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Identity and individuality in classical and Quantum physics

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IDENTITY AND INDIVIDUALITY IN CLASSICAL
AND QUANTUM PHYSICS

A Thesis Presented by Steven French
for the degree of
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and
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ABSTRACT

The aim of this thesis is to analyse the treatment which the notions of identity, individuality and indistinguishability have received from physicists in the two domains of classical and quantum physics, Thus in Chapter 1 we outline certain of the philosophical positions which have been adopted with regard to these concepts, emphasising those aspects which will be of relevance to our later discussions.

In Chapter 2 we consider classical physics and show that it is consistent with either of two views regarding particle individuality, which we call 'Transcendental Individuality' and 'Space-Time Individuality' respectively.

We begin Chapter 3 with the formal treatment of indistinguishable particles in quantum mechanics, suitably generalised to accommodate paraparticles as well as bosons and fermions. In particular we consider the statistical behaviour of paraparticles and conclude that it differs significantly from that exhibited by a 'paragas' of the kind first proposed by Gentile in 1940. These considerations are then used to support our conclusion that quantal particles can either be regarded as 'non-individuals' or as individuals subject to constraints on the set of states they can occupy.

Our final chapter re-examines certain philosophical questions in the light of our conclusions. In particular we discuss in detail the Principle of Identity of Indiscernibles and demonstrate that questions regarding its validity within quantum physics are dependent both on the form of the Principle which is taken and on the status of so-called 'mixed states' in the theory.

CONTENTS

<u>CHAPTER ONE</u>	<u>IDENTITY, INDIVIDUALITY AND INDISTINGUISHABILITY</u>	
1.1	Introduction	p. 8
1.2	A Brief History of the Philosophy of Individuality	p. 12
1.3	Leibniz	p. 21
1.4	Relative Identity and Proper Names	p. 32
1.5	Reidentification through Time	p. 45
<u>CHAPTER TWO</u>	<u>IDENTITY AND INDIVIDUALITY IN CLASSICAL PHYSICS</u>	
2.1	Particles .v. Fields	p. 51
2.2	Individuality in Classical Statistical Mechanics	p. 60
2.2.1	Transcendental Individuality	p. 61
2.2.2	Is T.I. Necessary ?	p. 62
2.2.3	Space-Time Individuality	p. 68
2.2.4	Space-Time Individuality and Space Time	p. 76
2.3	The History of Classical Statistical Mechanics	p. 81
2.3.1	The Introduction of Statistical Considerations into the Kinetic Theory of Matter	p. 82
2.3.2.	Boltzmann's Initial Attempts	p. 84
2.3.3.	The H-Theorem Approach	p. 86
2.3.4	Loschmidt's Criticisms	p. 91

2.3.5	Boltzmann's Combinatorial Approach	p.94
2.3.6	The Decline and Fall of the Combinatorial Approach	p.113
2.3.7.	Ergodic Theory - A Possible Bridge Between the Approaches	p.115
2.4	The Individuality of Fields	p.119
2.5	Conclusion	p.130

CHAPTER THREE

IDENTITY AND INDIVIDUALITY IN QUANTUM PHYSICS

3.1	Introduction	p.132
3.2	The Theoretical Context	p.133
3.2.1	The Generation of Basis Functions for Three Particle States	p.156
3.2.2	The Cluster Principle	p.162
3.2.3	The Statistics of Paraparticles	p.170
3.3	Identity and Individuality in Quantum Physics	p.191
3.3.1	Individual Particles + Acausal Forces	p.193
3.3.2	Non-Individual Particles	p.195
3.3.3	Non-Individual Particles + Individual States or Individual Particles + State Accessibility Restrictions	p.205
3.4	The History of Quantum Statistics	p.219
3.5	Individuality and Quantum Field Theory	p.266
3.6	The Gibbs Paradox	p.271
3.7	Conclusion	p.280

CHAPTER FOURA PHILOSOPHICAL EPILOGUE

4.1	Philosophical Criticism of T.I. and Space-Time Individuality	p.284
4.2	Physics and the Principle of Identity of Indiscernibles	p.294
4.3	Essentialism and Natural Kinds	p.323
4.4	The Underdetermination of Theories	p.331
4.5	Conclusion	p.336
	Appendix : Group Theory	p. 338
	Bibliography	p. 348

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CHAPTER ONE

IDENTITY, INDIVIDUALITY AND INDISTINGUISHABILITY

1.1 Introduction

The notions of identity, individuality and indistinguishability have been extensively discussed and analysed in the philosophical literature. The primary aim of this thesis is to consider the treatment these notions have received from physicists and to discuss the consequences of this for the traditional philosophical arguments about these matters. In their most basic form these arguments have centred on the following two questions:

- 1) What is it that confers individuality upon a thing? The desire to find some satisfactory answer to this has generated a long and profound search for a fundamental individuator or 'Principle of Individuation'. In this thesis we shall primarily be concerned with two candidates for this role: substance and location in space-time.
- 2) What is it that re-identifies a thing at one time as the 'same' thing at some other time? This is the problem of change and the various solutions to it have occupied almost every conceivable position between the two extremes of 'everything is changing' and 'nothing is changing'.

Of course, an answer to the first question may also provide an answer to the second. Thus it has been argued that substance is both the individuator and also that which allows us to re-identify a thing across time.

This chapter will be concerned with certain of the answers which have been given to these questions. Before discussing these however, we shall find it useful to give a preliminary sketch of how we shall use various terms in this thesis.

Identity is a relation that may exist between items, either particulars or universals, in some domain of discourse. The statement that item a is identical with

item b, written symbolically $\underline{a} = \underline{b}$, means informally that there are not in reality two distinct items at all but only one which may be referred to indifferently as a or b.

A physical individual, or just individual for short, is a particular which exists in the physical world and can be distinguished from other sorts of particulars such as events or states of affairs which may be said to occur or obtain rather than to exist. Individuals which have well defined spatial locations are commonly referred to in the literature as 'things'.

We are now confronted with the first of our questions above, what confers individuality upon physical individuals? If one rejects the view that the concept of individuality is metaphysically basic and cannot be explicated further, then a suitable 'individuator' must be put forward in answer to this question. One candidate for this role is the unknowable substantial substratum which 'supports', or to which is 'attached', the attributes of an individual. If an individual acquires its individuality by something which transcends its attributes in this way then we shall say that it exhibits 'Transcendental Individuality' or T.I. for short. As we shall see this notion has been strongly criticised by many philosophers, particularly those within the empiricist camp.

An alternative view is to argue that physical individuals are individuated by their location in space and time. If this position is adopted then we shall say that the individual exhibits 'space-time individuality' or S.T. for short. The problem of re-identification through time, of supplying the grounds for claiming that an individual b at time t_2 is the same individual a at an earlier time t_1 , is then solved in terms of the spatio-temporal continuity of the trajectory joining the location of a at time t_1 , and the location of b at time t_2 together with some requirement, such as that the two individuals must be impenetrable, to ensure that two or more trajectories do not intersect at the same spatial point at the same time. A proponent of T.I. might agree that spatio-temporal continuity is what allows one to infer a

re-identification across time but would claim that it is the persistent T.I. which, ontologically speaking, confers the re-identifiability,

In this context it is worth noting the point that that which confers individuality upon a physical individual may be regarded as distinct from that which distinguishes that physical individual from other individuals. Thus, if the position involving T.I. is adopted, one can argue that although the spatio-temporal location of an individual may allow us to distinguish that individual from others, it is the substance underlying the attributes which confers individuality upon it. T.I. may be regarded as intrinsic to an individual and unrelated to other physical individuals in a way that location in space-time may not be. On the other hand an advocate of S.T., recalling Ockham's famous 'razor', may argue that a metaphysical economisation is effected if that which distinguishes individuals from each other, their location, is also taken to be that which confers individuality upon them.

The S.T. view encounters difficulties also, principally with regard to what is meant by spatio-temporal location. On a relational view of space and time the locations of physical individuals involves their relations with other physical individuals. Thus, a circularity develops with regard to individuation. The circularity can be avoided by adopting an absolute theory of space and time but then one is required to give an account of what it is that confers individuality on the points of space and instants of time themselves. We shall discuss these problems in more detail in Chapter Two.

A ³third possibility is to argue that an individual is individuated by its attributes and that it can be reduced to nothing more than a bundle of properties or attributes. However, since these properties are universals and can be ^apredicted of more than one individual, there is then the possibility, at least in principle, that two (non-identical) physical individuals can have all the same attributes in common. We shall say that such individuals are indistinguishable. In order to be able to individuate on this view, this possibility must be eliminated.

This can be done by invoking Leibniz's famous Principle of the Identity of Indiscernibles which states that if two individuals are, apparently, indistinguishable then they are, in fact, identical in the sense of being actually just one individual, and so indistinguishable individuals cannot exist. The Principle thus denies that the distinction between identity and indistinguishability exists in fact. (Whether this is true or not it is important to draw a distinction in principle. Many authors, particularly non-philosophers, do not and thus create a great deal of confusion).

An important issue here is whether or not properties of spatio-temporal location are included in the collection of attributes possessed in common by two indistinguishable individuals. We can accordingly distinguish two forms of Leibniz's Principle, a strong form which states that two individuals cannot possess all properties, including spatio-temporal ones, in common and a weak form which says that two individuals cannot possess all properties, excluding spatio-temporal ones in common. It can immediately be seen that if the individuals are regarded as impenetrable then the strong form of the Principle is automatically satisfied. As we shall see in Chapter Four, the question of whether or not the Principle is violated by the results of physics, classical or quantum, turns precisely on the above distinction.

The entities we shall be concerned with in this thesis are the particles and fields of physics. In view of the points mentioned above it can be argued that indistinguishability for elementary particles should be taken to mean possessing all non-spatio-temporal properties in common. In our discussions we shall always make it clear whether we mean indistinguishability in this narrow sense or not. The non-spatio temporal properties of an elementary particle have also been called its 'intrinsic' properties. It is then an interesting question whether these 'intrinsic' properties are also the 'essential' properties which determine what it is for the particle to be a particle of that kind. We shall consider this question and some of the problems associated with essentialism and natural kinds in Chapter Four.

Finally we must briefly consider the notion of relative identity which is captured in the expression 'a is the same S as b' for some sortal predicate S. This immediately invites the question 'can a be the same S as b but not the same T?' which has provoked a great deal of discussion in the philosophical literature. We shall briefly consider some aspects of the controversy in a subsequent section of this chapter.

Instead of saying that 'a at time t_1 is the same individual as b at time t_2 ', this notion of relative identity allows us to make the claim that 'a at time t_1 is the same S as b is at the time t_2 ', for some sortal predicate S. This is the kind of reidentification that is involved for processes rather than things.

Having indicated what we mean by identity, individuality and indistinguishability, we shall now give a brief historical account of some of the positions which have been adopted with regard to these concepts.

It is not our intention, in what follows, to analyse these positions in any depth. We merely wish to draw attention to those aspects which are of relevance to this thesis. In particular, we shall emphasise the work of Locke, Leibniz, and certain points arising from the more recent discussions of these matters.

1.2 A Brief History of the Philosophy of Individuality etc

Perhaps the most fundamental problem facing the early thinkers was that of accounting for change in the observable world. The solutions to this problem have occupied a wide variety of positions between the two extremes of Parmenides and Heraclitus. The former condemned all change as illusory whereas the latter argued that everything was in a state of controlled change. In between these two views lies the class of solutions which derives from the claim that to say that something is changing presupposes something ^{which} remains the same. In other words change can be accounted for in terms of something invariant underlying the change. Within this broad approach we can further discern two opposing tendencies, which we may call Monism and Pluralism respectively. Thus the Pre-Socratic Milesian systems were dominated by the idea of a single primary material

(1) out of which all the others were produced, whose nature remained the same but whose qualities changed to produce the changing things observed in the world.

Empedocles and the Atomists however, explained change in terms of the rearrangement of a multiple of entities; for the former these were the four 'roots', Earth, Air, Fire and Water, whereas for the latter these were the atoms, distinct particles moving about in the void. The Aristotelian system represents a return to Monism with 'prime matter', unknowable and unqualified by any properties, being regarded not only as the substratum underlying change but also as the ultimate subject of predication and the principle of individuation for material substances. (2)

This system was subsequently taken up and extended by the Scholastics, although it is worth noting that Duns Scotus departed from the Aristotelian line by arguing that the principle of individuation was a primitive 'thisness' or *haecceity* by which the common nature, common to all individuals of the same kind, became the nature of this individual. (3) Suarez made the interesting distinction between individuality and distinguishability, the latter involving a dependence on the existence of other things whereas the former does not. Thus he argued that individuals are distinguished by their matter but that their individuality must be ascribed to their form.

The Cartesian system can be regarded as partly a continuation of the Scholastic tradition and partly a reaction against it based on the new scientific discoveries made by Descartes himself and others. These led him to take even further Galileo's attempt to geometrize mechanics and formulate its laws in terms of factors involving space and time only. Thus Descartes believed that the only essential properties of any corporeal substance were geometrical and kinetic (4)

1. For a discussion of these systems see G.S. Kirk and J.E. Raven (1957) or E. Hussey (1972)
2. Aristotle in D.Ross (1962), esp. Physics I, De Gen.et Corr. I and metaphysics. Also see D. Ross (1945) p.166-177.
3. Duns Scotus in C.R.S. Harris (1927) Vol.II p.115
4. Descartes Principles II 64, in E.S. Haldane and D. Ross (1955)

i.e. shape , size, position and motion-or-rest.

Substance itself was regarded as '... that which can exist by itself without the aid of any other substance.' (5) and also as that in which the attributes of things are embedded and which is known only by inference (6). In his famous argument concerning the heating of a piece of wax he concluded that the individuality of the wax itself, which he took to confer upon it its identity through time, must lie in its being something extended, flexible and changeable (7). This could be interpreted as an attempt by Descartes to isolate the substance, as a subject, from the properties which it supports. However, it is difficult to reconcile this with his final characterisation of the wax in terms of the properties of being extended flexible and changeable (8). An alternative view is that the argument is concerned with the essentialist doctrine that that which is essential to an individual is that which is necessarily involved in its being what it is, and what remains the same, however else the thing may change. On this interpretation the main point of Descartes' argument is that the essence of the wax does not consist in any of its sensible properties and that what remains invariant through change is merely its being something extended, flexible and changeable.

The physical consequences of the Cartesian system and their conflict with observation are well known. In these terms his programme was ultimately unsuccessful and was eventually superseded by the Newtonian mechanics which added mass as a fundamental concept to the Galilean duo of space and time. Metaphysically, however, it had a profound impact, for by regarding matter primarily as 'that-which-is-opposed-to-spirit' rather than the Aristotelian substratum underlying substance, Descartes took this concept out of the realm of the ontologically

5. Descartes Principles II 64, in E.S. Haldane and D. Ross
p.101 (1955)

6. Ibid p.53

7. Ibid p.154

8. B. Williams (1978) p. 213-225

indeterminate and epistemologically unreachable. Prior to Descartes there was no single general term covering the sorts of things to which physical science applied. Matter was now invoked to fulfill this role and came to be regarded as the subject of the new and powerful science of mechanics. The idea that it was a Platonic 'co-principle' with form was abandoned and explanations in terms of Aristotelian substantial forms gradually gave way to explanations in terms of metrical properties from which their motions could be logically derived.

Descartes can thus be seen as a major figure in the metaphysical movement away from the concepts and modes of thought of the Greeks, In particular this movement, and the rise of Newtonian mechanics in general, led to the adoption of a new way of individuating physical objects

The notion of state description was central to the new physics and for this to work physical objects had to be capable of being uniquely specified in terms of certain properties which in turn lent themselves to metrical specification. Thus geometrically defined space and time parameters came to be used to distinguish one physical body from another. In other words material bodies came to be distinguished and hence individuated by their separation in space and time. Spatio-temporal location thus came to be regarded as the principle of individuation.

This view rapidly took hold among the mechanical philosophers and has remained one of the fundamental metaphysical principles subscribed to by physicists ever since. The power of the new science was such that all talk of individuality virtually ceased among the mechanical philosophers, with the exception of Leibniz as we shall see. Thus the 17th century saw a steady diverging between those who were concerned with the new mechanics and its related concepts and problems, and those who remained concerned with the traditional philosophical puzzles, such as those regarding substance, change etc.

Thus we can discern two metaphysical lines of thought regarding the individuality of physical objects. One view holds that individuality is conferred through a material substratum underlying the object's properties,

whereas the other claims that it is location in space and time which fulfills the role of individuator. As we shall see in subsequent chapters an examination of the physics of elementary particles will reveal that this fundamental component of physical science is consistent with either of the above metaphysical positions.

The former, 'substantialist', position was restated by Locke, although his reappraisal of the notion of substance led, according to some, to its destruction as a metaphysical category. In order to fully grasp the nature of this re-examination it is important to distinguish between Locke's conception of particular substances and his notion of the general idea of substance. Both concepts were discussed in 'An Essay Concerning Human Understanding' (9) and nowhere was his empiricism more in conflict with his rationalism. He believed that we are conversant only with particular substances through experience yet his rationalism would not permit him to abandon completely the general idea of substance. Thus although he initiated the empiricist attack he failed to follow it though - something which was left to Berkeley and Hume to do.

Locke began with a consideration of particular substances, remarking that the mind often notices that a certain number of simple ideas go constantly together. Thus, he wrote '... not imagining how these simple ideas can subsist by themselves we accustom ourselves to suppose some substratum wherein they so subsist, and from which they do result, which therefore we call substance' (10). It is naive to interpret this passage as indicating that it was the notion of substance as the substratum necessary for the existence of qualities which was the important sense of the word for Locke at this point. (11)

9. J. Locke (1959)

10. Ibid p. 390-391

11. D. O'Connor (1967) p.75

Rather it is meant to show that the complex idea of a particular substance is produced in the mind by sensory phenomena being found to co-exist in aggregates in our experience (12). The idea of a particular substance is formed by the combination of many simple ideas together, a process completely opposite to that which produces the general idea of substance.

This latter idea is formed by the process of abstraction from particular substances and was discussed by Locke in the next section. Thus he wrote 'So that if anyone will examine himself concerning his notion of pure substance in general, he will find he has no other idea of it all, but only a supposition of he knows not what support of such qualities which are capable of producing simple ideas in us, which qualities are commonly called accidents!' (13) Thus, the general idea of substance is a 'supposition of, we know not what, support'. In other words all that we can know about substance is that it acts as a support for the accidents and that is all. We cannot know it in any way beyond that it is a mysterious 'something' to which the qualities attach (14).

This is clearly a major step along the road towards the complete abolition of substance from metaphysics altogether. Locke himself did not go this far although I think that his whole line of thought is clearly leading to it. Perhaps he saw that to reduce substance to a mere fiction would resolve reality with the sceptics into nothing more than a succession of impressions (as Hume later showed) and that this was the inevitable conclusion of his empiricism (15) - a conclusion from which he quickly and firmly recoiled (16).

Locke's distinction between the two notions of substance is closely connected to another he made between nominal and real essences. Thus the complex idea of the

12. See R. Woolhouse (1971) p.61 ff and R. Aaron (1963) p. 173-177

13. Locke op cit p. 391

14. Ibid p. 392

15. This was pointed out to Locke by Stillingfleet who accused him of having 'discarded substance'. See O'Connor op cit. p. 77

common characteristic of a type of thing is the nominal essence of that thing (17) which is abstracted by leaving out characteristics peculiar to individual things as individuals and retaining their common characteristics. The real essence of a thing is its real, but known, constitution which cannot be abstracted and from which comes the qualities of a thing which distinguishes it from other things. (18) Locke considered this to be no more useful than the analogous idea of substance as an unknowable 'something we know not what'. Real and nominal essences are related but are always different in substances (19).

The nominal essence is the abstract idea of the observable common characteristics to which corresponds a real essence of inner corpuscular constitution which is some configuration of insensible particles and which explains the co-instantiation in many particulars of the properties taken into account of in the nominal essence (20).

As regards reidentification through time Locke believed that this lay in the fact that the object in question could be uniquely described and identified in terms of space and time coordinates. Thus he wrote that when we observe an object in a certain place at a certain time we are sure that it is that object and not some other, similar or even indistinguishable object in another place at another time, ' For we never finding nor conceiving it possible that two things of the same kind should exist in the same place at the same time, we rightly conclude that whatever exists anywhere at any time excludes all of the same kind, and is there itself alone'. (21)

Thus Locke's notion of self-identity depends upon what we call the Impenetrability Principle, that no two things can exist in the same place at the same time. (22)

18. Locke op cit p. 27-28

19. Ibid p.29

20. Ibid p.61

21. Ibid p. 439-440

22. It is interesting to note that Locke restricts this principle to things of the 'same kind', thus allowing the possibility that things of different kinds may occupy the same place at the same time.

He then went on to say that it follows from this that one thing cannot have two beginnings of existence nor two things one beginning. (23) As we shall see this implies that a material individual cannot retain its identity through a creation and annihilation process.

Thus a thing's identity is related to the spatio-temporal point at which it began its existence. This led on to his idea of what constituted the principle of individuation, as he wrote, 'From what has been said it is easy to discover what is so much inquired after, the principium individuationis; and that it is plain is existence itself; which determines a being of any sort to a particular time and place, incommunicable to two beings of the same kind.' (24). Thus for Locke it is a thing's actual existence which confers individuality upon it and, by underpinning the Impenetrability Principle, allows us to infer its identity at a particular place and time. This is clearly very similar to the view we have called T.I.

Locke also gave a compositional account of the re-identification of inorganic compound objects. Thus he gave the example of a mass of matter formed by the joining of two or more atoms. This mass will remain 'the same' through time so long as the same atoms are joined together, but if one of them is taken away or a new one added then the mass is no longer the same mass as before and loses its identity. (25) This is obviously an atomist solution to the problem of the reidentification of compound objects and relies, for its efficacy, on an account of the reidentification through time of the atoms themselves - an account which Locke has already given as we have seen. (26)

Finally it is interesting to note that whereas Aristotle considered matter to be the indeterminate substratum and a constituent of substance, Locke inverted this relationship and made substance the substratum and constituent of matter.

23. Ibid. p.440

24. Ibid. p.441-442

25. Ibid. p.442

26. Thus Quinton's criticism that Locke's account is viciously regressive is incorrect. See A. Quinton (1973) p. 65.

Of course by his time matter had acquired its new meaning as the subject of mechanics. Substance was thus called upon to fill the metaphysical role of the substratum underlying the properties which matter had left vacant.

The sceptical consequences of Locke's analysis of substance were further unravelled by Berkeley and Hume. The former's 'immaterialist hypothesis' (27) urged the elimination of material substances that could never be directly apprehended to be replaced, as the cause of our sensations, by God. Hume went even further, and not content to write off substance alone as an unintelligible chimaera, applied the sceptical arguments to God and even 'the self', rationalising both out of existence in a similar way. (28). Causality also was relegated to the status of a myth and thus the world of events was reduced to nothing more than a series of fleeting perceptions with no external object, no enduring subject to which they could belong, and not themselves even bound to one another.

It is not surprising therefore, that Hume also regarded identity through time as ultimately a metaphysical illusion. Thus he argued that the idea of identity is simply the product of ^a mistake into which we naturally fall when we think about time. It arises, he believed, from a propensity of the mind to attribute invariableness or uninterruptedness to an object while tracing it, without a break in the attention of time, - a propensity which Hume considered to be nothing more than a fiction. (29).

We now turn our attention to the man whose rationalistic philosophy laid much of the foundation of the modern approaches to identity and whose Principle of the Identity of Indiscernibles must be included in any account of these matters.

27. Berkeley (1962) esp. p. 156 ff

28. D. Hume (1962)

29. Ibid p. 200-201.

1.3 Leibniz

As we have noted the new mechanics of the 17th century encouraged the view that material bodies could be individuated by their locations in space and time and subsequently all discussions of individuality virtually ceased among the more scientifically minded natural philosophers. It remained a problem for Leibniz however. He rejected Newton's conception of an absolute space and thus also the idea that position in this space could serve to individuate an object, but had difficulty in developing a completely coherent view of his own, as we shall see.

Leibniz's metaphysics can be taken to begin with his analysis of subject - predicate propositions in which he made the predicate part of the subject concept (30). When many predicates can be attributed to one and the same subject while this subject cannot be made the predicate of any other subject then the former was called an individual substance (31). Furthermore, he believed that every predicate, necessary or contingent, past present or future, is comprised in the notion of the subject. This is the concept of the Complete Notion of an Individual (32).

The principle according to which the states of a substance ^{developed} was called its 'activity', regarded as essential to every substance. The notion of an individual substance differed from a mere collection of general notions by being capable of wholly distinguishing its subject and involving circumstances of time and place. The nature of an individual substance was to possess so complete a notion as to suffice for comprehending and deducing all its predicates. Hence he concluded that no two substances could be exactly alike, which is the Principle of Identity of Indiscernibles. From this stage, with the help of the premiss that perception gives knowledge of the external world, the doctrine of monads easily follows. The monads themselves were immaterial, unextended substances.

30. See B. Russell (1904), L. Couturat (1903)

31. G.W. Leibniz in Mason (1967) p. 51

32. Broad called it the 'Predicate-in-Notion Principle' whereas Rescher terms it 'the program'. C.D. Broad (1975) p.6 ; N. Rescher (1967) p.14.

As they were unrelated to one another and could not interact any change had to come from within each monad itself (its activity). As we shall see there are problems as to how the monads are individuated.

Thus an important and fundamental part of Leibniz's system is the Complete Notion of an Individual (33) which can be thought of as the sum of all that can ever be said about an individual, including its relations and interactions with other individuals - so each individual is a world unto itself - and serves to distinguish an individual as that individual.

In one of his letters (34) Leibniz tries to explain what he means by this notion, using Adam as an example. Thus he wrote that the Complete Notion of an Individual (C.N.I.) as it applied to Adam was identical with the knowledge God had of Adam when he created him. 'Specific notions' such as that of the sphere, were distinguished from individual notions, such as that of Adam, on the grounds that the former applied to an indefinite number of actual or possible individuals and was therefore incomplete, in a way which the individual notion was not.

There is then the problem of alternative possible individuals with the same proper name, for example several alternative possible Adams. Arnauld remarked (35) that he found such phrases obscure and that the contention that God could actualize one out of several possible individuals and leave the rest as unrealized possibilities was meaningless. In his reply (36) Leibniz conceded the point that the phrase 'several alternative possible Adams' was meaningless if one took the word 'Adam' to be the proper name of a certain complete individual, but then claimed that when he used the phrase he took the word 'Adam' to connote only a

33. See Russell op cit p.29; Broad op cit p 6-9; Rescher op cit p. 14 ff.

34. Leibniz in Mason op cit p. 53-66

35. Arnauld in Mason ibid p. 24-34

36. Ibid p. 53-66.

certain limited collection of properties which are a sub-set of the larger group denoting the Complete Notion of an Individual. Because the name 'Adam' is only a label in this sense, it can also be applied to other individuals possessing the same subset of properties. These others will not actually be indistinguishable because they will differ as regards their other properties lying outside this subset which, together with the subset, go to make up their CNI. The CNI of Adam, the sum of all his predicates, is the only description which suffices to distinguish the actual Adam from all other individuals, actual or possible.

Elsewhere (37) Leibniz wrote that only God can completely and distinctly comprehend an individual's CNI, so human beings have to depend on experience or hearsay for their knowledge of many of the facts about individuals.

As this concept plays such an important role in Leibniz's work, it is worth making a few remarks about it.

Firstly, the idea is a profound and original way of characterising what it is to be an individual, but the CNI is of no practical use as a principle of individuation because it must be an infinite set of predicates (38). Therefore, as Leibniz himself noted, only God can have a full knowledge of any CNI and thus only God is truly capable of individuating objects.

Secondly, it can be seen how easily the Principle of Identity of Indiscernibles follows from this idea. If every individual has a distinct complete notion then two things which are completely indiscernible, and which therefore possess all predicates in common and so have the same complete notions, must in fact be the same thing. According to Leibniz, if we are presented with two things which appear indistinguishable then if we look long enough and hard enough we will discover a predicate which one

37. Leibniz 'Discours de Metaphysique' in Parkinson (1973) p. 18-19.

38. An individual persists for a finite time, so the notion of an individual involves an infinite number of propositions specifying its states at a continuous series of moments. See Russell op cit. p.29.

has and the other doesn't and which will serve to distinguish them.

However, it can be argued that Leibniz cannot simultaneously hold both this idea of a C.N.I. and a relativist theory of space. Adherence to the former implies rejection of the latter and the adoption of an absolute theory of space, something Leibniz would have sought to avoid given his professed belief in the relativist position. The argument is essentially similar to that put forward by Black (39).

Imagine a universe totally empty except for two distinct individuals which possess exactly the same set of predicates - volume, mass, colour, history etc. The only difference in their complete notions which could serve to individuate them is provided by their separation in space. But, in the absence of any other body, even an observer, the spatial relation of one object A, say, to the other B, is exactly the same as that of B to A. Therefore, if a relativist view of space is adopted, the objects' spatial separation cannot provide the difference in the set of predicates which is necessary to distinguish their complete notions. The only way in which the spatial separation can do this is if an absolutist view of space is taken, so that the position of A in this absolute space is different from that of B. Which difference then provides the difference in the set of predicates distinguishing their complete notions and thus individuating the two objects. Thus Leibniz cannot simultaneously hold both the idea of a complete notion and his relativist view of space. We shall consider this form of argument again in Chapter Four.

A fourth point is that Leibniz has clearly distinguished between universals ('specific notions'), proper names and individuals. Universal notions are incomplete because they apply to more than one individual and so do not completely individuate a substance. Proper names also do not serve to individuate since they are merely labels for a certain subset of properties out of all the innumerable properties which comprise a complete notion.

Individual notions are complete and completely individuate a particular substance. All that will ever happen to that substance is inherent in its complete notion.

Finally, let us consider the individuality of the monads - Leibniz's unextended immaterial substantial 'basic' building blocks' (to steal a phrase from modern physics). A fundamental problem with the doctrine of monads is how were they individuated? The C.N.I. implied that each and every monad was an individual. If two monads shared the same set of properties then it necessarily followed from the C.N.I., that they were in fact the same monad. This is his Principle of Identity of Indiscernibles, also derived, although only contingently, from his Principle of Sufficient Reason, as we shall see. However, the Principle of Identity of Indiscernibles resolves individuality with the predicates (as does the C.N.I. from which it is derived), and thus gives encouragement to those (for example Russell later) who would do away with substance altogether, as an individuator, and regard objects as mere bundles of properties. It is hard to see how Leibniz could have stomached this, committed as he was, through his subject-predicate distinction, to the notion of substance. He regarded substance as that which can only be a subject, not a predicate, which has many predicates and which, subsequently, persists through change. Eliminate substance and you eliminate the subject and hence the subject-predicate division upon which Leibniz's whole system was based.

But then what exactly constituted this substance which existed over and above the properties? In answering this question Leibniz would have been unable to appeal to some kind of Aristotelian prime matter, because he had stated that the monads were immaterial and spiritual. The only choice, it seems, would have been to individuate the monads in terms of spirit somehow. This, with its attendant whiff of Scholasticism, is another conclusion Leibniz would have rejected. Thus individuality remained a problem for him.

Of course, a completely coherent position could have been arrived at if he had used location in a Newtonian space as individuator, but given his firm adherence to the relativist view, this would have been impossible.

It is worth examining Leibniz's view of substance in a little more detail. We have noted that Descartes defined substance in two ways: as that which is the subject of various qualities and as that which is capable of independent existence. Leibniz believed that the relation to subject and predicate was more fundamental than the somewhat doubtful inference to independent existence, and, therefore, brought his concept of substance into dependence upon the former logical relation. He attacked Locke on the grounds that there is good reason to assume the existence of substance since we conceive several predicates in one and the same subject and this is all that is meant by the words 'support' or 'substratum' which Locke used as synonymous with 'substance'. (40)

However Leibniz agreed that the word 'substance' also meant the notion of a something persisting through change of a subject which preserves its identity whilst its qualities alter. He held that this idea is not, therefore, independent of subject and predicate, but is subsequent to it - it is the notion of subject and predicate applied to what is in time (41).

Furthermore, he wrote that '... the nature of an individual substance is to have a notion so complete that it suffices to comprehend and to render deducible from it, all the predicates of the subject to which this notion is attributed'. (42) In other words, an individual substance is not only the subject of various predicates and is not attributed to any other subject, but also possesses a Complete Notion. To say that all the states of a substance are involved in its Complete Notion is merely to say that the predicate is contained in the subject. It then follows that every substance, every monad, is a world apart, independent of all else except God. All it will ever do or be expressed in its Complete Notion and so nothing external to it can ever affect it, except God.

40. Leibniz 'New Essays ...' in Langley (1916) p. 225

41. See Russell *Op cit* p. 42-43

42. Leibniz, trans in Russell *Ibid* p. 213-214. It is interesting to note that Leibniz equates individual notion with haecceity here.

How then do the monads change? Leibniz believed that there must be, in every state of a substance, some element or quality in virtue of which that state is not permanent but tends to pass into the next, and this element is what Leibniz meant by 'activity'. (43)

This view is similar to the Heraclitean doctrine that all things are laws. Indeed Leibniz stated that a monad can be reidentified through time '... by the persistence of the same law of the series or of continuous simple transition, which leads us to the opinion that one and the same subject or monad is changing. That there should be a persistent law, involving the future states of that which we conceive as the same is just what I assent to constitute it the same substance' (44).

However, a substance is not, for Leibniz, merely the sum of its states, but on the contrary, those states cannot exist without a substance in which to inhere. (45) The important point is that the actual basis for assuming substance is purely logical. All that we observe are states of substance which are the only things given in experience. One makes the inference that they are states of substance because they are held to be of the logical nature of predicates, and thus to demand subjects of which they may be predicated.

Russell has raised the criticism that a consequence of this view, particularly as regards the reducibility of time and place to predicates, is that the substance remains on its own, distinct from its predicates and totally destitute of meaning (46). One then has to face the Lockean problem of how such a meaningless term can be of any use.

One of the most important aspects of Leibniz's philosophy from the point of view of this thesis anyway, is the Principle of Identity of Indiscernibles, which was stated

43. Leibniz in Duncan (1890) p. 117

44. Leibniz in Russell op cit p. 264

45. Leibniz in Duncan op cit p. 118

46. Russell op cit p. 50

in different ways in different places. Thus he wrote that it asserts '... that there are not in nature two indiscernible real absolute beings' (47), or '... that it is not possible for there to be two individuals entirely alike or differing in number only' (48) or, again, that '... no two substances are completely similar, or differ solo numero' (49). It should immediately be noted that this Principle applies only to substances; existent attributes may, in fact, be indiscernible, as Leibniz explained in discussing place. (50)

This principle occurs frequently in the letters to Clarke, where it is used in connection with the controversy between the absolute and relational theories of space. However, there is some confusion as to which of the following two alternatives Leibniz meant to assert : 1) That the very supposition that there might be two things exactly alike in all their qualities is self-contradictory and meaningless; i.e. the principle is necessary. 2) That although the supposition is logically possible we can be sure that God would not create two such things i.e. the principle is contingent. As Clarke pointed out, Leibniz asserted first one view and then the other. Thus in the Fourth letter he wrote that the proposition that there could exist two indistinguishable things was self-contradictory and meaningless. (51) On the other hand, in the Fifth letter Leibniz took the second alternative and said, explicitly, that it was not absolutely impossible for there to be two things exactly alike, but that it would be contrary to God's wisdom to create two ^{such} bodies and therefore we can be certain that there are none (52).

A plausible explanation for this inconsistency is that this discourse shows Leibniz shifting and moderating his position in response to criticism exposing its weaknesses. (53). It may also not be unconnected with the fact that

47. Leibniz in Duncan op cit p. 259

48. Leibniz in Mason op cit p. 45

49. Leibniz in Russell op cit p. 433

50. Leibniz in Duncan op cit p. 266

51. Leibniz in Alexander (1956) p. 37

52. Ibid P. 61

53. It may be possible for this 'Darwinian' approach to be applied to other areas of Leibniz's work also. See H. Castañeda (1974) p. 383.

Leibniz deduced the Principle in two ways. One was from the C.N I., which makes the principle necessary, and the other was from the Principle of Sufficient Reason, which causes the Principle to be contingent. (54)

Leibniz himself said the P.I.I. could be deduced from C.N.I. but never actually gave this deduction in full. Thus after outlining the concept of the C.N.I. he wrote 'From this follow several considerable paradoxes, as, amongst others, that it is not true that two substances resemble each other completely and differ only numerically.' The argument can easily be reconstructed. Each individual substance has a unique Complete Notion which is the set of all its predicates and contains all of its qualities and relations (as every extrinsic denomination every relation, has an intrinsic foundation, a corresponding predicate), past, present and future. To say that two, apparently distinct, things are completely indiscernible means that they must possess the same set of predicates, qualities and relations. Therefore they must possess the same Complete Notion and therefore, by uniqueness, they must in fact be one and the same individual. Thus if two things are indiscernible they must in fact be identical, in the sense of being the same thing. Indiscernibility implies identity, and C.N I. implies P.I.I. It is impossible for two, 'true', individuals to be indistinguishable because they^{MUSE} have unique Complete Notions to be individuals and so there must be a difference in predicates somewhere which could serve to distinguish them. Thus P.I.I. is a necessary consequence of C.N.I.

Russell also outlines the deduction and presents the following example: suppose A and B were two indiscernible substances. A differs from B in the sense that they are different substances. But to be thus different is to have a relation to B. This relation must have a corresponding predicate of A. But since B does not differ from itself B cannot possess the same predicate. Hence A and B will differ as to predicates contrary to hypothesis. (55)

54. Leibniz in Russell p. 433. See also Leibniz in Mason op cit p. 45

55. Russell op cit p. 58

Thus two individuals cannot be indiscernible. If the premiss of the C.N.I. is accepted, and if it is agreed that every external relation corresponds to some internal predicate, then P.I.I. is necessary. (56)

Elsewhere Leibniz deduced P.I.I. from the Principle of Sufficient Reason (57). The argument runs as follows. Suppose that there co-existed two material particles a and b which were exactly alike as regards all their qualities and relations. They would have to be in different places at any moment of their co-existence. It does not matter whether an absolute or relational view of space is adopted here. If x and y are points of an absolute space then there could be no reason for preferring to put a at x and b at y rather than vice versa. However, a similar consequence follows on the relational account. In this case x is defined by certain spatial relations to a certain set of material objects taken as a system of reference and y is defined by certain other spatial relations to the same set of objects. Now if a and b are exactly alike in all their qualities and dispositional properties then there can be no reason for preferring to put a into the former relation and b into the latter rather than the other way round.

If, then, on either view, God were to create two such particles he would 1) be bound to put them into different places, and yet 2) have no reason for choosing between the two alternatives which would arise by imagining the two particles being interchanged. Now God never acts without a sufficient reason (Principle of Sufficient Reason). So we can conclude either that the supposition is meaningless, or that, if it is not, God will never create two exactly similar particulars and therefore there will never be two such particles. Thus when P.I.I. is deduced from P.S.R. in this way it is contingent. (58)

This argument, resting as it does on the supposition of

56. The difficulty in fact is to prevent the argument from proving that there cannot be two substances at all, and degenerating into Spinozism. See Russell *Ibid* p. 58
57. Leibniz in Alexander *op cit* p. 36-45 and also p.62.
58. This form of the argument gets round Russell's criticism of it.

some external creative agent (*i.e.* God) who acts according to the Principle of Sufficient Reason, is far less cogent and persuasive than the previously necessary deduction from C.N.I. Clarke was certainly not satisfied with it and argued that by Leibniz's own principle God would actualise neither a nor b simply because he cannot actualise both, and has no reason to prefer one to the other. (59)

The Identity of Indiscernibles has been discussed by several authors (60) of whom we take Quinton to be an example. He argues (61) that there are two forms of the principle. If position is included as one of the properties of a thing then the principle is necessarily true; this is the 'wide' form of the principle. If the set of properties of a thing includes only qualitative properties then the principle is only contingently true; this is the narrow form. Quinton then writes that Leibniz was compelled to take the principle in its narrow qualitative sense since by denying that space and time were fully real he was committed to the view that positional properties were the more or less misleading appearances presented by underlying qualities. A symptom of this is the fact that he attempted to derive the principle from the law of sufficient reason, which he took to rule out the possibility that there could be two qualitatively indistinguishable things in distinct places. (62).

