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IDENTITIES OF THE NATURAL REPRESENTATION OF THE INFINITELY BASED SEMIGROUP

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ABSTRACT. An equational theory of a very small semigroup may fail to be finitely presented. A well-known example of such a semigroup was studied in detail by Peter Perkins some twenty years ago. We prove that the *natural representation* of his semigroup has a finite basis of identical relations and discuss this fact in a general context of universal algebra.

1. INTRODUCTION

Let k be a field and let e_{ij}, $1 \le i, j \le 2$, be the matrix units of the algebra $M_2(k)$ of 2×2 matrices over k. These four matrix units together with the zero matrix O and the identity matrix E form the semigroup Π that does not possess a finite base of identities (see [1]). Nevertheless, in this paper we prove the following

Theorem 1. *The natural representation* $\mathrm{id}\colon \Pi \to M_2(k)$ *of the semigroup* Π *is finitely based. A particular basis of identities of this representation consists of the identities* (1)–(7).

The semigroup Π acts by (left) multiplications on $M_2(k)$. The regular representation of Π in its semigroup algebra splits up into the direct sum of this four-dimensional representation and two one-dimensional representations. Hence we have

Corollary 1. *Any representation of the semigroup* Π *is finitely based.*

2. BASIC DEFINITIONS

Identities of representations of semigroups can be naturally defined as follows (see [2]). Let $F \equiv F(X)$, $X = \{x_1, x_2, \dots, y_1, y_2, \dots, z_1, z_2, \dots\}$, be the free semigroup with the countable set of free generators X, and let kF be its semigroup algebra. Take a representation of a semigroup S by the linear transformations of a vector space V. A polynomial $P \equiv p(x_1, \dots, x_t) \in kF$ is said to be an identity of such $r : S \to \operatorname{End} V$ (of the pair $(\operatorname{End} V, S)$) if $p(r(s_1), \dots, r(s_t)) = 0$ for all $s_1, \dots, s_t \in S$.

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Example. If an identity $f = g$ holds in a semigroup then the identity $f - g$ holds in any of its representations (for faithful representations the converse is also true). More generally, if the universal disjunctive formula (pseudoidentity) $w \equiv f_1 = g_1 \vee \cdots \vee f_m = g_m$ holds in a semigroup $S(f_i, g_i \in F, \ i = 1, \dots, m)$ then, clearly, the identities

$$u(w) = (f_1 - g_1)x_1^{\nu_1} \cdots x_{m-1}^{\nu_{m-1}}(f_m - g_m), \qquad \nu_1, \dots, \nu_{m-1} \in \{0, 1\},$$

hold in the regular representation (kS, S).

In what follows the word "identity" will mean an identity of a representation. Let polynomials $p_1, \dots, p_n$ be identities of $(\operatorname{End} V, S)$. Then for any endomorphisms $\varepsilon_i : F \to F$ and any $a_i, b_i \in kF \ (i = 1, \dots, n)$ the polynomial

$$a_1\varepsilon_1(p_1)b_1 + \cdots + a_n\varepsilon_n(p_n)b_n \tag{2.1}$$

is also an identity of $(\operatorname{End} V, S)$. Let $I \subset kF$ be the set of all the identities of the pair $(\operatorname{End} V, S)$. Clearly, I is an ideal in kF. Moreover, it is a *verbal* ideal, which means that all the expressions (2.1) with $p_1, \dots, p_n \in I$ also belong to I. A set $B \subset I$ is called a *basis* of the identities of $(\operatorname{End} V, S)$ (a basis of I) if any $q \in I$ can be written in the form (2.1) with $p_1, \dots, p_n \in B$. The problem is to identify the situations in which such a B can be finite. In this latter case, a representation (or the verbal ideal of its identities) is called *finitely based*.

