda t)}\left\{\frac{(-1)}{(1+\lambda t)}\left(F-F^2\right) - 2F\left(\frac{(-1)}{1+\lambda t}\left(F-F^2\right)\right)\right\} \\ &= \frac{(-1)^2\lambda}{(1+\lambda t)^2}\left(F-F^2\right) + \frac{(-1)^2}{(1+\lambda t)^2}\left(F-F^2\right) + \frac{(-1)^3 2!}{(1+\lambda t)^2}\left(F^2-F^3\right) \\ &= \frac{(-1)^2(\lambda+1)}{(1+\lambda t)^2}\left(F-F^2\right) + \frac{(-1)^3 2!}{(1+\lambda t)^2}\left(F^2-F^3\right), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} F^{(3)} &= \frac{dF^{(2)}}{dt} \\ &= \frac{\lambda(-1)^3 2!(\lambda+1)}{(1+\lambda t)^3}\left(F-F^2\right) + \frac{(-1)^2(\lambda+1)}{(1+\lambda t)^2}\left(F^{(1)} - 2FF^{(1)}\right) \\ &\quad + \frac{(-1)^4 2!2\lambda}{(1+\lambda t)^3}\left(F^2-F^3\right) + \frac{(-1)^3 2!}{(1+\lambda t)^2}\left(2FF^{(1)} - 3F^2F^{(1)}\right) \\ &= \frac{(-1)^3(1+\lambda)(2\lambda+1)}{(1+\lambda t)^3}\left(F-F^2\right) + \frac{(-1)^4(\lambda+1)2}{(1+\lambda t)^3}\left(F^2-F^3\right) + \frac{(-1)^4 2!2\lambda}{(1+\lambda t)^3}\left(F^2-F^3\right) \\ &\quad + \frac{(-1)^4 2!2}{(1+\lambda t)^3}\left(F^2-F^3\right) + \frac{(-1)^5 2!3}{(1+\lambda t)^3}\left(F^3-F^4\right) \\ &= \frac{(-1)^3(1+\lambda)(2\lambda+1)}{(1+\lambda t)^3}F + \frac{(-1)^4(2\lambda+7)(\lambda+1)}{(1+\lambda t)^3}F^2 + \frac{(-1)^5 3!(\lambda+2)}{(1+\lambda t)^3}F^3 \\ &\quad + \frac{(-1)^6 3!}{(1+\lambda t)^3}F^4. \end{aligned} \tag{2.6}$$

Thus we are led to set

$$F^{(N)} = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} a_i(N,\lambda)(-1)^{i-1} F^i, \quad (N \in \mathbb{N}), \tag{2.7}$$

where

$$F^{(N)} = \frac{d^N F}{dt^N}(t) = \underbrace{\frac{d}{dt} \times \cdots \times \frac{d}{dt}}_{N-\text{times}} F(t).$$

To determine the coefficients $a_i(N,\lambda)$ in (2.7), we take the derivative of (2.7) with respect to t as follows:

$$\begin{aligned} F^{(N+1)} = \frac{dF^{(N)}}{dt} &= \frac{(-1)^{N+1}\lambda N}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+1} a_i(N,\lambda)(-1)^{i-1} F^i \\ &+ \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} a_i(N,\lambda)(-1)^{i-1} iF^{i-1}F^{(1)}. \end{aligned} \tag{2.8}$$

From (2.4) and (2.8), we have

$$\begin{aligned} F^{(N+1)} &= \frac{(-1)^{N+1}\lambda N}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+1} a_i(N,\lambda)(-1)^{i-1} F^i \\ &+ \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+1} a_i(N,\lambda)(-1)^{i-1} i\left(F^i - F^{i+1}\right) \\ &= \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \left\{ \sum_{i=1}^{N+1} (\lambda N a_i(N,\lambda) + i a_i(N,\lambda))(-1)^{i-1} F^i \right. \\ &\left. + \sum_{i=2}^{N+2} a_{i-1}(N,\lambda)(-1)^{i-1}(i-1)F^i \right\} \\ &= \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \left\{ (\lambda N a_1(N,\lambda) + a_1(N,\lambda)) F \right. \\ &+ \sum_{i=2}^{N+1} (\lambda N a_i(N,\lambda) + i a_i(N,\lambda) + (i-1) a_{i-1}(N,\lambda))(-1)^{i-1} F^i \\ &\left. + a_{N+1}(N,\lambda)(-1)^{N+1}(N+1)F^{N+2} \right\}. \end{aligned} \tag{2.9}$$