This misses the point that Leibniz did try to reduce position, as a relation, to a predicate of the individual (63) and more importantly, that he gave a more fundamental deduction of P.I.I. from his idea of the C.N.I.

P.I.I. has been considered in some detail here in order to prepare the ground for our later discussions of its validity within classical and quantum physics.

We shall now outline some aspects of the modern approaches to these problems.

59. Clarke in Alexander *op cit* p. 45-53

60. See Black (1952); Strawson (1965); T. Vinci (1974) p.195. D. Armstrong (1978), Also Russell, Broad and Rescher *op cit*.

61. Quinton *op cit* p. 24

62. *Ibid* p. 25

63. Leibniz in Langley *op cit* p. 238.

1.4 Relative Identity and Proper Names

Frege can be regarded as one of the principal architects of the modern approach to identity and initiated its characteristic involvement with the theory of meaning. Thus he pointed out that if identity is interpreted simply as a relation between an object and itself then one is faced with the following paradox. It is clear that the sentences 'a = a' and 'a = b' generally have different cognitive significance, as can be seen by comparing 'The Morning Star is identical with the Morning Star' ^{with 'The Morning Star is identical with the Evening Star!'}. However if a and b are the same object and identity is interpreted as above, then it is impossible to explain how the two sentences can differ in cognitive content. Frege therefore concluded that identity should be regarded as a relation holding between the names or signs of objects (64).

Frege's solution to the above problem thus went straight to the heart of the theory of meaning. He argued that as well as names and the objects they refer to one must distinguish a third element, the meaning or sense of the name in virtue of which, and only in virtue of which, it refers to the object. Thus he wrote 'It is natural, now, to think of there being connected with a sign (name, combination of words, letter) besides that to which the sign refers, which may be called the reference of the sign, also what I should like to call the sense of the sign, wherein the mode of presentation is contained'. (65)

Thus in the above example the reference of 'the Morning Star' would be the same as that of the 'Evening Star', i.e. the planet Venus, but not the sense. The sense gives the mode of presentation of the object and the object is illuminated, (partially), by the sense of the expression. It is because the two expressions 'Morning Star' and 'Evening Star' have different senses that factual information can be conveyed and thus the statement differs in cognitive significance from the analytic uninformative sentence 'The Morning Star is identical with the Morning Star', in which

64. G. Frege in P. Geach and M. Black (1952) p. 56 ff

65. Ibid p. 57

both of the denoting expressions have not only the same referent but has the same sense also.

This work had two profound consequences which we shall consider here - one concerning the status of 'Leibniz's Law of the Indiscernibility of Identicals' in 'classical' identity theory and the other concerning the philosophy of proper names. We shall consider the former first.

The line of research initiated by Frege led, in the early part of this century, to widespread agreement among philosophers and logicians that the notion of meaning was ambiguous and that a distinction should be drawn between the meaning of an expression in the intensional and extensional senses. Consideration of Frege's problem also led to questions concerning synonymy - under what circumstances could expressions have the same meaning. Consequently, it was shown that Leibniz's law, which can be taken as saying that synonymous expressions may be interchanged in any context without change of truth value, does not hold generally and is true only of extensional contexts. (66)

This law is central to the 'classical' theory of identity and can be expressed thus : 'If two things are identical then whatever is true of one is true of the other'. (67) or in second order predicate calculus

$$(\forall x)(\forall y)[x = y \rightarrow (\forall \phi)(\phi(x) \leftrightarrow \phi(y))]$$

where 'x' and 'y' are syntactical individuated variables and ' ϕ ' is a syntactical predicate variable.

Written thus the Law is broad enough to be the converse of the Principle of Identity of Indiscernibles.

From this can be derived all the valid theorems or propositions of the 'classical' theory of identity, such as the Identity of Indiscernibles:

$$(\forall x)(\forall y)[(\forall \phi)(\phi(x) \leftrightarrow \phi(y)) \rightarrow x = y]$$

Reflexivity: $(\forall x)(x = x)$

Symmetry: $(\forall x)(\forall y)(x = y \rightarrow y = x)$

and Transitivity: $(\forall x)(\forall y)(\forall z)(x = y \text{ and } y = z \rightarrow x = z)$

66. See L. Linsky (1952), especially the essays by Goodman p.67, Quine p. 77 and Mates p.111

67. Mates (1965) p. 145

68. P.T. Geach (1972) p. 239

This 'classical' theory is also known as the 'absolute identity theory' (69) because it follows from Leibniz Law that if two objects are 'identical' then they are 'identical' in all respects - they are 'absolutely identical' (70)

This theory is a well established branch of logic (71) and its principles are employed in many philosophical reasonings in which questions of identity arise. However, although the theory is both powerful and simple it has been attacked on various grounds, particularly recently.

Thus it is argued that it fails to capture the notion of identity as used in ordinary language where the relation 'x is the same such-and-such as y' is frequently used and in ways which are not adequately encompassed by the statement 'x is identical with y' or 'x=y'. (72) A possible avenue of escape for the absolute identity theorist is to say that identity is being loosely or even incorrectly used in these situations and that it is actually 'similarity' that is being referred to.

There are other difficulties, however, Leibniz's Law makes no allowance for an individual changing its properties over a period of time, yet remaining 'the same' individual. In other words the absolute theory faces problems with regard to reidentification through time (73). The Law also displays what Quine has called 'referential opacity' (74) in leading from true propositions to false ones when, for example, non-extensional predicates are substituted for ' $\phi(x)$ '.

These, and other difficulties, have led to a variety of alternative theories being proposed in which the classical two place identity relation is replaced by a three place one, the third place being taken up by a general noun which specifies the respect in which identity is

69. N. Griffin (1977) p. 2 ff

70. If spatial, positions or relations are included as, or reduced to, predicates of the objects then 'numerical identity', two objects being 'identical' in all respects but separate in space, is not true identity in this sense. Indeed we would prefer to call it indistinguishability rather than identity.

71. See P.T. Geach (1972) p. 239

72. Griffin op cit p. 10

73. Ibid p.5

74. W.O. Quine (1961) p. 139

intended. These alternatives can broadly be classified as 'relative identity theories'.

A central tenet of these approaches is that the x and y of an identity statement must be taken to fall under, or be instances of, the same substantial concept of a thing or natural kind. (75) Thus Geach has written 'When one says 'x is identical with y' this I hold is an incomplete expression, it is short for 'x is the same A as y' where 'A' represents some count noun understood from the context of utterance ... ' (76) He also wrote 'On my own view of identity I would not object in principle to different A's being one and the same B' (77). This position receives support from examples of the use of 'identity' statements in ordinary language, such 'that box is the same colour as this box' or 'these two boxes have the same colour but different volumes', and is therefore regarded as an improvement over the absolute theory.

The above two quotes introduce two important theses. The first has been called 'D' by Wiggins (78) and it consists of the following : absolute identity statements of the form 'a is the same as b' are incomplete and a term A must be introduced to give a statement of the form 'a is the same A as b'. The completing term A must be a general term which is either a noun or noun phrase. There are several varieties of general nouns which lead to several variants of 'D' each different in scope or restrictiveness (79).

The second of Geach's remarks introduced the 'R' thesis which holds that 'a may be the same A as b but not the same B'. There are at least as many variations of R as there are of 'D' since each variant of 'D' will give a different way of completing 'A is the same such-and-such as B'.

Setting aside these variations these two theses in combination generate three broad positions according to whether one or other or both are accepted. Thus some philosophers have accepted both 'D' and 'R', others have

75. P.T. Geach (1968) p.39

76. Ibid p. 238

77. Ibid p. 157

78. D. Wiggins (1967) p. 1 and (1980) p. 15

79. See Griffin op cit p. 14-15

accepted only 'D', while others have accepted only 'R'.

Typical of the first position is Geach whose view we have already touched upon. Wiggins can be regarded as a representative of the second alternative. In his earlier work he presented 'R' in the form of a question, 'Can a be the same x as b and not the same y as b? (80) and noted that the possibility of this, the 'R' thesis, is commonly taken to provide the principal ground for 'D'. He then argued that 'R' could not be the rationale of 'D', if 'D' had any rationale at all, in fact 'R' cannot be a possibility at all, i.e. he rejected it entirely.

According to Wiggins the only defence of 'D' which is correct is the one that states '... to say what a is is automatically to provide an f which determines the truth grounds of a=b'. (81) The grounds for rejecting 'R' rest on a demonstration that the thesis is incompatible with the formal requirements of classical identity theory.

In particular Wiggins argued that if Leibniz Law were true, then 'R' must be false. (82)

The third of our three positions is advocated by Griffin in an attempt to meet the difficulties of both the relative and absolute identity theories. His aversion to 'D' rests on the grounds that '... the arguments in its favour are very weak and often rely on unacceptable postulates...' (83) Nevertheless, he believes, the 'D' relativists have uncovered an important point in that '... the notion of an individual item is incoherent without reference to a set of individuating principles. It is the role of sortals in natural language to provide these principles'. (84) This forms the core of his own theory in which it is argued that we cannot individuate items in a domain without reference to some sortal. Thus, equating individuation with counting, it is instances

80. Wiggins (1967) p.1.

81. Ibid p. 41

82. Wiggins (1967) & (1980). For criticisms of this view see H.W. Noonan (1976) p. 559 & S. Shoemaker in M. Munitz (1971) p.103, Griffin has recently pointed out that in order to carry this last argument through Wiggins must amend the classical theory, which leaves him open to the claim that 'R' can be retained provided this argument is dropped. Griffin op cit p.19 & (1978) p.576.

83. Griffin (1977) p.21

84. Ibid p.131

of F's and G'S (where F and G are sortal constants), rather than items that are counted and so relative identities are regarded as fundamental. It is only after this individuation process is complete that one can establish whether an individual F and an individual G possess all properties in common and thus whether they are 'absolutely identical'.

It follows from Griffin's view that a proper name cannot be significantly assigned to an individuated item except by reference to a sortal because, he argues without the latter we have no means of knowing what we have named (85). This brings us on to the role of proper names in problems of identity and individuality and thus the second consequence of Frege's work.

We have noted how in the name-referent relation, Frege introduced a third element, the sense of the name. He believed that all names have senses and wrote 'A proper name expresses its sense, stands for or designates its reference ' (86). Thus he argued that names essentially have a sense but only contingently have a reference - they refer if and only if there is an object which satisfies their sense. This encapsulates the 'sense theory' of proper names which holds that proper names are only disguised definite descriptions. Every proper name is equivalent in meaning to such a description, namely the one which gives an explicit formulation of its sense.

The opposing view is the 'no-sense'theory' which argues that proper names simply stand for objects and have no sense or meaning other than standing for objects. Thus a proper name only denotes its bearer and names necessarily have a reference but no meaning whatsoever. Proper names are regarded as the special connection between words and the world.

One can summarise the difference between these two theories thus: on the no-sense theory naming is prior to describing, on the sense theory describing is prior to naming for^a name only names by describing the object it names. Representatives of the latter view include Frege, obviously, and Quine who used Russell's theory of singular

85. Ibid p.198

86. Frege op cit p.61

descriptions to reduce proper names to descriptions. In Quine's view Russell's theory shows clearly how one can meaningfully use seeming names without supposing that there exist the entities allegedly named, by reducing the seeming name to a description. And, 'where descriptions are concerned there is no longer any difficulty in affirming or denying being' (87). Quine believes that when Russell's theory goes to work on a statement of being or non-being such a statement '... ceases to contain any expression which even purports to name the alleged entity whose being is in question so that the meaningfulness of the statement no longer can be thought to presuppose that there be such an entity' (88). Thus by reducing proper names to descriptions in this way a statement of existence or non-existence does not contain any expression naming the entity whose existence is in question so that whether the statement means anything or not no longer presupposes that such an entity exists. Quine wrote 'We need no longer labour under the delusion that the meaningfulness of a statement containing a singular term presupposes an entity named by the term. A singular term need not name to be significant (88). As we have noted such a split between meaning and naming had been made previously by Frege. Thus Quine holds that proper names can ultimately be reduced to descriptions (and then removed à la Russell).

Advocates of the no-sense position include Wittgenstein and Mill. Wittgenstein believed that the meaning of a proper name is simply the object for which it stands and wrote 'A name means an object. The object is its meaning.' (90). He also contended that 'Only propositions have sense, only in the nexus of a proposition does a name have meaning.' (91)

87. W.O. Quine (1964) p.7; see also N. Goodman (1951) p.160 and A.J. Ayer (1954) p.16

88. Quine op cit p.7

89. Ibid p. 9

90. L.Wittgenstein (1963) p.23

91. Ibid p.25

Mill held the view that proper names have denotation but not connotation (92). For him a common noun like 'horse' had both a connotation and a denotation - it connotes those properties which would be specified in a definition of the word 'horse' and it denotes all horses. A proper name on the other hand only denotes its bearer.

Both views are open to criticism. It would seem to be intuitively obvious that proper names as they are ordinarily used are not equivalent to definite descriptions because to call something by its name is not to describe it. Naming prepares the way for describing, not vice versa. Not only do we not possess definitions of most proper names, it is unclear how such definitions could be obtained. If one tried to give a complete description of the object as the sense of the name a consequence would be that any true statement about the object that used the name as subject would be analytic and any false one would be self-contradictory. The meaning of the name and perhaps the identity of the object would change every time there was a change in the object, and the same name would have different meanings for different uses of the name.

Frege himself realized that different people may attribute a different sense to the same proper name (93). Furthermore if the Fregean view is being held that a single description can be substituted for a name, then it may not be clear which description out of a whole set of possible descriptions of the object in question should be so substituted. In particular care must be taken to ensure that a contingent property is not selected for this purpose. Thus, for example, it might be decided that the name 'Planck' means 'the person who first regarded energy as quantised'. But then this would imply that the statement 'Planck was the first person who regarded energy as quantised' is a tautology, which is counter-intuitive as it expresses something which, with a little historical investigation, could be shown to be false.

92. J.S. Mill (1843) Ch. 2

93. G. Frege (1949) p.86

Some philosophers have attempted to meet such objections by suggesting that a cluster of descriptions, rather than any single description, determines the referent of a name. (94). The referent of a particular name is then that object which satisfies most (in some sense) of the descriptions in this set. Strawson appears to believe that all the properties in the set should be given equal weight in determining the referent (95). This seems inherently implausible; surely some properties should count for more' than others? Even if this is accepted there are still problems with this view. It is still held that it is a necessary truth that an object, or a person, has the properties which are usually attributed to it or them. Thus Searle wrote '... it is a necessary fact that Aristotle has the logical sum inclusive disjunction, of properties commonly attributed to him ...' (96). Again this seems intuitively implausible. Surely it is possible for Aristotle not to have been the teacher of Alexander? If this question is answered in the affirmative then one must provide some justification for referring to this person who has not the teacher of Alexander as 'Aristotle'.

Kripke has attempted to do this in terms of his notion of a 'rigid designator'. A rigid designator is something which designates the same object in every possible world, whereas a non-rigid, or accidental designator is something for which this is not the case (97). Kripke believes that proper names are rigid designators because they satisfy this 'intuitive test'. Thus he wrote '... although someone other than the U.S. President in 1970 might have been the U.S. President in 1970 (e.g. Humphrey might have) no one other than Nixon might have been Nixon'. (98) and also '...although the man (Nixon) might not have been the President it is not

94. J.R. Searle (1958) p. 166-173.

95. P.F. Strawson (1964) p. 191-192

96. Searle (1969) p.173

97. S. Kripke (1980) p. 48 and also in Munitz op cit p. 135

98. Ibid p. 49

the case that he might not have been Nixon (though he might not have been called 'Nixon'). (99)

We can fix the reference of a proper name by a property, essential or accidental, in this, the actual world. The proper name then rigidly designates the object possessing that property in the actual world, in all possible worlds (100). Even if, in these counterfactual situations, the object no longer possesses the property used to fix the reference in the actual world, the proper name still rigidly designates that object. In other words once its reference is fixed in the actual world the designation of the proper name is unaffected by possible changes in the actual world which we can imagine (101).

Thus we may pick out the reference of 'Aristotle' in this, the actual world as the man who taught Alexander. In a possible world in which someone else gained this position we would not refer to that someone else as 'Aristotle'. If Aristotle had never become a scholar he would not have possessed this property which we use in the actual world to fix the reference of his name. This is precisely because the proper name 'Aristotle' rigidly designates a certain man. Eventhough the property we have used to fix the reference of the name is an accidental one we still use the name to designate the man in all possible worlds.

Having outlined some of the criticisms of the sense theory, we shall now briefly consider some possible failings of the no-sense approach.

Firstly, as Frege demonstrated this view does not account for the occurrence of proper names in identity statements that convey information. Secondly, Quine showed that the theory is unable to give a decent account of the occurrence of proper names in existential statements. Thus in 'Pegasus does not exist' the proper name cannot be said to refer because no subject of an existential statement can really refer. Existence is not a predicate. An affirmative existential statement expresses a concept and states that it is instantiated, rather than referring to an object and

99. Ibid p.49

100. We shall not discuss the question of the status of possible worlds. See D.K. Lewis (1968) p. 114, and (1973).

101. Kripke (1980) p.75

stating that it exists. So if a proper name occurs in an existential statement it must have some conceptual or descriptive content. But if so, then this lends support to the Fregean view because the descriptive content could only be the sense of the proper name.

The nature of the existence of the referents of proper names also raises certain problems. A change such as the destruction of an object cannot destroy the meaning of words, because any change in the world must still be describable by words. Thus if one adopts Wittgenstein's position the existence of those objects which are named by genuine proper names cannot be an ordinary contingent fact. This then forces one into the view that there is a class of objects in the world possessing a necessary existence - those objects which are the meaning of the real proper names. But if this is accepted then it would make no sense to assert or deny the existence of the objects named by genuine proper names.

Thus the no-sense view, although supported by our initial intuitive feelings as to how proper names work, runs into difficulties with the existence of referents and identity and existential statements (102).

Although it seems implausible to claim *that* proper names are merely a kind of shorthand for definite descriptions, the sense theory does at least give an account of the latter kinds of statements.

Strawson adopts a version of the Fregean theory as we have said, and discusses proper names with particular regard to individuality. (103) He argues that the fundamental presupposition underlying the use of names is that in our linguistic and conceptual structure we presuppose that all particulars to which we refer are individuals in the sense that they are unique. And they are unique because there is something about each particular that is true of it and nothing else. Particulars which are sensibly present can, he believes, be demonstratively identified.

102. Yagisawa has recently tried to get round these difficulties, and also solve Fregean paradoxes without using disguised descriptions, by regarding proper names as bound variables. See T. Yagisawa (1984) p.195

103. Strawson op cit p.20 ff.

Particulars which are not sensibly present cannot be identified simply by names because '... it is no good using a name for particular unless one knows who or what is being referred to by the use of the name.' (104) Descriptions in general terms alone are also rejected on the grounds that there is no guarantee that such a description is unique and the possibility of massive reduplication remains open,

Strawson's solution is that a particular which cannot itself be demonstratively identified can be '... identified by a description which relates it uniquely to another which can be demonstrably identified.'

(105). Thus a linguistic chain is set up in which particulars which are not sensibly present are uniquely related to those which are and which can be pointed to (verbally and/or physically) by the speaker.

It might be asked if there are any basic particulars or if they are all on an equal standing in this chain. Strawson's answer is that material objects are the basic particulars. The general conditions of particular identification require a unified system of spatio-temporal entities that can be publicly observed. Such a system is formed by the material universe and material objects can therefore be identified independently of the identification of particulars in other categories but the latter particulars cannot be identified without reference to material objects (106).

Quinton has put forward a view which is similar to Strawson's in the respect that things are individuated by demonstrably referring them to the 'here and now' (107). The basic argument here is that individuals owe their unique individuality necessarily to their position in space and time (108). Qualitatively indiscernible distinct things can always be distinguished by reference to their respective spatial positions. The proof of this, Quinton believes, lies in the metaphysical truth that no two things can be in the same place at the same time.

104. Ibid p.20

105. Ibid p. 21

106. Ibid p.39

107. Quinton op cit p. 12-20

108. Ibid p.17. Similarly Goodman has proposed that place & time be considered as essential individuator. See Goodman op cit.

(109). In other words individuals are impenetrable. Thus Quinton's view of individuality is crucially dependent upon the Impenetrability Assumption. As we shall see in subsequent chapters there are things described by the laws of physics to which this assumption does not apply, and to this extent Quinton's scheme is fatally flawed, (as indeed are all others which try to argue that spatio-temporal location is a necessary individuating property. There can be no guarantee that the points of space-time are either vacant or only uniquely occupied and indeed a case can be made for saying this is precisely what is not true in quantum physics).

Thus, according to Quinton to state the spatio-temporal position of a thing is to predicate a conjunction of properties of it and is necessarily to individuate it (110). This would seem to support the 'bundle theorists' claim that a thing is simply the collection of its properties sans substance, because one can now solve the problem of individuation by including position in this collection. However, it is not quite so straight forward as Quinton further believes that positional predicates are different from other properties in the sense that '... to ascribe a position to an individual in space or time involves an essential and ineliminable reference to another individual or position (111). Thus again a kind of individuating chain is set up relating individuals to other individuals and thus transforming them into more than mere bundles of properties.

Of course an infinite regress can only be avoided if the chain ends with some basic or primary positional predicate which specifies an individual's unique position in space and time. Such a primary positional predicate is, according to Quinton, provided by the 'here and now'. Thus he writes 'The position where I am at the present moment is, then, the absolute point of origin of all my positional characterisations of things. It is the one

109. Ibid p.17

110. Ibid p.17

111. Ibid p. 17-18

position I do not have to pick out by its relation to something else and by their relation to which in the end everything else is individuated. The linguistic correlate of this absolute point of origin or absolute position is the demonstrative part of language, such terms as 'here', 'now', 'this', and 'I', which require for their understanding in any particular employment a knowledge of the context in which they are uttered. There is a demonstrative aspect to the understanding of every singular term that purports to refer to a unique individual. It may be practically sufficient to explain a singular term to someone by enumerating the properties of its bearer but misidentification can be ruled out conclusively only by bringing in the position of the thing in question. But any reference to position that is sufficient to individuate must connect it to the here-and-now.' (112).

One can easily criticise this view on the grounds that it implies a commitment to a theory of absolute space, using again our Blackian universe in which there exists two things which have all their qualities in common and yet which are distinct. The only thing which can individuate them, according to Quinton, is their spatial positions. However, their spatial relations are only to each other and are identical. Thus if spatial position is the principle of individuation then the two things cannot be individuated on a relational view of space. Therefore, if Quinton is to maintain his position he must also hold an absolute theory of space. An absolute spatial substratum will provide the difference in positional predicates necessary to individuate the things.

We shall consider this problem again in Chapter Four, but now we turn our attention to the problem of ^{the} reidentification of individuals through time.

1.5 Reidentification Through Time

This problem arises from the obvious fact that both the spatial positions and certain, accidental, qualities of individuals change with time. How then is

the individual to be reidentified through these changes? Whether we adopt the T.I. (113) or S.T. view of individuality this problem remains, namely, what grounds can be supplied for claiming that an individual b at time t_2 is the same individual a at an earlier time t_1 ?

The most obvious solution is in terms of the spatio-temporal trajectory joining the location of a at time t_1 with the location of b at time t_2 . Thus Quinton writes 'Two things can be identical through time ... only if there is a continuous spatial path between the places at which they occur. (114)

Shoemaker has also noted that spatio-temporal continuity is a logically necessary condition of 'identity through time' (115) and Coburn has given a detailed analysis of this criterion in terms of the set theoretic concept of the linear continuum (116).

Although necessary this condition is not sufficient, because there are an infinite number of continuous spatio-temporal paths between any two points in space-time separated by a non-vanishing interval. This leads to the suggestion that a criterion of qualitative continuity is required. Thus a necessary condition for a particular space-time path to correspond to the career of some individual is that any individual-stage on the path should be qualitatively similar to a neighbouring, or adjacent, individual stage on the path (117). This is a very vague condition, the vagueness residing principally in the word 'similar', but it is plausible. As it stands this condition limits the range of admissible qualitative variation of the individual. Thus its acceptance implies the rejection of the assumption that an individual can retain its identity whatever qualitative changes it undergoes. Drastic qualitative

113. Shoemaker has discussed the connection between the substantivalist approach and the analyzability of identity statements. S. Shoemaker (1963) p. 57-63 and p. 254-260.

114. Quinton op cit p. 66. See also Strawson op cit p.57

115. Shoemaker op cit p. 4-5. See also B.Williams (1956-57) p. 230

116. Coburn in Munitz op cit p. 51-103.

117. Quinton op cit p. 67 Also see E. Hirsch (1982) p.10-15.

changes are effectively forbidden.

However spatio-temporal and qualitative continuity are still not enough. The analysis given so far would not prevent us from tracing an individual in such a way that we combine stages of the individual with stages of some of its parts. (118) Thus a further criterion is needed. This is provided by the so called 'Sortal Constraint' which states that for a succession of individual stages to correspond to a persisting individual there must be some sortal term S such that every individual stage in the succession comes under, or is an instance of, S . (119) It is not our intention to discuss what a sortal, or substantial term is. Wiggins has given a very detailed analysis (120) as has Hirsch who defines a sortal in terms of the above constraint. Thus he gives the following definition :

'The general term F is a sortal' means: It is a conceptual truth (a rule of language) that any spatio-temporally and qualitatively continuous succession of F -stages corresponds to (what counts as) stages in the career of a single persisting F -thing.' (121). Examples of sortals include nouns like car, tree or mountain, but not adjectives like green or hard. However, it is difficult to establish a clear distinction between what does and what does not count as a sortal term, as Hirsch demonstrates.

Underlying the sortal constraint is the less exact but conceptually more basic idea that for a succession of thing stages to correspond to stages in the career of a single persisting thing the succession must minimize change, except of course mere change in location (122). An objects career should be traced by following a spatio-temporally and qualitatively continuous path which minimises change as much as possible.

Thus we have the following three conditions which are necessary for a space-time path to correspond to the space-time path or world line, of λ^a single persisting individual or for a succession of individual stages to correspond to the career of a single persisting individual.

118. Hirsch Ibid p. 32-33

119. Quinton op cit p. 67. Hirsch op cit. p. 36

120. Wiggins (1967) p. 35 ff

121. Hirsch p. 37-38

122. Hirsch ibid p. 72-82

- 1) the path or succession must be spatio-temporally continuous.
- 2) the path or succession must be qualitatively continuous in the sense given above, and
- 3) there is a sortal term S such that the succession is a succession of S-stages, or the path underlies such a succession. (123)

These are still only necessary and not sufficient conditions as there are further criteria which may be required. For example a compositional criteria might be needed to deal with cases such as that of a watch which is taken apart, has some parts replaced and is then reassembled again. More importantly from the point of view of this thesis, two further conditions have been suggested, in addition to the three above, as being necessary for the identity of a thing. These are compositional continuity, which states that the matter of which a thing is composed at one moment should be the same, or almost the same, as the matter of which it is composed at the next moment, and the causality condition which holds that there must be a causal relationship between the states of a thing at successive moments (124)

The first of these extra criteria can be rejected on the grounds that it assumes the acceptance of a concept of matter, a concept which modern physics has to a certain extent cast doubt upon (125). The second can not be eliminated quite so easily. Hirsch thinks it can but his counter examples are at best borderline cases. For example he believes that successive stages of a shadow are causally independent because each stage is the effect of the body which has the shadow and not an effect of previous stages. (126) However, while the shadow-stages are not directly causally related they are indirectly so related through the body casting the shadow. There is a direct causal relationship between the body and a shadow stage and therefore an indirect relationship between each

123. Quinton op cit p. 67; Hirsch op cit p. 36.

124. Shoemaker in French, Uehling and Wettstein (1979) p. 326.

125. Hirsch op cit p. 217

126. Ibid p.219

stage. In the case of the identity through time of a classical elementary particle there is certainly a causal link between successive particle-stages as expressed by the laws of motion. Indeed these laws embody both causal and spatio-temporal continuity.

The situation is not quite so transparent in the case of quantum physics as here there is some doubt about whether the fundamental laws describing the motion of particles could be described as causal or not. Electron transitions within the atom might be candidates for events violating this causality requirements, but given the Q.F.T. description of such events in terms of particle creation and annihilation, they could just as easily be regarded as violations of the criterion of spatio-temporal continuity also. In this case it is not possible to refer to 'the same' electron before and after the transition. We shall examine these points in greater detail in Chapters Two and Three.

Thus we conclude that b at time t_2 is the same individual as a at time t_1 , if we can trace a spatio-temporally and qualitatively continuous succession of stages between a and b which all fall under some sortal term thus minimizing change. Of course if circularity is to be avoided the stages themselves must not be extended in time. The S-Stages must be momentary in the sense that they themselves do not depend on any criteria for reidentification through time. (127)

This concludes our account of the problems of individuality, identity and reidentification. The point we wish to emphasise is that historically, as we have seen, there were two main candidates proposed for the role of individuator. The first is an unknown and unknowable 'something', Aristotelian prime matter, or Lockean substance, which supports the properties possessed by an individual. The other is location in space and time, with the individual reidentified through time by its continuous spatio-temporal trajectory. Proponents of the former view might agree as we have said that

reidentifiability can be inferred by spatio-temporally continuity but that it is conferred, ontologically speaking, by the persistent substantial substratum. In the following chapter we shall examine the extent to which the individuality of elementary particles, as described by classical physics, can be analysed in terms of the above two positions.

CHAPTER TWO

IDENTITY AND INDIVIDUALITY IN CLASSICAL
PHYSICS

2.1 Particles.v.Fields

Classical physics can be regarded as comprising two kinds of theories : particle theories and field theories. The distinction between the two is most obvious when one considers the question of matter and its interactions. Thus one account holds that matter is composed of particles which interact via fields of force. Various alternative views have, of course, been propounded and vigorously supported, such as the Boscovichian 'field' theory of matter and the 'action at a distance' view of interactions (usually associated with a 'pure' particle theory of matter), (1) and we acknowledge that the distinction as expressed here is a crude one.

As it stands it is clearly linked to a difference in the underlying ontologies of the theories; in particular as regards their fundamental individuals. A field theory can be defined as one in which certain properties are associated with every point of space-time. There are then two possibilities. One is to argue that the field is ontologically independent of these points, in much the same way as particles can be said to be, but with some correspondence existing between the points and the field quantities. Fields are then granted the status of independent, real entities which, furthermore, may be regarded as substantial if the term 'substance' is taken to mean that which possesses, or is the repository of, energy. Thus it can be argued that fields exhibit T.I.

The alternative view is to say that, on the contrary, fields are not independent of the points of space-time since they are nothing more than properties of these points. On this interpretation it is the space-time points

1. For a good account of these various theories see M. Hesse (1961)

which are the primary individuals and which stand in the same relation to the fields as a subject does ^{to} its predicates. We shall discuss the individuality of classical fields in more detail in section 2.4 of this chapter.

A particle theory, on the other hand, is one in which various properties, including spatio-temporal ones, may be attributed to certain individuals, the particles. These characterisations bring out the different role which the points of space-time may play in particle and field theories respectively. In one interpretation of the latter, the second given above, they are the individuals possessing certain properties whereas in the former they are regarded as the properties possessed by individuals.

This distinction between particle and field theories does not, however, uniquely and rigidly classify the subject matter of classical physics since it can be shown that the classical particle mechanics of equal point masses can, in fact, be rewritten in terms of fields. (2) This underdetermination (3) of the particle and field interpretations is obviously accompanied by a similar ontological 'underdetermination' in the sense that one can regard either the particles, considered to be substantial, or the points of space-time as one's primary individuals. This 'underdetermination' as regards individuality which, we shall argue, occurs in both classical and quantum physics, represents our central thesis.

The realisation that a particle theory could always be reformulated in terms of fields occurred very early in the history of classical physics in the work of Newton. (4) The theory of particle mechanics associates with each particle a set of properties,

2. M. Redhead (1983) Section 2.

3. See W.O. Quine (1970) p.179 and, in particular, W. Newton-Smith (1980) p. 68-73 and p. 231-242, for a good discussion of this concept. We shall examine it in more detail in section 5.4 of Chapter Four.

4. I. Newton in H.R. Hall and M.B. Hall (1962) p.138-140.

including their positions and momenta which can be represented as the particle's location in a 6-dimensional phase space. The dynamical state of a system of N particles can then be represented in $6N$ -dimensional phase space with $q_i, p_i, i=1, \dots, N$, as the coordinates and momenta respectively. Such a theory can be rewritten in field theoretic terms in the following way. (5)

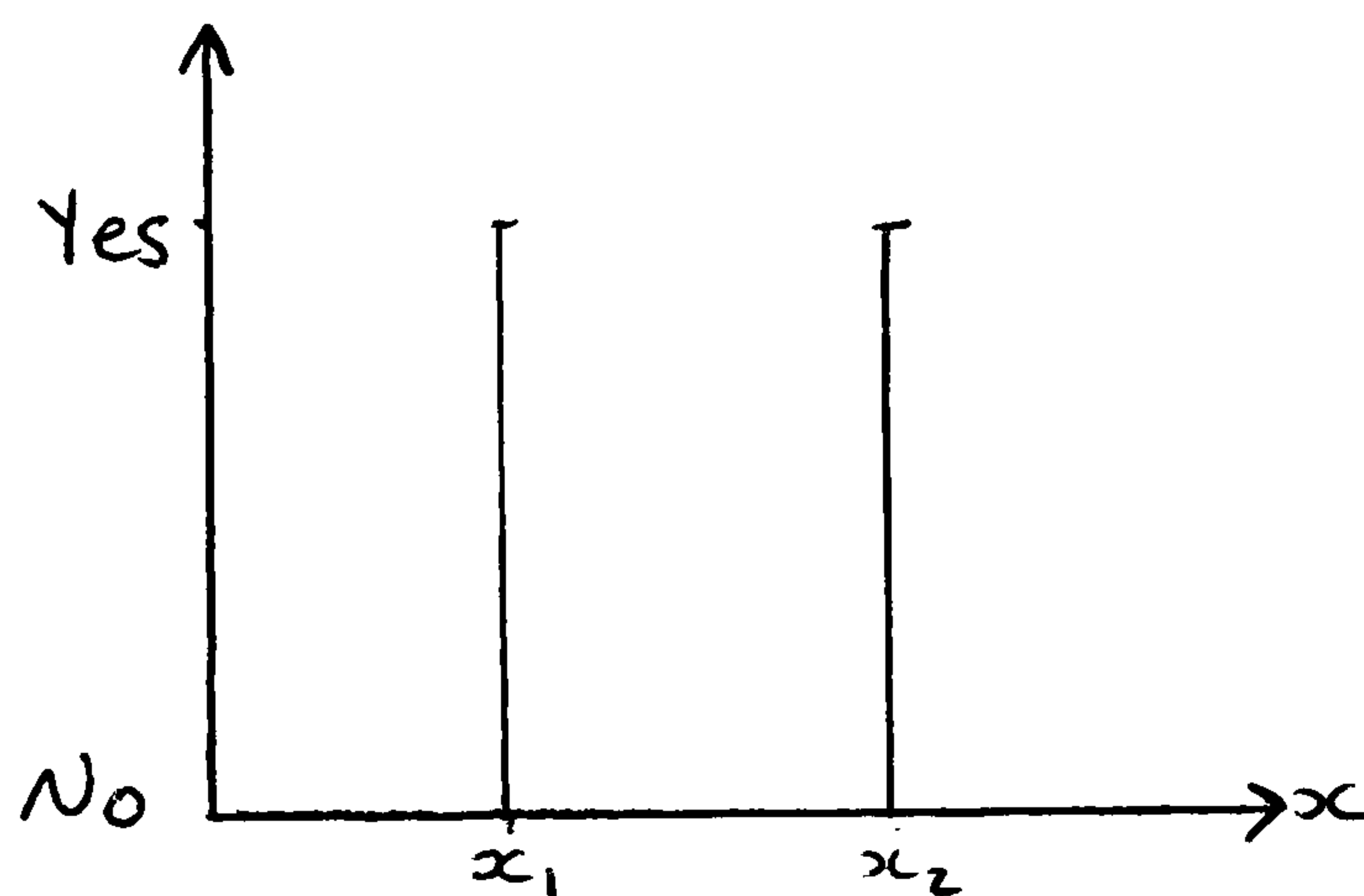
A particle can be represented by a dichotomic field $\psi(\underline{r}, t)$ which can possess only two values: $\{ \text{Yes, No} \}$. The value 'Yes' indicates that a particle is present at that particular space-time location and 'No' indicates that a particle is not present. Newton identified 'particle' in this field formulation with the property of impenetrability. The possession of the value 'Yes' by the field at the point (\underline{r}, t) simply meant that this point had associated with it the property of impenetrability, in the sense that no other 'particle' could occupy this point.

Thus, if the field has the value 'Yes' at a point (\underline{r}, t) then this is taken to mean that there is a particle disturbance at this point. However, a field is only completely specified if the field amplitude at all points is specified (unlike the particle case, where spatial locations other than the one at which the particle is observed are not considered). Thus if the field has the value 'Yes' at some point then it must be specified to have the value 'No' everywhere else. A particle will therefore be represented by an infinitesimally thin, sharp 'spike' in the field and its motion will be translated into motion of this spike in the field.

If we consider for simplicity motion in one dimension labelled x , then this can be discussed in terms of 'blip space' which can be represented graphically by taking the x -axis as the real number line and y -axis as having only two values, 'Yes' and 'No'. Our dichotomic field can now be represented by a mathematical function $F(x)$ whose domain is the real numbers

and whose range is the two-element set $\{\text{Yes}, \text{No}\}$;
 i.e. $F: \mathbb{R} \rightarrow \{\text{Yes}, \text{No}\}$; $F(x)$; $\text{dom } F = \mathbb{R}^3$, $\text{Range } F = \{\text{Yes}, \text{No}\}$
 In the particle description motion is represented in terms of the particle at time t_1 being at location x_1 and moving at time t_2 to location x_2 . In terms of the dichotomic field in 'blip space' this can be represented as a change in field configuration from the spike at x_1 to the spike at x_2 (see Fig.1.)

In other words the movement of particles in space-time can be reformulated as the movement of spikes or 'blips' in the dichotomic field in 'blip space'. In the particle description an individual, the particle, has moved from x_1 at t_1 to x_2 at t_2 . In the dichotomic field representation some property, impenetrability say, has been passed from location x_1 at t_1 to location x_2 at t_2 .