3. Proof of Theorem 1

Consider a monomial

$$M = p_1(a_1^2 - a_1)p_2(a_2^2 - a_2) \cdots p_r(a_r^2 - a_r)p_{r+1} \in kF, \tag{3.1}$$

where $p_1, \dots, p_{r+1} \in F$ are possibly empty products of the squares of free variables, $a_1, \dots, a_r \in X \ (r \geq 0)$. We will use the following notations: $A(M) := \{a_1, \dots, a_r\}$, $Y(M) := \{x \in X, \ x^2$ occurs in at least one of the monomials $p_1, \dots, p_{r+1}\}$; $l(M) := r$; $o(M) := \{x \in X, \ x$ equals some of the a_i with i odd$\}$; $e(M) := \{x \in X, \ x$ equals some of the a_i with i even$\}$. For $x \in A(M)$ let $n_x(M) := \{$the number of occurrences of $x^2 - x$ in $M\}$. For an arbitrary $h \in kF$ let $X(h) := \{x \in X, \ x$ occurs in $h\}$. Clearly, $X(M) = A(M) \cup Y(M)$ and $A(M) = o(M) \cup e(M)$. The homomorphism $kF \to M_2(k)$ induced by a map $\sigma : X \to \Pi$ will be denoted by the same letter. Set $U := \{E, e_{11}, e_{22}\}$, $N := \{e_{12}, e_{21}\}$, $U_0 := U \cup 0$, and $N_0 := N \cup 0$.

Lemma 1. *A map $\sigma : X \to \Pi$ such that $\sigma(M) \neq 0$ exists iff the following conditions are satisfied:*

(i) *$\sigma(A(M)) \subset N$ and $\sigma(Y(M)) \subset U$; in particular, $A(M) \cap Y(M) = \varnothing$;*

(ii) *$\sigma(o(M)) \cap \sigma(e(M)) = \varnothing$, i.e., there are only two possibilities: either $\sigma(o(M)) = \{e_{12}\}$, $\sigma(e(M)) = \{e_{21}\}$, and $\sigma(M) \in ke_{12} \oplus ke_{11}$ or $\sigma(o(M)) = e_{12}$, $\sigma(e(M)) = e_{12}$, and $\sigma(M) \in ke_{21} \oplus ke_{22}$; in particular, $o(M) \cap e(M) = \varnothing$.*

Lemma 2. *M is an identity of $(M_2(k), \Pi)$ iff $A(M) \cap Y(M) \neq \varnothing$ or $o(M) \cap e(M) \neq \varnothing$.*

The proofs of these lemmas are straightforward and rely on the following trivial observation.

Remark 1. If $s \in \Pi$ then either $s^2 - s = 0 \ (s \in U_0)$ or $s^2 = 0 \ (s \in N_0)$.

Lemma 3. *The pair* $(M_2(k), \Pi)$ *satisfies the following identities*:

(1) $$x^3 - x^2;$$

(2) $$x^2y^2(x^2 - x), \qquad (x^2 - x)y^2x^2;$$

(3) $$(x^2 - x)y^2(x^2 - x);$$

(4) $$(x_1^2 - x_1)y^{2\nu}(x_2^2 - x_2)(x_3^2 - x_3) - (x_3^2 - x_3)y^{2\nu}(x_2^2 - x_2)(x_1^2 - x_1);$$

(5) $$(x_1^2 - x_1)y^{2\nu}(x_2^2 - x_2)z^2 - z^2(x_1^2 - x_1)y^{2\nu}(x_2^2 - x_2);$$

(6) $$(x^2 - x)z^{2\nu}(y^2 - y)(x^2 - x)(y^2 - y) = (x^2 - x)z^{2\nu}(y^2 - y);$$

(7) $$x^2y^2 = (xyx)^2 = y^2x^2$$

(*here* $\nu \in \{0, 1\}$ *so that each of the expressions* (4)–(6) *represents a pair of identities*).

All these identities can be easily verified with the use of Lemmas 1, 2.

Remark 2. It follows from (7) that the identities (2)–(6) remain valid if one substitutes any products of squares of free variables instead of y^2 and z^2.

Remark 3. The identities $(x^2 - x)^m$, $m \geq 2$, and $x^i - x^j$, $i, j \geq 2$, follow from (1).

Denote by V the verbal ideal generated by the identities (1)–(7).

Remark 4. Let an endomorphism $\varphi: F \to F$ be such that $\varphi|Y(M)\backslash A(M) = \mathrm{id}$, $\varphi(o(M)) = o(M)$, $\varphi(e(M)) = e(M)$. Then the identity (4) shows that $\varphi(M) \equiv M \pmod V$.