By (2.7) and (2.9), we easily get

$$\begin{aligned} &\frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \left\{ (\lambda N a_1(N,\lambda) + a_1(N,\lambda)) F + a_{N+1}(N,\lambda)(-1)^{N+1}(N+1)F^{N+2} \right. \\ &\left. + \sum_{i=2}^{N+1} (\lambda N a_i(N,\lambda) + i a_i(N,\lambda) + (i-1) a_{i-1}(N,\lambda))(-1)^{i-1} F^i \right\} \\ &= \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+2} a_i(N+1,\lambda)(-1)^{i-1} F^i. \end{aligned} \tag{2.10}$$

By comparing the coefficients on both sides of (2.10), we get

$$\begin{aligned} a_1(N+1,\lambda) &= \lambda N a_1(N,\lambda) + a_1(N,\lambda) \\ &= (\lambda N + 1) a_1(N,\lambda), \end{aligned} \tag{2.11}$$

$$a_{N+2}(N+1,\lambda)=(N+1)\,a_{N+1}(N,\lambda), \tag{2.12}$$

and

$$a_i(N+1,\lambda)=(\lambda N+i)\,a_i(N,\lambda)+(i-1)\,a_{i-1}(N,\lambda), \tag{2.13}$$

where $2\le i\le N+1$.

From (2.4) and (2.7), we have

$$\frac{(-1)}{1+\lambda t}\left(F-F^2\right)=F^{(1)}=\frac{(-1)}{1+\lambda t}\left(a_1(1,\lambda)F-a_2(1,\lambda)F^2\right). \tag{2.14}$$

Thus, by (2.14), we get

$$a_1(1,\lambda)=1,\quad \text{and}\quad a_2(1,\lambda)=1. \tag{2.15}$$

From (2.11) and (2.15), we can derive the following identities:

$$\begin{aligned} a_1(N+1,\lambda)&=(\lambda N+1)\,a_1(N,\lambda)\\ &=(\lambda N+1)(\lambda(N-1)+1)\,a_1(N-1,\lambda)\\ &=(\lambda N+1)(\lambda(N-1)+1)(\lambda(N-2)+1)\,a_1(N-2,\lambda)\\ &\ \vdots\\ &=(\lambda N+1)(\lambda(N-1)+1)\cdots(\lambda+1)\,a_1(1,\lambda)\\ &=(\lambda N+1)(\lambda(N-1)+1)\cdots(\lambda+1)\cdot 1\\ &=\lambda^{N+1}\left(\frac{1}{\lambda}\right)_{N+1}, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} a_{N+2}(N+1,\lambda)&=(N+1)\,a_{N+1}(N,\lambda)\\ &=(N+1)\,Na_N(N-1,\lambda)\\ &\ \vdots\\ &=(N+1)\,N(N-1)\cdots 2a_2(1,\lambda)\\ &=(N+1)!. \end{aligned} \tag{2.17}$$

We observe that

$$\begin{aligned} &a_1(1,\lambda)=1,\ a_1(2,\lambda)=(1+\lambda),\quad a_1(3,\lambda)=(1+\lambda)(1+2\lambda),\quad \cdots\\ &a_1(N,\lambda)=(1+\lambda)(1+2\lambda)\cdots(1+(N-1)\lambda)=\lambda^N\left(\frac{1}{\lambda}\right)_N, \end{aligned} \tag{2.18}$$

and

$$a_2(1,\lambda)=1,\ a_3(2,\lambda)=2!,\ a_4(3,\lambda)=3!,\ldots,a_{N+1}(N,\lambda)=N!. \tag{2.19}$$

That is, the matrix $(a_i(j,\lambda))_{1\le i\le N+1,1\le j\le N}$ is given by

$$N+1\left\{\begin{bmatrix} \overbrace{\begin{matrix} 1 & (1+\lambda) & (1+\lambda)(1+2\lambda) & \cdots & \lambda^N\left(\frac{1}{\lambda}\right)_N\\ 1! & \times & \times & \cdots & \times\\ & 2! & \times & \cdots & \times\\ & & 3! & \cdots & \times\\ & 0 & & \ddots & \times\\ & & & & N! \end{matrix}}^{N} \end{bmatrix}\right.$$