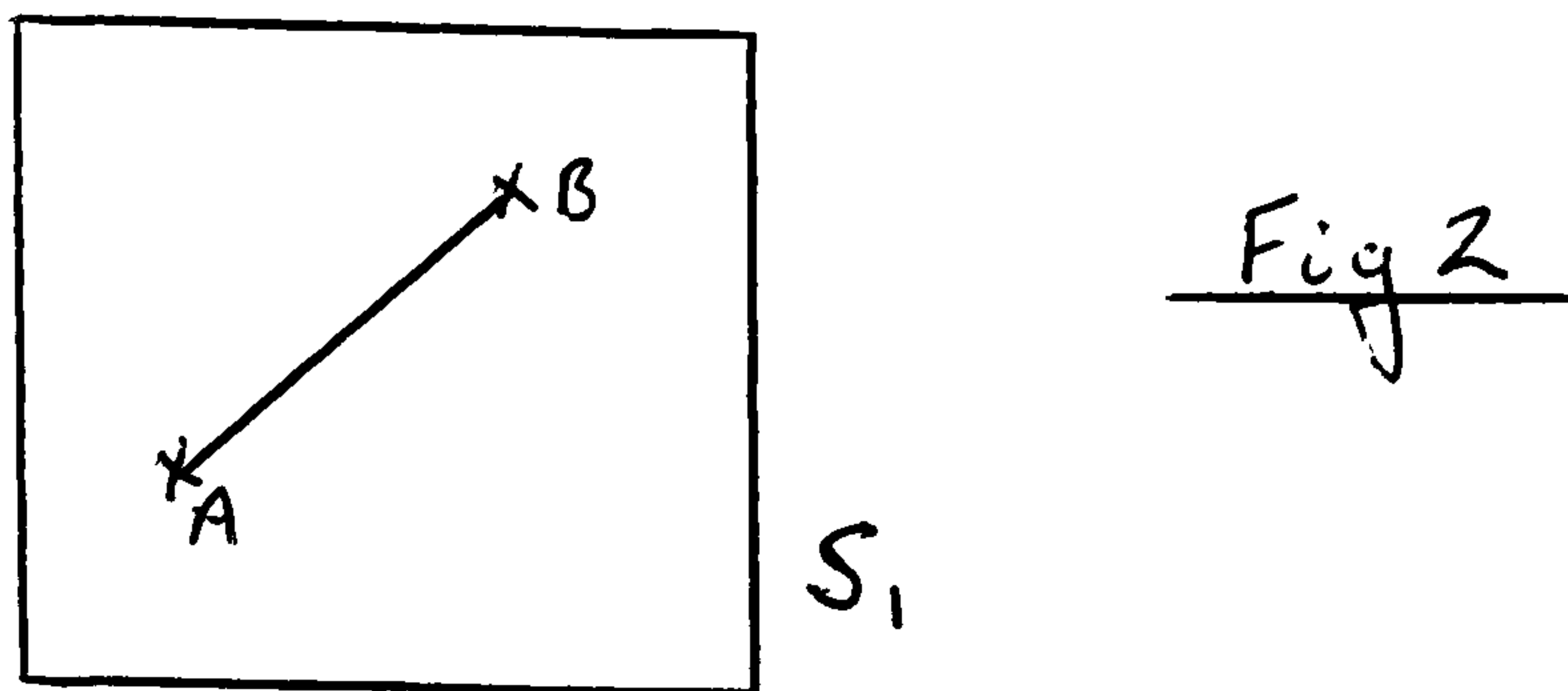


This is a typical field description and just as Quantum Field Theory (QFT) describes the creation and annihilation of particles in a very 'natural' way so this description allows one to introduce these concepts into classical physics also. This is done by introducing Fock space into classical mechanics, in which the motion of particles is described in terms of creation and annihilation operators operating on the spikes in the dichotomic field (6)

We consider a single classical particle, the time development of the state of which takes place in a 6-dimensional phase space S_1 . The motion of the

6. See M. Redhead, *Ibid*, Section 3, for a discussion of the use of Fock space in QFT and Section 4, for its application to classical particle mechanics.

particle from A to B in S_1 can be illustrated thus:



Adapting the dichotomic field description above we can represent the field as the limiting case of a continuous distribution $\rho(p_i, q_i; t)$ where p_i and q_i are the momentum and position coordinates of the particle which is strongly peaked around the location of the particle in S_1 . The time development of ρ specifies a transformation which maps A in S_1 onto point B, and is determined by Liouville's equation,

$$\frac{\partial \rho}{\partial t} = \sum_i \left(\frac{\partial H}{\partial q_i} \frac{\partial \rho}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial \rho}{\partial q_i} \right) \quad 1.$$

where $H(p_i, q_i)$ is the Hamiltonian function.

An operator Q can be introduced to represent this transformation such that $Q \rho(A) = \rho(B)$

If we now introduce the Liouville operator, defined by

$$L = \sum_i \left(-i \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} + i \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right) \quad 2.$$

we obtain

$$i \frac{\partial \rho}{\partial t} = L \rho \quad 3.$$

This equation can be integrated to give

$$\rho(t) = e^{-iL(t-t_0)} \rho(t_0) \quad 4.$$

showing the value $\rho(t)$ of ρ at time t as produced by the action of the operator $e^{-iL(t-t_0)}$

on the value $\rho(t_0)$ of ρ at time t_0 . Thus we have

$$Q \equiv e^{-iL(t-t_0)} \quad (7). \quad 5.$$

We can illustrate the movement of the particle from A to B, or the passing of the value of the distribution ρ from $\rho(t_0)$ to $\rho(t)$ in 'blip' space, thus:

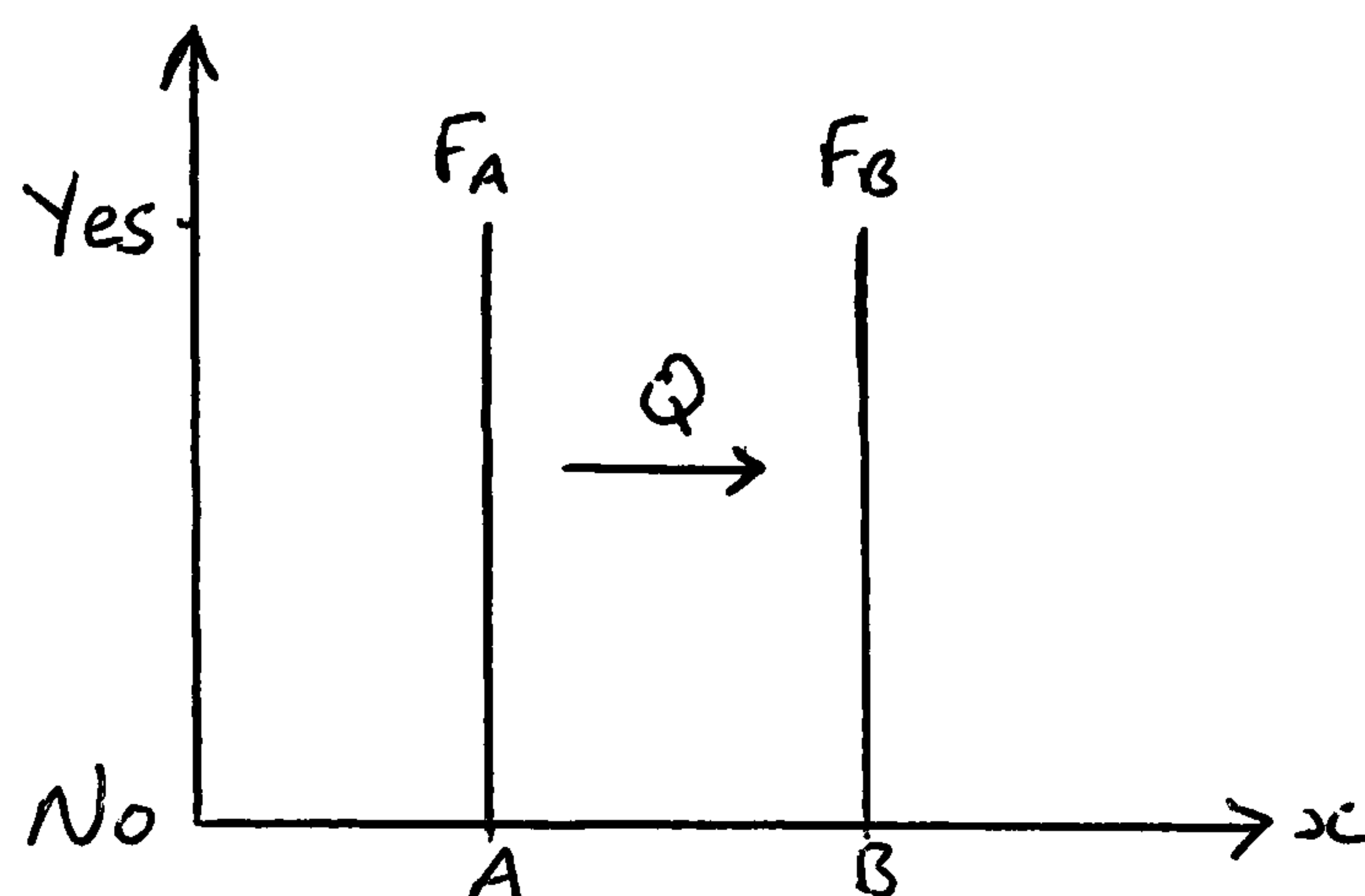


Fig 3

In the Fock space formulation of second quantisation the quantum mechanical operators can be decomposed into creation and annihilation operators. Likewise we can formally factorize our transformation operator Q

$$e^{-iH(t-t_0)} = \beta \cdot \alpha$$

$$\text{or } Q = \beta \cdot \alpha \quad 6.$$

where α is an annihilation operator which takes us from the initial field $\rho(t_0)$ to the vacuum, defined by $\rho=0$, and β is a creation operator, which recreates the particle from the vacuum in the final state $\rho(t)$. In other words α annihilates the blip at A and transforms it from the space S_1 to the Space S_0 , the vacuum state. β transforms the blip from S_0 back to S_1 and recreates it at point B.

Thus the motion of the particle from A at t_0 to B at t can be decomposed into two parts: first the particle is annihilated at A, second it is re-created at B. We can represent the whole process thus:

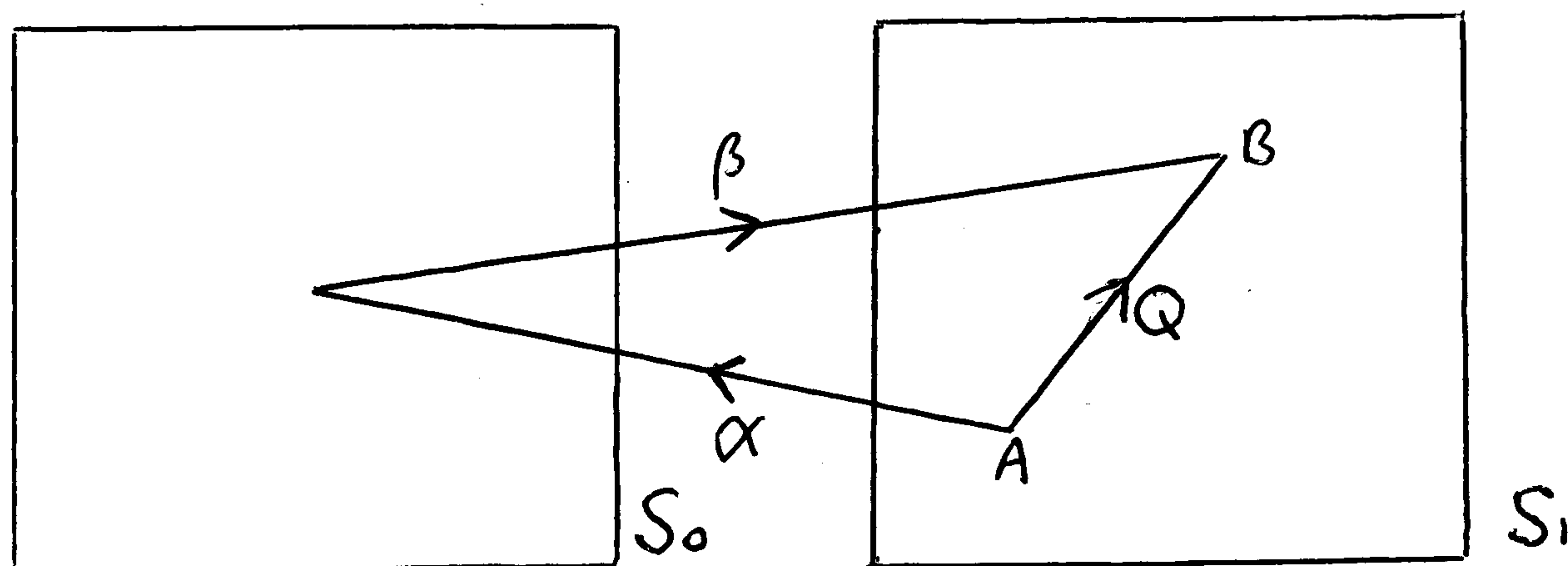


Fig 4

This procedure can be generalised to take account of more than ^{one} particle and in general the transformation operator Q can be written as a product of many creation and annihilation operators.

$$Q = \beta_N \dots \beta_1 \alpha_N \dots \alpha_1 \quad 7.$$

where N is the number of particles. The phase space in which this transformation takes place is then decomposed into the union of several phase spaces, of smaller dimension:

$$\mathcal{S} = S_0 + S_1 + S_2 + \dots + S_N \quad 8.$$

This is the classical Fock space in which the creation and annihilation of 'particles' takes place.

Returning to our single particle moving from A at time t_0 to B at time t , one can obviously regard the interval between t_0 and t as being composed of a denumerable infinity of instances $t_1, t_2, \dots, t_i, \dots$

The above description in terms of creation and annihilation operators can then be introduced for each successive pair of instants (t_i, t_{i+1}) . Our description therefore consists essentially in the claim that the particle is being destroyed at every instant t_i and recreated at the successive instant t_{i+1} .

This description is remarkably similar to the view of a group of 10th Century Islamic philosophers, called the Mutakallimim (8). The fundamental proposition of their system was that all things are composed of atoms which were indivisible, point like and indistinguishable.

(9) The number of atoms was not regarded as constant as the Mutakallimim believed they could be created and that their annihilation was therefore not impossible. (10). The motion of these atoms, which gives rise to change, together with a version of the Impenetrability Assumption (11), compelled them to postulate the

8. M. Maimonides (1904) p.120-133

9. Ibid p. 120

10. However no specific arguments were given for the reciprocity of these two processes, see J. Bromberg (1977) p.147-157.

11. Maimonides op cit p. 121

existence of a vacuum, devoid of all substance, in which the atoms move about.

The third proposition of their system was perhaps the most unorthodox of all : time, they believed, was discrete, composed of indivisible 'time elements'. This notion of 'time atoms' was held to be a logical consequence of the first proposition, and it was also believed that the atomicity of space then necessarily followed from that of time. (12)

One can even obtain an estimate of the size of one of these time-atoms: 'An hour is e.g. divided into sixty minutes, the minute into sixty seconds, the second into sixty parts, and so on; at last after ten or more successive divisions by sixty, time elements are obtained ...' (13). This gives a maximum size, or rather duration, for a time-atom of $1/(60)^{10}$ of a second or 1.65×10^{-8} s !

The main conclusion which the Mutakallemim drew from these three propositions concerned the nature of motion. This, they believed, consisted in the translation of each atom of a body from one space element and time atom to the next. (14) Motion was thus explained in terms of a series of momentary leaps, each atom occupying in succession different individual space - and time-atoms. Physical motion was thus reduced to a discontinuous process.

Furthermore, since by the sixth proposition 'Accidents do not continue in existence during two time atoms' (15) this process is one of creation and annihilation. The essential characteristic of an accident was believed to be in incapability of enduring over two time atoms. Immediately (16) after its creation it is utterly destroyed, and another accident of the same kind created, this process continuing for as long as God pleases.

12. Ibid p.121

13. Ibid p.121

14. Ibid p.121

15. Ibid p.120 and p.124

16. It is difficult to reconcile the term 'immediately' with the belief in time atoms. It would perhaps be more consistent to say that an accident persisted during a time atom but was then destroyed about 1.65×10^{-8} of a second after its creation.

Thus we see a remarkable parallel between the views of the Mutakallemim and those of modern physics. The Islamic scholars also believed that atoms were indistinguishable, variable in number, and could be created and ^dannihilated. Space and time were regarded as discrete and motion was considered to be a process of continuous creation and annihilation. The similarities with the fundamental precepts of QFT are quite striking!

It is not our intention here to examine the history of creation and annihilation processes, although it is worth noting the point that this history did not begin with Dirac's theory of electrons. (17) As Bromberg has demonstrated (18) the *concept* of the annihilation of pairs of oppositely charged elementary particles dates from the turn of the 20th century when it was offered as an explanation of radioactivity. (19) This process subsequently became important in astrophysics where it was posited as a possible source of stellar energy (20), and was thus regarded as a familiar and acceptable concept by quantum theorists. The advent of quantum mechanics did change the situation in so far as prior to its development a belief in annihilation did not entail a belief in creation whereas afterwards it did. The great value of Dirac's work, according to Bromberg, was to substitute a math^eatically precise handling of these processes for qualitative speculations. We shall return to some of these points in section 3.8 of Chapter Three.

Thus we conclude that particle mechanics can be re-written as a field theory and there is no observable difference between these two descriptions in the sense that no experiment will ever decide which is correct. This conclusion will be used in Section 2.2.2. to show that it is not necessary, in classical physics, to

17. P.A. Dirac (1927) p. 243-265

18. Bromberg op cit.

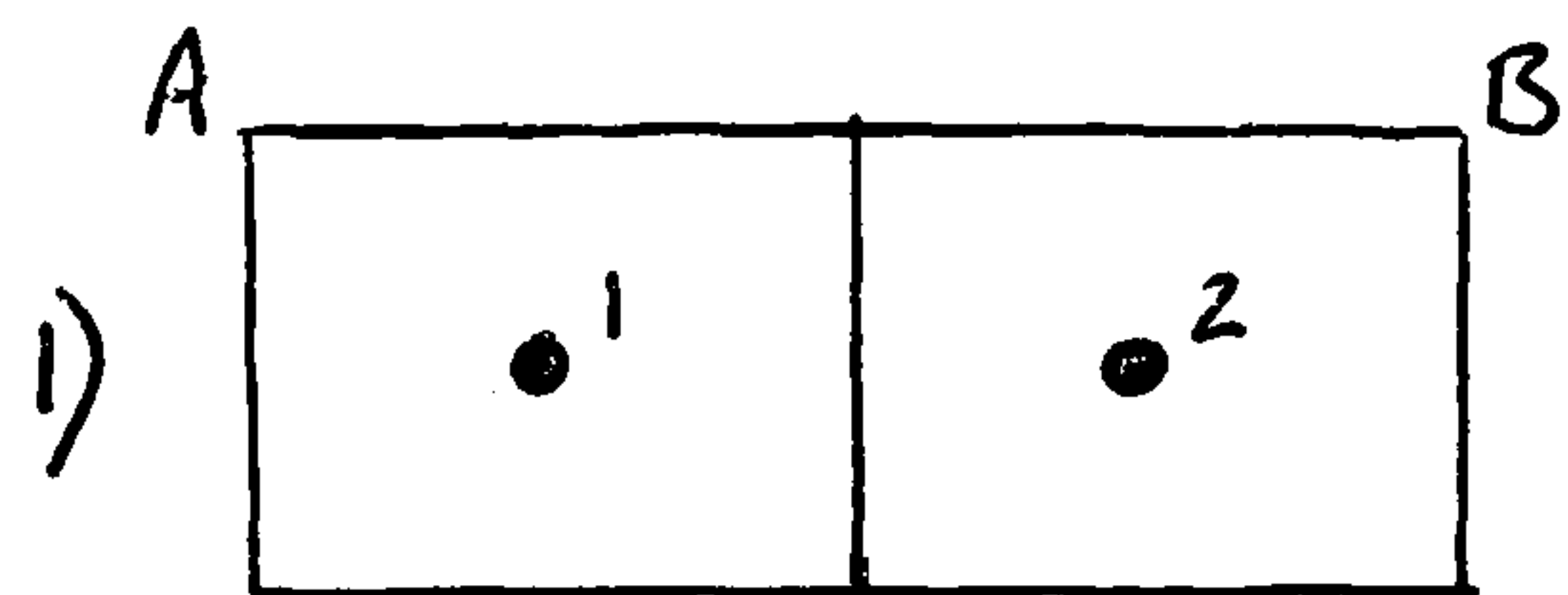
19. J. Jeans (1904) p.101

20. Eddington (1921) p. 351

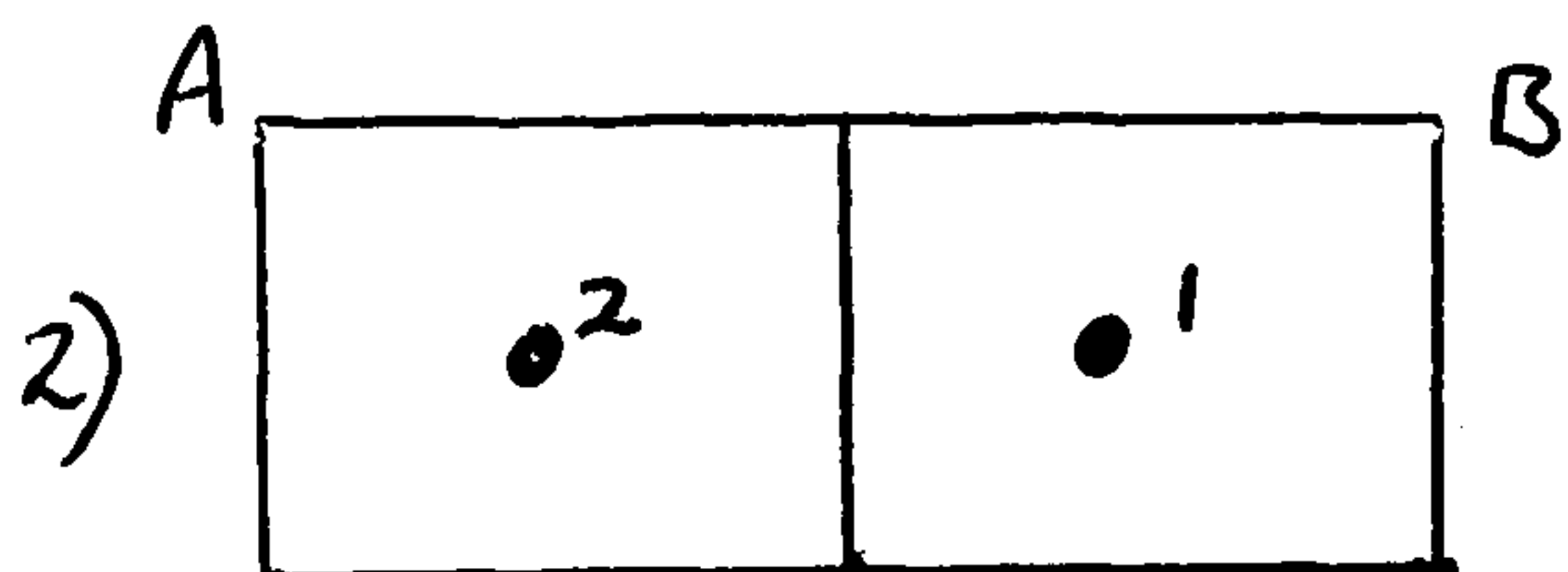
regard the individuality of particles in terms of (Transcendental Individuality'. Before we do this, however, we must consider this view as it applies to classical particles.

2.2 Individuality In Classical Statistical Mechanics

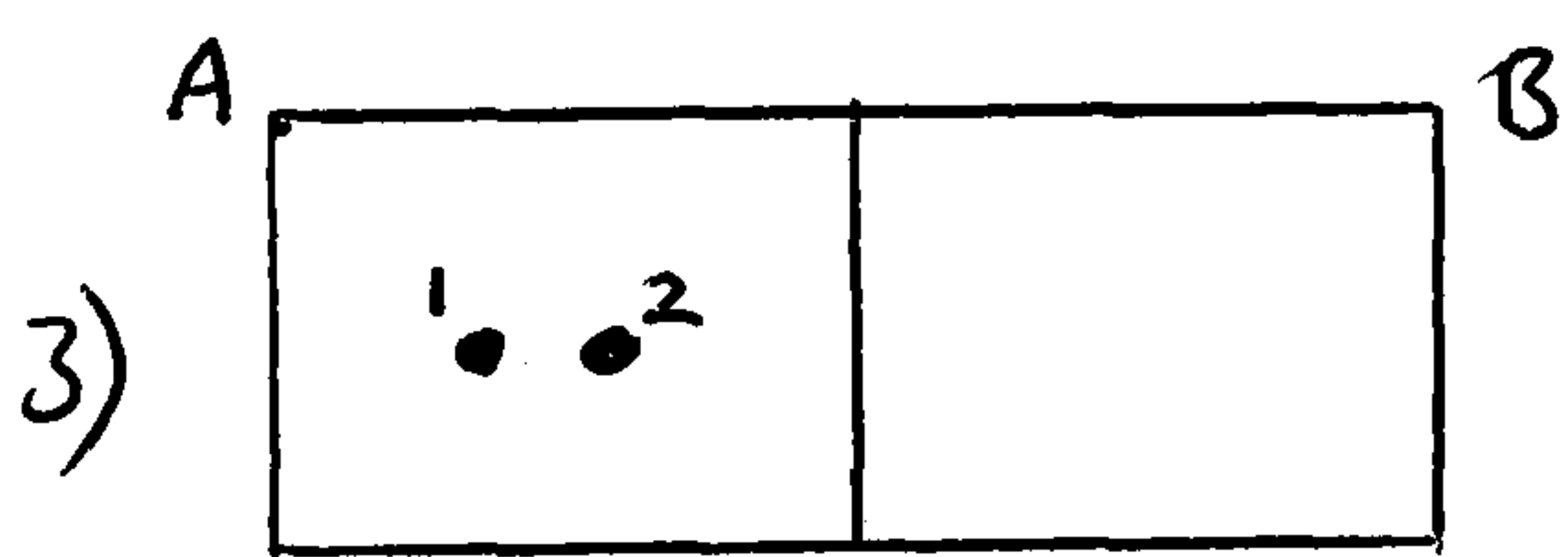
We begin by considering the distribution of two indistinguishable (in the sense of possessing the same set of intrinsic properties, such as mass, charge etc.) particles over two one particle energy states. In classical statistical mechanics the following four arrangements would be produced:



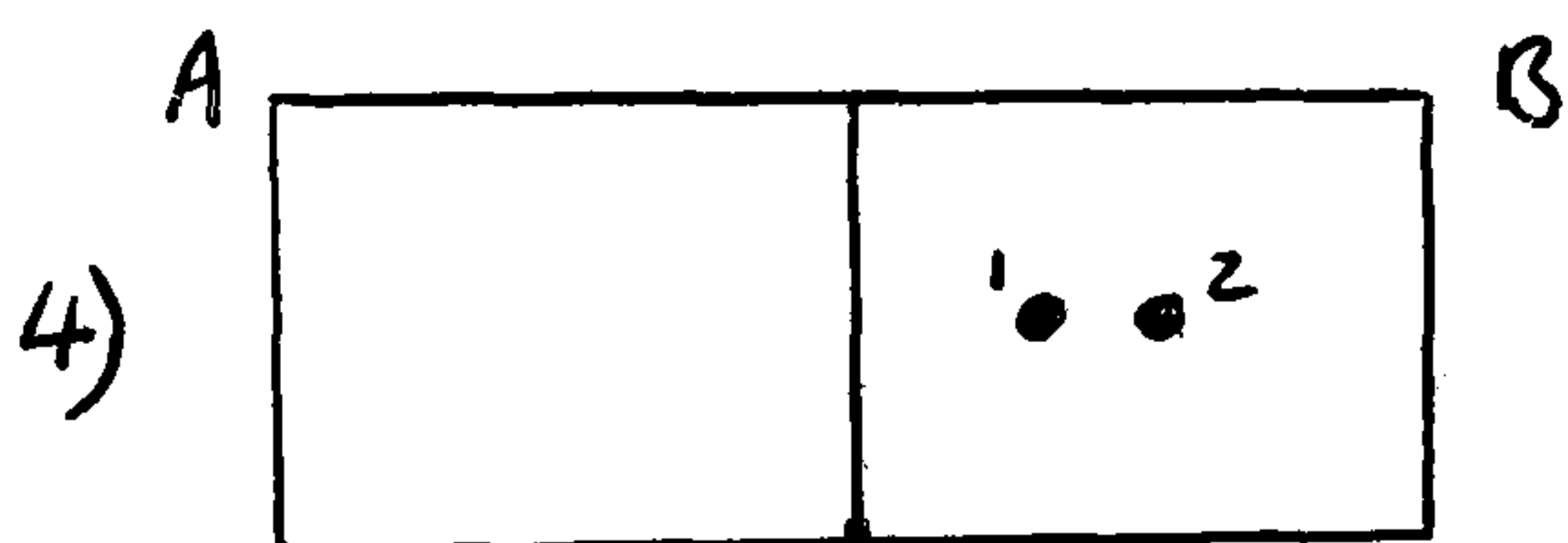
particle 1 in state A, particle 2 in state B



particle 2 in state A, particle 1 in state B.



particles 1 and 2 in state A, none in state B



none in state A, particles 1 and 2 in state B.

Each of these is assigned equal a priori weights in the counting procedure used to calculate the probability of a particular distribution.

Classical, Maxwell-Boltzmann, statistics clearly distinguishes between arrangements 1) and 2) and counts them as distinct; that is, the arrangement in which the two particles have been permuted is regarded as distinct in some way, although not observably distinguishable from the unpermuted arrangement. This implies that the particles are regarded as being distinct individuals capable of being identified as such through a permutation.

There are two ways in which individuality can be granted to these particles, corresponding to the

particle and field approaches in classical physics respectively. We shall consider each in turn.

2.2.1 Transcendental Individuality

The first way essentially corresponds to a naive Lockean substance approach. An individual is regarded as a thing composed of an underlying substantial substratum bearing certain properties. The bare substratum is indicated by a proper name or label, which picks out that individual as itself at one particular time and reidentifies it as the same individual at different times. The labels/proper names stand for or designate that which individuates the particles i.e. the underlying substance.

This kind of individuality has been called Transcendental Individuality (or T.I. for short) by Post, because it resides in something, the substratum, which transcends the properties of a thing. Thus he writes '... we mean by individuality something that transcends observable differences - what I will call 'transcendental individuality'.(21) Arrangements 1) and 2) above are observably indistinguishable but are counted as distinct. Therefore, that which allows them to be so counted, their individuality, must reside in something over and above their observable properties. This 'some-thing' ('we know not what' perhaps, although we shall discuss this in Chapter 4) is the substantial substratum which remains invariant through a change in the properties and allows a thing, a particle in this case, to be individuated and reidentified through such changes.

We shall consider the philosophical arguments surrounding the nature of this substantial substratum in Chapter Four, where we shall argue that the way proper names function in classical physics accords with the view that we termed the 'no-sense theory' in Chapter One. For the moment we merely wish to note that

21. H. Post (1963) p.15 See also Reichenbach (1956) p.225

according to the view involving T.I. a particle is individuated and reidentified by an underlying substance, in which the properties predicated of the particle in here, and which is designated by a label or proper name.

However distasteful this concept may be philosophically, it is usually taken to be the way in which classical physics treats individuals. Thus it is commonly accepted that T.I. is implicit in Boltzmann's Combinational Approach, which is the foundation of classical equilibrium statistical mechanics. Post, for example, has said 'Boltzmann, who founded statistical mechanics at the end of the last century, extended this notion of transcendental individuality implicitly to the atomic realm'. (22) As we shall see historical evidence can be adduced in favour of T.I., but the support is perhaps not so extensive as is usually thought; principally because of the crucial role played by Liouville's Theorem, which is essentially dynamical in character. It is also open to question whether one can in fact 'do' classical statistical mechanics solely in terms of T.I., as we shall now demonstrate.

2.2.2. Is T.I. necessary?

The assignment of weights to the macrostates in classical statistical mechanics (the arrangement

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 for example is given weight two, corresponding to the two ways in which it can be realised, through a permutation of the particles) can be regarded in either one of two ways. One view takes the weighting assignments to be part of the axioms of the theory, with no justification required for these assignments. In this case T.I. would be sufficient for classical statistical mechanics. Alternatively, it can be argued

that some justification of the weighting assignments must be given, by employing some sort of ergodic approach together with Liouville's theorem, with its inherent reliance on spatio-temporal continuity of path, in which case T.I. must be supplemented with some notion involving spatio-temporal continuity of the particle trajectories. However, even if the weighting assignments are taken to be axiomatic and T.I. is taken to be sufficient it can be shown that it is not necessary to classical statistical mechanics.

One can do this by rewriting the whole theory in terms of fields and representing the particles as dichotomic Yes-No fields as outlined above. A two particle system can then be represented as two 'blips' in 'blip space' which move around in this space as the particles move. Obviously, T.I. cannot be introduced into 'blip space' because now there is no underlying substance, merely a property (which Newton took to be impenetrability) passing from one location to another. The problem now is how such a 'blip' can be reidentified.

The solution involves the notion of relative identity outlined in Chapter One. We recall that a and b are described as being identical in the relative sense if a at time t_1 is the same F as b at time t_2 for some sortal concept F. In this case the term 'blip' is acting as the sortal concept and the grounds for referring to the 'same' blip at t_1 and t_2 rest on the continuous well defined spatio-temporal trajectory of the blip.

Similarly, in our two particle case, if the two blips are permuted then the change in the situation is revealed through the change in the blips trajectories in space-time. The permutation can be represented in 'blip space' thus:

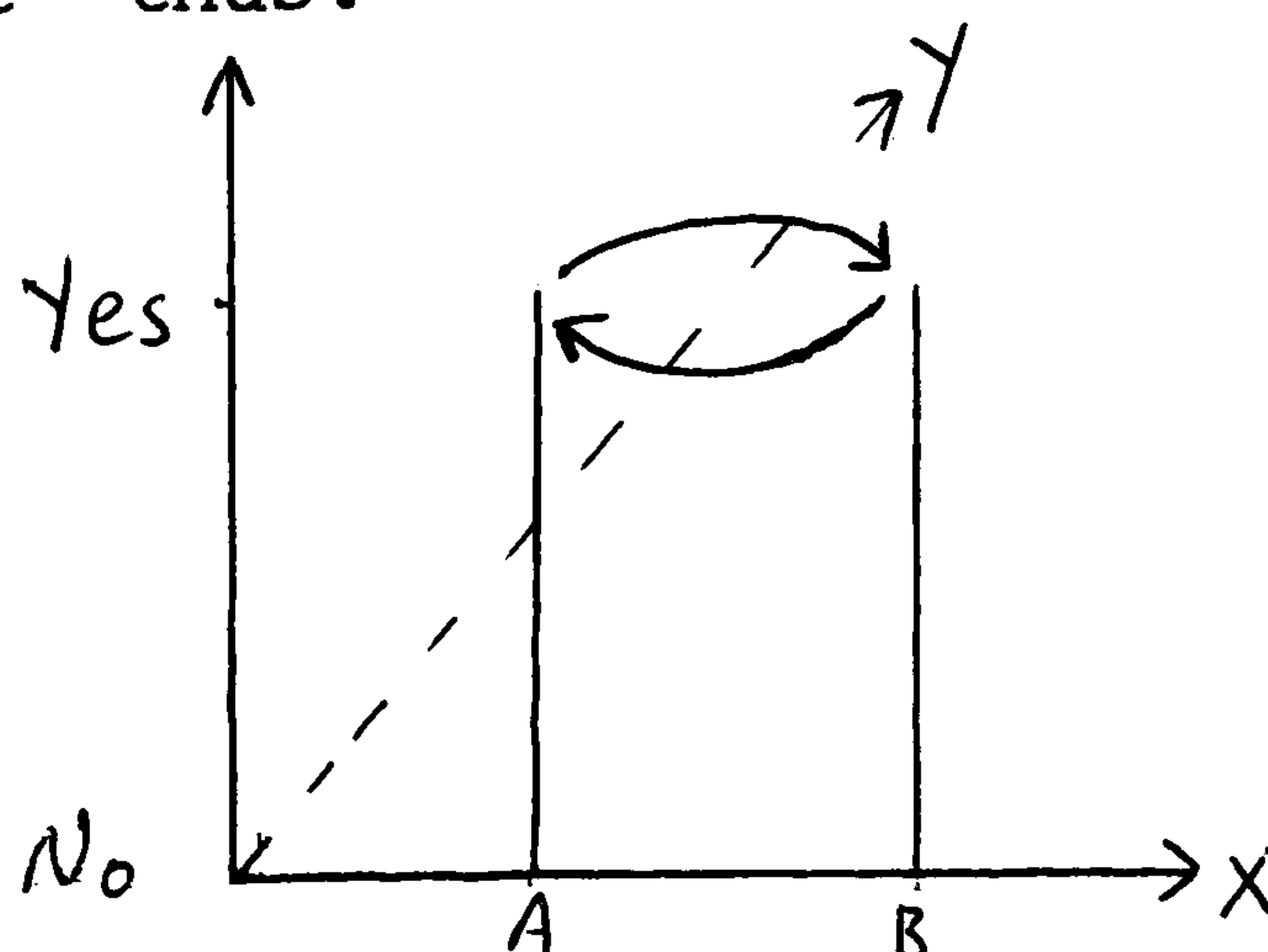
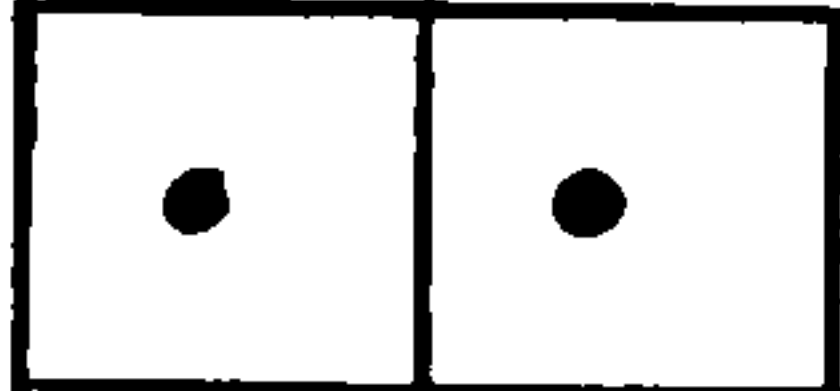


Fig 5

where we go to three dimensions in order that the particles may actually move around one another. Thus it is their trajectories in space-time which confer individuality upon the blips.

The next problem is how to obtain the correct weighting assignments, while working within this field description of classical mechanics.

In the particle approach the configuration  is given the weight two because it can be achieved in two ways, obtained by permuting the particles. T.I. is then introduced to indicate that the two complexions are regarded as distinct. In the field, or 'blip' approach, however, the two configurations are the same and, in the absence of an underlying substance, there is no way of ontologically distinguishing them. How then can the correct weights be obtained?