Take an identity $f = f(x_1, \dots, x_t) \in kF$ of $(M_2(k), \Pi)$ and write $x_i = x_i^2 - (x_i^2 - x_i)$, $i = 1, 2, \dots, t$, in order to get

(3.2) $$f = \sum_{i=1}^{n} \alpha_i M_i,$$

where $\alpha_i \in k$ and M_i are monomials of the form (3.1). Assume that (3.2) is minimal, i.e., $\sum_{i \in I} \alpha_i M_i$ is not an identity of $(M_2(k), \Pi)$ for any proper subset $I \subset \{1, \dots, n\}$. To prove the theorem it is sufficient to show that $f \in V$. If $n = 1$ then this follows from Lemma 2 and Remarks 2, 3, and 4. So we assume that $n > 1$.

Lemma 4. *For any* $i, j \in \{1, 2, \dots, n\}$ *the following conditions hold*:

(i) $X(M_i) = X(M_j)$;
(ii) $A(M_i) \cap Y(M_i) = \varnothing$, $o(M_i) \cap e(M_i) = \varnothing$;
(iii) $A(M_i) = A(M_j)$, $Y(M_i) = Y(M_j)$;
(iv) $o(M_i) = o(M_j)$, $e(M_i) = e(M_j)$;
(v) $l(M_i) = l(M_j) \pmod 2$.

Proof. Condition (i) is obvious—it means that f is "blended" in the sense of [5, p. 15].

Condition (ii) follows from the minimality assumption and Lemma 2.

To prove (iii) suppose that $x \in A(M_i)\backslash A(M_j)$ and let

$$f' = \sum_{x \in A(M_t)} \alpha_t M_t \quad \text{and} \quad f'' = \sum_{x \notin A(M_t)} \alpha_t M_t.$$

By Lemma 1, $\sigma(x) \in N$ for any $\sigma: X \to \Pi$ such that $\sigma(f') \neq 0$, and $\sigma(x) \in U$ for any $\sigma: X \to \Pi$ such that $\sigma(f'') \neq 0$. Hence $\sigma(f') = 0$ or $\sigma(f'') = 0$ for every $\sigma: X \to \Pi$. This contradicts the minimality assumption (note that $f = f' + f''$). Further, if $o(M_i) \neq o(M_j)$ then again write

$$f = f' + f'' = \sum_{o(M_t)=o(M_i)} \alpha_t M_t + \sum_{o(M_t)\neq o(M_i)} \alpha_t M_t$$

and suppose that $\sigma(f') \neq 0$ for some $\sigma: X \to \Pi$. Using (iii) and Lemma 1, one easily verifies that either $\sigma(o(M_i)) = e_{12}$, $\sigma(f') \in ke_{11} \oplus ke_{12}$, $\sigma(f'') \in ke_{21} \oplus ke_{22}$ or $\sigma(o(M_i)) = e_{21}$, $\sigma(f') \in ke_{21} \oplus ke_{22}$, $\sigma(f'') \in ke_{11} \oplus ke_{12}$. Both cases contradict $\sigma(f) = 0$. This proves (iv). The proof of (v) is similar.

In view of Lemma 4 we can use the notation $X(f)$, $A(f)$, etc.

Lemma 5. *There exists a polynomial $f' = \sum_{i=1}^n \alpha_i M_i'$ such that $f - f' \in V$, f' satisfies all the conditions of Lemma 4, and, in addition, $n_x(M_i') = n_x(M_j')$ for all $x \in A(f')$, $i, j \in \{1, 2, \dots, n\}$.*

Proof. Let $m_y = \max_{1\leq i\leq n}\{n_y(M_i)\}$, $y \in e(F)$. Fix a variable $x \in o(f)$. Set

$$M_i'' = \begin{cases} M_i \prod_{v\in e(f)}[(x^2 - x)(v^2 - v)]^{m_v - n_v(M_i)} & \text{if } l(f) \text{ is even}, \\ M_i \prod_{v\in e(f)}[(v^2 - v)(x^2 - x]^{m_v - n_v(M_i)} & \text{otherwise}. \end{cases}$$