From (2.13), we have

$$\begin{aligned}
a_2(N+1,\lambda) &= (\lambda N+2)\, a_2(N,\lambda) + a_1(N,\lambda) \\
&= (\lambda N+2)\{(\lambda(N-1)+2)\, a_2(N-1,\lambda) + a_1(N-1,\lambda)\} + a_1(N,\lambda) \\
&= (\lambda N+2)(\lambda(N-1)+2)\, a_2(N-1,\lambda) + (\lambda N+2)\, a_1(N-1,\lambda) + a_1(N,\lambda) \\
&= a_1(N,\lambda) + (\lambda N+2)\, a_1(N-1,\lambda) + (\lambda N+2)(\lambda(N-1)+2)\, a_1(N-2,\lambda) \\
&\quad + (\lambda N+2)(\lambda(N-1)+2)(\lambda(N-2)+2)\, a_2(N-2,\lambda) \\
&\ \vdots \\
&= a_1(N,\lambda) + \sum_{m_1=1}^{N-1} \left(\prod_{l=0}^{m_1-1} (\lambda(N-l)+2) \right) a_1(N-m_1,\lambda) \\
&\quad + (\lambda N+2)(\lambda(N-1)+2)\cdots(\lambda+2)\cdot 1 \\
&= \lambda^N \left(\frac{1}{\lambda}\right)_N + \sum_{m_1=1}^{N-1} \lambda^{m_1} \left(\frac{2}{\lambda}+N-m_1+1\right)_{m_1} \lambda^{N-m_1} \left(\frac{1}{\lambda}\right)_{N-m_1} \\
&\quad + \lambda^N \left(\frac{2}{\lambda}+1\right)_N \\
&= \sum_{m_1=0}^{N} \lambda^{m_1} \left(\frac{2}{\lambda}+N-m_1+1\right)_{m_1} \lambda^{N-m_1} \left(\frac{1}{\lambda}\right)_{N-m_1} \\
&= \lambda^N \sum_{m_1=0}^{N} \left(\frac{2}{\lambda}+N-m_1+1\right)_{m_1} \left(\frac{1}{\lambda}\right)_{N-m_1},
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
a_3(N+1,\lambda) &= (\lambda N+3)\, a_3(N,\lambda) + 2!a_2(N,\lambda) \\
&= 2!a_2(N,\lambda) + (\lambda N+3)\{(\lambda(N-1)+3)\, a_3(N-1,\lambda) + 2a_2(N-1,\lambda)\} \\
&= 2!a_2(N,\lambda) + 2!(\lambda N+3)\, a_2(N-1,\lambda) \\
&\quad + (\lambda N+3)(\lambda(N-1)+3)\, a_3(N-1,\lambda) \\
&= 2!a_2(N,\lambda) + 2!(\lambda N+3)\, a_2(N-1,\lambda) \\
&\quad + 2!(\lambda N+3)(\lambda(N-1)+3)\, a_2(N-2,\lambda) \\
&\quad + (\lambda N+3)(\lambda(N-1)+3)(\lambda(N-2)+3)\, a_3(N-2,\lambda) \\
&\ \vdots \\
&= 2!a_2(N,\lambda) + 2!\sum_{m_2=1}^{N-2} \left(\prod_{l=0}^{m_2-1} (\lambda(N-l)+3) \right) a_2(N-m_2,\lambda) \\
&\quad + 2!(\lambda N+3)(\lambda(N-1)+3)\cdots(3\lambda+3)(2\lambda+3) \\
&= 2!a_2(N,\lambda) + 2!\sum_{m_2=1}^{N-2} \lambda^{m_2} \left(N-m_2+1+\frac{3}{\lambda}\right)_{m_2} a_2(N-m_2,\lambda) \\
&\quad + 2!(\lambda N+3)(\lambda(N-1)+3)\cdots(3\lambda+3)(2\lambda+3) \\
&= 2!a_2(N,\lambda) + 2!\sum_{m_2=1}^{N-2} \lambda^{m_2} \left(N-m_2+1+\frac{3}{\lambda}\right)_{m_2} a_2(N-m_2,\lambda) \\
&\quad + 2!\lambda^{N-1} \left(\frac{3}{\lambda}+2\right)_{N-1}
\end{aligned} \tag{2.21}$$