If one does not ascribe T.I. to the particles then the 'natural' description of the system would be in what can be called 'occupation number space', in which one associates with each energy box the number of particles in that box, in the particle approach, or associates with a given spike configuration the number of particles in it, 0 or 1, in the blip approach. One can represent this occupation number space as a multi-dimensional function space (23) thus

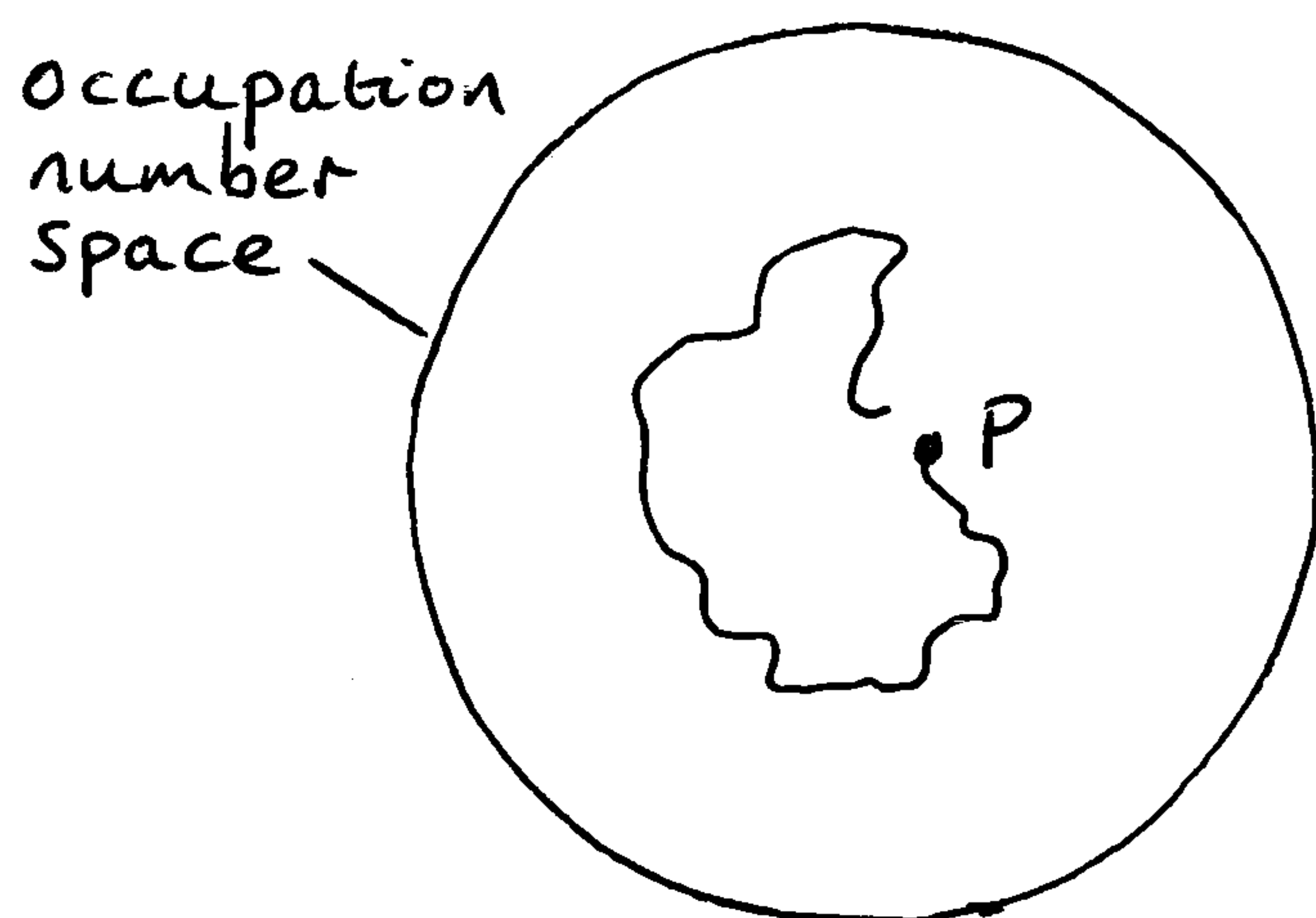


Fig 6

Representation of
occupation number
space as a multi-
dimensional function
space

(There are infinitely many axes in this space) The system is represented by a point P which moves about in this space as the system undergoes change. (By the Recurrence Theorem (24) it will eventually return arbitrarily close to the point from which it started). All the dynamics can be done in this space.

23. M. Redhead (1975) p.77

24. H. Poincaré (1890) p.1.

However, one also needs some way of assigning the weights to the accessible states in order to perform the necessary counting procedure. In occupation number space this counting procedure is applied to the particles occupying the states and permutations of particles can only be counted as distinct if T.I. is introduced, since these permutations do not cause any change in the position of P. Thus it would appear that to attach the correct weights to the states in occupation number space one must follow the particle approach, introduce T.I. and abandon the field/blip approach. One can, however, discover the required weights by transforming to another space in which particle permutations can be counted as distinct without necessarily resorting to T.I. This other space is an r -dimensional phase space, where r is the number of degrees of freedom of the system. Thus, considering more than just two states, our two blip system can be represented by a point in a 2-dimensional phase space:

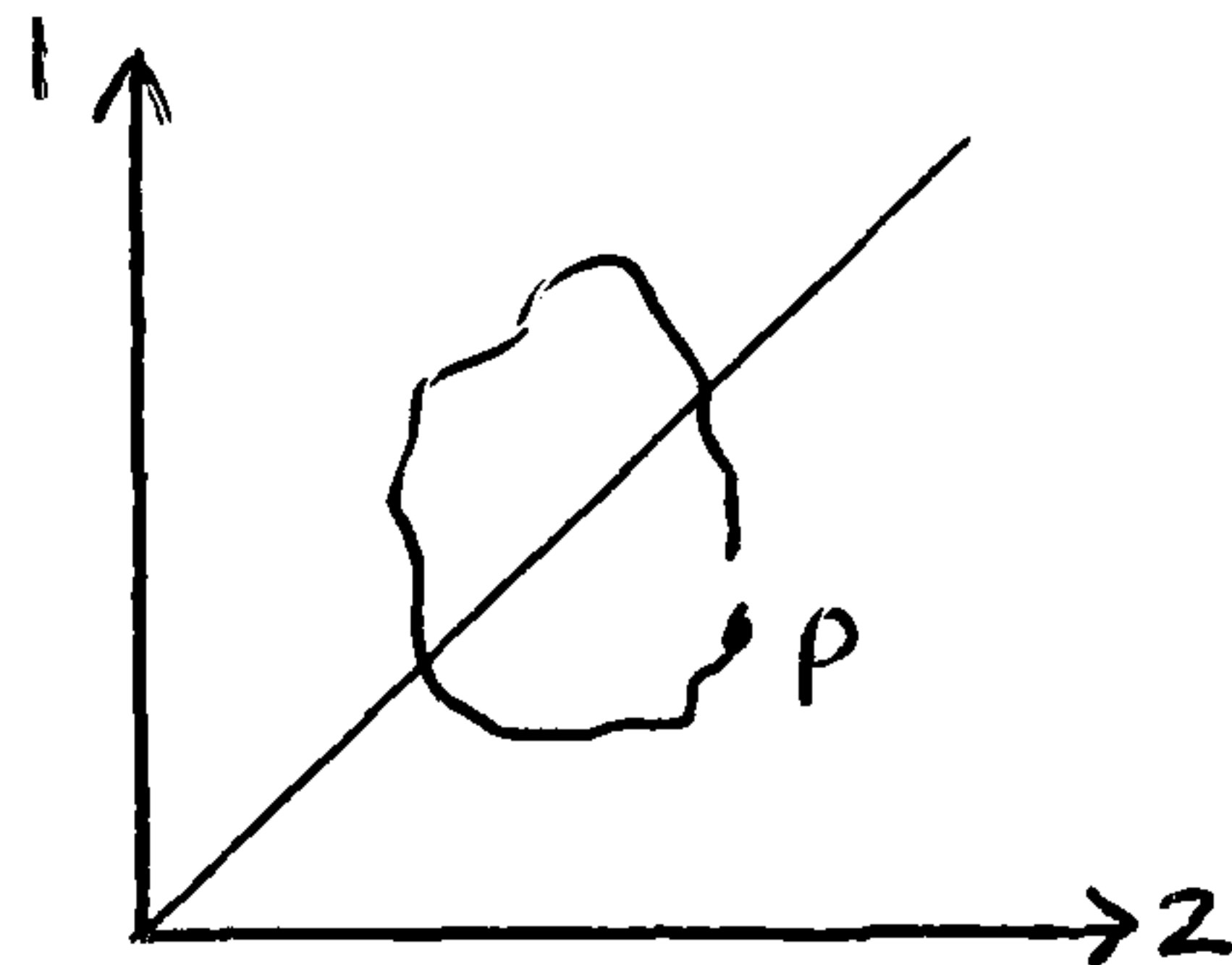


Fig 7
Representation of the
system in a 2-dimensional
phase space

The axes of this phase space have been labelled in this graphical representation. These axis labels could be used to ascribe T.I. to the particles/blips but the important point is that there is no need to do so (unlike the case of occupation number space) because there is another method available for individuating the blips as we shall now see.

The point representing the system will move around the phase space as the system changes. The line at 45° through this space represents the set of points where the blips pass each other as they move about. For any point i there will be a mirror image point f , the reflection of i through the 45° line. The permutation of two blips is then described by the movement of i

to f thus :

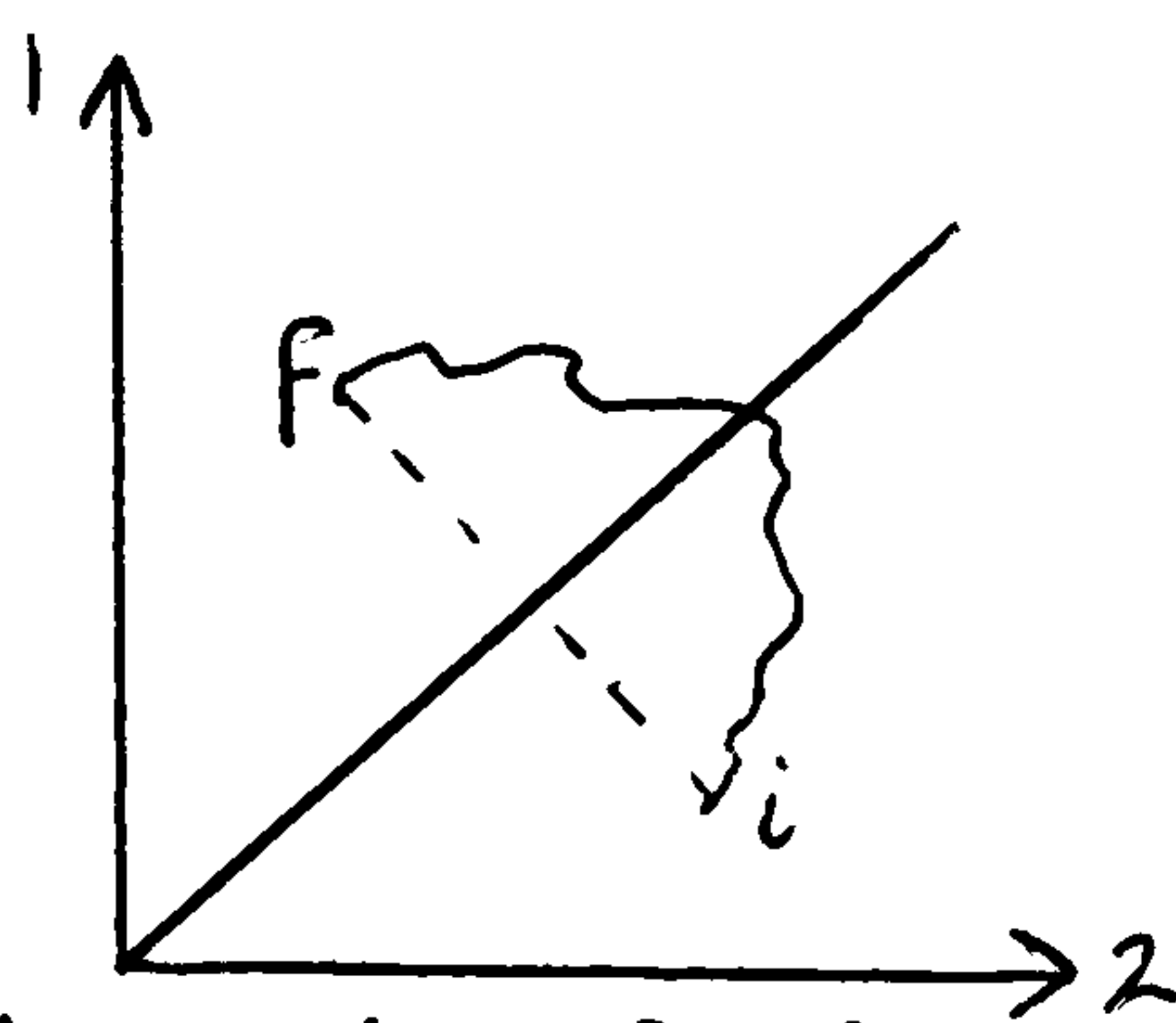


Fig 8

So in this two dimensional phase space the permutation of the blips can be represented and distinguished by a continuous trajectory which allows the permuted complexions to be counted without requiring T.I. Thus the individuality of the blips is now grounded in the continuity of a trajectory in this two-dimensional phase space.

The system can be represented in occupation number space but in order to obtain the correct weights one must either invoke T.I. or transform to this phase space. As regards a permutation of the blips when one makes such a transformation, the point in occupation number space is 'exploded' into the two points i and f thus: (25)

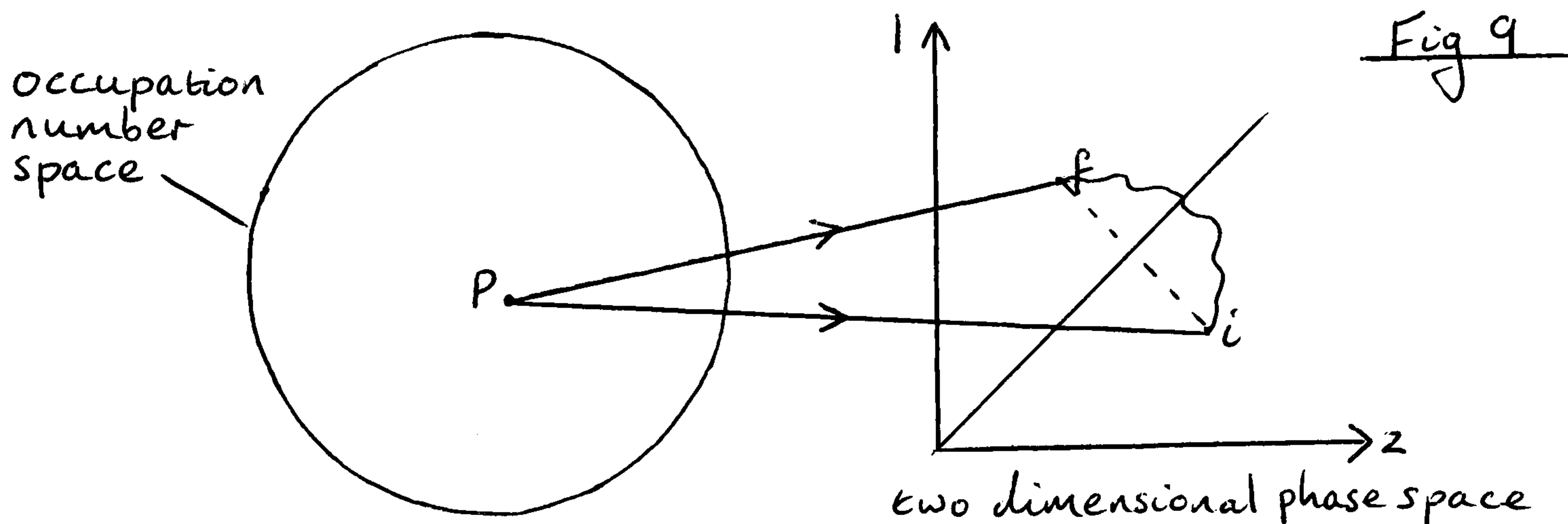


Fig 9

(Of course in the case of n particles, the point P in occupation number space will 'explode' into $n!$ i and f type points in phase space).

The problem now is how to assign the statistical mechanical weights to the state represented by the point in occupation number space. If an axiomatic approach is adopted then one simply assumes that each such point has a factor $n!$ assigned to it. If, however, one feel compelled to justify these axioms then one can transform to the r -dimensional phase

space where this point is 'folded out' into the i and f type points, thus giving the weighting factor without necessarily having to use T.I. In this case one must justify the (implicit) assignment of equal probabilities to the points in this phase space by adopting some sort of ergodic approach.

If one remains in occupation number space then the weight assignment for permutations can only be carried out by introducing T.I. If one moves to the phase space, however, such a procedure can be carried out by counting the 'exploded' points in this space and employing some form of ergodic theorem. Thus T.I. is not required because the permuted complexion is distinguished from the unpermuted one by the continuity of the system's trajectory in phase space, linking points i and f above, for example. This trajectory continuity allows the blips to be labelled and thus individuates them in this phase space.

Our conclusion is, therefore, two fold:

- 1) T.I. is sufficient but not necessary for classical statistical mechanics because the required weighting assignments can always be discovered in our multi-dimensional phase space, where the single representative point in occupation number space is 'exploded' into many points, and the individuality is conferred, not through T.I., but through distinct, continuous trajectories.
- 2) we can obtain the correct weights using the blip/field approach, and thus this obstacle to the adoption of a field theoretic description of classical mechanics has been removed. As we have noted before, although there is an ontological difference between the particle and field approaches, there is no observable difference between them, in the sense that no experiment would ever decide which is correct. This is an example of the underdetermination of theories by empirical data which we shall discuss in more detail in Chapter Four.

2.2.3 Space-Time Individuality

The above account gives us our second way of conferring individuality upon classical particles; they can be labelled and hence individuated by their space-time trajectories. Clearly one can distinguish the two particles, and hence arrangements 1) and 2) above, by their different trajectories. These allow a particle to be reidentified as the same individual at different times and through a permutation.

Such a re-identification depends on at least the following two criteria being satisfied (26):

- 1) the two particles must never share any part of their space time trajectories, i.e. they must not both occupy the same place at the same time. Thus we must make an Impenetrability Assumption in order that the trajectories be distinct and well defined.
- 2) the particles must not 'jump' instantaneously across space. The trajectories must be continuous, this is guaranteed by the equations of motion.

These criteria are usually taken to be satisfied for classical particles. However, as we shall see in Chapter Four, quantum physics presents us with examples of entities which fail to satisfy one or both of them.

We shall now examine these criteria in a little more detail.

Clearly the efficacy of a trajectory in space-time as an individuator depends on such a trajectory being unique. Two particles can share the spatial components of their trajectories, but only at different times. Likewise two particles can obviously exist at the same time but only if they occupy different spatial positions. Two space-time trajectories can never intersect, because given the indistinguishability of particles of the same species, it would then be impossible to decide which particle was which after the intersection, and hence the possibility of individuation and reidentification would be lost. (As we shall see later this is exactly the situation which arises in the case of fields).

26. Recall our discussion at the end of Chapter One.

Thus if their space-time trajectories are to be used to individuate the particles then it must be assumed that they are impenetrable in the sense that no two particles can occupy the same spatial position at the same time. We have called this the Impenetrability assumption and it can be stated in the following form, due to Quinton, 'A complete, that is to say spatial and temporal position is either monogamous or virginal, ontologically speaking' (27).

As well as being distinct the space-time trajectory of a particle should also be continuous, in order that one could always, at least in principle, follow or 'keep track' of the trajectory and hence individuate and reidentify the particle. At any particular time a particle can be individuated by its position in space, i.e. by specifying its spatial co-ordinates. Likewise at any particular space a particle can be individuated through its 'position' in time i.e. by specifying its temporal co-ordinates. Thus at any initial space and time a particle can be individuated by its initial space-time co-ordinates. These set down the initial conditions of the situation.

The particle's subsequent spatial and temporal positions are then given by the classical equations of motion governing it. These can be written in the Hamiltonian or canonical, form as

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p} \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \quad 9.$$

where t is the time, H the 'Hamiltonian', q the generalised co-ordinate and p the conjugate generalised momentum. The uniqueness of the solutions of these, equations of motion then guarantees the re-identifiability and individuality of the particle through space and time and, in particular, through a particle permutation. Thus what we shall call the 'space-time individuality' of the particles is conferred through the initial conditions together with the equations of

motion.

At the root of this notion lies the assumption that even in the absence of an intrinsic difference of quality, i.e. even if the particles are indistinguishable, the extrinsic difference of kinematical behaviour is a sufficient condition for their individuality and identifiability. This difference derives its operational meaning from the assumption of an unrestricted possibility of establishing an unbroken connection between the object at time t_1 with the object at time t_2 '... by continuous observation (either direct or indirect) through all intermediate time' (28). It is this unbroken connection which is effected by space time individuality.

Various authors have discussed this, and similar notions. Thus Pais has given the following illustration of what is meant by it: 'Suppose I show someone two identical balls lying on a table. Next I ask him to close his eyes and a few moments later to open them again. I then ask him whether or not I have meanwhile exchanged the two balls. He cannot tell, since the balls are identical. Yet I do know the answer. If I have exchanged the balls then I have been able to follow the continuous motion which brought the balls from the initial to the final configuration.' (29) Pais then makes the historical point that Boltzmann himself held this view of space-time individuality because in his 1897 'Lectures on Mechanics' he stated that the assumption that indistinguishable particles, which cannot come infinitely close to each other, can be distinguished by their initial conditions and by the continuity of their motion, '... gives us the sole possibility of recognising the same material point at different times. (30) We shall consider this point in more detail later when we discuss the history of classical statistical mechanics.

28. P.W. Bridgman (1927) p.92

29. A. Pais (1979) p.893

30. L. Boltzmann (1974) Vol.1. p.9

Pais's example is, of course, light hearted in the sense that space-time individuality requires continuous observation in principle only and blinking, sleeping, or closing the eyes in general are not regarded as forming a serious basis for any non-trivial counter argument, (sceptical arguments notwithstanding).

As regards the relation between space-time individuality and particle permutations, Tolman has noted that '... our procedure in regarding the interchange of two similar molecules as corresponding to a significant change in the mechanical state of a system, eventhough not in its condition, evidently implies the possibility of keeping a continuous observation on the system which would let us know whether two similar molecules do change roles or not. This, however, is in entire agreement with the point of view of the classical mechanics, which would permit such a continuous observation, at least in principle, ...'

(31).

The reidentification of particles has been discussed by Jammer who remarked that in classical mechanics '... it was claimed that two particles, once 'told apart', can always be 'told apart', for they can always be reidentified, thanks to the uniqueness of the solutions of the equations of motion' (32). He continued, 'If the motion of each particle is represented by its four-dimensional world line in space-time, a family of non-intersecting lines is obtained each of which can be used to characterize or label uniquely the particle whose trajectory it is. It was therefore thought it is always possible, at least in principle, to decide unambiguously whether or not a particle found at a certain time at a certain place is the same as a particle found at another time at the same or another place'. (33) Thus Jammer's interpretation of the classical view is that the world-line of a particle can be used to uniquely label and hence individuate it.

31 R.C. Tolman (1962) p.77
 32. M. Jammer (1966) p.341-342
 33. Ibid p. 342

On this view the space-time trajectory confers individuality upon the particles and hence allows them to be reidentified. It should be noted, however, that space-time individuality does not necessarily underpin particle reidentifiability. One may, of course, adopt the alternative view in which the particle is regarded as exhibiting T.I. with the underlying substance, ontologically speaking, conferring reidentifiability upon the particle. It is now T.I. which underwrites, as it were, the reidentification of the particles. Thus ter Haar writes 'In classical mechanics it was regarded as permissible to imagine labels attached to each of the ... particles, eventhough they are identical. In that way each of them could be followed along its orbit and localized at every moment'.(34). Particles are 'localized' in this sense, that we can label them separately and 'keep track' of them.

Reichenbach effectively subsumed both T.I. and space-time individuality under his term 'material genidentity' (35). Genidentity is the relation which connects different states of the same thing at different times and the properties of this relation are, according to Reichenbach, what the notion of physical identity is essentially based upon. There are two kinds of genidentity: material and functional. The former is defined in terms of three characteristics. 'First, we associate material genidentity with a certain continuity of change' (36). This is clearly equivalent to our requirement that the space-time trajectory of a particle be continuous. 'Second, it is a characteristic of material objects that the space occupied by one cannot be occupied by another' (36) . This is a statement of the Impenetrability Assumption. 'Third, we find that whenever two material objects exchange their spatial positions this fact is noticeable. We usually recognise this change of position by the use of specific marks on the objects These marks remain on the object in accordance with the continuity

34. D. ter Haar (1958) p.71

35. H. Reichenbach (1956) p.38 and p. 255 ff

36. Ibid p.225

criterion and permit an identification of the objects even when no observation during the change of spatial positions was made and the continuity criterion cannot be applied. In other words an inter-change of spatial positions is a verifiable change even though no records of the act of interchanging are available.

(36). It is not clear whether Reichenbach regarded these 'marks' as 'dynamically irrelevant unused structures' or not. If he did not, then they would render the particles distinguishable, and there would be no need to talk of T.I. or space-time individuality because this difference in their extrinsic properties would individuate the particles. If the 'marks' were regarded as dynamically irrelevant, like proper names, then Reichenbach is essentially advocating T.I. in this passage.

If, however, he is restricting his discussion to an outline of what we have called space-time individuality then this third 'characteristic' is unnecessary. The stated justification of this requirement is to allow reidentification over periods when no actual observation is made. However, a commitment only to continuous observation at least in principle is sufficient, and if the above, rather than any advocacy of T.I. is the only justification then this last characteristic is clearly redundant.

It is easy to give examples to show that these three characteristics are necessary but not sufficient. Reichenbach himself gives the following: if a house is pulled down the resulting heap of rubble is not called the same thing as the original house even though all three criteria are satisfied (37). This is clearly a problem of relative identity and one needs to supplement Reichenbach's three characteristics with some view of relative identity, involving the kind of sortal concept which, in the above example, would

cover both the house and the heap of rubble.

'Functional genidentity' is associated with processes such as water waves. Things possessing this kind of genidentity may easily violate the second and third characteristics but not, usually the first. We shall discuss this in a little more detail when we come to consider individuality and fields.

Reichenbach believed that 'We can define 'genidentity to suit our purposes' (38), but that, nevertheless, it does make sense to distinguish between different kinds of genidentity, such as the material genidentity of a billiard ball and the functional genidentity of the associated travelling kinetic energy. The problem now is how are these two forms of genidentity to be distinguished in the case of elementary particles, where the macroscopic means of identification are inapplicable. Reichenbach correctly identifies the solution (39) as lying in the statistical mechanical view which replaces the examination of particles with inferences based on the statistical properties of an assemblage of such particles. A central notion in this programme is that of the arrangement of particles situated in different cells (in some parameter space), two such arrangements being regarded as distinct when they result from one another by an interchange of particles between two cells. Reichenbach then writes 'The third characteristic of material genidentity, the requirement that exchange of spatial position should lead into an observable difference, is thus translated into a statistical property; it supplies the definition of arrangement.' (40)

The parallel with our notion of T.I. is obvious. Reichenbach goes further to argue that experimental tests of the physical laws involving entropy constitute a test for the assignation of 'material genidentity' to the particles because such laws are based, fundamentally on the classical Maxwell-Boltzmann distribution formula

38. Ibid p.226

39. Ibid p. 228-229

40. Ibid p. 229

which in turn is based on the above definition of arrangement. Such tests, he claims have been positive in the realm of classical physics, but negative in quantum physics. The conclusion is that in the latter material genidentity must be replaced by functional genidentity. As we have said, we shall discuss this in more detail in a later section,

It should be noted that, contrary to Reichenbach's claims, these 'experimental tests' do not constitute a test for material genidentity as a whole but only for his 'third characteristic', corresponding to what we have called T.I. They are not, therefore, also tests of space-time individuality and no inferences can be drawn from them as regards the validity of this notion.

We shall conclude this outline of the views of other writers by noting that Newton characterised the elements of his 'proto' - theory of classical mechanics in terms of 1) Impenetrability, 2) mobility, and 3) the ability to excite the senses. Again, the parallel between our characterisation of space-time individuality and Newton's criteria 1) and 2) is clear.

We must now examine the ontology underlying this second kind of individuality for classical particles.

On this view there is no underlying substance 'carrying' the individuality. There is merely the set of properties predicated of the particle moving along a well defined, continuous path in space-time. Now it is not necessarily true that the entire set of properties remains invariant between one space-time point and the next. Some of the properties may change during the particle's history. However, if the particle is to be reidentified as a particle of the same kind then a certain subset of properties must remain invariant over space and time. This subset may be identified with the particle's essential properties, where an 'essential property' is taken to be one which a particle must

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possess at all times if it is to remain a particle of that kind. (41) We shall discuss essentialism and natural kinds in Section 4.3 of Chapter Four.

Thus if an electron 'becomes', through some peculiar interaction, a proton, say, then we must say that at a certain point an electron was destroyed and a proton created. As we have noted, the formalism of Quantum Field Theory allows us to handle creation and annihilation processes.

On the other hand if an electron merely changes its velocity then we refer to the particle before and after the change as 'the same' electron. Properties such as this have been called 'extrinsic' properties by Jauch, (42) since they are dependent upon the state of the particle. Properties which are not dependent upon the state, such as charge, rest-mass, etc., are termed 'intrinsic' properties. Thus our subset of invariant properties may be referred to as the intrinsic properties of the particle.

In the last section of Chapter One we noted that for an object to be reidentified through time the three sufficient conditions of spatio-temporal continuity, qualitative continuity and the so-called 'sortal constraint' must be satisfied. In terms of these conditions we can discuss the reidentification of elementary particles through changes in their extrinsic properties as follows :-

First of all, as we have emphasised in the discussion at the beginning of this section, classical elementary particles satisfy the requirement that their trajectories be spatio-temporally continuous. Secondly, the extrinsic properties of the particles change in a continuous manner and 'drastic', or discrete, changes do not occur. Thus the condition of qualitative continuity is also satisfied, and can be formulated for the specific case of classical particles thus: for any succession of particle stages to correspond to the career of a particle any particle stage in ^{the} succession must be qualitatively similar as regards the particle's extrinsic properties, to any neighbouring particle stage in the succession.

41. E. Hirsch in M.K. Munitz (1971) p.31

42. J.M. Jauch (1966) Ch 15.

Of course, this requirement may not be fulfilled in quantum physics where discontinuous phenomena, of the sort prohibited by the condition of qualitative continuity, do occur. However, in these cases particle reidentification is precluded by the failure of the requirement of spatio-temporal continuity anyway, and it is precisely in these situations (such as electron transitions within atoms for example) that creation and annihilation processes are introduced.

Finally, we note that, whereas the requirement of qualitative continuity is concerned with the extrinsic properties of the particle, the Sortal Constraint is concerned with its intrinsic properties. Thus the relatively small set of intrinsic properties which may be taken to define what it is to be that kind of particle, also define the sortal term covering a particle. The sortal 'electron', for example, is *explicated* in terms of a certain rest mass, certain charge, etc. which are precisely those properties used, together with spatio-temporal continuity to reidentify an electron through time. (43). Thus the distinction between intrinsic and extrinsic properties reveals how much closer the relationship is between sortal terms and natural kinds for elementary particles compared with macroscopic objects.

So, if classical particles are taken to exhibit 'Space-Time' Individuality then, on this view, there is simply an underlying trajectory of space-time points along which the set of properties predicated of the particle is moving. There is nothing more than a set of properties and an underlying sequence of space-time points at which this set is instantiated. There is nothing extra, nothing attributed to the particle, over and above its set of properties. So by grounding the individuality of the particle in space-time in this way, this view has, in effect, transferred this individuality from the particle to a

43. See P.E. Hodgson (1980) for a discussion of the particle identification criteria actually used by practising high energy physicists.

point, or set of points, in space-time. This point, or set of points, then form the primary individuals to which the particles are related and in terms of which the particles are individuated.

Clearly a description of classical particles in these terms is, recalling our definitions in Section I of this chapter, equivalent to the field description. The particles are now regarded as nothing more than properties of the points of space-time, with their individuality residing at the most fundamental level, in these points. The question, now of course, is how is individuality to be attributed to the space-time points themselves? We shall return to a brief consideration of possible answers to this question in Chapter Four.

Thus we conclude that the individuality of classical particles can be described in terms of either of the two positions outlined in the introduction to Chapter One, namely T.I. and S.T. Individuality. These two views correspond to, in the sense of ontologically underpinning, the two interpretations of classical mechanics in terms of particles or fields respectively. There is no way of experimentally distinguishing between these two interpretations, although they are clearly metaphysically quite distinct.

In the next section we shall examine the relationship between our second view of particle individuality above and the nature of space-time itself.

2.2.4 Space-Time Individuality and Space-Time

Broadly speaking considerations of the nature of space and time fall into two opposing camps, the relationalist and the absolutist, or substantivalist. We shall briefly consider each of these in turn.

If a relational account of space-time is adopted then it is difficult to see how one can simultaneously hold the view that it is our notion of space-time individuality which supplies the Principle of

Individuation for material particles. The naive relationist position claims that space-time is given simply by the relations between things no more and no less, and that ultimately reality rests solely with the latter. In particular it is argued that there is no underlying substratum of space-time in which the individuality of a material object can be embedded. A relationist cannot therefore ground the individuality of an object on the continuity of its spatio-temporal trajectory and must look elsewhere for a 'Principle of Individuation'. (44)

It is equally clear that if the alternative, absolutist, position is taken up then space-time individuality can be consistently invoked, although it is not necessary that it should be since one could say that T.I. could be attributed both to the material objects and the points of space-time with perhaps some sort of 'matching' between them. The naive absolutist view holds that space-time is a fixed substratum or some sort of 'aether', in which material objects are somehow embedded. Both the latter and space-time itself are granted the status of real entities, existing, to a certain extent which will be made clear shortly, independently of one another. This is the common element of all such absolutist views, whether of space alone or time, or space-time, namely '... the claim that the structure in question can be said to exist and to have specified features independently of the existence of any ordinary material objects...' (45) Space-time is thus regarded as a substratum underlying material objects and is the 'arena' in which events take place.

Clearly an absolutist can ground the individuality of an object in this underlying space-time via its spatio-temporal location and can reidentify it through time by the set of points which compose the object's trajectory. One could even go further and regard

44. One possibility is to attribute T.I. to the objects, another is to invoke the Principle of Identity of Indiscernibles and individuate them by the set of properties which they possess. Leibniz's Complete Notion of an Individual represents a particularly extreme version of this alternative, illustrating perhaps the lengths which relationists may be forced to go to.

45. I. Sklar (1977) p.161

space-time as the primary particular in the sense that all others are derived from it. This would be a version of the view in which material objects are regarded as mere manifestations of the underlying spatio-temporal plenum, (c.f. certain interpretations of the field equations of General Relativity). If material objects are regarded as properties of the points of a substantial space-time in this way then the view involving space-time individuality must be adopted, if any such position is to be sought for.

The notion of 'support' is used here in the same way as in the substantivalist account of material particulars and is thus open to the same philosophical objections which we shall rehearse in Chapter 4. Indeed the absolutist account of space-time in general falls into the substantivalist camp and arguments for and against the former bear a close resemblance to those produced in discussions of the latter. (46)

It is important to note that we are considering here space-time, not space or time alone. Special Relativity compels one to reject an absolute theory of space, in which it is taken to be separate from time. Thus observers in relative motion to one another will take their space and time axes at different angles to one another. 'slicing' space-time at different angles. However, S.R. is quite consistent with either an absolute or a relational account of space-time since the fact that it can be sliced at different angles does not prevent it from being regarded as some thing existing independently of material objects.

The situation is somewhat more complicated when we come to the space time of General Relativity since there now exists a lawlike connection between its structure and the distribution of non-gravitational mass-energy 'contained' in it. This connection is established through the field equations.

$$R_{ik} - \frac{1}{2} g_{ik} R = -\kappa T_{ik} \quad 10.$$

where g_{ik} is the g-function, R the scalar curvature, R_{ik} the Ricci tensor, κ is a constant and T_{ik} is the 'stress-energy' tensor. The terms on the left hand

side give the structure of the gravitational field regarded as the intrinsic geometry of space time, and the term on the right gives the distribution of mass-energy over this space-time.

The extent to which G.R. can be used to attack the absolutist position then depends on whether or not these equations are regarded as completely ruling out the term 'independently' in the characterisation of this view. If they do, if, for example, they are interpreted as implying that the curvature of any portion of space-time is 'produced' by the matter in it, then it is difficult to see how the absolutist account can be retained and some alternative must be adopted. However, it is not at all clear that they do imply this.

First of all the equations can equally well be read from the left or the right. Thus it could just as well be said that the structure of space-time 'produces' the mass-energy distribution, as the reverse that the latter 'causes' the former. Indeed the interpretation that matter consists simply in regions of special curvature of space-time lies at the heart of the geometrodynamics programme in *relativistic* cosmology (47), which can be regarded as the ultimate in substantivalist programmes.

Secondly, there exist solutions of these equations which predict that there would be a curvature and hence a structure of space-time even if there were a total absence of matter. The following historical point is worth noting. In the years immediately following the publication of the field equations Einstein believed that in their correct form they should have no solutions at all in the absence of matter (48). However, in 1917 de Sitter found a solution

47. J. Wheeler (1962) p.361; A. Grunbaum (1957) p.525
see also the discussions by Reichenbach and
Robertson in P.A. Schilpp (1951)
48. A. Pais (1982) p.287.

in which the mass density is equal to zero (49) and although Einstein looked for ways in which this solution could be eliminated, he quickly came to realise that it could not.

In general G.R. is compatible with a number of different possible space-times, of varying metric structure, each of which could be the space-time of a universe devoid of non-gravitational mass-energy. These solutions weigh heavily in favour of the absolutist position since a relationist clearly encounters severe difficulties in talking about or even conceiving of, totally empty space (50), counterfactually or otherwise.

Finally the relationship between the mass-energy distribution and space-time structure as expressed by these equations is not a simple causal one, in either direction. The stress energy tensor takes into account not just the amount of mass distributed over space-time but also how it is distributed and thus this tensor itself expresses metric features of the mass-energy distribution. It is therefore somewhat naive to argue that the latter causes the structure of space-time in a way in which the relationist would like it to do.

These brief points suggest that G.R. provides little evidence to support the relationist position. As regards the independence of space time from matter, presupposed by the absolutist view, all that one can deduce from the field equations is that the structure of space-time is not causally independent from the mass energy distribution. However, it may be ontologically independent and, indeed the points outlined above indicate that this may be so. Whether this is enough to save the latter view is a matter for further,

49. W.de Sitter (1917a) p.1217 and (1917b) p.229

50. As we have seen the relationist account also has difficulty in discussing a universe containing only two indistinguishable objects. For a more recent discussion of the differences between the absolutist and relationist positions see M. Friedman (1983) esp. p.216 ff.

more detailed, consideration.

Finally, it is worth noting that it may be possible to adopt a view of space-time which falls into neither the relationist nor the absolutist camp. Thus although Newton is most commonly identified as belonging to the latter, in a manuscript published after his death he argued that space and time may be neither substantial nor mere relations between objects but may belong to a different ontological category of their own. This argument was based on the claim that '... it is only by their mutual order and positions that the parts of time and space are understood to be the very same which in truth they are, and they do not possess any principle of individuation apart from that order and those positions.' (51) In other words the points of space-time can only be individuated up to an automorphism of the space-time structure. We shall return to a consideration of this last point in Chapter 4. For the moment it is sufficient to note that neither S.R. nor G.R. conclusively rule out the absolutist view of space-time, nor, consequently, space-time individuality.

2.3 The History of Classical Statistical Mechanics

Our intention in this section is to show that our philosophical conclusion is supported by an historical analysis of the development of statistical mechanics in the 19th century. In particular we shall consider Boltzmann's two approaches to this subject : 1. The H-Theorem Approach, based on a consideration of collisions between molecules traversing well defined continuous spatio-temporal trajectories, and 2. The Combinatorial Approach, based on a consideration of the distribution of individual molecules over cells in phase space, or energy states and free, to an extent which will be made clear, from 'dynamical' underpinning. These approaches implicitly attributed space-time and transcendental individuality, respectively,

51. Newton in A.R. Hall and M.B. Hall (1978) p.100. It was a review article by Torretti which drew my attention to this paper. R. Torretti (1984) p.280.

to the atoms whose behaviour they described.