If $y \in e(f)$ then, clearly, $n_y(M_i'') = m_y$, $i = 1, 2, \dots, n$. Let $m_v'' = \max_{1\leq i\leq n}\{n_v(M_i'')\}$, $v \in o(f)$. Fix a variable $y \in e(f)$ and again set

$$M_i' = \begin{cases} \prod_{v\in o(f)}[(y^2 - y)(v^2 - v)]^{m_v'' - n_v}(M_i'') & \text{if } l(f) \text{ is odd}, \\ \prod_{v\in o(f)}[(v^2 - v)(y^2 - y)]^{m_v'' - n_v(M_i'')} & \text{if } l(f) \text{ is even}. \end{cases}$$

If $x \in o(f)$ then $n_x(M_i') = m_x$, $i = 1, \dots, n$, as above. On the other hand, if $x \in e(f)\backslash y$ then $n_x(M_i') = n_x(M_i'') = m_x$, $i = 1, 2, \dots, n$. Finally,

$$n_y(M_i') = m_y + \sum_{v\in o(f)} (m_v'' - n_v(M_i'')),$$

but

$$\sum_{v\in o(f)} n_v(M_i'') = \sum_{u\in e(f)} n_u(M_i'') + \nu = \sum_{u\in e(f)} m_u + \nu,$$

where $\nu = 0$ if $l(f)$ is even and $\nu = 1$ otherwise. So we have $n_x(M_i') = n_x(M_j')$ for all $i, j = 1, 2, \dots, n$. Moreover, it follows from (6) and Remark 4 that $M_i' \equiv M_i \pmod V$. Note also that $X(M_i') = X(f)$, $A(M_i') = A(f)$, etc.

Lemma 6. *Modulo the ideal V, the identity f equals*

$$\sum_{i,j} \alpha_{ij} M_{ij} \equiv \sum_{i,j} \alpha_{ij} p_i (a_1^2 - a_1) q_j (a_2^2 - a_2) \cdots (a_t^2 - a_t),$$

where $\alpha_{ij} \in k$, p_i, q_j are the products of squares of the variables belonging to $Y(f)$, $a_r \in o(f)$ if r is odd, and $a_r \in e(f)$ otherwise.

Proof. Apply (4), (5), and Remark 4 to an identity f' that satisfies the conditions of Lemma 5.

Obviously $X(p_i) \cup X(q_j) = Y(f)$ for all i, j. We may suppose also that $\langle X(p_i), X(q_j)\rangle = \langle X(p_i'), X(q_j')\rangle$ iff $i = i'$, $j = j'$. To conclude the proof of the theorem, we will show by induction on $\mathrm{Card}(X(p_i)\cap X(q_j))$ that all α_{ij} in (4.3) are zeros. Let

$$\sigma_{ij}(a_r)=\begin{cases} e_{12} & \text{if } r\equiv 1 \pmod 2,\\ e_{21} & \text{if } r\equiv 0 \pmod 2;\end{cases}$$

$$\sigma_{ij}(y)=\begin{cases} E & \text{if } y\in X(p_i)\cap X(q_j),\\ e_{11} & \text{if } y\in X(p_i)\backslash X(q_j),\\ e_{22} & \text{if } y\in X(q_j)\backslash X(p_i).\end{cases}$$

Note that this definition is correct because of Lemmas 4 and 5.

Suppose that $\alpha_{rs}=0$ if $\mathrm{Card}(X(p_r)\cap X(q_s))<\mathrm{Card}(X(p_i)\cap X(q_j))$. Clearly $\sigma_{ij}(M_{ij})=e_{12}$ if t is odd and $\sigma_{ij}(M_{ij})=e_{11}$ otherwise. On the other hand, if $\sigma_{ij}(M_{rs})\neq 0$ then $X(p_r)\subset X(p_i)$, $X(q_s)\subset X(q_j)$. Therefore $X(p_r)\cap X(q_s)\subset X(p_i)\cap X(q_j)$ and by induction $X(p_r)\cap X(q_s)=X(p_i)\cap X(q_j)$. All this means that $X(p_r)=X(p_i)$, $X(q_s)=X(q_j)$, and $\langle r, s\rangle=\langle i, j\rangle$. But $\sigma_{ij}(f)=0$, and hence $\alpha_{ij}=0$. □

4. Concluding remarks

4.1. Theorem 1 implies in particular that the infinite set of identities of the semigroup Π that was described in [1] can be derived from the identities (1)–(7) (in the sense of §2). It is not very difficult to show this directly.