$$
\begin{aligned}
&= 2! \sum_{m_2=0}^{N-1} \lambda^{m_2} \left(N - m_2 + 1 + \frac{3}{\lambda}\right)_{m_2} a_2\left(N - m_2, \lambda\right) \\
&= 2!\lambda^{N-1} \sum_{m_2=0}^{N-1} \sum_{m_1=0}^{N-m_2-1} \left(N - m_2 + 1 + \frac{3}{\lambda}\right)_{m_2} \\
&\quad \times \left(N - m_2 - m_1 + \frac{2}{\lambda}\right)_{m_1} \left(\frac{1}{\lambda}\right)_{N-m_2-m_1-1}.
\end{aligned}
$$

From (2.13), we note that

$$
a_4\left(N+1, \lambda\right) = \left(\lambda N + 4\right) a_4\left(N, \lambda\right) + 3a_3\left(N, \lambda\right). \tag{2.22}
$$

Thus, by (2.21) and (2.22), we get

$$
\begin{aligned}
a_4\left(N+1, \lambda\right) &= 3a_3\left(N, \lambda\right) + \left(\lambda N + 4\right)\left\{\left(\lambda\left(N-1\right)+4\right) a_4\left(N-1, \lambda\right) + 3a_3\left(N-1, \lambda\right)\right\} \\
&= 3a_3\left(N, \lambda\right) + 3\left(\lambda N + 4\right) a_3\left(N-1, \lambda\right) \\
&\quad + \left(\lambda N + 4\right)\left(\lambda\left(N-1\right)+4\right) a_4\left(N-1, \lambda\right) \\
&= 3a_3\left(N, \lambda\right) + 3\left(\lambda N + 4\right) a_3\left(N-1, \lambda\right) \\
&\quad + 3\left(\lambda N + 4\right)\left(\lambda\left(N-1\right)+4\right) a_3\left(N-2, \lambda\right) \\
&\quad + \left(\lambda N + 4\right)\left(\lambda\left(N-1\right)+4\right)\left(\lambda\left(N-2\right)+4\right) a_4\left(N-2, \lambda\right) \\
&\;\vdots \\
&= 3a_3\left(N, \lambda\right) + 3\sum_{m_3=1}^{N-3}\left(\prod_{l=0}^{m_3-1}\left(\lambda\left(N-l\right)+4\right)\right) a_3\left(N-m_3, \lambda\right) \\
&\quad + 3!\left(\lambda N + 4\right)\left(\lambda\left(N-1\right)+4\right)\cdots\left(3\lambda+4\right) \\
&= 3a_3\left(N, \lambda\right) + 3\sum_{m_3=1}^{N-3}\lambda^{m_3}\left(\frac{4}{\lambda} + N - m_3 + 1\right)_{m_3} a_3\left(N-m_3, \lambda\right) \\
&\quad + 3!\lambda^{N-2}\left(\frac{4}{\lambda}+3\right)_{N-2} \\
&= 3\sum_{m_3=0}^{N-2}\lambda^{m_3}\left(\frac{4}{\lambda} + N - m_3 + 1\right)_{m_3} a_3\left(N-m_3, \lambda\right) \\
&= 3!\sum_{m_3=0}^{N-2}\lambda^{m_3}\left(\frac{4}{\lambda} + N - m_3 + 1\right)_{m_3} \lambda^{N-m_3-2} \\
&\quad \times \sum_{m_2=0}^{N-m_3-2}\sum_{m_1=0}^{N-m_3-m_2-2}\left(N - m_3 - m_2 + \frac{3}{\lambda}\right)_{m_2} \\
&\quad \times \left(N - m_3 - m_2 - m_1 - 1 + \frac{2}{\lambda}\right)_{m_1}\left(\frac{1}{\lambda}\right)_{N-m_3-m_2-m_1-2}.
\end{aligned} \tag{2.23}
$$