2.3.1. The Introduction of Statistical Considerations into the Kinetic Theory of Matter

The kinetic theory of matter can be described as the attempt to explain the empirical regularities occurring in the macroscopic properties of material 'things' in terms of the microscopic behaviour of their atomic and molecular constituents which obeyed Newtonian mechanical laws. The early studies in the revival of kinetic theory which occurred in the second half of the 19th century (52), assumed that all the atoms travelled at the same velocity and that the influence of intermolecular impacts and forces were negligible and were therefore limited in explanatory and predictive power. The development of the subject took a major step forward when Maxwell allowed intermolecular collisions to play a role and demonstrated that their effect was to produce a statistical distribution of molecular velocities in which all velocities would occur with a known probability. (53) The nature of this distribution was given by Maxwell's distribution law:

$$f(x) = \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2} \quad (54) \quad 11.$$

where a is some constant and x is the component of velocity in the x direction. The original proof of this law was widely regarded as unsatisfactory and subsequent attempts to derive it more rigorously can be divided into four types: 1). If a Maxwellian distribution is already established then conservation of energy implies that further collisions will leave it unchanged and hence this distribution is the only one which is stable. This was the line taken by Maxwell

52. This revival was dependent upon a conceptual change from the idea of heat as substance to that of heat as motion of atoms and also the development of mathematically tractable models of the motion of particles. See Brush (1970)

53. J.C. Maxwell (1860)

54. J.C. Maxwell (1890) p. 380.

himself in 1867. (55). 2). One may define a quantity, H , which depends on the velocity distribution and then show that the effect of molecular collisions is always to decrease H , unless the distribution is Maxwellian in which case H remains constant. This constitutes the essence of Boltzmann's 'H-Theorem approach'. 3). The molecular velocities may be regarded as random quantities and the best possible estimate of their distribution, given fixed total energy, mass and momentum, obtained using probability theory. Maxwell's derivation may be improved by simply calculating all possible ways of dividing the energy amongst the molecules, given finite total energy and number of molecules. This was the approach suggested by Boltzmann in 1872 and Maxwell in 1879, and which subsequently developed into the 'Combinatorial Approach'. 4). The Maxwellian distribution may simply be treated as a primitive postulate of the theory and justified on the basis of its predictive power (56). In what follows we shall be almost exclusively concerned with the second and third approaches.

Using this law Maxwell was able to demonstrate that the fundamental linear transport processes in a gas - viscosity, heat conduction and diffusion - could all be conceptualized as special cases of a generalised transport process in which a physical quantity is carried by molecular motion and transferred from one molecule to another by collisions. (57) Maxwell distinguished this theory, based on the statistical method of explanation, from one concerned with the motions of individual molecules and, therefore based on the 'dynamical' method, arguing that such motions were unobservable and impossible to compute in practical

55. J.C. Maxwell Ibid Vol LL p.26

56. See J.P. Andrews (1928) p.118

57. J.C. Maxwell (1890) Vol.II p. 33/374. See also P.M. Heimann (1970) p. 201

terms. This formed the basis for his later argument that the Second Law of Thermodynamics was a statistical, rather than a dynamical, law.

Nevertheless his transport theory was explicitly grounded on a detailed study of intermolecular collisions and at the base level the molecules were still, of course, regarded as traversing distinct, continuous space-time trajectories, obeying Newtonian mechanics in a strictly deterministic manner. The statistical analysis was introduced purely as a matter of convenience, to overcome the practical problems in handling enormously large numbers of molecules. (58)

This work had a great influence on Boltzmann and completely changed the direction of his attempts to arrive at an explanation of the irreversibility of natural processes as expressed by the Second Law of Thermodynamics.

2.3.2. Boltzmann's Initial Attempts

In 1865 Clausius restated the Second Law in terms of his newly introduced entropy function. (59). The task of kinetic theory was then two fold: first, to demonstrate the existence of an entropy function satisfying Clausius's definition and secondly to show that this function can only increase in an irreversible adiabatic process.

Boltzmann's first attempt at providing a kinetic explanation of the Second Law began with the stated intention of deriving Clausius's results as purely mechanical theorems. (60) Although he claimed to have done just this, in fact he only managed to construct a mechanical counterpart of the entropy for systems which were strictly periodic, and his later studies compelled him to renounce his assertion that the irreversible aspect could be identified with the least action principle.

58. A belief in determinism was not held quite so strongly by the classical physicists as is usually thought. Thus Maxwell himself repudiated this doctrine on several occasions. See Brush (1983) p.83 and K. Popper (1950) p.117 and p.173.

59. R. Clausius (1865) p.

60. L. Boltzmann (1968) Vol.I p.9

After reading Maxwell's 1867 paper Boltzmann realised that the key to a successful reduction of the Second Law lay in the molecular distribution function rather than the complete set of molecular variables. In a series of papers in 1868 and 1871 he derived and extended Maxwell's results and also, in 1868, sketched an alternative derivation which was free from any assumptions regarding inter-molecular collisions. (61) Thus he simply assumed that there was a fixed total amount of energy to be distributed over a finite number of molecules in such a way that all combinations of energies were equally probable. By regarding the total energy as being divided into small but finite packets, Boltzmann could treat this as a problem in combinatorial analysis and thus he obtained a complicated expression which reduced to the Maxwell distribution law in the limit of an infinite number of molecules and infinitesimal energy elements. This is the first indication of what was to develop into his Combinatorial Approach of 1877. (62)

However he did not immediately pursue this line of thought but went on to lay the foundations of his 'H-Theorem Approach'. It is interesting to note that in two papers published in 1871 Boltzmann introduced what was later called the 'ergodic hypothesis' in order to establish the equivalence of two meanings which could be given to the distribution function:

- 1) that it determines the fraction of any suitably long time interval during which the velocity of any particular molecule has values within prescribed limits;
- 2) that it determines the fraction of the total number of molecules in the case which has velocities within the prescribed limits at any given moment. Initially Boltzmann thought that no analysis of this equivalence

61. Ibid p.49

62. Boltzmann subsequently realised that dividing up the energy continuum, as he did in 1868, leads to the Maxwell distribution in two dimensions only. To treat the three dimensional case a division of the three-dimensional velocity space must be considered.

was necessary but he subsequently realised that some hypothesis was called for, to the effect that in the course of time, the molecular coordinates and momenta take on all values consistent with the fixed total energy. (63) As we shall see this hypothesis plays an absolutely crucial role in the justification of the weighting assignments in the Combinatorial Approach and in establishing a connection between this and the H-Theorem Approach.

In the same year, 1871, Boltzmann began a new assault on the Second Law. (64) By using the general distribution law to distinguish the heat added to the system from the work done on it in any process, something which was not possible when the system was described directly in terms of its atomic variables, he was able to write down an expression for the entropy of a system in purely mechanical terms. Furthermore this specified a definite procedure for calculating the entropy of a system in thermodynamic equilibrium. Thus Boltzmann not only demonstrated that an entropy function exists, he also established a procedure for finding it. (65) The second half of the reduction of the Second Law-giving a mechanical explanation of irreversibility- was to be completed the following year.

2.3.3. The H-Theorem Approach

In 1872 Boltzmann discussed the behaviour of the entropy function in irreversible processes by considering the way in which the velocity distribution changed with time due to intermolecular collisions (66). This work can be regarded either as an attempt to complete the reduction of the Second Law, or as an alternative approach to the problem of thermal equilibrium, by assuming that the gas was not in equilibrium and then attempting to show that the effect of collisions would

63. Ibid p.50,p.284.

64. Ibid p.288

65. For a detailed discussion of this work see E.E. Daub (1969) p. 326.

66. L. Boltzmann (1872) p.275, (1968) Vol. I p.316.

be to bring equilibrium about. Viewed in this way two fundamental aspects of this work can be linked together: 1) the attempt to demonstrate the uniqueness of the Maxwell velocity distribution, i.e. that the distribution law is both a necessary and sufficient condition for equilibrium; 2) the derivation of an integro-differential equation for the non-equilibrium velocity distribution function, now known as 'the Boltzmann transport equation'. From this it immediately follows that the Maxwellian distribution represents the equilibrium state, in the sense that one obtains a zero value ^{for $\frac{\partial f}{\partial t}$} by direct substitution.

Thus Boltzmann began his 1872 study with a detailed analysis of the collision processes by means of which the molecules in a gas can change their energies. (67), restricting himself to the case of a spatially homogenous gas, with no external forces and of sufficiently low density that only binary collisions need to be considered. On the basis of '... an exact consideration of the collision process', (68) he determined the rate at which the energy distribution changes due to such collisions, by deriving an expression for the rate of change of the number of molecules having a given energy x , due to these encounters. This equation for $\frac{\partial f(x,t)}{\partial t}$ ^{is} in fact a special case of the Boltzmann transport _{equation} (69). If the distribution is Maxwellian then the right hand side of the equation vanishes identically for all values of the variables and $\frac{\partial f}{\partial t} = 0$. Thus in this sense the Maxwellian distribution is indeed stationary. Once it is achieved no further change in the distribution function can occur and the system will remain in the equilibrium state.

This is little more than an elaboration of previous work, but now, with an explicit formula for $\frac{\partial f}{\partial t}$,

67. Boltzmann chose energy rather than velocity as the basic variable. This merely introduces square root terms in the resulting equation which do not appear in its modern form.

68. L. Boltzmann op cit p. 329

69. Ibid p. 334

Boltzmann was able to show that $f(\alpha, t)$ always tends towards the Maxwellian form, (the emphasis on the word 'always' is important as we shall see). Thus to demonstrate the uniqueness of this distribution he introduced (70) an auxiliary quantity E, (later called H), defined by

$$E = \int_0^\infty f(\alpha, t) \left\{ \log \left[\frac{f(\alpha, t)}{\sqrt{2\pi}} \right] - 1 \right\} d\alpha \quad (71) \quad 12.$$

By considering the symmetrical character of the collision and the possibility of inverse collisions, and assuming as Maxwell had before him, that the velocities of the two molecules before they collide are statistically independent, (this is the randomness assumption known as the Stosszahlansatz). Boltzmann demonstrated (72) that E could only decrease with time i.e.

$$\frac{dE}{dt} \leq 0 \quad 13.$$

E cannot decrease to infinity and so the distribution function must approach a form for which E has a minimum value and its derivative vanishes. At this value the function has, and can only have, the Maxwellian form. The equation $\frac{dE}{dt} \leq 0$ encapsulates the essence of the H-Theorem. Thus the 'H-function' always decreases under the effect of collisions, unless the distribution is Maxwellian, in which case it remains constant.

Since E, or H, decreases in a strictly monotonic fashion with time, -E clearly increases in the irreversible approach to equilibrium and therefore behaves like the entropy function in the form given by Boltzmann the previous year. Furthermore he was able to produce an explicit form for the minimum value for E which differed by only a multiplicative and additive constant from the expression for the equilibrium entropy obtained in the 1871 study. Thus -E not only behaved like the entropy, it was actually proportional to it in the equilibrium state. This implied, as

70. Probably through a combination of educated guesswork and trial and error. See S.G. Brush (1976) Vol.II p.351 & 449.

71. L. Boltzmann op cit p. 335

72. The proof is quite straightforward and relies on the fact that the quantity $(a-b) \log (b/a)$ is always negative if a and b are real positive numbers. Ibid p. 335-345.

Boltzmann wrote (73), an entirely new approach to proving the Second Law, one that could deal with the increase in entropy in irreversible processes as well as with its existence as an equilibrium state function. Thus the H-Theorem effectively extended the definition of entropy to non-equilibrium situations (74) and thus completed the kinetic proof of the Second Law.

However, this derivation contains a major flaw, as was subsequently realised. In the form as given here the H-Theorem is strictly deterministic. The distribution function, however, is coarse grained in the sense that an infinite number of different possible arrangements of the molecules within each cell of phase space are compatible with the same form for f . This combination of a statistically based distribution function and an apparently fully deterministic theory gave rise to a tension within Boltzmann's work in which statements describing the statistical nature of the H-Theorem are juxtaposed with more frequent passages describing its results in purely deterministic terms.

To carry his derivation through Boltzmann wrote f in a form explicitly dependent upon time, developed a differential equation for its time dependence and then treated its form at t_0 as an initial condition which the appropriate solution of the equation must satisfy. However this treatment, although adequate for a non-statistical function of the system's coordinates, is inappropriate for the problem Boltzmann was considering. Although the form of $f(\alpha, t_0)$ describes the actual distribution of molecular energies at t_0 , its coarse grained nature prevents it from functioning as an initial condition in the strict classical sense. Each of the possible

73. Ibid p. 345-346.

74. See Brush op cit p. 443; Klein (1970) p.102
For a note of dissent see Kuhn op cit p. 42

molecular arrangements which it allows corresponds to a different initial condition for completely specified mechanical system and each results in a different trajectory for that system over time. The actual form given to f at t_0 does not therefore determine its form at later times. (75) Boltzmann's equations for f and H actually determine only average or most probable values and many other rates of change are compatible both with the given initial distribution and mechanics.

Although Boltzmann eventually recognised, through outside criticism, many of the problems which resulted from his initial misconceptions, he never actually revised his proof to eliminate the explicit dependence of f upon t , or to permit a family of f curves to be generated from a given initial form for f . His notable failure to consider fluctuations is due, in large part, to the deterministic way of regarding the time dependence of f and H , which obscured the realisation that physical conditions could exist under ^{which} the theory would break down, such as very high and low gas densities for example.

The alternative approach, first hinted at in 1868, also briefly appears in this

1872 work and Boltzmann considered it to be '... much clearer and much more intuitive'. (76) Again the energy was treated as a discrete variable so that the possible energy values of the system were restricted to the set $0, \epsilon, 2\epsilon, \dots, p\epsilon$, where p is an integer and ϵ is a small energy unit. The Boltzmann Equation was then replaced by a set of ordinary differential equations in time (77), and the forms of f and $\frac{\partial f}{\partial t}$ derived. This is clearly another precursor of the Combinatorial Approach, although the technique, as it stood, was not so general as to be used to obtain an expression for the entropy.

75. Except of course for times so close to t_0 that few, if any, intermolecular collisions have taken place.

76. Boltzmann op cit p. 346

77. i.e. non-linear coupled stochastic equations.

With the exception of this digression this 1872 analysis was based upon considerations of binary collisions between molecules traversing distinct continuous trajectories which, implicitly at least, individuated them. The introduction of the coarse grained distribution function prevents an exact calculation of the separate trajectories but insofar as it is still grounded on a consideration of such trajectories, this only makes the space-time individuality implicit rather than explicit.

2.3.4. Loschmidt's Criticisms

Boltzmann's conclusion, that the H function always decreased, was recognised by Maxwell, Tait and Thompson, as a result that could not, in fact, hold with universal validity, (78) Unaware of these discussions, Loschmidt presented a theorem intended to demonstrate the impossibility of deriving the Second Law from mechanics. He began by noting that in any system '... the entire course of events will be retraced if at some instant the velocities of all its parts are reversed.' (79) If entropy is a specifiable function of the positions and velocities of the particles of a system, and if that function increases during some particular motion of the system, then reversing the direction of time in the equations of motion will specify a trajectory through which the entropy must decrease. For every possible motion that leads towards equilibrium there is another, equally possible, that leads away and is therefore incompatible with the Second Law. Loschmidt concluded that if kinetic theory were true then the Second Law could not be universally valid and therefore Boltzmann's proof of the H-Theorem could not be correct.

78. See the discussions in M.J. Klein (1970) p.84
and E.E. Daub (1970) p.213
79. J. Loschmidt (1876) p.139

Boltzmann himself was quick to realise this last point and replied to this suggestion of a 'reversibility paradox' the following year. (80) In print he conceded little, arguing that the criticism could easily be accommodated within his H-Theorem Approach. In fact, however, he conceded a great deal, and from that moment on was forced to explicitly confront the following problem : how can the manifest irreversibility and uniform entropy increase observed in all natural processes and codified in the Second Law, be reconciled with the view that all thermodynamic systems consist of small particles obeying the time-reversible laws of mechanics?

Thus he admitted that one could not, in fact, prove that entropy increased 'with absolute necessity' and that, according to probability theory, even the most improbable non-uniform distribution is still not absolutely impossible. (81) However, he then argued that the existence of improbable entropy decreasing situations did not contradict the fact that for the overwhelming majority of initial states the entropy could be counted on to increase and that the improbabilities associated with the former case were for all practical purposes impossibilities.

The force of Loschmidt's argument was therefore neutralised through the claim that it actually illuminated the fundamentally probabilistic aspect of the H-Theorem. Boltzmann argued in fact that it followed from this theorem that the number of states leading to a uniform distribution after a certain time interval, must be much larger than the number leading to a non-uniform distribution, since there are infinitely many more uniform than non-uniform states (82). No justification was given for this argument although Boltzmann intimated where such justification could be found: 'One could even calculate the probabilities of the various states from the ratios of the number of ways in which their distributions

80. L. Boltzmann (1877a) p.62, in (1968) Vol.II p.112

81. Ibid p. 122

82. Ibid p.120

could be achieved, which could perhaps lead to an interesting method of calculating thermal equilibrium' (83). This 'interesting method' was subsequently elaborated later that same year.

Thus Boltzmann accepted Loschmidt's deterministic assumption but evaded the criticism with his probabilistic reasoning, providing a further example of the tension noted above. A second, related, point is that the impression given that the paradox has not forced him to introduce any new concepts into his work is very misleading. There is absolutely no indication in the 1872 study that any exceptions to the H-Theorem could be conceived of and indeed Boltzmann had claimed that he had given a rigorous proof that whatever the initial state of the system the entropy must necessarily increase (84). The reply to Loschmidt is therefore deceptive in containing no acknowledgement of this or that a conceptual change has been introduced, expressed, in part by the difference between the H-Theorem and Combinatorial Approaches.

Boltzmann's shift in position regarding his use of probability can be characterised as a move from microprobabilistic considerations, of the probability of the velocity of a single molecule lying within a certain range, for example, to macroprobabilistic ones involving collections of particles. This change was necessary for the further development of the Combinatorial Approach, as we shall see.

Finally, similar conclusions were reached by Tait, Thompson, Maxwell and Gibbs, the latter writing that '... the impossibility of an uncompensated decrease of entropy seems to be reduced to improbability' (85).

83. Ibid p. 120

84. Boltzmann (1968) Vol.I p.345

85. J.W. Gibbs (1961) Vol. I p.167

2.3.5. Boltzmann's Combinatorial Approach

In 1877 Boltzmann presented a new and radical alternative to this 1872 Approach, (86) the goal of which was '... not to limit ourselves to thermal equilibrium but to investigate the relations between these probability theorems and the second law of the mechanical theory of heat (87). This Combinatorial Approach was elaborated in a slow and labourious fashion by taking the reader through a series of computations of increasing complexity and closer approximation to the real physical situation.

His first model was the oversimplified and explicitly fictional discrete energy model previously introduced in 1868 and 1872. Thus he considered a collection of n molecules whose individual energies were restricted to the finite set $0, \epsilon, 2\epsilon, \dots, p\epsilon$. The total energy of the gas was taken to be fixed and equal to $\lambda\epsilon$ where λ , and p , are integers. If ω_k is the number of molecules in the collection with energy $k\epsilon$ then the set of numbers $\omega_0, \omega_1, \dots, \omega_p$ is sufficient to define a particular macro-state (Zustandverteilung) of the gas. (88) Boltzmann noted that such a macro-state could be achieved in many different ways, each of which he called a complexion (89), depending on which molecules possess which energy values. In general if a complexion was specified by a set of numbers each fixing the energy $k_i\epsilon$ of the i 'th molecule, then, he wrote, a second complexion belonging to the same macro-state would be achieved by any permutation of two molecules i and j which have different energies. Thus a permutation of particles between energy states was regarded as giving rise to a different macro-state of the system as a whole.

86. L. Boltzmann (1877b) p. 373, in (1968) Vol.II p.164. We cannot agree with Klein's claim that this study was merely a continuation of Boltzmann's previous work. Klein in Cohen and Thirring (1973) p.77

87. Boltzmann op cit p. 166

88. Ibid p. 168

89. Ibid p. 169

The number of such complexions for a given distribution can be found using well known combinatorial techniques, and was written thus:

$$\mathcal{P} = \frac{n!}{(\omega_0!)(\omega_1!)\dots} \quad (90) \quad 14.$$

where \mathcal{P} is the 'permutability' of the macro-state, not the probability. The most probable macrostate was then found by maximising \mathcal{P} subject to the constraints on the total number of particles and total energy, in the well known way.

So, with this discrete model the new method is very straightforward. A complete specification of a micro-state, or complexion, would require a listing of the energy of each molecule. A macro-state, specified only by the occupation numbers $\omega_0, \omega_1, \dots, \omega_p$ is generally compatible with a number of different complexions, which number was called the permutability measure and is given by the above expression.

As far as this thesis is concerned, the most important point to note about this derivation is that by counting as distinct those complexions obtained by permuting any two indistinguishable particles, Boltzmann clearly, although only implicitly, attributed some form of individuality to the particles. The usual interpretation (91) is that this must be T.I. because, it is claimed, the Combinatorial Approach is free from dynamical considerations. However, as we shall see, this is not strictly correct because in order to justify the assignment of equal a priori probabilities to the complexions, a form of Liouville's Theorem was used.

As for Boltzmann himself, in his later 'Theory of Mechanics' he advocated a version of what we have called

90. Ibid p.176

91 H.R. Post (1963), for example

'space-time' individuality, as we noted in section 2.2.3.

To illustrate what was meant by 'complexion' and 'permutability measure', Boltzmann invoked the image of a large urn filled with numbered slips, the number on each slip determining the number of energy elements to be assigned to a particle. A drawing of all the slips from the urn determined a complexion and the most probable state will be that for which \mathcal{P} is largest.

What Boltzmann was doing here was taking the permutability measure as proportional to the probability of a distribution (92), $\{\omega_0, \omega_1, \dots, \omega_p\}$. To be exact, he set the probability W equal to the ratio of \mathcal{P} for a given distribution to the sum of all values for all allowed distributions. (93) When he took the number of complexions compatible with a given distribution as a measure of the probability of that distribution Boltzmann emphasised that any particular complexion was as likely to occur as any other i.e. all complexions are equally probable.

Returning to the general problem Boltzmann noted that finding the most probable state by maximizing \mathcal{P} was equivalent to minimizing the denominator of $\frac{n!}{(\omega_0!) (\omega_1)! \dots}$ since n , the total number of molecules, is fixed, or, for the sake of computational ease, to minimizing its logarithm. Stirling's approximation and standard variational techniques lead directly to the conclusion that, for $p \gg n$,

\mathcal{P} will be a maximum if the occupation numbers are given by $\omega_k = \frac{nE}{\mu} e^{-sE/\mu}$ (94), where μ is the average energy of a molecule and s is 'Boltzmann's constant'. This specifies the most probable distribution which is what Boltzmann was after.

Giving a realistic interpretation to the finite energy elements, i.e. not going to the limit of infinitesimal

92. In fact, as we shall see, he took \mathcal{P} to be proportional to the logarithm of the probability.

93. Boltzmann op cit p. 169

94. Ibid p. 186

ξ - as Planck, at least implicitly, did in 1900, takes this model from the realm of fiction to that of reality. Thus the above concepts and techniques are, virtually all that is needed to establish quantum statistical mechanics.

Boltzmann however, could only regard the ξ 's as a convenient device and was required to move towards a classically more realistic model by allowing the molecules to possess continuous values of energy.

Thus he began by again dividing up the energy continuum into finite intervals $k\varepsilon$ to $k(\varepsilon + 1)$ and wrote the occupation numbers as $\varepsilon f(k\varepsilon)$ (95) where $\varepsilon f(k\varepsilon)$ is the number of particles with energies in the range $k\varepsilon$ to $(k+1)\varepsilon$, and f is the molecular distribution function. Thereafter he proceeded much as before except that he took the limit $\varepsilon \rightarrow 0$ and wrote the sums as integrals. Standard manipulations then give for the most probable distribution: $f(E) dE = C e^{-hE} dE$ (96) 15.

Boltzmann immediately pointed out that this corresponds to the Maxwellian distribution in two rather than three dimensions. In order to treat the latter one must divide up the three dimensional velocity space rather than the energy continuum, and thus Boltzmann commenced to elaborate the third of his models.

The permissible velocity components u, v, w were specified to lie in the ranges $a\varepsilon$ to $(a+1)\varepsilon, b\zeta$ to $(b+1)\zeta, c\eta$ to $(c+1)\eta$ where a, b, c , and integers. The occupation numbers then become

$$\omega_{abc} = \varepsilon \zeta \eta f(a\varepsilon, b\zeta, c\eta) \quad (97) \quad 16.$$

and the permutability measure is given by

95. Ibid p.187

96. Ibid p.190. See also Kuhn op cit p.50

97. Ibid p. 191

$$\mathfrak{B} = \frac{n!}{\prod_{a=+p}^{a=-p} \prod_{b=+q}^{b=-q} \prod_{c=+r}^{c=-r} \omega_{abc}!} \quad (98) \quad 17.$$

Boltzmann then changed the interpretation of \mathfrak{B} so that it now became the relative frequency of occurrence of the corresponding distribution in velocity space, \mathfrak{B} was then maximised as before, by minimizing the product in the denominator and introducing logarithms for convenience. The problem was thus reduced to that of maximizing the expression

$$\Omega = \iiint_{-\infty}^{+\infty} f(u, v, w) \log f(u, v, w) du dv dw \quad 18.$$

subject to the usual constraints. Expressed in this form the problem had already been solved because Ω , the 'permutability measure' (Permutabilitatmass) (99) is just the negative of the H function introduced five years previously. At that time Boltzmann had proved that H reaches a minimum when f corresponds to the Maxwellian distribution and he felt that he did not need to repeat this in 1877. Reference to this demonstration then completed his proof that the case of thermal equilibrium, corresponding to a minimum for H, also corresponded to the most probable state of the gas.

However, as Boltzmann himself realised, in going over from the discrete to the continuous case, the previous basis for the assignment of a priori probabilities, or statistical weights, had been lost (100). A new procedure is then required and Boltzmann introduced the assumption that equal weights were now to be assigned to equal volumes of molecular phase space, using a form of Liouville's Theorem as a guiding principle for selecting the correct weight function. However, as we shall see, not only did Boltzmann use this theorem incorrectly, the mere fact that it was introduced erodes the claim that this

98. Ibid p. 191

99. Ibid p.192

100. Ibid p.195

approach is entirely free of dynamical considerations.

Thus after progressing through three models of a gas, (101) Boltzmann had arrived at the conclusion that the most probable distribution was the Maxwellian one, already familiar as the description of the equilibrium state. The equilibrium distribution was not only the unique stationary distribution, as had been shown in 1872, it was now revealed as the one most likely to occur, in the sense that it could be achieved in the largest number of ways. The state of thermal equilibrium thus corresponds both to a minimum of the H function and also to the most probable state of the gas.

It has been pointed out (102) that Boltzmann has not fulfilled his promise given in reply to Loschmidt, of justifying the claim that there are infinitely more uniform than non-uniform states. The initial states from which H must increase are not even referred to in this paper, much less shown to be highly improbable. It may be that no theorem like the one he promised to justify could follow from the notion of probability he had used, and indeed that Boltzmann was aware of this.

Boltzmann then took the first of a series of steps leading to the conjecture that the entropy was proportional to the probability of achieving a given macro-state. Referring to his formula for Ω , Boltzmann wrote 'We shall call the magnitude Ω the permutability measure. It differs from the logarithm of the permutability only by an additive constant and it has a special importance for the material to follow. I also note an advantage of suppressing the constant : the total permutability measure of the union of a pair of bodies is then equal to the sum of their individual permutability measures.' (103)

The final part of this quote is important, not only as regards the relationship between permutability measure, and hence probability, and entropy, but also as regards

101. This is consistent with Boltzmann's general philosophical attitude towards physical theories. See L. Boltzmann (1905)

102. By Kuhn op cit p. 52

103. L. Boltzmann (1968) Vol.11. p.192

the difference between Boltzmann's and Planck's conception of this relationship, as we shall see. It is worth noting that the permutability measure is not identical to the probability - for one thing,

Planck's probability is multiplicative whereas the permutability measure, like the entropy, is additive.

Thus we can already see from the statement above, how Ω is to be related to the entropy, and Boltzmann made this relationship explicit in the closing section of his study entitled 'The Relation of the Entropy to the Quantity I have Called the Probability of Distribution'. (104)

He began by substituting an explicit form for f , written in terms of molecular velocity coordinates, in the expression for Ω .

Integrating then gave $\Omega = \frac{3N}{2} + N \ln [V(\frac{4\pi T}{3m})^{3/2}] - N \ln N$ (105) 19.
Using the standard expressions $dQ = N dT + P dV$ and $PV = \frac{2}{3} NT$

Boltzmann wrote down the entropy of the gas as

$$\int \frac{dQ}{T} = \frac{2}{3} N \ln (VT^{3/2}) + C \quad (106) \quad 20.$$

Ignoring the additive constant C (usually considered to be of no physical significance since thermodynamic arguments determine entropy differences only)

he then equated this with the expression for Ω and obtained:

$$\int \frac{dQ}{T} = \frac{2}{3} \Omega \quad (107) \quad 21.$$

Thus the entropy of the gas is equal to two-thirds of the permutability measure.

It should be noted that Boltzmann does not, in this work, equate entropy and probability via $S = k \log W$, as is commonly believed. Rather he equated the entropy with Ω , which was proportional to the log of the permutability, which in turn was related to the probability of a particular distribution, i.e.

$$\begin{aligned} S &= C_1 \Omega + C_2 \\ \Omega &= \log \mathcal{P} + C_3 \\ \therefore S &= C_1 \{ \log \mathcal{P} + C_3 \} + C_2 \quad 22. \end{aligned}$$

where $C_1 = 2/3$, C_2 is the thermodynamic difference, and C_3 is the constant difference mentioned above. It is

104. Ibid p. 215
105. Ibid p. 216
106. Ibid p. 216
107. Ibid p. 216

interesting to note that Planck proceeded by the opposite route in 1900, arguing that the entropy is proportional to the log of the probability, which is proportional to the number of complexions, i.e. the permutability. Thus we have:

Boltzmann: $S \propto \Omega$, $\Omega \propto \log \mathcal{P}$, $\mathcal{P} \propto \text{probability}$

Planck: $S \propto \log(\text{probability})$, $\text{probability} \propto \mathcal{P}$

Thus it seems that Boltzmann had achieved a successful characteri^{sation} of the thermodynamic concept of equilibrium entropy in terms of statistical mechanical principles, obtained through combinatorial techniques. For the first time the entropy associated with a molecular distribution had been explicitly equated with the probability of obtaining that distribution.

The next step was to extend the Combinatorial Approach to non-equilibrium situations and explain irreversible behaviour using the same principles. Thus Boltzmann emphasised that Ω and hence $\log \mathcal{P}$ was well defined whether or not the system was in equilibrium so this function could serve as a suitable generalisation of the entropy (108). In equilibrium situations its behaviour must match that of the entropy as given by the Second Law, i.e. it must ^{either} increase or, for reversible processes, remain constant. This characteristic was then extended to transitions between non-equilibrium states, which do not obey the Maxwellian distribution, and for which the entropy had not previously been well defined. He pointed out that Ω was well defined for both classes of states and that it '... can always be computed; and its value surely will be necessarily greater after the change that before.' (109).

Boltzmann then asserted, as a general theorem, that, for an arbitrary change between states that need not be characterised by equilibrium, '... the total

108. Ibid p.218

109. Ibid p.217

permutability measure of all the bodies will increase continuously during the change of state and it can at most remain constant if all the bodies throughout the transformation approximate thermal equilibrium infinitely closely (a reversible change of state.)¹ (110). This was Boltzmann's reformulation of the Second Law as the statement that systems go from less to more probable states (111), thus extending the meaning of the entropy concept into contexts where it was thermodynamically ill-defined.

However, it has been argued (112), correctly, that this extension is essentially ill-founded, because of a circularity in Boltzmann's arguments. After demonstrating that the state of thermal equilibrium is the most probable state of the gas, and establishing the relationship between the equilibrium entropy and Ω he then argued that for systems in equilibrium Ω must increase or remain constant because it was known from the Second Law that the entropy function does so. As an argument to establish the behaviour of a mechanical concept from thermodynamical considerations this is perfectly valid, but not if it is taken in reverse and used to justify the claim that a mechanical explanation of the Second Law has been provided.

For transitions between non-equilibrium states the extension was based upon two claims : 1) that Ω is always well defined; and 2) that its value will necessarily be greater after the change of state than before. The first is certainly true, but the second stands in need of some supporting argument, which is not given - Boltzmann merely asserts its truth, without reason. It may be that the relationship between the permutability measure and the probability of a distribution convinced him that his arguments were stronger than they actually were. Certain statements

110. Ibid p. 218

111. See Klein op cit p.82

112. By Planck and later Kuhn, op cit p.53

in this paper and in the later 'Gas Theory' (113), suggest that he had in mind the plausible assertion that all natural changes proceed from states of low probability to those of higher probability.

There are one or two other points worth noting. First, as we said above, Boltzmann's relationship was between the entropy and the permutability measure, the latter being related to the probability of a distribution. Planck rewrote this as $S = k \log W$, referring to W as the 'thermodynamic probability' (114).. This is an unfortunate choice of terminology since W cannot be summed or integrated over any conceivable set or space of values to give unity and is thus inconsistent with one of the fundamental axioms of probability theory. The problem arises from the imposition of discrete counting procedures over the continuous range of positions and velocities permitted by classical mechanics. Classical attempts at a resolution required the abandonment of absolute entropies in favour of relative ones. (115) However, quantum theory produced a more satisfactory solution and provided a justification of absolute entropies through the introduction of a unit volume of phase space. Given these difficulties it would perhaps have been more appropriate to call W the permutability measure as Boltzmann did, and reserve the term probability for the quantity relating to molecular distributions.

However, Boltzmann himself created confusion among his contemporaries by switching from micro- to macro- probabilities without drawing attention to the change. (116) Clarification was subsequently provided by the Ehrenfests who characterised the shift in terms of a move from a theory using the 6-dimensional phase space of a single molecule, (' μ - space'), to one

113. L. Boltzmann (1910) p.38-47. It is significant that here the step involving the permutability measure is omitted.

114. M. Planck (1913) English trans. (1959) p.120

115. J.W. Gibbs (1902) Ch. XV. Also see R.H.Fowler (1936) p,203,230

116. See Brush op cit Ch.10 and Klein (1972) p.108-112.

involving the $6N$ -dimensional phase space (Γ -space) of the whole gas of N molecules. (117) They noted that this was essentially a shift from a theory emphasising dynamics and based on the special assumptions about collisions underlying the Boltzmann equation (118), to one emphasising combinatorial methods and independent from collision analyses. The move from micro- to macro-probabilities thus characterises the change from the H-Theorem to the Combinatorial Approach.

A further problem, involving the above distinction, has been identified concerning Boltzmann's use of a weighting function in this 1877 work. (119) We have already noted the importance Boltzmann attached to selecting the correct weight function, in his discussion of the urn model, and his use of a form of Liouville's Theorem (120) to do this. Thus he argued that if the molecules are contained within a cell of a certain volume in velocity space at time $t=0$ then, if arbitrary forces act on them, they will remain in a cell of the same volume as the gas evolves in time. (121)

However, this argument is incorrect for two reasons. Firstly, the molecules of the gas will not remain within the same volume of velocity space as time goes on, but will in fact disperse throughout this space. It is only by going to a higher dimensional space in which the entire gas is represented by a single point—a move corresponding to the shift from μ -space to Γ -space that Liouville's Theorem can be taken to apply and one can say that this point will remain within a phase space volume of the same size at all times, thus supplying the necessary justification of the weighting procedure. Secondly, the theorem does not hold in a phase space constructed to represent the generalized

117. P. & T. Ehrenfest (1912) p.17-39

118. Such as the 'Stosszahlansatz'.

119. See Kuhn op cit p. 54-57

120. Introduced by Boltzmann himself in 1868 (1968)
Vol. 1. p.49

121. Ibid Vol. II p.193

coordinates and velocities, (122) so Boltzmann would also have had to replace his space constructed in terms of the latter by one involving generalised coordinates and momenta. Further discussion should make these points clear.

The mechanical state of a system of N indistinguishable molecules can be represented by an assembly of N phase points in a 6-dimensional (123) ' μ -space', the cartesian coordinates of a point in which is given by components of the molecular position and linear momentum vectors. The distribution of these phase points over μ -space at equilibrium is determined by the Maxwell-Boltzmann distribution law and the distribution function $f(r, p, t)$ gives the density of molecules in this space, in the sense that the number of molecules which at time t are located in the volume element $d^3r d^3p$ constructed about each point (r, p) in this space, is equal to $f(r, p, t) d^3r d^3p$. It is important to realise that this function changes with time because molecules constantly enter or leave a given volume element, or cell, in μ -space (124).

The state, or phase, of the system, as a whole can be represented by a representative point P in a $6N$ -dimensional space, the ' Γ -space', spanned by the vectors $(q_1, \dots, q_N; p_1, \dots, p_N)$. (125) Given some arbitrary position of P at some given time t_0 , it's position at any other time is determined by Hamilton's equations of motion and thus, as time goes on, P describes a trajectory in Γ -space, lying entirely on the energy hypersurface. (126)

122. This is one reason why Hamilton's equations of motion are used. Yourgrau van der Merve and Raw (1966) p.62

123. If the molecules have s degrees of freedom, rather than just 3, then this space has $2s$ dimensions.

124. One can show that the cell volume remains invariant in time only if certain conditions are met. See K. Huang (1963) p. 57-58.

125. Again if the molecules have s degrees of freedom the space is $2sN$ dimensional.

126. P. and T. Ehrenfest (1912) p.18

In order to calculate average values of phase functions it is useful to introduce a large ensemble of virtual copies of the real system and to consider their distribution over this space. (127) This is given by the density function $\rho(p, q, t)$ where (p, q) is an abbreviation of $(p_1, \dots, p_N, q_1, \dots, q_N)$ defined so that $\rho(p, q, t) dpdq$ (128) gives the number of representative points which at time t are contained in the volume element $dpdq$ of Γ -space centred about the point (p, q) . This function completely specifies an ensemble (129) and given some initial value its subsequent values are determined by the dynamics of molecular motion.

Now it can be shown that Liouville's Theorem implies that the rate of change of ρ at a point moving along a trajectory in Γ -space - the 'convected' rate of change- is given by $\frac{d\rho}{dt} = 0$. 23. Thus, ρ , the density in phase, remains constant in the neighbourhood of a representative point as it moves along its trajectory. (130) Furthermore it then follows that the volume element $dpdq$ called by Gibbs the 'extension in phase', occupied by a collection of representative points, remains constant in time. (131) Thus Liouville's Theorem can be taken as stating that if the representative point of some system is contained within a $6N$ -dimensional volume $dpdq$ of Γ -space at time t , then it will be contained in a phase space volume of the same size at all future times. Together with the ergodic hypothesis this theorem can be used to prove that equal volumes of phase space are equiprobable. Some such justification of the 'correct' - in the sense that it leads to a theory accounting for the observable phenomena - weighting assignment is necessary if a representative ensemble is to be constructed and the

127. This approach is due mainly to Gibbs although elements of it also appear in works by Maxwell and Boltzmann. Each system differs in phase and so the ensemble spreads out in a cloud over a region of Γ -space.

128. Yourgrau et al op cit p.60

129. Huang op cit p. 76

130. Thus the distribution of P points moves in Γ -space like an incompressible fluid.

131. P. and T. Ehrenfest op cit p.20 & p.87. TerHaar op cit p. 102

average values of phase functions, corresponding to measurable properties of the system, are to be calculated. (132)

An analysis of the possible weight functions which were allowed, and of this justification procedure in general, was given by the Ehrenfests in their 1912 article (133). They considered two ensembles in

Γ -space and supposed that the representative points of the first occupy, at successive times, the regions A_1, A_2, A_3, \dots in this space and that those of the second ensemble occupy, at the same times, the regions B_1, B_2, B_3, \dots . The measures of the relative probabilities A_i and B_i , i.e. the measures for the relative frequency of occurrence of a distribution were then written as $M(A_i)$ and $M(B_i)$ respectively. Clearly only those measures which give the same value for these relative probabilities for all times are acceptable and this requirement should also hold for any arbitrary choice of the two initial regions A_1 and B_1 . The possible choices of M must therefore satisfy the requirement

$$\frac{M(A_1)}{M(B_1)} = \frac{M(A_2)}{M(B_2)} = \dots = \frac{M(A_i)}{M(B_i)} \quad (\text{I}) \quad (134)$$

Boltzmann's choice for this measure of relative probability was the volume of the corresponding region of Γ -space; thus $M(A_i) = V(A_i)$; $M(B_i) = V(B_i)$ where $V(A_i)$ and $V(B_i)$ are respectively, the volumes of the regions A_i and B_i i.e.

$$V(A_i) = \int_{A_i} dp dq, \quad V(B_i) = \int_{B_i} dp dq$$

Now Liouville's Theorem implies that

$$\frac{V(A_1)}{V(B_1)} = \frac{V(A_2)}{V(B_2)} = \dots = \frac{V(A_i)}{V(B_i)} \dots \text{ and condition}$$

(I) follows immediately by substitution. Thus if Liouville's Theorem is true then Boltzmann's assignment of equal probabilities to equal volumes of phase space satisfies the invariance requirement (I).