4.2. Consider the semigroup $\Pi'=\Pi\backslash E$. The natural representation of this semigroup satisfies the identity

$$y^2(x^2-x)y^2. \tag{8}$$

Minor changes in the above proof of Theorem 1 (note that $X(p_i)\cap X(q_j)=\varnothing$ because of (8)) yield

Corollary 2. *The identities* (1)–(8) *constitute a basis of identities of the natural representation of* Π'.

Corollary 3. *All representations of* Π' *are finitely based. A finite basis of identities of the semigroup* Π' *was written down in* [6].

4.3. Theorem 1 provides an illustration for a more general situation that we will briefly discuss. Let A be an algebraic system of signature Ω (Ω-algebra). A representation of A is a map $r\colon A\to B$ into a Σ-algebra B such that for any $w\in\Sigma$

$$r(w(a_1,\dots,a_t))=\sigma_w(r(a_1),\dots,r(a_t)), \tag{4.1}$$

where $\sigma_w\in\Sigma$ and $a_1,\dots,a_t\in A$. Identities of representations can be defined in this general setting (cf. [3]).

Conjecture. Any algebraic system possesses a faithful finitely based representation into an algebra of (reasonably) extended signature.

Example. Let K be an algebraically closed field. Call a Σ-algebra *polynomial* if there exists an injective map $\varphi\colon A\to K^n$ such that for any $w\in\Omega$

$$\lambda_s(\varphi(w(a_1,\dots,a_t)))=P_s^w(\lambda_1(\varphi a_1),\dots,\lambda_n(\varphi a_1),\dots,\lambda_n(\varphi a_t)), \tag{4.2}$$

where P_s^w is a polynomial in nt variables, $\lambda_s\colon K^n \to K$ is the sth coordinate function, $s = 1, 2, \dots, n$, and $a_1, \dots, a_t \in A$. Fix a basis $e_1, \dots, e_n \in K^n$ and consider the two-carrier algebra (K^n, K), which apart from the usual vector space operations, includes the following:

$e_1, \dots, e_n \in K^n$-nullary operations;

$P^w\colon (K^n)^t \to K^n$, $w \in \Omega$—the operations defined by the right-hand sides of the relation (4.2) (i.e., the sth coordinate of $P^w(x)$ equals P_s^w(coordinates of x));

$\lambda_s\colon K^n \to K$, $s = 1, 2, \dots, n$—the coordinate functions.

The map $\varphi\colon A \to K^n$ satisfies the conditions (4.1) with $\sigma_w = p^w$. Hence one has the representation (triple) (A, K^n, K).

Theorem 2. *The triple (A, K^n, K) is finitely based.*

We mention also some of the natural open questions that arise in connection with Theorem 2.

(a) What is the 'minimal' extension of a signature that ensures a finite basis of identities?

(b) What remains of Theorem 2 when K^n is replaced by an infinite-dimensional space?

(c) Does every semigroup (group) possess a finitely based linear representation?

4.4. It is possible that any linear representation of a finite semigroup is finitely based. We will state here without proof one partial result in this vein. The result shows that the example from §2 is a rather general one.

Theorem 3. *Let B be a basis of pseudoidentities of the semigroup S. Then some power of any identity of the representation (kS, S) belongs to the verbal ideal generated by the set $\{u(b), b \in B\}$.*

Corollary. *Let S be a finite semigroup. Then the verbal ideal I of identities of its regular representation contains a finitely based (verbal) ideal I_0 such that I/I_0 is a nil-algebra.*

Proof. It is well known (see, e.g., [4]) that the positive universal theory of a finite algebraic system is finitely based.

The question of whether the regular representation of the semigroup Π is finitely based was asked by Plotkin in the late seventies. A connection between positive universal formulas and identities of group representations was studied in a joint paper of Plotkin and Kushkuley (unpublished).

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