By (2.23), we see that

$$
\begin{aligned}
a_4\left(N+1, \lambda\right) &= 3!\lambda^{N-2}\sum_{m_3=0}^{N-2}\sum_{m_2=0}^{N-m_3-2}\sum_{m_1=0}^{N-m_3-m_2-2}\left(\frac{4}{\lambda} + N - m_3 + 1\right)_{m_3} \\
&\quad \times \left(N - m_3 - m_2 + \frac{3}{\lambda}\right)_{m_2} \\
&\quad \times \left(N - m_3 - m_2 - m_1 - 1 + \frac{2}{\lambda}\right)_{m_1}\left(\frac{1}{\lambda}\right)_{N-m_3-m_2-m_1-2}.
\end{aligned} \tag{2.24}
$$

Continuing this process, we get

$$\begin{aligned} a_i(N+1,\lambda) &= (i-1)!\lambda^{N-i+2}\sum_{m_{i-1}=0}^{N-i+2}\sum_{m_{i-2}=0}^{N-m_{i-1}-i+2}\cdots\sum_{m_1=0}^{N-m_{i-1}-\cdots-m_2-i+2}\left(\frac{i}{\lambda}+N-m_{i-1}+1\right)_{m_{i-1}} \\ &\times\left(N-m_{i-1}-m_{i-2}+\frac{i-1}{\lambda}\right)_{m_{i-2}}\cdots\left(N-m_{i-1}-\cdots-m_1-i+3+\frac{2}{\lambda}\right)_{m_1} \\ &\times\left(\frac{1}{\lambda}\right)_{N-m_{i-1}-m_{i-2}-\cdots-m_1-i+2}. \end{aligned} \tag{2.25}$$

Therefore, by (2.7) and (2.25), we obtain the following theorem.

Theorem 2.1. *For $N\in\mathbb{N}$, let us consider the following non-linear differential equation with respect to t:*

$$F^{(N)}=\frac{(-1)^N}{(1+\lambda t)^N}\sum_{i=1}^{N+1}a_i(N,\lambda)(-1)^{i-1}F^i, \tag{2.26}$$

where

$$\begin{aligned} a_i(N,\lambda) &= (i-1)!\lambda^{N-i+1}\sum_{m_{i-1}=0}^{N-i+1}\sum_{m_{i-2}=0}^{N-m_{i-1}-i+1}\cdots\sum_{m_1=0}^{N-m_{i-1}-\cdots-m_2-i+1}\left(N-m_{i-1}+\frac{i}{\lambda}\right)_{m_{i-1}} \\ &\times\left(N-m_{i-1}-m_{i-2}-1+\frac{i-1}{\lambda}\right)_{m_{i-2}}\cdots\left(N-m_{i-1}-\cdots-m_1-i+2+\frac{2}{\lambda}\right)_{m_1} \\ &\times\left(\frac{1}{\lambda}\right)_{N-m_{i-1}-m_{i-2}-\cdots-m_1-i+1}. \end{aligned}$$

Then $F=F(t)=\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}$ is a solution of (2.26).

Now, we observe that

$$\begin{aligned} F^{(N)} &= \frac{1}{2}\frac{d^N}{dt^N}\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right) \\ &= \frac{1}{2}\frac{d^N}{dt^N}\sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}\frac{t^m}{m!} \\ &= \frac{1}{2}\sum_{m=N}^{\infty}\mathcal{E}_{m,\lambda}\frac{m(m-1)\cdots(m-N+1)}{m!}t^{m-N} \\ &= \frac{1}{2}\sum_{m=0}^{\infty}\mathcal{E}_{m+N,\lambda}\frac{t^m}{m!}. \end{aligned} \tag{2.27}$$

Thus, by (2.27), we get

$$\begin{aligned} (1+\lambda t)^N F^{(N)} &= \left(\sum_{l=0}^{\infty}\binom{N}{l}\lambda^l t^l\right)\left(\frac{1}{2}\sum_{m=0}^{\infty}\mathcal{E}_{m+N,\lambda}\frac{t^m}{m!}\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{N}{l}(n)_{\underline{l}}\,\lambda^l\mathcal{E}_{n-l+N,\lambda}\right)\frac{t^n}{n!}, \end{aligned} \tag{2.28}$$

where $(x)_{\underline{n}}=x(x-1)\cdots(x-n+1)$, $(n\geq 0)$.