It is also interesting to note that if the

132. Ibid p.129-130

133. P. and T. Ehrenfest op cit p. 95-97

134. Ibid p.96. Ter Haar op cit p.100 and p.380

generalized velocities are used in the construction of Γ -space, rather than the momenta, then requirement (I) would not, in general, be satisfied. Thus one must use not only Γ rather than μ -space, but p rather than \dot{q} , and Boltzmann's justification fails on both counts.

Liouville's Theorem is not all that is required however. One could take for the probability measure

$$M = \int F(E, \varphi_2, \varphi_3, \dots, \varphi_{2SN-1}) dp dq \quad (\text{II})^{(135)}$$

which also satisfied (I) for any arbitrary F function, or phase function, dependent on the energy E and $2SN-2$ integrals of motion φ_i , provided that the F is chosen once and for all. It can then be shown that this expression gives the most general way of selecting M . (136). Thus Liouville's Theorem and (I) taken together exclude all possible probability measures except those in the class defined by (II). This still leaves a range of possible choices and to narrow this down further to Boltzmann's measure other restrictions must be introduced.

Thus the ergodic hypothesis, which states that there exists systems which are ergodic in the sense that the trajectory of the representative point of such a system in Γ -space, taken from $t = -\infty$ to $t = +\infty$, passes through all points on the energy hypersurface, (as we shall see a proof of the impossibility of such systems was given in 1913 and attention then shifted to the quasi-ergodic hypothesis in which the trajectory on the energy hypersurface comes arbitrarily close to all points on this surface), restricts the choice of density functions to those dependent on the energy only and therefore the class of possible measures is restricted to those for which

$$M = \int F(E) dp dq$$

If the further requirement is added that the systems are isolated so that the ensembles possess the same

135. Ibid p.96 Ter Haar op cit p.100 and p.380

136. Ehrenfest p.21 and ter Haar p. 103

energy E , then results identical to those obtained using the general ergodic measure $F=F(E)$ can be obtained by taking $F=1$. (137) In this case $M = \int dpdq$ and Boltzmann's weighting assignment is finally arrived at.

The complete justification for this assignment of equal probabilities to equal volumes of phase space is therefore as follows: Liouville's Theorem and the plausible requirement (I) exclude all probability measures except those in the class given by (II), of which Boltzmann's measure is the simplest case. The ergodic hypothesis then further restricts the set of possible measures to those given by $M = \int F(E) dpdq$. Finally, if it is assumed that the systems are isolated then this reduces to $M(A_i) = V(A_i)$ and $M(B_i) = V(B_i)$ as required. (138)

Boltzmann's justification for his particular choice of probability measure rested on the claim that if there were initially N molecules in a cell in μ -space then they would all move together with the passage of time and be contained within a cell of the same volume always. This is only true, however, if the molecules are restricted to interactions with fixed scattering centres and intermolecular forces are zero, which is completely unrealistic. (139). Liouville's Theorem then applies, not to the cell in μ -space but to the cell containing the representative point specifying the state of the entire system in Γ -space. It is the volume of this cell which remains invariant with time, whereas molecules initially in the same cell in μ -space will be widely dispersed over this space as the system evolves. This is the whole crux of the matter.

137. Ehrenfest p.96

138. Since ergodic systems cannot actually exist some alternative justification must be sought. Thus one could use the observable consequences of Boltzmann's assignment to justify it, or simply include this assignment as part of the axioms of the theory.

139. In a real gas all molecules can interact and none may be treated as fixed during a collision.

As it stands, therefore, Boltzmann's argument permits no conclusions to be drawn concerning the relative probability of the different possible space locations of the molecules. This is a serious deficiency in his theory as a whole which was not noticed by his contemporaries nor corrected in the 'Gas Theory' (140). Indeed Kuhn has suggested that Boltzmann's attribution to the cells of phase space of the non-statistical behaviour which could not be attributed to the molecules is somehow symptomatic of his reluctance to completely relinquish deterministic dynamical considerations in general (141).

Boltzmann's switch in attention from the molecules to the phase space cells surrounding them, which allowed him partially to circumvent Loschmidt's criticism and preserve something akin to his original deterministically phrased version of the H-Theorem, is an expression of a general conflation of three different notions concerning the distribution of molecules. The first, which can be termed 'molecular,' consists of those concepts determined by the precise specification of the position and velocity of each particle within a cell. The second notion applies to the distribution of molecules within cells and considers collectives of trajectories rather than the individual molecular trajectories themselves. Such concepts include, for example, the f-function and are concerned with micro-probabilities. The third group applies to the distribution of molecules over cells, again with nothing specified about molecular positions within cells and includes macro-probabilistic concepts like the permutability measure. (142)

Clearly the first and second sets are underpinned, strongly and weakly respectively⁽¹⁴³⁾, by the notion of space-

140. L. Boltzmann (1910) p.40

141. Kuhn op cit p. 57

142. In 1896 Boltzmann subsumed both of our second and third sets under the general term of 'molar' concepts.

143. Concepts in the second set involve groups of trajectories taken together, and in this sense are weakly underpinned by s-t individuality.

time individuality whereas the third is not necessarily, and thus opens itself to an alternative, such as T.I. It was only by moving back and forth between these three sets of concepts that Boltzmann was able to preserve for so long a predominantly deterministic way of discussing his H-Theorem in conjunction with a micro-probabilistic way of considering it, together with the macro-probabilistic Combinatorial Approach. There was thus a three-way tension in his work, (144) which is manifested in both his replies to Loschmidt and the strict failure of the mechanical reduction of the Second Law.

It is also worth noting that by *eliminating* time as a variable in the description of a system the Combinatorial Approach replaces a consideration of the deterministic dynamics of a system by a process involving random choice. Thus in carrying out the calculation of state probabilities Boltzmann assumed '... that the kinetic energy of each individual molecule is determined, as it were, by a lottery which is selected completely impartially from a collection of lotteries which contains all the kinetic energies than can occur in equal numbers' (145). This is where the power of this approach lies; it is not restricted to particular molecular models for which collision mechanics must be worked out in detail, but can be used for any system for which the spectrum of possible energies is known. To a large extent this explains why this Approach was almost exclusively (146) used in the development of quantum statistical mechanics. The above difference between the two approaches as regards time is reflected in that between a 'timeless' T.I. and space-time individuality.

144. Compare with Kuhn op cit p.57, who argues for a two-way tension.

145. L. Boltzmann (1968) Vol. II p. 172

146. We say 'almost exclusively' because, as we shall see in the next chapter, attempts were made to base quantum statistics on the alternative H-Theorem Approach, although with less than total success

In conclusion we note that by stating that a distribution is compatible with a number of different complexions and by counting as distinct those obtained through a permutation of the indistinguishable molecules, Boltzmann, at least implicitly, attributed some form of individuality to these particles. We have noted that in his later work on mechanics he argued in favour of some form of space-time individuality as he believed that the assumption that indistinguishable particles which cannot come infinitely close to one another can be distinguished by their initial conditions and by the continuity of their space-time trajectories'... gives us the sole possibility of recognising the same material point at different times.' (147) However, on the basis of the 1877 paper one can perhaps conjecture that, at this time, he actually had something more like T.I. in mind. For example he wrote that the Combinatorial Approach enabled one to determine the probability of a distribution in a way '... completely independent of whether or how that distribution has come about' (148). In other words the history of a distribution does not determine the probability with which it can be realised.

T.I. would have been sufficient if Boltzmann had accepted the weight assignments as simply part of the axioms of the theory. However, he did not and by appealing to Liouville's Theorem (albeit incorrectly!), he introduced a dynamical component based on considerations of continuity of path. Thus the Combinatorial Approach required some consideration of the history of the assembly as a whole, but not of the individual molecules in determining the probability of a distribution.

Thus we conclude that the principle of space-time individuality receives clear historical support from

147. L. Boltzmann (1974) Vol. II p.9.
 148. L. Boltzmann (1968) Vol. II p.168

the H-Theorem Approach but that the situation regarding T.I. and the Combinatorial Approach is rather more ambiguous. If the assignment of weights was included in the axioms of the theory then T.I. would be required by this Approach as a Principle of Individuation. If, on the other hand, it is felt that some justification of these weights must be sought for, then an historical component must be introduced, but only for the whole system not the molecules themselves. Thus space-time individuality is again required, but now as a principle of individuation for the entire assembly. It is in this sense then, that both of the classical views of individuality receive historical support.

2.3.6. The Decline and Fall of the Combinatorial Approach

After the publication of the 1877 paper it seems that Boltzmann again felt that he had settled the problem of the foundations of the Second Law and had done so by introducing a new approach to the subject which he then proceeded to apply to a number of interesting problems.

Thus in two papers in 1878, Boltzmann used this approach to give a direct and rigorous method for deriving the Maxwellian distribution, (149), and also to obtain, for the first time, an expression for the entropy of diffusion in gases which led to agreement with thermodynamic calculations (150). It is precisely because it led to novel theoretical predictions not contained in the laws it sought to explain that the work based on the Combinatorial Approach was a much more successful reduction of the Second Law than that

149. L. Boltzmann Ibid p. 250

150. Ibid p. 289

produced in 1871 (151).

However despite the obvious power and success of this approach Boltzmann did not develop it further. When he returned to statistical mechanics in the 1890's, amidst the furore generated by the 'English School' and the criticisms of Zermelo, (who argued that, left to itself, any mechanical system must eventually return to a configuration arbitrarily close to the one from which it began, and therefore it was not possible to give a mechanical proof of the Second Law) (152), it was the H-Theorem which he turned to and further developed in an effort to counter his critics (153). Thus the Combinatorial method was essentially just a response to Loschmidt's criticism and although it found some applications in 1878, it was not thereafter significantly used or developed any further.

Indeed it was not until the importance of Planck's derivation of the black body law came to be acknowledged, with its explicit use of combinatorics and references to Boltzmann's 1877 paper, that the Combinatorial Approach really began to attract attention. BY 1911, however, the Ehrenfests were able to report that '... the last few years have seen a sudden and wide dissemination of Boltzmann's ideas.' (154) With the realisation of its great computational power and independence from specific molecular models, this approach was called upon to play a predominant role in the development of quantum statistical mechanics, as we shall shortly see. What still requires explanation however, is why it received so little attention at the end of the 19th century, particularly from the very

151. Daub has given an excellent account of both reductions, and of the development of Boltzmann's ideas in general. E.E. Daub (1969)
152. E. Zermelo (1896) p. 485
153. Kuhn op cit p. 70 Klein (1970) p. 125
154. P. T. Ehrenfest op cit p. 67. Ehrenfest must also take a lot of the credit for this sudden surge of interest.

man who created it. (155)

2.3.7. Ergodic Theory - A Possible Bridge Between the Approaches

One can distinguish the Combinatorial and H-Theorem Approaches by the meanings which Boltzmann gave to the term 'probability' within each. In the former he meant a quantity measured by phase volume, whereas in the latter, at least by the time of the 'H-curve' discussions of the 1890's, he meant something measured by frequency in time. If it could be shown that these two meanings were in fact equivalent then the conceptual gap between the two approaches could be closed. It is just such an equivalence which is established by the ergodic hypothesis, which states, as we have noted, that in the course of time a system will pass through every point on the energy hypersurface in Γ -space. It then follows that the fraction of time spent by the system in some region of this space will be proportional to the volume of that region and the average over an ensemble of systems all with the same energy will be equal to the time average taken over a suitably long interval.

Thus, if the existence of ergodic systems could be demonstrated then the two meanings of 'probability' above would be interchangeable and the combinatorial arguments of 1877 could then be reinterpreted in a kinetic sense to explain the evolution of a system in time. One would then be justified in claiming that the Maxwellian distribution will predominate overwhelmingly in time over all other appreciably different state distributions.

155. Boltzmann mentioned it briefly in only four papers after 1878: (1879) p.653, (1880) p. 529. (1881) of which two are brief replies to Mayer, one is a short elaboration of some aspects of probability theory and only one contains a physical application.

A programme for constructing just such a bridge between the two approaches was laid by the Ehrenfest's in 1911. Essentially one must relate the H-function written in the form

$$H(t) = \int d\underline{v} f(\underline{v}, t) \ln f(\underline{v}, t) \quad 24.$$

where $f(\underline{v}, t) d\underline{v}$ is the number of molecules per unit volume which at time t possess velocities in the range \underline{v} to $\underline{v} + d\underline{v}$ to the volume $[z]$ in Γ -space given by

$$[z] = \frac{N!}{a_1! a_2! \dots a_k! \dots} \omega^N \quad 25.$$

where ω is the volume of the cells in μ -space, each cell containing at any time t a_k of the N points representing the phases of the molecules. Thus the set of occupation numbers $\{a_k\}$ determines the distribution Z at time t and is equivalent to the molecular distribution function f .

Clearly although Z is uniquely determined when the representative point in Γ -space is specified, the converse is not true. Each Z corresponds to a domain in Γ -space because the molecular phase points are only placed in finite cells when Z is given and also because Z specifies only the numbers of molecules in each cell and not which molecules are in which cell. Given this the volume $[z]$ corresponding to the distribution Z can be obtained from the equation above.

$[z]$ is then taken to be a measure of the probability of the distribution Z , as Boltzmann did in 1877, with the equilibrium distribution possessing a larger Γ volume $[z]$ than any other for the same value of energy. This is the sense in which the equilibrium distribution can be regarded as the most probable one. Boltzmann had also argued but had not proved that the value of $[z]$ corresponding to any distribution other than this would be very much less and thus that the equilibrium distribution was overwhelmingly the most probable. These results were then appealed to in order to justify his arguments concerning the behaviour of a system over time thus

shifting the meaning of probability from the first to the second sense above.

Thus Boltzmann himself crossed from one approach to the other (156) and indeed used a form of ergodic hypothesis in several of his works, (157) although he never actually built the kind of bridge envisaged by Ehrenfest.

If this hypothesis is accepted, so that the two meanings of probability could be regarded as interchangeable, then H could be expressed in terms of Z , as given by the $\{a_k\}$ thus,

$$H(z) = \sum_k a_k \ln a_k \quad 26.$$

The change in time of this function as effected by changes in the a_k 's then no longer continuous but given in terms of discrete steps. A plot of H against time is now a step function and Boltzmann's H -curve can be interpreted as a discrete set of points chosen from this function at regular intervals Δt , taken to be short compared with any experimental scale but long compared with the mean time between collisions (158). On this interpretation Boltzmann's macroprobabilistic version of the H -Theorem, as expressed in his later works, is really a series of assertions about the properties of the bundle of H -curves corresponding to a given initial distribution which radiate from a point $t = t_A$, the initial time in the (t, H) plane.

Thus a proof of the ergodic hypothesis would allow H to be related to Z through the equation above, which constitutes the basic structure of the bridge between the Combinatorial and H -Theorem Approaches.

However, such a proof cannot be given, as is well known.

Although versions of this hypothesis had been introduced by both Boltzmann and Maxwell, (159) and had been extensively discussed towards the end of the

156. This may have been a factor contributing towards his neglect of the Combinatorial Approach.

157. L. Boltzmann (1968) Vol II p. 582 and Vol. III p.122

158. C.f. the Ehrenfests discussion of the urn model in 1906.

159. Boltzmann op cit ; J.C. Maxwell (1879) p.547

19th century, it was the Ehrenfests who sharply formulated the existence problem for ergodic systems and who posed the question: can any deterministic mechanical system be ergodic as distinct from quasi-ergodic? (160)

In 1913 this was answered in the negative by Plancheral and Rosenthal who independently produced proofs of the impossibility of ergodic systems.

(161) Both proofs hinged on the fact that the ergodic hypothesis implies that a multi-dimensional region of the energy hypersurface must be mapped onto a line of finite length, continuously and one to one. Rosenthal showed that this is impossible according to Brouwer's proof of the invariance of dimensionality under continuous one to one mappings, whereas Plancheral argued that Lebesgue's theory of measure also forbids such a mapping since a line is a set of measure zero with respect to a region of two or more dimensions.

Thus it was shown to be impossible for the representative point of a mechanical system to pass through every point on the energy hypersurface in Γ -space and so such a system cannot be ergodic.

Quasi-ergodicity and the question whether, or what, actual systems could be described as quasi-ergodic, then became the subject of interest, but we shall not discuss these developments here. (162)

Our conclusion then is that Boltzmann's two meanings of probability cannot be formally equated and thus, strictly speaking the Combinatorial and H-Theorem

160. The 'quasi-ergodic' hypothesis states that the phase point of the system passes arbitrarily near each point of the energy hyper-surface in Γ -space.

161. M. Plancheral (1913) p. 1061 Eng. trans in Brush (1971) p. 287.

A. Rosenthal (1913) p. 796 Eng. trans in Brush (Ibid)

162. See ter Haar (1955) p. 302 ff and ^{IE} Farquhar (1964). Lavis also gives an excellent discussion of ergodicity and quasi-ergodicity in the context of more recent programmes in ^{IE} statistical mechanics. D. Lavis (1977) p. 255.

Approaches remain separated, conceptually and formally.

2.4 The Individuality of Fields

Although, as we have noted in the sections above, classical particle mechanics can be given a field theoretic formulation, one would expect there to be profound differences between a principle of individuation for fields and one for particles. It is our intention in this section to examine these differences and to briefly discuss the individuality of fields in general.

In particle mechanics corresponding to a discrete physics, the physical quantities, such as positions and momentum, are functions of time only. In a field theory, however, corresponding to continuum physics, the physical quantities in the field, such as energy, are functions of both time and space. As we noted at the beginning of this chapter, any point in the field is associated at a given time with certain quantities which may be scalars or vectors, depending on the type of field considered. It can then be argued either that the fields are no more than properties of the points of space-time, and therefore can only be regarded as individuals if the latter can be so regarded, or that they are substantial in a sense to be made clear shortly and can thus be taken to exhibit T.I. In the material to follow we shall consider both positions in more detail and in particular we shall present a series of arguments against the latter view.

The argument that fields are individuals derives from the idea, noted in the Introduction to Chapter One, that the concept of an individual captures what we mean by a particular. Fields are particulars - we speak of the electro-magnetic field, for example, and hence should be regarded as individuals. The fact that they are 'global' particulars, in the sense of not being

localised in space is not an argument against this view since the size, or spatial extent of a thing is merely a property of that thing and thus cannot preclude a thing being regarded as an individual. It is not necessary for a particular to have a definite spatial position at a given time (163).

Thus according to this view a field is an individual and can be assigned a label, or proper name, which designates the substance underlying the properties. Fields can be regarded as substantial and thus can be taken to possess T.I. However, an objection now arises because one of the most ~~striking~~ developments in late 19th and early 20th century physics was the emancipation of the field concept from the notion of a material substance (164). Arguments based on this historical observation tend to be of an essentialist nature because they utilise a possible answer to the question 'what is the essence of substance?' to justify the exclusion of fields from the category of substance. Thus, for example, one possible answer would be to say that an essential characteristic of substances is that they are, at their most fundamental level, impenetrable. Fields are not impenetrable therefore they cannot be regarded as substantial.

However, although the premise is certainly true, one can avoid the conclusion by rejecting this essentialist line and consider instead the role which the concept of substance has played in physics.

Thus, whereas Descartes regarded spatial extension as the essential attribute of matter, Newton extended the set of universal properties to include extension, hardness, impenetrability, mobility and inertia (165).

163. According to this argument, ^{samples of} the chemical elements can also be regarded as individuals with the distinction between elements and atoms being one of continuity and discontinuity rather than non-individuality and individuality.

164. Regarded as the elastic fluid aether, in various forms. P.M. Harman (1982) p.72, gives a good account of these developments.

165. I. Newton in A.R. Hall and M. Boas Hall eds. (1962) p. 132-144.

The last, in particular, was given an elevated status and was considered to be both an essential and a universal property. Leibniz argued that both active and passive forces, such as impenetrability, were phenomenal manifestations of primitive forces which characterised the nature of substances. (166)

This emphasis on the status of force as a defining characteristic was then taken up by Kant (167) who argued that forces were essential properties of substances. Gradually this rather vague notion came to be split in two, with one aspect developing into the dynamic concept of force as used today and the other evolving into the concept of energy. Thus during the 19th century energy took over impenetrability's role as the essential characteristic of substance.

Faraday rejected Newtonian atomism and argued that the forces between bodies were mediated by an ambient field between them - a concept which was explicitly based on a metaphysics of substance in which force was regarded as the defining property of matter (168). This idea was then further elaborated by Maxwell who produced a highly mathematised and coherent field theory, with a correspondingly complicated metaphysics. As far as this gloss is concerned it is sufficient to note the following points. Maxwell distinguished the substantial concept of matter as the material substratum constituting physical reality, from the functional concept as defined by the symbolism of mechanical principles. Thus his analytical formalism of the latter did not give a representation of the structure of the hidden mechanism constituting the electromagnetic field, nor did it consider the forces between the particles of this mechanism. (169)

166. G.W. Leibniz in C.I. Gerhardt ed. (1849-63)
 Vol. 6 p.236-8, p.241
 167. I. Kant (1902) Vol.I p.139
 168. M. Faraday (1843)
 169. J.C. Maxwell (1896) p.527

Maxwell also emphasised that the field was a repository of energy, maintaining that energy could only exist in connection with material substances and concluding that the aether was the repository of the energy of the electromagnetic field. Clearly Maxwell regarded this field as a material substance of some kind; unlike the other mechanical analogues of electromagnetic quantities the energy of the field was to be taken literally. (170) It is interesting to note that in his work on gas theory he had rejected '... the doctrines that all matter is extended and that no two portions of matter can coincide in the same place ...' (171).. If the Impenetrability Assumption is rejected for material substance in the form of atoms then it is obviously just a small step to go even further and declare that fields, which also violate I.A., are also substantial.

Maxwell carefully distinguished the substantial concept of matter from the dynamical, and believed that although the former was unknowable, the latter, regarded as the recipient of momentum and energy, formed the basis of physics. (172) In his view energy could not be conceived of as independent of the substance in which it existed and therefore as field theory is concerned with the distribution of energy in the field, this field must be substantial.

Thus the role of substance within physics slowly changed from that which is impenetrable to that which is the recipient and carrier of energy. Subsequent developments after Maxwell continue to support this view. Thus the monistic theories of Thomson and Larmor reduced matter to a particular state of the aether, (173) a programme which led to the electro-

170. Maxwell Ibid Vo.I p. 529, and p. 564

171. Ibid p.33 and p.448

172. Maxwell (1873) Vol.II p.181

173. See, for example, B.G. Doran (1975) p.133

magnetic theories of matter of Abraham, Wien, Mie et al, in which matter was regarded as simply a local concentration of the electromagnetic field. The opposing dualistic tendency resulted in theories such as Lorentz's (174) involving both charged particles of matter and a non-mechanical aether. The latter was just the Maxwell fields of electric and magnetic force and was also regarded as substantial because it carried energy.

Although Einstein's Special Theory of Relativity demonstrated the redundancy of the aether for electromagnetic field theory it also gave support to the substantial nature of energy by associating it with inertial mass via $E = mc^2$. The Equivalence Principle of General Relativity then took this association one step further by relating energy to gravitational mass. G.R. initiated a monistic programme to construct a unified field theory in which the electro magnetic field, as well as the gravitational, would be given geometrical significance, and matter would again be identified with local concentrations of this unified field.

Thus if substance is regarded as that which carries energy, then, since fields carry energy they must be substantial. Fields then become amenable to a Lockean description in which they have T.I. attributed to them. Against this conclusion, however, one can argue that closer examination of the nature of energy reveals serious deficiencies in the view that it should be regarded as substantial. This can then be used to support the alternative view that fields are no more than the properties of the points of space-time and one could no more give a field an individuating label than one could so label the colour red, for example.

With a view to rescuing fields from this conclusion, it may be asked whether properties are, or can be, individuals. The answer is no. To say otherwise

174. See R. McCormach (1970) p. 459

would be to confuse two categories which should be kept distinct. Properties are not perceived on their own, disembodied; they are merely general predicates instantiated in certain cases by individuals. This distinction is reflected linguistically in the separation of subject from predicate which suggests that the individual is related to 'its' properties as an owner is to his/her possessions.

The correctness of this suggestion is confirmed by the inability of a thing's properties, which are general and applicable to many things, to individuate it, i.e. to pick that thing out as the unique individual that it is (175).

Properties cannot even serve to individuate, for the reason already given. They are predicable of many individuals and so cannot serve to individuate any one of them (176). A possible exception is position in space-time, but this involves an ineliminable reference to another individual, such as the 'here-and-now', which grants this property a special status. This if fields are regarded as properties of space-time points then they cannot be individuals, nor can they serve to individuate these points. As regards the argument that fields are substantial because they are characterised by the presence of energy, the following points should be noted.

Firstly, it has been argued that the energy of a system can be used to reidentify that system (177) and that energy therefore provides the fixed point of reference which is invariant through change and which is necessary for the world to be comprehended.

175. If things are regarded as bundles of properties instantiated at points in space-time then either these points must be taken as the individuator in which case our conclusion stands, or the Principle of Identity of Indiscernibles must be invoked, in which case it is the collection of all properties which individuates, not the separate properties themselves.

176. See Quinton op cit p. 12 ff or K. Popper (1959) p.66

177. D.Theobald (1966) p. 151 and ^RFowler (1926)

(178). It is then argued that since ⁱⁿvariance through change is a fundamental characteristic of the substantial substratum, energy should be regarded as substantial and as serving to ^{re}identify.

However relativity theory implies that energy is a conserved scalar quantity ^{only} within a given frame of reference and is not itself universally conserved under general coordinate transformations. As far as Lorentz covariant physics is concerned the true scalar invariant is a function of momentum and energy (179), Although the energy of a system may not vary with time with respect to any chosen frame of reference it will vary if the reference frame is changed. In particular energy is not an invariant of the field; the invariant quantities associated with the electro-magnetic field are in fact scalar quantities such as $\underline{E} \cdot \underline{H}$ and $E^2 - H^2$, where \underline{E} and \underline{H} are the electric and magnetic vectors respectively, obtained from the six component antisymmetric field tensor.

This argues against regarding the energy ^{as} substantial, at least in ^{the sense in} which it is taken above (180). In particular the energy of a system cannot always serve to reidentify that system because a change of reference frame will produce a change in the observed energy value of a system, which cannot therefore be used to 'tag' and hence reidentify that system.

A second argument is that energy is at least like the substantial substratum in being a primitive concept which cannot be represented by a model or said to possess any structure. However, this does not necessarily imply that energy must be regarded as substantial, since it could be viewed as merely a property of systems. Thus we speak of particular system possessing so much energy and being possessed by something may be taken to

178. E. Mach (1942) p.607

179. In the case of a free particle this function is $E^2 - C^2 p^2$ where E is the energy and p the momentum.

180. See P.W. Bridgman (1961) Ch.1.

imply being a property of that thing. Substance is not possessed in this way and so energy should not be regarded as substantial.

Thermodynamical considerations of a 'flow of energy' may also suggest that energy is substantial (181) and in this case problems associated with a change in reference frames do not arise since heat is random kinetic energy and is unaffected by such changes. But then statements concerning the v_{λ}^e velocity of heat 'flow' and energy 'flow' in general make no sense. Assignments of absolute values to energy would also support the substantivalist position but in thermodynamics these depend on the assumption that substances can be obtained in their lowest energy forms at an absolute zero of temperature, and it is possible that these substances may possess other forms of lower energy at this temperature.

Electromagnetic field theory appears to provide firmer support for absolute energy values since the field energy apparently has a natural zero when $\underline{E} = \underline{H} = 0$. However, this also is not conclusive since only those processes in the field involving energy differences are of physical interest and therefore no physical consequences result from the addition of an arbitrary constant to the expression for the energy density. Obviously the energy will be zero in the above situation since then there is no field anyway! (This conclusion is true in classical field theory but not in QFT where the energy eigenvalue contains a constant term giving the so called 'zero point energy' of the field, which is present even when the excitation numbers of the field modes are zero).

One can also cast doubt upon the supposed acquisition by energy of a substantial nature through Einstein's mass-energy relation. (182) This does not, in fact,

181. This suggestion was embodied in the caloric theory of heat.

182. See Ruddick's discussion in Munitz op cit p. 241

explicitly assert the identity of mass and energy, since such an identification is patently false. All that Special Relativity requires is that a change in energy corresponds to a change in mass and vice versa. It does not demand that they possess the same set of properties and clearly they do not. (183) Energy is a measure of matter's ability to do work and undergo change, whereas mass, being a measure of the inertia of matter, can do no work by itself and is thus a measure of matter's ability to resist change. The two concepts are thus concerned with contrary aspects of reality and the fact that high concentrations of energy display inertial properties and that mass can be transformed into energy does not justify the statement that they are identical, at least not in the sense of being predicatively equivalent.

Although the electromagnetic field is characterised by the presence of ^{localised} energy it is not clear whether one can say the same for the gravitational field. Thus it has not yet been conclusively established whether matter in motion propagates energy as charges in motion do and whether this energy is radiated in the form of waves. However, the gravitational field may acquire a substantial status in another way. General Relativity identifies this field with the structure of space-time itself, and as we have seen, there are good grounds for adopting the substantivalist position with regard to the latter. This identification then presents an obstruction to any putative unification of the gravitational and electromagnetic fields since this would require the unification of the two forms of waves involved, the electromagnetic waves in space and the gravitational waves of space. It would have to be decided whether the space-time metric depends totally upon the matter present or not. If it does then it would not be possible to distinguish between

183. We are denying here the unification of mass and energy, both in the sense of identification and as being different manifestations of something else. For a brief discussion of the different meanings of 'unification' see M. Redhead (1984) p.274

matter as the source of the field and matter as dependent upon the field, and the energy field would then become the most fundamental 'ultimate' (184) of reality.

We conclude then that strong arguments can be given against the view that energy, and hence, also the electro-magnetic field, (185) can be regarded as having, in some sense, a substantial nature. This leaves open the possibility of adopting the alternative namely that fields are simply properties of the points of space-time and it is the latter which should be regarded as substantial. Thus space-time and the electro-magnetic field stand in the same relation to one another as do substance and property in the traditional accounts of these notions.

Just as fields cannot be said to possess T. I neither can waves in the field, nor can they have space-time individuality attributed to them, at least not in the same form with which it can be attributed to particles.

Waves in the electro magnetic field, for example, clearly satisfy the first requirement of space-time individuality, that of continuity. Maxwell's equations can be regarded as the equations of motion of the electromagnetic field and thus characterise the fact that as the wave propagates it traverses a continuous space-time trajectory. (186) The field equations ensure continuity, just as the equations of motion do for classical particles. (187)

However it is equally clear that such waves do not satisfy the Impenetrability requirement, since in general any particular point of space-time can support more than one field. For fields a complete, spatial and temporal, position can be polygamous, to use Quinton's terminology. This failure thus destroys the efficacy of position as individuator since to state the spatio-temporal location of a field will not uniquely distinguish it and will not therefore necessarily

184. Quinton devotes a whole chapter to this idea of 'ultimates' which is good philosophically but rather thin on the physics. op cit p,81.

185. Presumably the same conclusions apply to the weak nuclear field also.

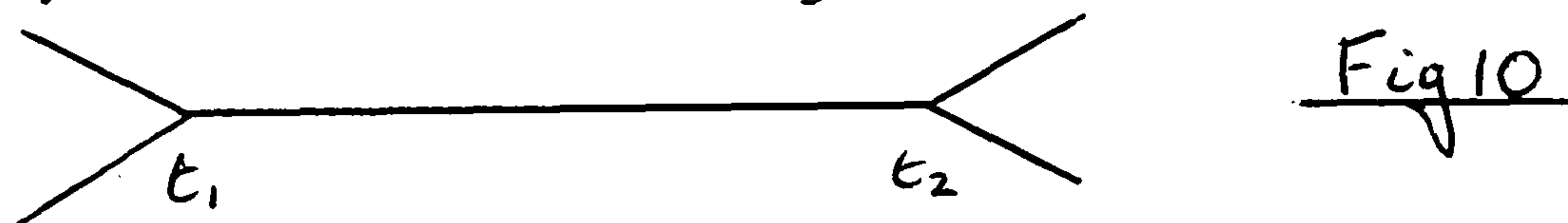
186. J. Jackson (1975)

187. A. Einstein (1961) p.146

individuate it. Obviously the form of space-time individuality attributable to particles cannot be attributed to fields or to waves in fields.

The latter also encounterⁿ severe problems with regard to their reidentification through time, principally because of this failure to satisfy I.A. For example, if two indistinguishable waves meet, pass through one another and then separate then simple observation cannot supply an answer to the question which wave is which after this interaction. The criterion of spatio-temporal continuity although satisfied, cannot help because it is consistent with either of the two answers. In other words in situations where two, or more world lines intersect because I.A. is not satisfied, continuity alone cannot ensure reidentification. This is because at the actual point of superposition the two waves cannot be distinguished and this it is no longer possible to uniquely trace the career of each one as a succession of wave stages under some sortal term.

Thus, at the actual point of crossing there is no way of telling which wave is which and one cannot individuate two waves which share, for some period of time, their trajectories, as in the following situation (188):



This behaviour of waves and fields compelled Reichenbach to develop the notion of 'functional genidentity' (189). This applies to processes or chains of events where the impenetrability and labelling characteristics of 'material genidentity' are violated and only the continuity requirements is satisfied. Thus he put forward water waves and energy as examples of things which possess functional genidentity. As regards the relationship between the two forms of genidentity Reichenbach believed

188. Strawson considered similar arguments with regard to the reidentification of sounds. He concluded that sounds could not be reidentified at different times and used this to support his view that material objects embedded in space-time form the primary individuator.

that 'We can define genidentity to suit our purposes' (190), in the sense that we can move from one level of description to another as we wish.

Redhead has captured much the same idea in the suggestion that waves belong to a new category of entities called 'ephemerals' (191). These can be distinguished from one another at the same time but cannot be reidentified through time via T.I. as 'continuants' can, such reidentifications only being possible if the continuity criterion is satisfied. As we have indicated however, continuity alone may not be enough, given ephemerals' violation of I.A. Redhead particularly emphasises the ease with which creation and annihilation processes can be envisaged in terms of ephemerals as compared to continuants and therefore this is a category which is most readily applied to the entities of Quantum Field Theory.

Our conclusion then is that there are good reasons for rejecting the view that fields are substantial, and can therefore have T.I. attributed to them, and the form of space-time individuality that applies to the waves in this fields is metaphysically weaker than that which can be attributed to particles, justifying the inclusion of, waves in a different category of entities.

2.5 Conclusion

We have argued, in this chapter, that there are two metaphysical positions which can be adopted with regard to the individuality of classical particles. The first, corresponding to a naive Lockean approach, casts an underlying substance into the role of individuator and this form of individuality has been called 'Transcendental Individuality' as the individuator transcends the properties predicated of the particles.

190. Ibid p. 226

191. M. Redhead (1983) p.39

By considering the way in which classical particle theory can be rewritten in field theoretic terms we were then led to the view that particles could be individuated in terms of their location in space-time. This view can be most easily accommodated within an absolutist, or substantivalist, approach to space-time as the relativist alternative induces circularities.

These results were then supported by an historical account of the development of Boltzmann's statistical mechanics which examined in detail the differences between his Combinatorial and H-Theorem approaches. In particular we showed that just as T.I. is not necessary to classical physics, so the former approach may not be entirely free from spatio-temporal considerations, as they must be invoked if some justification of the weighting assignments is sought. Finally we discussed the individuality of fields, arguing that they should properly be regarded as properties of the space-time points and that waves in these fields can have only a weak form of individuality attributed to them.

Thus just as there is an underdetermination for the particle and field descriptions, so classical physics is essentially agnostic with regard to our two views of particle individuality. It is philosophical rather than physical, arguments which must be called upon if we want to choose one position over the other.

In the next chapter we shall consider quantum physics and we shall see that a similar situation arises there also, although rather more analysis is required to draw out those metaphysical positions regarding particle individuality with which the theory is consistent.

CHAPTER THREE

IDENTITY AND INDIVIDUALITY IN QUANTUM
PHYSICS

3.1 Introduction

In the previous chapter it was shown that classical physics can be taken to support either of two views regarding particle individuality, which we called 'Transcendental' and 'Space-Time' - Individuality respectively. It is our intention in this chapter to demonstrate that a similar situation exists in quantum physics also. The two positions which we shall arrive at, namely individual particles plus accessibility restrictions imposed upon the individuated states and 'Non-individual' particles plus individual states, are related to, though obviously not identical with, the two views above.

There are two points worth noting here. The first is that the situation as regards particle individuality is somewhat more complicated in quantum mechanics (Q.M.) than in classical mechanics (C.M.). This is because in the latter there is only one form of particle statistics, Maxwell-Boltzmann statistics, whereas in the former theory there are three: Fermi-Dirac, Bose-Einstein and para-statistics. The type of statistics obeyed in Q.M. is embodied in the symmetry restrictions imposed upon the particle wave-functions. As we shall see it is through a consideration of these underlying symmetry principles that we are led to our two views above concerning particle individuality.

Our second point is that the majority of discussions of particle individuality in Q.M. have tended to emphasize only one view, namely that quantum mechanical particles should be regarded as 'non-individuals' in some sense. The ancestry of this position can be traced back to Ehrenfest, who realised very early in the development of Q.M. that the 'new physics' implied a

conception of particle individuality which was very different from the classical view. However, we shall try to show there is an alternative view which can equally well be held.

In 3.2 we shall lay down the basic theoretical structure in the context of which particle individuality will be discussed. In particular we shall emphasise the crucial role played by the 'Indistinguishability Postulate in these discussions.

In the next section we consider the implications of this postulate and its various interpretations. This leads to the explication of our two views of particle individuality, supported by theoretical considerations from the previous sections. Thus we establish the second half of our central thesis.

We then present a historical outline of the conception and development of the three forms of quantum statistics, drawing attention to those points which support our philosophical position above. We also note that, with only one exception, the early work in this area used a suitably adapted form of Boltzmann's Combinatorial Approach.

In the final two sections we briefly consider quantum field theory and the Gibbs Paradox in the light of our previous arguments.

3.2 The Theoretical Context

We begin by considering a system of N indistinguishable (in the sense given in the introduction) particles, taking them all, for simplicity, to be of the same kind and with zero spin (1). Using Dirac's notation (2) we can distinguish the states which one particle of the system can be in by writing the kets thus: $|a^1\rangle, |a^2\rangle, \dots$. Here we have labelled the states with the superscripts

1. These restrictions will not affect the argument. The problems which arise when particle production is allowed will be touched upon later and are discussed in O. Greenberg and A. Messiah (1964). p.253
2. P.A.M. Dirac (1978)

1,2,... and clearly they are being regarded as individuated. They can be distinguished and assigned labels in this way because different states are characterised by different wave-functions. These labels will be called state, or place, labels.

A second set of labels can be introduced which specify which particles are in which states. Thus the particles can be distinguished by writing the kets thus:

$|a_1^1\rangle, |a_2^1\rangle, \dots$ Here the particles have been labelled with the subscripts 1,2, ... and can thus be regarded as individuals. As we shall see it is the permutation of these labels in the Schrodinger formulation which leads to the claim that the particles have somehow lost their individuality and must therefore be regarded as 'non-individuals'.