From (1.2), we have

$$\begin{aligned}F^i &= \frac{1}{2^i}\underbrace{\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)\times\cdots\times\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)}_{i-\text{times}} \\ &= \frac{1}{2^i}\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(i)}\frac{t^n}{n!}.\end{aligned} \tag{2.29}$$

Therefore, by Theorem 2.1, (2.28), and (2.29), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, $N \in \mathbb{N}$, we have*

$$\begin{aligned}&\sum_{l=0}^{n}\binom{N}{l}(n)_{\underline{l}}\,\lambda^l\mathcal{E}_{n-l+N,\lambda} \\ &= \sum_{i=1}^{N+1}(i-1)!\lambda^{N-i+1}\sum_{m_{i-1}=0}^{N-i+1}\sum_{m_{i-2}=0}^{N-m_{i-1}-i+1}\cdots\sum_{m_1=0}^{N-m_{i-1}-\cdots-m_2-i+1}\left(N-m_{i-1}+\frac{i}{\lambda}\right)_{m_{i-1}} \\ &\quad\times\left(N-m_{i-1}-m_{i-2}-1+\frac{i-1}{\lambda}\right)_{m_{i-1}}\cdots \\ &\quad\times\left(N-m_{i-1}-m_{i-2}-\cdots-m_1-i+2+\frac{2}{\lambda}\right)_{m_1}\left(\frac{1}{\lambda}\right)_{N-m_{i-1}-\cdots-m_1-i+1} \\ &\quad\times(-1)^{N+i-1}\,\frac{1}{2^{i-1}}\mathcal{E}_{n,\lambda}^{(i)},\end{aligned}$$

where $(x)_{\underline{n}} = x(x-1)\cdots(x-n+1)$.

Let

$$F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}-1}. \tag{2.30}$$

Then, by (2.30), we get

$$\begin{aligned}F^{(1)} = \frac{dF}{dt} &= \frac{(-1)}{(1+\lambda t)}\left\{\frac{(1+\lambda t)^{\frac{1}{\lambda}}}{\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^2}\right\} \\ &= \frac{(-1)}{1+\lambda t}\left(F+F^2\right),\end{aligned} \tag{2.31}$$

$$\begin{aligned}F^{(2)} = \frac{dF^{(1)}}{dt} &= \frac{(-1)^2\lambda}{(1+\lambda t)^2}\left(F+F^2\right)+\frac{(-1)}{1+\lambda t}\left(F^{(1)}+2FF^{(1)}\right) \\ &= \frac{(-1)^2(\lambda+1)}{(1+\lambda t)^2}F+\frac{(-1)^2(\lambda+3)}{(1+\lambda t)^2}F^2+\frac{(-1)^2\,2}{(1+\lambda t)^2}F^3\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}F^{(3)} &= \frac{dF^{(2)}}{dt} \\ &= \frac{(-1)^3(\lambda+1)(2\lambda+1)}{(1+\lambda t)^3}F+\frac{(-1)^3(2\lambda+7)(\lambda+1)}{(1+\lambda t)^3}F^2 \\ &\quad+\frac{(-1)^3\,3!\,(\lambda+2)}{(1+\lambda t)^3}F^3+\frac{(-1)^3\,3!}{(1+\lambda t)^3}F^4.\end{aligned} \tag{2.33}$$

So we are led to put

$$F^{(N)} = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} a_i(N,\lambda) F^i. \tag{2.34}$$

Thus, by (2.34), we get

$$\begin{aligned} F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\ &= \frac{(-1)^{N+1}\lambda N}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+1} a_i(N,\lambda) F^i \\ &\quad + \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} a_i(N,\lambda)\, iF^{i-1}F^{(1)} \\ &= \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+1} (\lambda N + i)\, a_i(N,\lambda) F^i \\ &\quad + \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \sum_{i=2}^{N+2} a_{i-1}(N,\lambda)(i-1) F^i. \end{aligned} \tag{2.35}$$

From (2.34) and (2.35), we note that

$$\begin{aligned} F^{(N+1)} = \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} &\Big\{ (\lambda N+1)\, a_1(N,\lambda) F + a_{N+1}(N,\lambda)(N+1) F^{N+2} \\ &+ \sum_{i=2}^{N+1} \left((\lambda N + i)\, a_i(N,\lambda) + (i-1)\, a_{i-1}(N,\lambda) \right) F^i \Big\} \\ = \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} &\sum_{i=1}^{N+2} a_i(N+1,\lambda) F^i. \end{aligned} \tag{2.36}$$

By comparing the coefficients on the both sides of (2.36), we get

$$a_1(N+1,\lambda) = (\lambda N+1)\, a_1(N,\lambda), \quad a_{N+2}(N+1,\lambda) = (N+1)\, a_{N+1}(N,\lambda), \tag{2.37}$$

and

$$(\lambda N + i)\, a_i(N,\lambda) + (i-1)\, a_{i-1}(N,\lambda) = a_i(N+1,\lambda), \quad (2 \le i \le N+1). \tag{2.38}$$

Also, we observe that

$$\begin{aligned} F^{(1)} &= \frac{(-1)}{1+\lambda t} \left\{ a_1(1,\lambda) F + a_2(1,\lambda) F^2 \right\} \\ &= \frac{(-1)}{1+\lambda t} \left(F + F^2 \right). \end{aligned} \tag{2.39}$$

Thus, from (2.39), we get

$$a_1(1,\lambda) = 1, \quad \text{and} \quad a_2(1,\lambda) = 1. \tag{2.40}$$

Therefore the relations in (2.37), (2.38), and (2.40) are the same as the ones in (2.11), (2.12), (2.13), and (2.15). Hence, from (2.25), we obtain the following theorem.

Theorem 2.3. *For $N \in \mathbb{N}$, the following non-linear differential equation*

$$F^{(N)} = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} a_i(N,\lambda) F^i \tag{2.41}$$

has the solution $F = F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}$, where

$$a_i(N,\lambda) = (i-1)!\lambda^{N-i+1} \sum_{m_{i-1}=0}^{N-i+1} \sum_{m_{i-2}=0}^{N-m_{i-1}-i+1} \cdots \sum_{m_1=0}^{N-m_{i-1}-\cdots-m_2-i+1} \left(N-m_{i-1}+\frac{i}{\lambda}\right)_{m_{i-1}}$$
$$\times \left(N-m_{i-1}-m_{i-2}-1+\frac{i-1}{\lambda}\right)_{m_{i-2}} \cdots \left(N-m_{i-1}-\cdots-m_1-i+2+\frac{2}{\lambda}\right)_{m_1}$$
$$\times \left(\frac{1}{\lambda}\right)_{N-m_{i-1}-m_{i-2}-\cdots-m_1-i+1}.$$

For $r \in \mathbb{N}$, the degenerate Bernoulli polynomials of order r are defined by Carlitz as

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see [3]}). \tag{2.42}$$

When $x=0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$ are called the degenerate higher-order Bernoulli numbers. In particular, $r=1$, $\beta_{n,\lambda} = \beta_{n,\lambda}^{(1)}$ are called the degenerate Bernoulli numbers. Note that $\beta_{0,\lambda}=1$.

We observe that

$$\begin{aligned} F = F(t) &= \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}-1} \\ &= \frac{1}{t}\sum_{n=0}^{\infty}\beta_{n,\lambda}\frac{t^n}{n!} = \sum_{n=1}^{\infty}\beta_{n,\lambda}\frac{t^{n-1}}{n!}+\frac{1}{t} \\ &= \sum_{n=0}^{\infty}\frac{\beta_{n+1,\lambda}}{n+1}\frac{t^n}{n!}+\frac{1}{t}. \end{aligned} \tag{2.43}$$

Thus, by (2.43), we get

$$\begin{aligned} F^{(N-1)} &= \frac{d^{N-1}}{dt^{N-1}}\left(\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right) \\ &= \sum_{n=N-1}^{\infty}\frac{\beta_{n+1,\lambda}}{n+1}\frac{t^{n-N+1}}{(n-N+1)!}+\frac{(-1)^{N-1}}{t^N}(N-1)! \\ &= \sum_{n=0}^{\infty}\frac{\beta_{n+N,\lambda}}{n+N}\frac{t^n}{n!}+\frac{1}{t^N}(-1)^{N-1}(N-1)!. \end{aligned} \tag{2.44}$$