Thus the two essential ingredients of both views, the individuated states and the individuality of particles, are present in our initial theoretical structure. (3)

Let $|a_1^1\rangle, |a_2^1\rangle, \dots$ be the kets for the first particle considered as a dynamical system by itself. There will be corresponding kets $|a_1^2\rangle, |a_2^2\rangle, \dots$ for the second particle by itself and so on. A representation of the complete system can then be built up and a ket for the whole assembly obtained by taking the tensor product, \otimes , of kets for each particle by itself:

$$|a_1^1\rangle \otimes |a_2^2\rangle \dots \otimes \dots \otimes |a_N^N\rangle \quad (4)$$

Kets of this form shall be denoted quite generally by

$|f\rangle$. These product functions $|f\rangle = |a_1^1 \dots a_N^N\rangle \equiv |a_1^1\rangle \otimes \dots \otimes |a_N^N\rangle$ and suitably permuted variants, span a Hilbert space \mathcal{H}_N which can be constructed as follows. (5)

Considering only one species of spinless particle and supposing for the moment that the number of particles is fixed, we first of all assume that the one-particle states are in one-to-one correspondence with the rays of the space \mathcal{H}_1 made up of all L^2 wave functions of one co_ordinate. The states of a system of N

3. cf J. Hartle and J. Taylor (1969) p.2044

4. Dirac op cit p. 207

5. Hartle and Taylor op cit p. 2044

indistinguishable individual particles, assigned labels $r_i = 1, \dots, N$ corresponding to the subscripts in the ket $|1\rangle$, would correspond to the vectors of the space \mathcal{H}_N which is the tensor product of N single particle spaces.

$$\mathcal{H}_N = \mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_1 \quad (N \text{ times}) \quad 3$$

This is the space of all L^2 functions of N coordinates which can be spanned by our product functions and the permuted variants. It should be noted that the space \mathcal{H}_N is defined in such a way that the order of the particle labels in a product vector is relevant. Thus the vectors $|uv\rangle$ and $|vu\rangle$ are quite distinct unless $u = v$.

This construction can be generalised to allow for systems with an arbitrary number of particles of different species in the following manner.

In this case state vectors and observables are defined in the Fock space \mathcal{F} which is the direct sum of spaces \mathcal{H}^N , each of which is spanned by vectors representing states with a fixed number N_s of particles in each species s : $\mathcal{F} = \sum_{\mathcal{N}} \mathcal{H}^{\mathcal{N}}$ [$\mathcal{N} = (N_1, N_2, \dots, N_s, \dots)$] ⁴

$\mathcal{H}^{\mathcal{N}}$ is a direct product of spaces which are themselves direct (or tensor) products of single particle spaces for each species. Thus, denoting the space associated with a single particle of species s by \mathcal{H}_1^s we have

$$\mathcal{H}^{\mathcal{N}} \equiv \mathcal{H}_{N_1}^1 \otimes \mathcal{H}_{N_2}^2 \otimes \dots \otimes \mathcal{H}_{N_s}^s \otimes \dots \quad 5$$

$$\begin{aligned} \mathcal{H}_{N_s}^s &= \mathcal{H}_1^s \otimes \mathcal{H}_1^s \otimes \dots \otimes \mathcal{H}_1^s \quad (N \text{ times}) \\ &= \bigotimes_{N_s} \mathcal{H}_1^s \quad 6 \end{aligned}$$

Clearly the \mathcal{H}_N of 3 corresponds to the $\mathcal{H}_{N_s}^s$ of 5 and 6. (6)

In most of what follows we shall restrict our attention to only one of the subspaces \mathcal{H}_N defined above and consider for simplicity only systems containing one species of particle whose number is fixed. However, we shall have occasion to move to the more general Fock space description, in particular when we come to discuss Quantum Field Theory.

The basic elements in any quantum mechanical description of a system of particles are the Hilbert space-of-state vectors and the operators corresponding to observables whose expectation values are the measurable numbers. Having outlined the structure of the first of these, we shall, in what follows, consider the nature of the second.

The ket given by 1 corresponds to a particular kind of state for the assembly which can be described by saying that each particle is in 'its own state' (the values of the subscripts and superscripts within each ket are the same). The general ket for the assembly is of the form of a weighted sum or integral of kets such as that given in 1. If the kets $|a_1^1\rangle, |a_1^2\rangle, \dots$ are a set of basic kets for the first particle by itself, then the kets $|a_2^1\rangle, |a_2^2\rangle, \dots$ will be a set of basic kets for the second particle by itself, and so on, and the kets of the form given in 1 will be a set of basic kets for the assembly. The representation provided by such kets for the assembly treats all the particles on an equal footing. (7)

Another ket for the assembly may be obtained by simply interchanging the kets for the first two particles in 1, giving

$$|a_2^1\rangle \otimes |a_1^2\rangle \otimes |a_3^3\rangle \otimes \dots \otimes |a_n^n\rangle \quad 7$$

This process of interchanging the first two particles is a linear operator which can be applied to any kets for the assembly. In fact the process of interchanging any two particles is a linear operator and by repeated application of such interchanges one can obtain any permutation of the particles appearing as a linear operator which can be applied to kets for the assembly.

In general if the particle labels are permuted than another ket of the form 1 in the tensor product space \mathcal{H}_N is obtained. Thus inside each \mathcal{H}_N one can define unitary operators describing the permutation

of particles belonging to the same species. These permutation operators form a group usually denoted by S_N and can be defined by

$$P_i = \begin{pmatrix} i \\ r_i \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \dots & i \\ r_1 & r_2 & r_3 & \dots & i \end{pmatrix} \quad 8$$

where P_i is the particle permutation operator.

Let us consider the application of such a permutation to a ket for the assembly of particles. We shall apply the operator P_i to a ket of the type given in 1 noting that P_i operates on the particle labels, corresponding to the subscripts in the kets. Thus,

$$P_i |a_1^1\rangle |a_2^2\rangle \dots |a_N^N\rangle = |a_{r_1}^1\rangle |a_{r_2}^2\rangle \dots |a_{r_N}^N\rangle \quad 9$$

(omitting the tensor product signs for convenience)

In general one will obtain $N!$ such states, corresponding to the number of permutations of N particle labels among themselves.

A particle permutation is essentially just a re-shuffling of the particle labels and so, because the particles of the assembly are all indistinguishable, such a permutation cannot lead to any observable effects. Thus we now introduce

The Indistinguishability Postulate : If a particle ^{permutation} P_i is applied to any ket for the assembly then there is no way of distinguishing the resultant permuted ket from the original unpermuted one by any means of observation at any time. (9)

This assumption lies at the heart of the theory of indistinguishable particles and characterises the basic requirement that indistinguishable particles, possessing all intrinsic properties in common, cannot be observably distinguished. Thus the observables for such particles are those for labelled, individuated, particles which do not distinguish the labels. (10)
A more formal expression of this postulate can be given as follows

8. H. Margenau and G.M. Murphy (1956) p. 558;
M. Hammermesh (1962)
9. Greenberg and Messiah op cit p. 250
10. Hartle and Taylor op cit p. 2044

The observables which do not distinguish the particle labels are those whose ^{se} expectation value for any vector $|f\rangle$ in \mathcal{H}_N is the same as that for $P_i|f\rangle$. Thus the requirement that dynamical states represented by vectors $|f\rangle$ and $P_i|f\rangle$ cannot be distinguished in any measurement can be expressed by the relation

$$\langle f|P_i Q|P_i f\rangle = \langle f|Q|f\rangle$$

where the state $P_i|f\rangle$ is abbreviated to $|P_i f\rangle$

Thus, since P_i is unitary, $\langle f|Q|f\rangle = \langle f|P_i^{-1} Q P_i|f\rangle$ 10

where Q is some suitable defined Hermitian operator. (11) This is the formal statement of the Indistinguishability Postulate (IP).

It is sometimes said that this postulate necessarily implies what is known as the 'Symmetrization Postulate' (S.P.), which says that states containing several indistinguishable particles are, according to the species, either symmetric (bosons) or antisymmetric (fermions), (12). Now if it is supposed that a particle permutation applied to the ket $|f\rangle$ gives only symmetric or antisymmetric states then clearly this implies equation (10) i.e. $P_i|f\rangle = \pm|f\rangle \Rightarrow \langle f|P_i^{-1} Q P_i|f\rangle = \langle f|Q|f\rangle$ 11 So a sufficient condition for I.P. is that a particle permutation gives only symmetric or antisymmetric states, for all observables Q .

However, this is not a necessary condition, as Messiah and Greenberg made clear. S.P. absolutely forbids those states which cannot be represented by either symmetric or anti-symmetric wave functions, and is thus a very strong selection rule. As Messiah and Greenberg remarked 'This is an extremely strong condition, very much stronger than what is implied by the indistinguishability of identical particles'. (13) They then went on to demonstrate that the arguments usually employed to insert S.P. into the Q.M. formalism are ad hoc in nature and do not proceed unavoidably from first principles as is

11. Hartle and Taylor op cit p. 2044; Greenberg and Messiah op cit p. 251

12. Greenberg and Messiah Ibid p. 248

13. Ibid p. 248

sometimes thought. They also showed that many experimental facts which are apparently tests of the S.P. selection rule are in fact merely tests of I.P.

The difference between the two postulates can be expressed thus:

S.P. is a restriction on the states for all observables Q , whereas I.P. is a restriction on the observables Q for all states. The latter claim can be demonstrated as follows. Equation 10 must hold for any vector $|f\rangle$ and so will hold with $|f\rangle$ replaced by any linear combination of vectors. Applying this relation to the two superpositions $|f\rangle + \alpha |g\rangle$ and $|f\rangle + i\alpha |g\rangle$ gives

$$\langle f | Q | g \rangle = \langle f | P_i^{-1} Q P_i | g \rangle \quad f, g \in \mathcal{H}_{N12}$$

or equivalently $[P_i, Q] = 0$ 13

This must hold for all the possible particle permutations of the N indistinguishable particles. Thus all physical observables, in the sense of all observables associated with an actual measurement, must be particle permutation invariant. (14)

What we are doing here is identifying or 'picking out' the observables of a system of N indistinguishable particles. Since 10 holds with $|f\rangle$ replaced by any linear combination of vectors it also holds for off diagonal matrix elements and so is an operator identity. There^{fore} the observables for N indistinguishable particles are that subset of observables for N labelled particles which commute with the particle permutation operators. I.P. thus restricts the possible observables to this subset.

Having thus identified the observables of the system the next step is to establish a correspondence between the states and vectors of \mathcal{H}_N . We shall consider this

14. It can further be shown that the condition imposed on the evolution operator $U(t)$ by the requirement that dynamical states represented by $|f\rangle$ and

$P_i |f\rangle$ should not exhibit any observable difference at any time t , $[P_i, U^\dagger(t) Q U(t)] = 0$ is in fact automatically fulfilled. See Greenberg and Messiah Ibid p. 251

later, where the concept of paraparticles will be developed, but for the moment we wish to discuss some further points of interest.

First of all the permutation invariance expressed in 13 is a rigorous invariance property holding for all physical observables and the evolution operator. Contrary to what is often assumed the state vector space \mathcal{H}_N is not irreducible with respect to the algebra of physical observables and it is through consideration of the manner in which \mathcal{H}_N can be decomposed into irreducible subspace that one is lead to paraparticles.

It is also important to realise that 10 implies that the operators denoted by Q must be symmetric functions of the particle labels, as can easily be shown thus

$$\langle f | Q | f \rangle = \langle f | P_i^{-1} Q P_i | f \rangle \Rightarrow P_i^{-1} Q P_i = Q$$

$$\Rightarrow Q P_i = P_i Q$$

Thus Q is unchanged by the application of any particle permutation, acting on the particle labels and must therefore be a symmetric function of these labels. This result is the basis of the claim that I.P. should be interpreted as a restriction on the observables, to the effect that they must be symmetric functions as just stated.

The most significant observable of a Q.M. system is of course, the Hamiltonian and the indistinguishability of particles is usually taken to require that this Hamiltonian be a symmetrical function of the dynamical variables (15). The importance of Greenberg and Messiah's work lies not so much instating the above restriction but in placing it within the context of what is meant by 'indistinguishability', as given by I.P. and in clearly separating it from S.P.

Having defined the particle permutation operators P_i , which permute the particle labels in our general ket, it is now necessary to introduce a place permutation operator \bar{P}_i which permutes the state labels (i.e. it operates on the superscripts). These operators are defined by

$$\bar{P}_i = \begin{pmatrix} i \\ i \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \dots & N \\ \tau_1 & \tau_2 & \tau_3 & \dots & \tau_N \end{pmatrix} \quad 14$$

Consider the application of such a place permutation to a ket for the assembly of particles. Let us apply the operator \bar{P}_i to a ket of the type given in 1 noting that \bar{P}_i operates on the state labels. Thus we obtain:

$$\bar{P}_i |a_1^1\rangle |a_2^2\rangle \dots |a_n^n\rangle = |a_1^{r_1}\rangle |a_2^{r_2}\rangle \dots |a_n^{r_n}\rangle \quad 15$$

The difference between the particle and place permutation operators has been expressed by Dirac thus: 'Let us consider a permutation in the general sense, say that consisting of the interchange of 2 and 3. This may be interpreted either as the interchange of the objects 2 and 3 or as the interchange of the objects in the places 2 and 3, these two operations producing in general quite different results. The first of these interpretations is the one that gives the operators P [the particle permutation operators] the objects concerned being the similar particles. A permutation P can be applied to an arbitrary ket for the assembly. A permutation with the second interpretation [i.e. a place permutation] has a meaning, however, only when applied to a ket of the form

$$[P\psi = |a_1^1\rangle |a_2^2\rangle \dots |a_n^n\rangle]$$

for which each of the particles is in a place specified by an α or to a sum of kets of the form $[P\psi = \text{etc}]'$ (16)

The place permutation, unlike the particle permutation, operators depend upon the basis with respect to which they are defined.

The exact nature of the relationship between these two kinds of operator can be elicited as follows:

Let us consider a state function $|f\rangle$ given by

$$|f\rangle = |a_1^1 a_2^2 \dots a_n^n\rangle \quad 16$$

and let us operate on this function with the particle permutation operator P_i as defined above:

$$P_i |f\rangle = |a_1^{r_1} a_2^{r_2} \dots a_n^{r_n}\rangle \quad 17$$

If we now apply a place permutation \bar{P}_i again as defined above, to this new state function we obtain

$$\begin{aligned}\bar{P}_i P_i |f\rangle &= \bar{P}_i |a_{r_1}^1, a_{r_2}^2, \dots, a_{r_N}^N\rangle \\ &= |a_{r_1}^1, a_{r_2}^2, \dots, a_{r_N}^N\rangle \\ &= |a_1^1, a_2^2, \dots, a_N^N\rangle \\ &= |f\rangle\end{aligned}$$

Thus

$$\bar{P}_i P_i |f\rangle = |f\rangle$$

This gives

$$P_i |f\rangle = P_i^{-1} |f\rangle$$

18

In other words, the place permutation operators are simply the inverses of the particle permutation operators, when acting on a ket of the form given by $|f\rangle$.

A plausibility argument can be given for this result using a 'ball- and-box model in which there are a number of boxes, representing the states and a number of balls representing the particles. The P_i 's will permute the balls among the boxes and the \bar{P}_i 's will permute the boxes among, or rather, around the balls. Suppose initially there is ball a in box 1 and ball b in box 2. A particle permutation is then applied and a and b are interchanged, giving b in box 1 and a in box 2. Now a place permutation is applied and boxes 1 and 2 interchanged, giving the original arrangement. Thus the result of first applying a P_i followed by applying a \bar{P}_i is simply identity, the initial arrangement. Hence $\bar{P}_i = P_i^{-1}$.

Since the P 's are linear operators which can be applied to any ket for an assembly of N particles, they can be regarded as dynamical variables in the N particle system. However, as Dirac has pointed out, they are not, in general, real dynamical variables because their Hermitean conjugates are equal to their reciprocals. (17)

The \bar{P} 's can be considered as dynamical variables, again only with states obtainable by superposition of the states given by 14 .

It must now be asked whether these two kinds of permutation operator can be regarded as observables. The

condition for an operator Q to be an observable is that it must commute with the particle permutation operators i.e. $[P_i, Q] = 0$ for all observables Q .

Taking the P_i 's first, it can be shown that these do not in general commute with one another i.e.

$[P_i, P_j] \neq 0$ This is because the Symmetric group S_N is non-Abelian i.e. the elements do not in general commute. Therefore given the above condition, the particle permutation operators cannot be regarded as observables.

An entirely different situation holds for the \bar{P}_i 's. It can be shown that these do commute with the particle permutation operators i.e. $[\bar{P}_i, P_j] = 0$ (18). The P_i 's act without regard to place and the \bar{P}_i 's act without regard to the identities of the particles. Another plausibility argument can be given for this commutability in terms of the ball and box model. Suppose initially ball a is in box 1, ball b in box 2, ball c in box 3 and ball d in box 4. A particle permutation is then applied interchanging balls c and d. This gives a in 1 b in 2 c in 4 d in 3. A place permutation is then applied, interchanging boxes 1 and 2. This gives the final arrangement a in 2, b in 1, c in 4, and d in 3.

Returning to the initial arrangement, suppose a place permutation, interchanging boxes 1 and 2 was applied first, followed by the particle permutation interchanging balls c and d. This gives a in box 2, b in 1, c in 4 and d in 3, which is the same as the final arrangement obtained above. Thus applying a P followed by a \bar{P} gives exactly the same result as first applying a \bar{P} then a P . In other words the order in which these two operations are performed does not matter.

It therefore follows that any self adjoint function of the place permutation operators is an observable; (we shall return to this point in section 3.3.3). Further arguments based on physical grounds can be given to support this conclusion, and

it has been suggested that a general definition of indistinguishable particles should be that there are no other observables, apart from those which are functions of the \bar{P} 's, that distinguish between states differing only in the ordering of the indistinguishable particle variables. (19) This result, that any self-adjoint function of the \bar{P}_i 's should be considered observable, following as it does from the above commutation relation, means that I.P. is all that is needed to make the formalism of distinct particles suitable for the description of indistinguishable particles. (20)

These results have important implications for the individuality of particles in Q.M. The first, that the P 's are not observables, is what lies behind the non-classical counting of states in quantum statistics which has led some authors to say that quantal particles cannot be regarded as individuals in a classical sense. The second conclusion, that ^{functions of} the \bar{P} 's are observables, suggests that there is more to it than this, and that individuality resides in the states, in a sense. The distinction between the P 's and the \bar{P} 's, although of quite general validity, manifests itself only in the case of para-statistics of order greater than 2 i.e. not in the case of Fermi-Dirac or Bose-Einstein statistics, as we shall see.

It can also be shown that any symmetric function of the observables Q , can be expanded in terms of the \bar{P}_i 's (21) and thus that the place permutation operators are canonical observables of the system, in a sense which will become apparent shortly. This result follows directly from theorem (3.4 A) of Weyl's book 'The Classical Groups', Dirac has sketched a proof for the particular case of the observable being some perturbing energy V , defined by

$$\langle f | V P | f \rangle \quad (23)$$

19. P.V. Landshoff and H.P. Stapp (1967) p. 73 p. 77-81
20. R.H. Stolt and J.R. Taylor (1970) p.6.
21. C.F. Landshoff and Stapp p.90
22. H. Weyl (1946) ch. IV.
23. Dirac op cit p. 218

A more general proof can be given as follows: (24)

Theorem: $\langle f' | P_j Q P_i | f \rangle = \sum_i \langle f' | Q P_i | f \rangle \langle f | P_j \bar{P}_i P_i | f \rangle$ 19

where $|f'\rangle$ and $|f\rangle$ are kets for the assembly of the form given on p.134

Proof: We note first of all that

$$\langle f | P_j \bar{P}_i P_i | f \rangle = \begin{cases} 1 & \text{if } P_i = P_j P_i \\ 0 & \text{otherwise} \end{cases}$$

This is because

$\langle f | P_j \bar{P}_i P_i | f \rangle = \langle f | P_j P_i P_i^{-1} | f \rangle$ (\bar{P}_i commutes with P_i and $\bar{P}_i = P_i^{-1}$)
and $\langle f |$ and $|f\rangle$ are orthogonal so $\langle f | P_j P_i P_i^{-1} | f \rangle$ has to equal unity but can only do so if

$$P_j P_i P_i^{-1} = 1 \Rightarrow P_j P_i = P_i$$

Thus this expression $\langle f | P_j \bar{P}_i P_i | f \rangle$ reduces to a kind of Kronecker delta function:

There^{fore} using this and substituting for \bar{P}_i we have

$$\begin{aligned} \sum_i \langle f' | Q P_i | f \rangle \langle f | P_j \bar{P}_i P_i | f \rangle &= \sum_i \langle f' | Q P_i P_i | f \rangle \\ &= \sum_i \langle f' | P_j Q P_i | f \rangle \\ &\quad (\text{as } Q \text{ commutes with } P_j) \\ &= \langle f' | P_j Q P_i | f \rangle \\ &\quad \text{Q.E.D.} \end{aligned}$$

Thus we have shown that $\langle f' | P_j Q P_i | f \rangle = \sum_i \langle f' | Q P_i | f \rangle \langle f | P_j \bar{P}_i P_i | f \rangle$ which following Dirac, we can write as

$$Q \approx \sum_i c_i \bar{P}_i \tag{20}$$

where the $c_i = \langle f' | Q P_i | f \rangle$ and the sign \approx means an equation in the restricted sense, the operators on the two sides being equal so long as they are used only with kets of the form $|P_i f\rangle$ and their conjugate imaginary bras.

This result, that the observable Q is equal, in the restricted sense, to a linear function of the \bar{P}_i 's with coefficients c_i given as above, will play an important role in our later discussions of paraparticle theory in particular with regard to two-body transitions between 'triangular' paraparticle states.

Having identified the observables for N indistinguishable particles as being that subset of observables for N labelled particles which commute with P_i 's, this result

following from the I.P., the next step is to establish the correspondence between the states of the system and the vectors of the Hilbert space

The first point to note is that two vectors which give the same expectation values for all observables must represent the same state. Thus the I.P. implies that whenever a vector $|f\rangle$ in \mathcal{H}_N corresponds to some physical state then the vector $P_i|f\rangle$ for any i must correspond to the same state.

It is usually assumed that every physically distinct state of N indistinguishable particles must correspond to some unique ray in \mathcal{H}_N .

If $|f\rangle$ represents some state then $|f\rangle$ and $P_i|f\rangle$ must lie in the same ray and hence must be proportional.

This immediately leads to the Symmetrization Postulate.

However, as Hartle and Taylor emphasise: 'There is ... no a priori reason to insist that every state of N identical particles correspond to a unique ray in \mathcal{H}^N . The possibility that a single state of identical particles could correspond to some larger collection of vectors in \mathcal{H}^N must be considered, and it is this possibility which leads to the notion of paraparticles'.

(25)

The problem then is to decide what collection of vectors in \mathcal{H}_N can correspond to physical states and the solution lies in the fact that any state can be completely characterised by specifying the expectation values of all observables. Thus these sets of vectors must possess the following properties:

1) Two vectors $|f\rangle$ and $|f'\rangle$ representing the same state must give the same expectation value for all observables i.e.

$$\langle f|Q|f\rangle = \langle f'|Q|f'\rangle \quad (\text{all } Q \text{ with } [Q, P_i] = 0)$$

2) Two vectors $|f\rangle$ and $|f'\rangle$ representing different states must give different expectation values for some observable i.e.

$$\langle f|Q|f\rangle \neq \langle f'|Q|f'\rangle \quad (\text{some } Q \text{ with } [Q, P_i] = 0)$$

The apparatus of group theory can then be used to determine

25. Hartle and Taylor op cit p. 2045. Also see Stolt and Taylor op cit p.1.

the sets of vectors possessing these qualities. (26)

We begin by considering an arbitrary vector $|f\rangle$ in \mathcal{H}_N and the $N!$ dimensional subspace spanned by the vectors $P_i|f\rangle$ for all permutations i in the group S_N . This subspace can be decomposed into irreducible subspaces invariant under the P_i 's. Each subspace carries an irreducible representation D^μ of S_N and the number of subspaces carrying any irreducible representation D^μ is equal to the dimension N_μ of D^μ . The problem now is to construct basis functions for the various irreducible representations of S_N .

The following procedure for obtaining these functions is given by Hammermesh. (27)

We note first of all that in general any function $|f\rangle$ is expressible as the sum of functions which can act as base functions in the various irreducible representations:

$$|f\rangle = \sum_{\nu} \sum_{i=1}^{n_{\nu}} |f_i\rangle^{(\nu)} \quad 21$$

The base functions for the ν th irreducible (unitary) representation satisfy the equations

$$Q_R |f_i\rangle^{(\nu)} = \sum_j^{(\nu)} D_{ji}^{(\nu)}(R) |f_j\rangle^{(\nu)} \quad 22.$$

where Q_R is any operator of the group considered.

The necessary and sufficient condition which a given function must satisfy in order that it may belong to the i 'th row of a given representation is

$$\sum_R D_{ii}^{(\nu)*}(R) Q_R |f_i\rangle^{(\nu)} = \frac{h}{n_{\nu}} |f_i\rangle^{(\nu)} \quad 23$$

Thus given a function $|f_i\rangle^{(\nu)}$ which satisfies the above requirement, one can associate with it $(n_{\nu}-1)$ 'partners' given by

$$|f_l\rangle^{(\nu)} = \frac{n_{\nu}}{h} \sum_R D_{li}^{(\nu)*}(R) Q_R |f_i\rangle^{(\nu)} \quad 24$$

so that the set of functions satisfies equation 23.

Returning to ²¹ the question now is, given the function

$|f\rangle$, how does one find the $|f_i\rangle^{(\nu)}$ in the first place?

In other words, how does one resolve the given function into a sum of functions, each of which belongs to a

26. Hartle and Taylor op cit.; Appendix p. 338

27. Hammermesh op cit p. 85 and p. 111; cf Appendix p. 347

particular row of some irreducible representation?

Hammermesh shows that the operator

$$\Pi_{ii}^{(\mu)} = \frac{n_{\mu}}{h} \sum_R D_{ii}^{(\mu)*}(R) Q_R$$

is a projection operator i.e. $\Pi_{ii}^{(\mu)} |f_j\rangle^{(\nu)} = |f_i\rangle^{(\mu)} \delta_{\mu\nu} \delta_{ij}$ 25

Thus if this is applied to equation 21 one obtains the desired functions.

$$|f_i\rangle^{(\mu)} = \frac{n_{\mu}}{h} \sum_R D_{ii}^{(\mu)*}(R) Q_R |f\rangle \quad 26$$

Constructing the basis functions is therefore a two stage process. First the projection operator $\Pi_{ii}^{(\mu)}$ is

applied to the given function $|f\rangle$ to obtain the functions $|f_i\rangle^{(\mu)}$. Then the operator $\frac{n_{\nu}}{h} \sum_R D_{ii}^{(\nu)*}(R) Q_R$ is applied to find the $(n_{\nu}-1)$ partners of this function, such that the whole set satisfies equation

$$23, \text{ thus giving the functions } |f_i\rangle^{(\nu)}$$

which form a basis for the ν 'th irreducible representation.

Applying this procedure to the arbitrary state vector $|f\rangle$ we note first of all that the operators

Q_R will be the particle permutation operators P_i . The projection operator can then be written

$$\Pi_{ij}^{(\mu)} = \frac{n_{\mu}}{h} \sum_R D_{ij}^{(\mu)*}(R) P_i \quad 27$$

whilst the second operator given above, which we shall call the transfer operator, will be given by

$$\Pi_{ij}^{(\nu)} = \frac{n_{\nu}}{h} \sum_R D_{ij}^{(\nu)*}(R) P_i \quad 28$$

Applying these each in turn one then obtains the base functions $|f_i\rangle$:

$$|f_i\rangle^{(\nu)} = \Pi_{ij}^{(\nu)} \Pi_{jj}^{(\mu)} |f\rangle \quad 29$$

These satisfy the equation $P_i |f_i\rangle = \sum_j |f_j\rangle D_{ji}(R)$ 30 as we shall demonstrate.

At first sight this seems rather a complicated procedure but we can simplify it by eliminating one of the stages. We can in fact dispense with the projection operators altogether since $\Pi_{ij} \Pi_{jj} = \Pi_{ij}$ a result which we shall now prove, but which Hartle and Taylor clearly already knew.

To Prove $\prod_{ij} \prod_{ij} = \prod_{ij}$

$$\prod_{ij} = \sum_s D_{ij}^{(u)*}(s) P(s); \text{ and } \prod_{ij} = \sum_R D_{ij}^{(u)*}(R) P(R)$$

$$\begin{aligned} \therefore \prod_{ij} \prod_{ij} &= \sum_R D_{ij}^{(u)*}(R) P(R) \sum_s D_{ij}^{(u)*}(s) P(s) \\ &= \sum_R \sum_s D_{ij}^{(u)*}(R) D_{ij}^{(u)*}(s) P(R) P(s) \end{aligned}$$

Rewrite $P(R) P(s) = P(CT)$

$$\Rightarrow P(s) = P(R^{-1}) P(CT)$$

$$\Rightarrow S = R^{-1} T$$

$$\begin{aligned} \therefore \prod_{ij} \prod_{ij} &= \sum_R \sum_T D_{ij}^{(u)*}(R) D_{ij}^{(u)*}(R^{-1}T) P(CT) \\ &= \sum_T \left\{ \sum_R \sum_k D_{ij}^{(u)*}(R) D_{jk}^{(u)*}(R^{-1}) D_{kj}^{(u)*}(CT) \right\} P(CT) \\ &\quad \text{(} D_{kj}(CR) \text{ is unitary)} \\ &= \sum_T \sum_R \sum_k D_{ij}^{(u)*}(R) D_{kj}^{(u)*}(CR) D_{kj}^{(u)*}(CT) P(CT) \\ &= \sum_T \sum_k S_{ik} S_{ij} D_{kj}^{(u)*}(CT) P(CT) \\ &= \sum_T \sum_k S_{ik} D_{kj}^{(u)*}(CT) P(CT) \\ &= \sum_T D_{ij}^{(u)*}(CT) P(CT) \\ &= \prod_{ij} \end{aligned}$$

$$\therefore \prod_{ij} \prod_{ij} = \prod_{ij}$$

31

Thus to obtain the basis functions $|f_i\rangle$ one simply applies the transfer operator \prod_{ij} to the state function $|f\rangle$:

$$\begin{aligned} |f_i\rangle &= \prod_{ij} |f\rangle \\ \text{ie } |f_i\rangle &= \frac{\lambda u}{h} \sum_R D_{ij}^{(u)*}(R) P(R) |f\rangle \end{aligned} \quad 32$$

Before we actually use this method to obtain a set of basis functions there are two more results which must be established. We must show that these functions

$\prod_{ij} |f\rangle$ support representation of the P's and the \bar{P} 's.
To Show that $\prod_{ij} |f\rangle$ Supports a Representation of the P(s)'s

$$\prod_{ij} = \frac{\lambda u}{h} \sum_R D_{ij}^{(u)*}(R) O_R$$

Put $O_R = P(R)$ where $P(R)$ is a particle permutation operator. Then,

$$\Pi_{ij} |f\rangle = \frac{N\mu}{h} \sum_R D_{ij}^{(\mu)*}(R) P(R) |f\rangle$$

Fix j , allow i to be a variable (the significance of this will be revealed shortly). Apply a particle permutation operator $P(S)$ to the function $|f_i\rangle$ formed by applying Π_{ij} to $|f\rangle$:

$$P(S) \Pi_{ij} |f\rangle = P(S) \left\{ \frac{N\mu}{h} \sum_R D_{ij}^{(\mu)*}(R) P(R) |f\rangle \right\} = \frac{N\mu}{h} \sum_R D_{ij}^{(\mu)*}(R) P(S) P(R) |f\rangle$$

Write $P(S) P(R) = P(T) \Rightarrow P(R) = P(S^{-1}) P(T) \Rightarrow R = S^{-1} T$.

$$\begin{aligned} \therefore P(S) \Pi_{ij} |f\rangle &= \frac{N\mu}{h} \sum_{S^{-1}T} D_{ij}^{(\mu)*}(S^{-1}T) P(T) |f\rangle \\ &= \frac{N\mu}{h} \sum_{S^{-1}} \sum_T D_{li}^{(\mu)*}(S^{-1}) D_{lj}^{(\mu)*}(T) P(T) |f\rangle \\ \sum_R &= \sum_{S^{-1}T} = \sum_T \quad (S \text{ fixed}) \end{aligned}$$

$$\begin{aligned} \therefore P(S) \Pi_{ij} |f\rangle &= \frac{N\mu}{h} \sum_T \sum_l D_{li}(S) D_{lj}^{(\mu)*}(T) P(T) |f\rangle \\ &= \sum_l D_{li}^{(\mu)*}(S) \frac{N\mu}{h} \sum_T D_{lj}^{(\mu)*}(T) P(T) |f\rangle \\ & \quad \quad \quad (D_{li} \text{ is unitary}) \\ &= \sum_l D_{li}(S) \Pi_{lj} |f\rangle \end{aligned}$$

$$\therefore P(S) \Pi_{ij} |f\rangle = \sum_l D_{li}(S) \Pi_{lj} |f\rangle, \text{ or } P(S) |f_i\rangle = \sum_l |f_l\rangle D_{li}(S)$$

Therefore the functions $\Pi_{ij} |f\rangle$ support a representation of the $P(S)$'s i.e. they satisfy the equation

$$P(S) |f_i\rangle = \sum_j |f_j\rangle D_{ji}^{(\mu)}(S)$$

33

To Show that $\Pi_{ij} |f\rangle$ Supports a Representation of the $\bar{P}(S)$'s

Again we have

$$\Pi_{ij} = \frac{N\mu}{h} \sum_R D_{ij}^{(\mu)*}(R) P(R)$$

and

$$\Pi_{ij} |f\rangle = \frac{N\mu}{h} \sum_R D_{ij}^{(\mu)*}(R) P(R) |f\rangle$$

Now fix i , allow j to be variable and apply a place permutation operator $\bar{P}(S)$ to $\Pi_{ij} |f\rangle$

$$\begin{aligned} \bar{P}(S) \Pi_{ij} |f\rangle &= \bar{P}(S) \left\{ \frac{N\mu}{h} \sum_R D_{ij}^{(\mu)*}(R) P(R) |f\rangle \right\} \\ &= \frac{N\mu}{h} \sum_R D_{ij}^{(\mu)*}(R) P(R) \bar{P}(S) |f\rangle \quad (\text{since } [P(R), \bar{P}(S)] = 0) \\ &= \frac{N\mu}{h} \sum_R D_{ij}^{(\mu)*}(R) P(R) P^{-1}(S) |f\rangle \quad (\text{since } \bar{P}(S) = P^{-1}(S)) \end{aligned}$$

Rewrite $P(R) P^{-1}(S) = P(T) \Rightarrow P(R) = P(T) P(S) \Rightarrow R = TS$

$$\begin{aligned} \therefore \bar{P}(S) \Pi_{ij} |f\rangle &= \frac{N\mu}{h} \sum_T D_{ij}^{(\mu)*}(TS) P(T) |f\rangle \\ &= \frac{N\mu}{h} \sum_T \sum_l D_{li}(T) D_{lj}^{(\mu)*}(S) P(T) |f\rangle \\ &= \sum_l D_{lj}^{(\mu)*}(S) \Pi_{li} |f\rangle \end{aligned}$$

$$\therefore \bar{P}(s) \pi_{ij} |f\rangle = \sum_k D_{kj}^{(\mu)}(s) \pi_{ik} |f\rangle \quad (\text{since } D \text{ is real})$$

$$\therefore \bar{P}(s) |f_i\rangle = \sum_k |f_k\rangle D_{kj}^{(\mu)}(s)$$

Therefore the base functions $\pi_{ij} |f\rangle$ support a representation of the $\bar{P}(s)$'s i.e. they satisfy the equation

$$\bar{P}(s) |f_i\rangle = \sum_k |f_j\rangle D_{ij}^{(\mu)}(s) \quad 34$$

Thus we have shown that the base functions $\pi_{ij} |f\rangle$ support representations of both the particle and place permutation operators.

Under the $P(R)$'s the base functions transform as

$$P(R) |f_i\rangle = \sum_j |f_j\rangle D_{ji}^{(\mu)}(R)$$

as we have just shown. This means that for fixed j and for $i = 1, \dots, N_\mu$, these functions span an irreducible subspace invariant under the $P(R)$. Within this subspace - or generalized ray as it has been called (28) - the function $|f_i\rangle$ transforms as the i 'th basis function of the irreducible representation $D^{(\mu)}$. The fact that the $P(R)$ leave any function in the same subspace, or generalized ray, reflects the fact that two state functions which differ by a permutation of the particle labels must represent the same state, as we have noted.

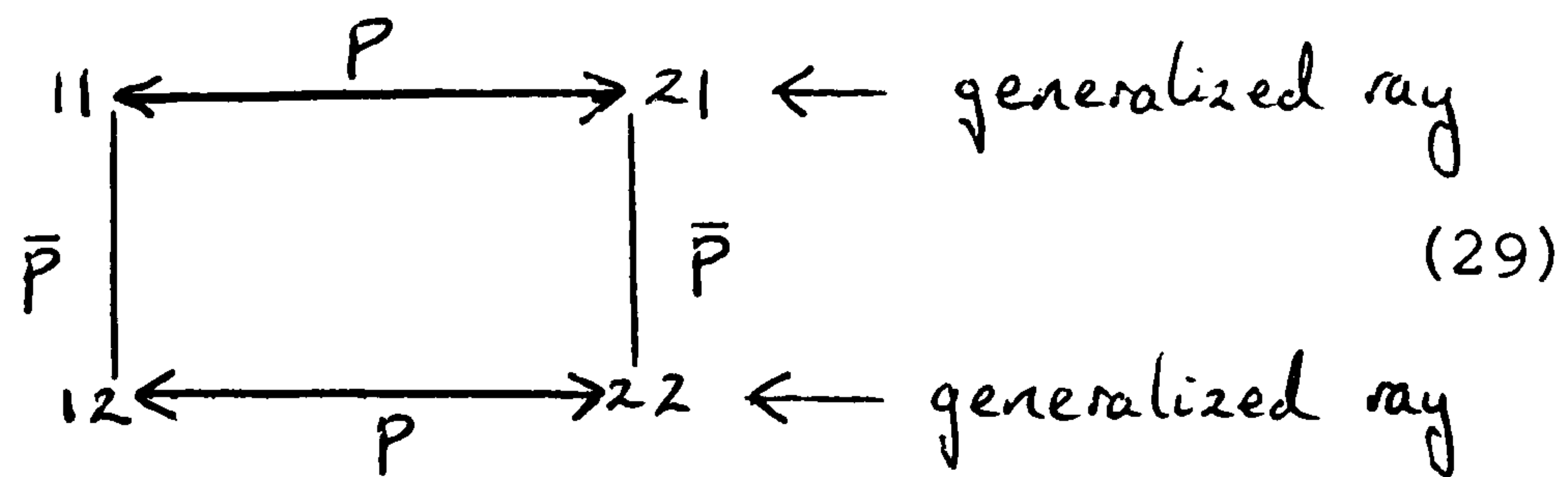
We have also shown that under the $\bar{P}(R)$'s the base functions satisfy

$$\bar{P}(R) |f_i\rangle = \sum_j |f_j\rangle D_{ij}^{(\mu)}(R)$$

That is, for fixed i and $j = 1, \dots, N$, the functions span a subspace invariant under the $\bar{P}(R)$ and within this space $|f_i\rangle$ transforms as the j 'th basis function of the irreducible representation $D^{(\mu)}$.