From (2.44), we have

$$\begin{aligned} t^N F^{(N-1)} &= \sum_{n=N-1}^{\infty}\frac{\beta_{n+1,\lambda}}{n+1}\frac{t^{n+1}}{(n-N+1)!}+(-1)^{N-1}(N-1)! \\ &= \sum_{n=N}^{\infty}\frac{\beta_{n,\lambda}}{n}\frac{t^n}{(n-N)!}+(-1)^{N-1}(N-1)!. \end{aligned} \tag{2.45}$$

Replacing N by $N+1$, we get

$$\begin{aligned} (1+\lambda t)^N t^{N+1} F^{(N)} &= (1+\lambda t)^N \sum_{n=N+1}^{\infty}\frac{\beta_{n,\lambda}}{n}\frac{t^n}{(n-N-1)!}+(-1)^N N!(1+\lambda t)^N \\ &= \sum_{n=N+1}^{\infty}\left(\sum_{l=0}^{n-N-1}\lambda^l\binom{N}{l}\frac{\beta_{n-l,\lambda}}{n-l}n(n-1)\cdots(n-l-N)\right)\frac{t^n}{n!} \\ &\quad +(-1)^N N!\sum_{n=0}^{N}(N)_{\underline{n}}\lambda^n\frac{t^n}{n!}, \end{aligned} \tag{2.46}$$

where $(x)_{\underline{n}} = x(x-1)\cdots(x-n+1)$.

From Theorem 2.3, we have

$$\begin{aligned}
(1+\lambda t)^N t^{N+1} F^{(N)} &= (-1)^N \sum_{j=1}^{N+1} a_j(N,\lambda) F^j t^{N+1} \\
&= (-1)^N \sum_{j=1}^{N+1} a_j(N,\lambda) \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^j t^{N+1-j} \\
&= (-1)^N \sum_{j=0}^{N} a_{N+1-j}(N,\lambda) t^j \sum_{m=0}^{\infty} \beta_{m,\lambda}^{(N+1-j)} \frac{t^m}{m!} \\
&= (-1)^N \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{\min\{n,N\}} a_{N+1-j}(N,\lambda) \frac{n!}{(n-j)!} \beta_{n-j,\lambda}^{(N+1-j)} \right\} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ (-1)^N \sum_{j=0}^{\min\{n,N\}} a_{N+1-j}(N,\lambda) n(n-1)\cdots(n-j+1) \beta_{n-j,\lambda}^{(N+1-j)} \right\} \frac{t^n}{n!}.
\end{aligned} \tag{2.47}$$

Therefore, by (2.46) and (2.47), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\begin{aligned}
&(-1)^N \sum_{j=0}^{\min\{n,N\}} a_{N+1-j}(N,\lambda) n(n-1)\cdots(n-j+1) \beta_{n-j,\lambda}^{(N+1-j)} \\
&= \begin{cases} (-1)^N N! (N)_{\underline{n}} \lambda^n & \text{if } 0 \leq n \leq N, \\ \sum_{l=0}^{n-N-1} \lambda^l \binom{N}{l} \frac{\beta_{n-l,\lambda}}{n-l} n(n-1)\cdots(n-l-N) & \text{if } n \geq N+1, \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
a_i(N,\lambda) &= (i-1)! \lambda^{N-i+1} \sum_{m_{i-1}=0}^{N-i+1} \sum_{m_{i-2}=0}^{N-m_{i-1}-i+1} \cdots \sum_{m_1=0}^{N-m_{i-1}-\cdots-m_2-i+1} \left(N-m_{i-1}+\frac{i}{\lambda}\right)_{m_{i-1}} \\
&\times \left(N-m_{i-1}-m_{i-2}-1+\frac{i-1}{\lambda}\right)_{m_{i-2}} \cdots \left(N-m_{i-1}-\cdots-m_1-i+2+\frac{2}{\lambda}\right)_{m_1} \\
&\times \left(\frac{1}{\lambda}\right)_{N-m_{i-1}-m_{i-2}-\cdots-m_1-i+1}.
\end{aligned}$$

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