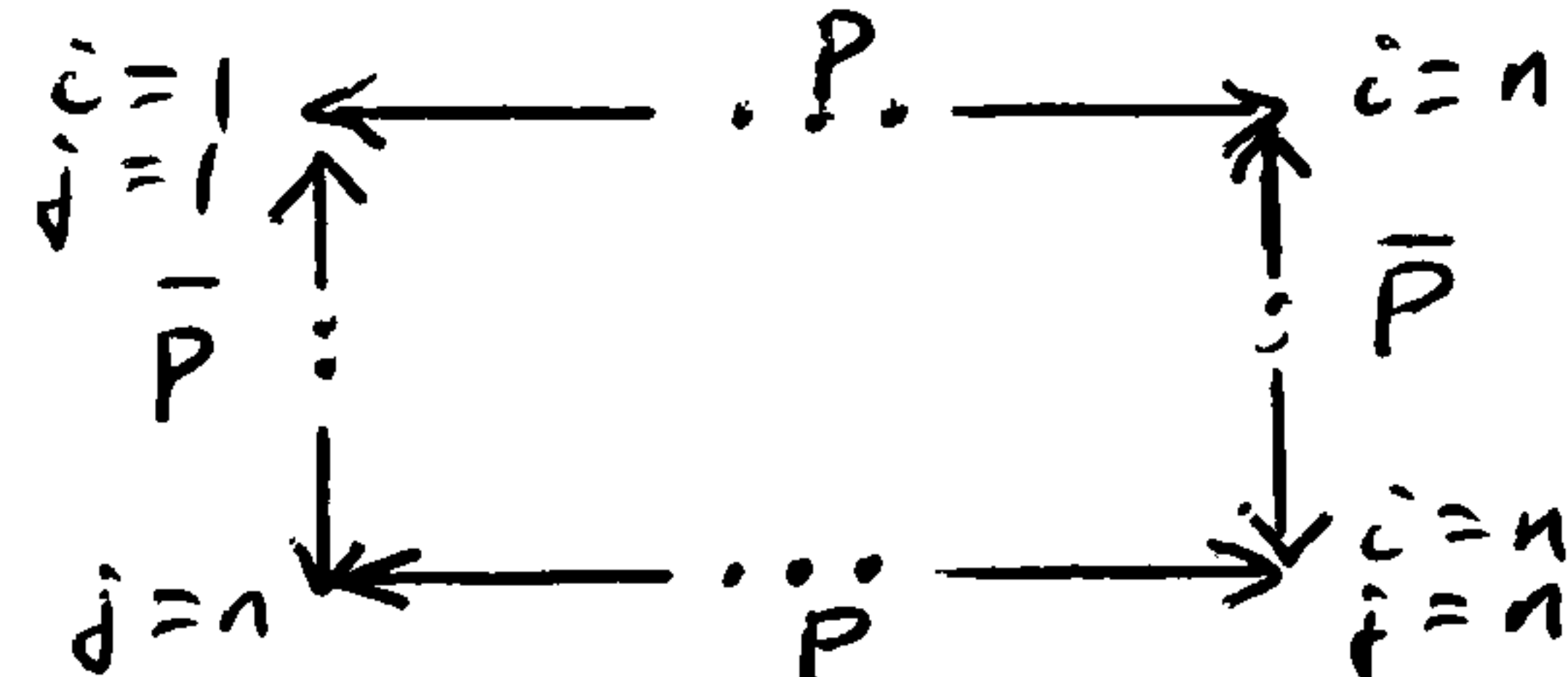
Clearly in general the place permutation operators carry a function from one subspace, or generalized ray to another, which is consistent with the fact that place permutations can change the physical state.

The complementary role of the particle and place permutations, in the case of a two dimensional representation D^μ , can be illustrated thus:



In this case $i, j = 1, 2$. The $P(R)$ map the vectors of any row (fixed j) among themselves, whereas the $\bar{P}(R)$ map the vectors of any column (fixed i) among themselves. The subspace spanned by the two vectors, or basis functions, in any row is a generalized ray and all the vectors in this subspace represent the same physical state.

Obviously in the case of an n -dimensional representation one would have



with j fixed and i variable the row number is fixed and one is ranging across the columns, which gives the particle permutations. With i fixed and j variable the column number is fixed and one is ranging up and down the rows, giving the place permutations,

Before continuing it is worth noting that in the case of the one dimensional symmetric and anti-symmetric representations $D = 1$ or ± 1 . Thus $P(R)$ and $\bar{P}(R)$ are the same for the symmetric and anti-symmetric vectors, and so the distinction between particle and place permutation is not manifested in the case of ordinary bosons and fermions.

Thus we are now able to generate a set of basis functions for any representation. (30) Furthermore we have shown that this procedure can be reduced from a two-stage to a one-stage process and also that the basis functions so formed support representations of both the particle and place permutation operators.

Using I.P. and Schur's lemma (31) it can easily be shown that these basis functions, $|f_i\rangle$, do indeed satisfy the two requirements for vectors to represent

29. Stolt and Taylor op cit p.9

30. M. Tinkham (1964) p. 39-43; also see Appendix p.347.

31. See Appendix P.342.

states (32) Thus having shown, following Greenberg and Messiah and Hartle and Taylor, that every state of a system of N indistinguishable particles corresponds to some multi-dimensional subspace, or generalized ray, the next step is to discuss which irreducible representations correspond to states. In the case of ordinary bosons and fermions, for example, this is quite straightforward as only the rays of the totally symmetric or totally anti-symmetric representation actually represent states.

If \mathcal{E}_μ is defined to be the subspace spanned by the basis functions $|f_i\rangle$ (with μ fixed) then the Hilbert space \mathcal{H}_N can be decomposed into a number of subspaces thus:

$$\mathcal{H}_N = \bigoplus_{\mu} \mathcal{E}_\mu \quad 35$$

Although \mathcal{E}_μ is defined with respect to a definite basis it can be seen that it is actually basis independent. \mathcal{E}_μ contains all functions associated with the irreducible representation D^μ , and since every pure state must be associated with a definite irreducible representation, each such state is represented by a generalized ray of vectors contained in some \mathcal{E}_μ .

Although every pure state is associated with a definite irreducible representation, not every one of the latter is associated with an attainable state, as the example of ordinary bosons and fermions makes clear. In general the statistical type of a paraparticle is identified by specifying the set of all irreducible representations in each \mathcal{S}_N which corresponds to attainable states of the particle.

At this point it is perhaps worth making clear the difference between 'ordinary' particles and paraparticles. For the former the states correspond to one-dimensional sub-spaces, or rays, in the usual way because they correspond to the one-dimensional completely anti-symmetric and symmetric^{ir} irreducible representations. However, in general the irreducible representations of

the permutation group are multi-dimensional and thus the states of a paraparticle correspond to multi-dimensional subspaces, or generalized rays.

Hartle and Taylor demonstrated that not every family of irreducible representations corresponds to a possible statistical type. (33) These results were then extended by Stolt and Taylor who showed that all possible types of first quantized paraparticles can be divided into two kinds: those of finite order and those of infinite order (34). The former can be classified further into parabosons and parafermions of order $p = 1, 2, 3, \dots$. Parabosons of order p have states corresponding to those irreducible representations whose Young diagrams have no more than p rows, whereas for parafermions the same holds for the columns. There are infinitely many paraparticles of infinite order, all of which have states corresponding to Young diagrams of arbitrarily many rows and columns, (see Appendix p346).

The space appropriate to a paraparticle of a given statistical type T will be the subspace of \mathcal{H}_N containing just those vectors which correspond to allowed irreducible representations. If the set of all irreducible representations associated with the attainable N -particle states of a particle of type T is denoted by T_N then the appropriate space is simply the direct sum (35)

$$\mathcal{H}_N^T = \bigoplus_{\mu \in T_N} \mathcal{E}_\mu \quad 36$$

In particular for a paraboson or parafermion of order p this sum runs over all ^{states} whose Young diagrams have no more than p rows or columns respectively.

The next step is to eliminate the 'generalized ray' and restore the usual connection between states and rays in Q.M. (36)

- 33. Hartle and Taylor op cit p. 2050
- 34. Stolt and Taylor (1970b) p. 2226 and (1970c) p.1759
- 35. Stolt and Taylor (1970a) p.10
- 36. Hartle and Taylor op cit p. 2046-2047; Stolt and Taylor op cit 10-11

A particle of type T will have N -particle states corresponding to the generalized rays in each \mathcal{E}_μ such that $\mu \in T_N$. Any state associated with the irreducible representation D^μ corresponds to a n_μ -dimensional subspace δ of \mathcal{E}_μ , any vector of which can equally well be taken to represent that state. Each subspace \mathcal{E}_μ can be further decomposed thus

$$\mathcal{E}_\mu = \bigoplus_{i=1}^{n_\mu} \mathcal{E}_{\mu,i}$$

37

where $\mathcal{E}_{\mu,i}$ is the space spanned by the $|f_i\rangle$ for fixed μ and i . Since all observables commute with the P 's each subspace $\mathcal{E}_{\mu,i}$ is invariant under all observables. In particular since this applies to the Hamiltonian all representative vectors will remain in $\mathcal{E}_{\mu,i}$ for all times if they are chosen there initially. Since each generalized ray δ has a unique one-dimensional intersection with each $\mathcal{E}_{\mu,i}$ one could agree always to label states of symmetry D^μ by vectors from one specific $\mathcal{E}_{\mu,i}$, $\mathcal{E}_{\mu,1}$ say, and then each state is labelled by a unique ray. (37) The $\mathcal{E}_{\mu,i}$ for $i \neq 1$ can be ignored because of the invariance of $\mathcal{E}_{\mu,1}$ under all observables.

Thus the formalism in which states of symmetry D^μ are represented by generalized rays in \mathcal{E}_μ can be replaced by one in which the same states are represented by rays in $\mathcal{E}_{\mu,1}$. For a particle of type T the new smaller space is

$$\mathcal{H}_{N,ray}^T = \bigoplus_{\mu \in T_N} \mathcal{E}_{\mu,1} \quad 38$$

The essential point behind this elimination is that any state represented by a multi-dimensional subspace δ with basis $\{|f_i\rangle\}$ can be represented arbitrarily by the number one basis vector $|f_1\rangle$. Every state can then be labelled by a unique ray in $\mathcal{E}_{\mu,1}$ rather than a generalized ray in $\mathcal{E}_{\mu,i}$, thus restoring the normal connection between states and rays.

This restoration is achieved however at the expense of symmetry under the particle permutation operators

since these carry the space $\mathcal{E}_{\mu,1}$ onto the other spaces $\mathcal{E}_{\mu,i}$ (38) (i variable for the $P(R)$'s, ranging across the columns). In other words, with attention restricted to $\mathcal{E}_{\mu,1}$ the P 's are undefined (39) This is to be expected however, as the redundancy of the generalized ray is caused by the complete symmetry under the P 's and to remove this redundancy we have moved to a smaller subspace on which the P 's are not defined.

The \bar{P} 's on the other hand, are well defined on $\mathcal{H}_{N,ray}^T$ alone since they map each $\mathcal{E}_{\mu,i}$ onto itself (i fixed, j variable). Again this is to be expected since the self adjoint functions of the P 's are *observables*.

With this restoration of the 1-1 correspondence between states and rays the first quantized formalism can be compared directly with that of the second quantized theory. Stolt and Taylor have shown that for every particle of type T of finite order there is a natural isomorphism between $\mathcal{H}_{N,ray}^T$ and the N -particle space for the corresponding parafield (40). Thus they established that every type of second quantized parafield is equivalent to a unique first quantised paraparticle of finite order and vice versa.

Having outlined the general theory of indistinguishable particles, embracing paraparticles as well as ordinary bosons and fermions, we shall now discuss its consequences through a consideration of three particle states.

3.2.1 The Generation of Basis Functions for Three Particle States

We begin by recalling that to obtain the basis functions

$|f_i\rangle$ one simply applies the transfer operator Π_{ij} to the state function $|f\rangle$ thus,

$$|f_i\rangle = \Pi_{ij} |f\rangle \quad \text{i.e. } |f_i\rangle = \frac{1}{h} \sum_R \rho_{ij}^{(w*)}(R) P(R) |f\rangle$$

Obviously the first step is to write down the real orthogonal matrices of the irreducible representations

38. Ibid p. 2047 and p.11

39. Unless of course D^μ is one-dimensional in which case $\mathcal{H}_{N,ray}^T$ is the same as \mathcal{H}_N^T and the P 's are defined.

40. Stolt and Taylor Ibid.

of the permutation group S_3 i.e. the 2×2 matrix representatives of R in the irreducible representation D^μ denoted^{above} by $D_{ij}^{(\mu)*}(R)$ ($D_{ij}^{(\mu)}(R)$ is real so $D_{ij}^{(\mu)*}(R) = D_{ij}^{(\mu)}(R)$). These are given by Hammermesh with their corresponding permutations, written in cyclic notation (e.g. (123) or (13) indicating that we permute the first and third 'objects') :

Matrix	Corresponding Permutation
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	(1)(2)(3)
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	(12)
$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	(23)
$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	(13)
$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	(123)
$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	(213)

Remembering our previous notation, and the fact that the P 's permute the particles regardless of the state they are in, and the \bar{P} 's permute the states regardless of which particles are occupying these states, the effect of these permutation operators on our basis functions can be expressed as follows :

Particle Permutations P_i (Permuting the subscripts)

$$\begin{aligned}
 P_1 |a_1^1, a_2^2, a_3^3\rangle &= |a_1^1, a_2^2, a_3^3\rangle && (1)(2)(3) \\
 P_2 |a_1^1, a_2^2, a_3^3\rangle &= |a_2^1, a_1^2, a_3^3\rangle && (12)(3) \\
 P_3 |a_1^1, a_2^2, a_3^3\rangle &= |a_3^1, a_2^2, a_1^3\rangle && (13)(2) \\
 P_4 |a_1^1, a_2^2, a_3^3\rangle &= |a_1^1, a_3^2, a_2^3\rangle && (23)(1) \\
 P_5 |a_1^1, a_2^2, a_3^3\rangle &= |a_2^1, a_3^2, a_1^3\rangle && (123) \\
 P_6 |a_1^1, a_2^2, a_3^3\rangle &= |a_3^1, a_1^2, a_2^3\rangle && (213)
 \end{aligned}$$

Place Permutations \bar{P}_i (Permuting the superscripts)

$$\begin{aligned}
 P_1 |a_1^1, a_2^2, a_3^3\rangle &= |a_1^1, a_2^2, a_3^3\rangle && (1)(2)(3) \\
 P_2 |a_1^1, a_2^2, a_3^3\rangle &= |a_1^2, a_2^1, a_3^3\rangle && (12)(3) \\
 P_3 |a_1^1, a_2^2, a_3^3\rangle &= |a_1^3, a_2^2, a_3^1\rangle && (13)(2) \\
 P_4 |a_1^1, a_2^2, a_3^3\rangle &= |a_1^1, a_2^3, a_3^2\rangle && (23)(1) \\
 P_5 |a_1^1, a_2^2, a_3^3\rangle &= |a_1^2, a_2^3, a_3^1\rangle && (123) \\
 P_6 |a_1^1, a_2^2, a_3^3\rangle &= |a_1^3, a_2^1, a_3^2\rangle && (213)
 \end{aligned}$$

The equation above, rewritten in the form

$$|a_1^1, a_2^2, a_3^3\rangle_i^u = \left(\frac{1}{n}\right)^{1/2} \sum_r D_{ij}^{(u)*} P(r) |a_1^1, a_2^2, a_3^3\rangle$$

$$\text{or } |a_1^1, a_2^2, a_3^3\rangle_i^u = \Pi_{ij} |a_1^1, a_2^2, a_3^3\rangle^u$$

39

says that to obtain each basis function each particle permutation should be multiplied by the (ij) 'th element of the matrix to which that permutation corresponds, the sum of these products is then taken and multiplied by some normalisation factor. Thus to obtain the first basis function $|a_1^1, a_2^2, a_3^3\rangle_i^u$ we multiply P_1 by the element i in the first row, the first column of the first matrix above, multiply P_2 by the element in the first row first column of the second matrix and so on. We then add all these products and multiply the sum by $\sqrt{\frac{1}{n}}$. To obtain the second basis function we multiply the P_i 's by the elements in the first row second column of the matrices, and so on.

Then applying this projection operator method, our four

basis functions, chosen to be eigenvectors of P_2 are:

$$\begin{aligned} |a_1^1 a_2^2 a_3^3\rangle_{11}^{\mu} &= \Pi_{11} |a_1^1 a_2^2 a_3^3\rangle = \frac{1}{\sqrt{3}} \left\{ |a_1^1 a_2^2 a_3^3\rangle + |a_2^2 a_1^1 a_3^3\rangle - \right. \\ &\quad \left. \frac{1}{2} |a_3^3 a_2^2 a_1^1\rangle - \frac{1}{2} |a_1^1 a_3^3 a_2^2\rangle - \right. \\ &\quad \left. \frac{1}{2} |a_2^2 a_3^3 a_1^1\rangle - \frac{1}{2} |a_3^3 a_1^1 a_2^2\rangle \right\} \\ &= \frac{1}{2\sqrt{3}} \left\{ 2 |a_1^1 a_2^2 a_3^3\rangle + 2 |a_2^2 a_1^1 a_3^3\rangle - |a_3^3 a_2^2 a_1^1\rangle - \right. \\ &\quad \left. |a_1^1 a_3^3 a_2^2\rangle - |a_2^2 a_3^3 a_1^1\rangle - |a_3^3 a_1^1 a_2^2\rangle \right\} \end{aligned}$$

$$\begin{aligned} |a_1^1 a_2^2 a_3^3\rangle_{21}^{\mu} &= \Pi_{11} |a_1^1 a_2^2 a_3^3\rangle = \frac{1}{\sqrt{3}} \left\{ 0 + 0 + \frac{\sqrt{3}}{2} |a_3^3 a_2^2 a_1^1\rangle \right. \\ &\quad \left. - \frac{\sqrt{3}}{2} |a_1^1 a_3^3 a_2^2\rangle - \frac{\sqrt{3}}{2} |a_2^2 a_3^3 a_1^1\rangle \right. \\ &\quad \left. + \frac{\sqrt{3}}{2} |a_3^3 a_1^1 a_2^2\rangle \right\} \\ &= \frac{1}{2} \left\{ |a_3^3 a_2^2 a_1^1\rangle - |a_1^1 a_3^3 a_2^2\rangle - |a_2^2 a_3^3 a_1^1\rangle + |a_3^3 a_1^1 a_2^2\rangle \right\} \end{aligned}$$

$$\begin{aligned} |a_1^1 a_2^2 a_3^3\rangle_{12}^{\mu} &= \Pi_{12} |a_1^1 a_2^2 a_3^3\rangle = \frac{1}{\sqrt{3}} \left\{ 0 + 0 + \frac{\sqrt{3}}{2} |a_3^3 a_2^2 a_1^1\rangle \right. \\ &\quad \left. - \frac{\sqrt{3}}{2} |a_1^1 a_3^3 a_2^2\rangle + \frac{\sqrt{3}}{2} |a_2^2 a_3^3 a_1^1\rangle \right. \\ &\quad \left. - \frac{\sqrt{3}}{2} |a_3^3 a_1^1 a_2^2\rangle \right\} \\ &= \frac{1}{2} \left\{ |a_3^3 a_2^2 a_1^1\rangle - |a_1^1 a_3^3 a_2^2\rangle + |a_2^2 a_3^3 a_1^1\rangle - |a_3^3 a_1^1 a_2^2\rangle \right\} \end{aligned}$$

$$\begin{aligned} |a_1^1 a_2^2 a_3^3\rangle_{22}^{\mu} &= \Pi_{22} |a_1^1 a_2^2 a_3^3\rangle = \left\{ |a_1^1 a_2^2 a_3^3\rangle - |a_2^2 a_1^1 a_3^3\rangle \right. \\ &\quad \left. + \frac{1}{2} |a_3^3 a_2^2 a_1^1\rangle + \frac{1}{2} |a_1^1 a_3^3 a_2^2\rangle \right. \\ &\quad \left. - \frac{1}{2} |a_2^2 a_3^3 a_1^1\rangle - \frac{1}{2} |a_3^3 a_1^1 a_2^2\rangle \right\} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2\sqrt{3}} \left\{ 2 |a_1^1 a_2^2 a_3^3\rangle - 2 |a_2^2 a_1^1 a_3^3\rangle + |a_3^3 a_2^2 a_1^1\rangle \right. \\ &\quad \left. + |a_1^1 a_3^3 a_2^2\rangle - |a_2^2 a_3^3 a_1^1\rangle - |a_3^3 a_1^1 a_2^2\rangle \right\} \end{aligned}$$

Following Hartle and Taylor we shall refer to these as f_{11}'' , f_{21}'' , f_{12}'' , f_{22}'' respectively.

These are the 'mixed symmetry' basis functions spanning the 'triangular' subspaces. To obtain the symmetric and anti-symmetric functions we follow the same procedure. The matrix representatives are simple. In the symmetric case it is just (1) for all permutations, and in the anti-symmetric case it is (1) for the permutation (1) (2) (3) and (-1) for permutations given by (ij). Thus the symmetric basis

function is $f_s = \frac{1}{\sqrt{6}} \{ |a_1^1 a_2^2 a_3^3\rangle + |a_2^1 a_1^2 a_3^3\rangle + |a_3^1 a_2^2 a_1^3\rangle + |a_1^2 a_3^3 a_2^1\rangle + |a_2^3 a_1^1 a_3^2\rangle + |a_3^2 a_2^1 a_1^3\rangle \}$
and the antisymmetric function is

$$f_A = \frac{1}{\sqrt{6}} \{ |a_1^1 a_2^2 a_3^3\rangle - |a_2^1 a_1^2 a_3^3\rangle - |a_3^1 a_2^2 a_1^3\rangle - |a_1^2 a_3^3 a_2^1\rangle + |a_2^3 a_1^1 a_3^2\rangle + |a_3^2 a_2^1 a_1^3\rangle \}$$

Operating on these functions with the P's or \bar{P} 's can give the eigenvalue for the operator concerned. Thus the eigenvalues of the particle permutation operator P_2 for our basis functions can be obtained as follows.

Consider for example, the function f_{11}' :

$$\begin{aligned} P_2 f_{11}' &= \frac{1}{\sqrt{3}} P_2 \{ 2 |a_1^1 a_2^2 a_3^3\rangle + 2 |a_2^1 a_1^2 a_3^3\rangle - |a_3^1 a_2^2 a_1^3\rangle - |a_1^2 a_3^3 a_2^1\rangle - |a_2^3 a_1^1 a_3^2\rangle - |a_3^2 a_2^1 a_1^3\rangle \} \\ &= \frac{1}{\sqrt{3}} \{ 2 |a_1^1 a_2^2 a_3^3\rangle + 2 |a_2^1 a_1^2 a_3^3\rangle - |a_3^1 a_2^2 a_1^3\rangle - |a_1^2 a_3^3 a_2^1\rangle - |a_2^3 a_1^1 a_3^2\rangle - |a_3^2 a_2^1 a_1^3\rangle \} \\ &= \frac{1}{\sqrt{3}} \{ 2 |a_1^1 a_2^2 a_3^3\rangle + 2 |a_2^1 a_1^2 a_3^3\rangle - |a_3^1 a_2^2 a_1^3\rangle - |a_1^2 a_3^3 a_2^1\rangle - |a_2^3 a_1^1 a_3^2\rangle - |a_3^2 a_2^1 a_1^3\rangle \} \\ &= + f_{11}' \end{aligned}$$

Similarly the eigenvalue of P_2 for

$$f_{21}, f_{12}'', f_{22}'', f_s \text{ and } f_A$$

are -1, +1, -1, +1 and -1 respectively

Thus we have obtained a set of basis functions, chosen to be eigen-vectors of P_2 , spanning a six-dimensional subspace \mathcal{E} of \mathcal{H}_3 which can be decomposed into irreducible subspaces invariant under S_3 thus

$$\mathcal{E} = \mathcal{E}_s \oplus (\mathcal{E}_e' \oplus \mathcal{E}_e'') \oplus \mathcal{E}_a \quad 40$$

\mathcal{E}_s and \mathcal{E}_a are the one-dimensional subspaces defined, respectively, by the symmetric and anti-symmetric functions

f_s and f_A . The remaining four-dimensional space splits into two irreducible subspaces transforming under the same two-dimensional representation of S_3 , i.e. that of the triangular Young diagram (41). Our choice for these two subspaces is given by f_{11}' and f_{21}' spanning \mathcal{E}_e' , and f_{12}'' and f_{22}'' spanning \mathcal{E}_e'' . (42).

As $|f\rangle$ runs over \mathcal{H}_3 it can be decomposed in this way into three sectors

$$\mathcal{H}_3 = \mathcal{H}_s \oplus \mathcal{H}_e \oplus \mathcal{H}_a \quad 41$$

The irreducible subspaces of \mathcal{H}_s and \mathcal{H}_a are one-dimensional whereas those of \mathcal{H}_e are two-dimensional.

41. See Appendix P.346

42. Compare with Hartle and Taylor's basis functions.

Pure states may be represented by any irreducible subspace in any of these sectors. Two states represented by subspaces of different representations are physically distinct as are the states represented by different subspaces of the same representation. These points will play an important role in our determination of the statistics of paraparticles.

Given the relation $[P(\mathcal{R}), Q] = 0$, Schur's Lemma implies that all matrix elements of the observable Q connecting different representations are zero. It then follows (43) that if there exist states corresponding to subspaces of different representations of S_N then they must be separated by a super-selection rule. In other words, transitions between such states, for example those corresponding to the symmetric and anti-symmetric representations, or the symmetric and triangular representations and absolutely forbidden. Thus a boson cannot, by some peculiar mechanism become a fermion nor a fermion a paraparticle, for example.

However, states corresponding to subspaces of the same irreducible representation, such as the two triangular subspaces \mathcal{E}_{ϵ}' and \mathcal{E}_{ϵ}'' where there is the same irreducible representation repeated twice, can be connected. Transitions can therefore occur between such states. (44)

The situation in the general N -particle case is quite analogous. The Hilbert space then decomposes into a number of subspaces and transitions between states corresponding to certain of these subspaces i.e. those of different representations are not allowed. Thus restrictions have been imposed such that certain states are rendered inaccessible to particles of the same species. If we consider the time evolution of the system, as effected by some Hamiltonian, then we can conclude that what the I.P. (as expressed by $[P(\mathcal{R}), Q] = 0$) effectively says is that if the system starts in one irreducible representation then it will always stay in that representation. Bosons will always be bosons, fermions, fermions etc.

43. Greenberg and Messiah op cit p. 251

44. Although with ^{careful} two body collisions these transitions will also be forbidden as we shall see.

The I.P. can thus be thought of as an extra postulate of QM, or an initial condition in the specification of a situation. We shall return to these accessibility restrictions in section 3.6 where its consequences will be considered in more detail.

3.2.2. The Cluster Principle

In order to obtain the eigenvalue for various permutations it is convenient to rewrite the basis functions in terms of the P's given above. Thus as

$|a_1^1 a_2^2 a_3^3\rangle = P_1 |a_1^1 a_2^2 a_3^3\rangle$, $|a_2^1 a_1^2 a_3^3\rangle = P_2 |a_1^1 a_2^2 a_3^3\rangle$,
etc, we have, omitting the $|a_1^1 a_2^2 a_3^3\rangle$ for convenience:

$$f_s = \frac{1}{\sqrt{6}} (P_1 + P_2 + P_3 + P_4 + P_5 + P_6) ; f_A = \frac{1}{\sqrt{6}} (P_1 - P_2 - P_3 - P_4 + P_5 + P_6)$$

$$f'_{11} = \frac{1}{2\sqrt{3}} (2P_1 + 2P_2 - P_3 - P_4 - P_5 - P_6) ; f'_{21} = \frac{1}{2} (-P_3 + P_4 - P_5 + P_6)$$

$$f''_{12} = \frac{1}{2} (-P_3 + P_4 + P_5 - P_6) ; f''_{22} = \frac{1}{2\sqrt{3}} (2P_1 - 2P_2 + P_3 + P_4 - P_5 - P_6)$$

The result of applying two permutations consecutively can be evaluated very straightforwardly and expressed in the following two tables, where the columns give the permutation applied first and the rows the permutation applied second:

	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_1	P_6	P_5	P_4	P_3
P_3	P_3	P_5	P_1	P_6	P_2	P_4
P_4	P_4	P_6	P_5	P_1	P_3	P_2
P_5	P_5	P_3	P_4	P_2	P_6	P_1
P_6	P_6	P_4	P_2	P_3	P_1	P_5

TABLE 1:

P_i 's

	\bar{P}_1	\bar{P}_2	\bar{P}_3	\bar{P}_4	\bar{P}_5	\bar{P}_6
\bar{P}_1	\bar{P}_1	\bar{P}_2	\bar{P}_3	\bar{P}_4	\bar{P}_5	\bar{P}_6
\bar{P}_2	\bar{P}_2	\bar{P}_1	\bar{P}_6	\bar{P}_5	\bar{P}_4	\bar{P}_3
\bar{P}_3	\bar{P}_3	\bar{P}_5	\bar{P}_1	\bar{P}_6	\bar{P}_2	\bar{P}_4
\bar{P}_4	\bar{P}_4	\bar{P}_6	\bar{P}_5	\bar{P}_1	\bar{P}_3	\bar{P}_2
\bar{P}_5	\bar{P}_5	\bar{P}_3	\bar{P}_4	\bar{P}_2	\bar{P}_6	\bar{P}_1
\bar{P}_6	\bar{P}_6	\bar{P}_4	\bar{P}_2	\bar{P}_3	\bar{P}_1	\bar{P}_5

TABLE 2:

\bar{P}_i 's

Thus the result of operating on the basis functions with P_2 , for example, can be discovered very quickly and simply. The same eigenvalues are obtained as in the longer method above.

It is of some interest to discover the eigenvalues of \bar{P}_2 for our basis functions. These can be rewritten in terms of the \bar{P} 's thus

$$f_S = \frac{1}{\sqrt{6}}(\bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \bar{P}_4 + \bar{P}_5 + \bar{P}_6) ; f_A = \frac{1}{\sqrt{6}}(\bar{P}_1 - \bar{P}_2 - \bar{P}_3 - \bar{P}_4 + \bar{P}_5 + \bar{P}_6)$$

$$f'_{11} = \frac{1}{2\sqrt{3}}(2\bar{P}_1 + 2\bar{P}_2 - \bar{P}_3 - \bar{P}_4 - \bar{P}_5 - \bar{P}_6) ; f'_{21} = \frac{1}{2}(-\bar{P}_3 + \bar{P}_4 + \bar{P}_5 + \bar{P}_6)$$

$$f''_{12} = \frac{1}{2}(-\bar{P}_3 + \bar{P}_4 - \bar{P}_5 + \bar{P}_6) ; f''_{22} = \frac{1}{2\sqrt{3}}(2\bar{P}_1 - 2\bar{P}_2 + \bar{P}_3 + \bar{P}_4 - \bar{P}_5 - \bar{P}_6)$$

The eigenvalues of \bar{P}_2 for these basis functions can now be found using Table 2) and they are as follows:

$$f_S : +1 ; f_A : -1 ; f'_{11} : +1 ; f'_{21} : +1 ; f''_{12} : -1 ; f''_{22} : -1.$$

We can summarise these results and those for P_2 in the following table:

TABLE 3

Basis functions	Eigenvalue of P_2	Eigenvalue of \bar{P}_2
f_S	+1	+1
f_A	-1	-1
f'_{11}	+1	+1
f'_{21}	-1	+1
f''_{12}	+1	-1
f''_{22}	-1	-1

An immediate consequence of these results is that eigenvalues of the place permutation \bar{P}_2 serve to distinguish the symmetry types for 'clusters' of paraparticles whereas the eigenvalues of the particle permutation operator do not. (45)

Hartle and Taylor believed that it is possible to have basis vectors for a triangular representation of S_3 which are eigenvectors of P_2 with the same eigenvalues. (46) However this is not true, as our

45. It can be shown that the basis functions chosen by Hartle and Taylor are not eigenvectors of P_2 . It is possible that their basis vectors were misprinted. This does not affect the substance of their arguments.
46. Hartle and Taylor op cit p. 2048

results above demonstrate, and as one can prove in the following way.

It is well known that any group can be separated into classes of elements which are conjugate to one another. (47) Thus the permutation group S_3 separates into three classes, $K_1 = \{P_1\}$, $K_2 = \{P_2, P_3, P_4\}$, $K_3 = \{P_5, P_6\}$ having the cycle structure (1^3) , (12) and (3) respectively.

It is also well known that the traces of the matrices $D_{ij}^{(\mu)}(R)$ are invariant under a change of basis (i.e. a transformation of the co-ordinate axes). With regard to group representations the trace $\sum_i D_{ii}(R)$ is called the character of R in the representation D and is denoted by $\chi(R) = \sum_i D_{ii}(R)$. It can also be shown that conjugate elements in the group always have the same character (48). Hence when the group is described by listing the characters of its elements in a given representation, the same number is assigned to all the elements in a given class. Thus the representation will be described by a set of characters χ_1, \dots, χ_ν where ν is the number of classes in the group. Thus S_3 will be described by three characters, χ_1, χ_2, χ_3 for the symmetric, triangular and anti-symmetric representations respectively.

The characters of a group can be deduced using Frobenius's theorem (49) or by applying graphical methods involving Young tableaux (50). For S_3 the values of the characters for the three classes K_1, K_2, K_3 , are given in the following character table : (51)

<u>TABLE 4</u>	χ_1	χ_2	χ_3
K_1	1	2	1
K_2	1	0	-1
K_3	1	-1	1

47. Hammermesh op cit p. 23-25; Appendix p.338.

48. Ibid p. 79; Appendix p.340

49. Hammermesh op cit p. 182-197,

50. Ibid p. 201-208; Appendix p.346

51. Ibid p. 186; Also Margenau and Murphy op cit p.100

Let us suppose that f_1 and f_2 are basis functions supporting an irreducible representation of S_3 which transforms under the triangular representation. We assume that f_1 and f_2 are eigenvectors of P_2 with the same eigenvalue, +1 say. Thus $P_2 f_1 = f_1$ and $P_2 f_2 = f_2$. So the matrix representing P_2 is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D(P_2)$. The trace of this matrix is equal to 2 i.e. $\text{Tr} D(P_2) = 2$ 42. But by our assumptions $D(P_2)$ is related to the triangular representation by a similarity transformation which does not alter its trace. Thus, since P_2 belongs to the class K_2 we have

$$\text{Tr} D(P_2) = \chi_2^2 = 0 \quad 43$$

Clearly 42 and 43 contradict one another and thus it can be concluded that f_1 and f_2 cannot be eigenvectors of P_2 with the same eigenvalue.

Since $D(P_2)$ can always be diagonalized and the eigenvalues must be different and since $P_2^2 = I$, the eigenvalues of P must be ± 1 . Hence $D(P_2)$ can always be brought to the form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which has the correct trace of zero. Then $P_2 f_1 = f_1$ and $P_2 f_2 = -f_2$. Hence f_1 is symmetric under exchange of particles 1 and 2 and f_2 is antisymmetric under this interchange. This is indeed the case for our basis functions as we have seen.

Paraparticle theory came under attack by Steinmann (52), following a result suggested by Pauli, (53) on the grounds that it was inconsistent with the Cluster Principle. This states that systems sufficiently separated in space may be treated as isolated systems or, in other words, that the presence of particles on Mars, say, should not affect the results of experiments performed on Earth.

The argument runs as follows. Steinmann considered a three particle system consisting of a two particle cluster on Earth and the other particle a long way away, on the Moon. He stated correctly, that the general triangular state made up from one-particle wave functions will appear to an Earth bound observer as a statistical

52. O. Steinmann (1966) p. 755

53. W. Pauli (1958) p. 110

mixture of boson and fermion two-particle states (54). However, he then went on to argue that the observer can make a measurement on the two particle system to determine its symmetry type and that such a measurement changes the observed two-particle state to either pure symmetric or pure antisymmetric. Thus, he claimed if the history of the whole three particle system is considered then the measurement has introduced a distinction between basis functions which are supposed to be indistinguishable. Hence the inconsistency.

In other words the argument is that if the symmetry type of particles 1 and 2 on Earth, when 3 is on the Moon, is measured then f_1 and f_2 can be distinguished, yet they are supposed to be indistinguishable since they support the same irreducible representation of S_3 . The logical structure of this reasoning can be illustrated thus

$$\begin{array}{l}
 CP \rightarrow (\Delta \rightarrow (\sim IP)) \leftrightarrow (\sim \Delta) \vee (\sim IP) \\
 \text{But } CP. \\
 \hline
 (\sim \Delta) \vee (\sim IP) \leftrightarrow IP \rightarrow (\sim \Delta) \\
 \text{But } IP \\
 \hline
 \sim \Delta
 \end{array}$$

where CP denotes the Cluster Principle, Δ the triangular symmetry of the functions and IP denotes the Indistinguishability Postulate.

The basic fallacy in this argument is the supposition that the symmetry type of a two-particle cluster is determined by the eigenvalue of P_2 . It is assumed that because P_2 has opposite eigenvalues for f'_{11} and f'_{21} , for example, that the symmetry types of two-particle clusters are different for f'_{11} and f'_{21} . However, this is not true. It can be shown that both f'_{11} and f'_{21} give symmetric type functions for two-particle clusters and furthermore that f''_{12} and f''_{22} both give anti-symmetric types although they also possess opposite eigenvalues for P_2 .

54. It can be shown that a particle which has three particle states corresponding to the triangular Young diagram must have both boson and fermion-type two particle states. See Hartle and Taylor op cit p. 2048

The three particle wave function can be written

$$|a_1^1 a_2^2 a_3^3\rangle \quad \text{where } a_1^1 \text{ and}$$

a_2^2 are functions localized on Earth, say, and a_3^3 is localized on the Moon. With subscripts 1 and 2 as the Earth coordinates we have $a_1^3 = a_2^3 = 0$.

Operating on this wave function with P's gives:

$$P_1 |a_1^1 a_2^2 a_3^3\rangle = |a_1^1 a_2^2 a_3^3\rangle, \quad P_2 |a_1^1 a_2^2 a_3^3\rangle = |a_2^2 a_1^1 a_3^3\rangle$$

$$P_3 |a_1^1 a_2^2 a_3^3\rangle = 0, \quad P_4 |a_1^1 a_2^2 a_3^3\rangle = 0$$

$$P_5 |a_1^1 a_2^2 a_3^3\rangle = 0, \quad P_6 |a_1^1 a_2^2 a_3^3\rangle = 0$$

Considering now a state Ψ where

$$\Psi = (\alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_6 P_6) |a_1^1 a_2^2 a_3^3\rangle$$

then the probability distribution for finding one particle localized at 1, one at 2 and one at 3 is

$$\omega(1,2,3) = |P_1 \Psi|^2 + |P_2 \Psi|^2 + \dots + |P_6 \Psi|^2$$

Ψ is now set equal to our four basis functions in turn and the probability distribution for the case 1 is considered in each case.

Thus setting Ψ equal to f_{11} , using the expression for this function in terms of the P's and Table 1) we obtain

$$\Psi = f_{11} = \frac{1}{2\sqrt{3}} (2P_1 + 2P_2 - P_3 - P_4 - P_5 - P_6)$$

$$P_1 \Psi = \frac{1}{2\sqrt{3}} (2P_1 + 2P_2 - P_3 - P_4 - P_5 - P_6)$$

$$P_2 \Psi = \frac{1}{2\sqrt{3}} (2P_2 + 2P_1 - P_6 - P_5 - P_4 - P_3)$$

$$P_3 \Psi = \frac{1}{2\sqrt{3}} (2P_3 + 2P_5 - P_1 - P_6 - P_2 - P_4)$$

$$P_4 \Psi = \frac{1}{2\sqrt{3}} (2P_4 + 2P_6 - P_5 - P_1 - P_3 - P_2)$$

$$P_5 \Psi = \frac{1}{2\sqrt{3}} (2P_5 + 2P_3 - P_4 - P_2 - P_6 - P_1)$$

$$P_6 \Psi = \frac{1}{2\sqrt{3}} (2P_6 + 2P_4 - P_2 - P_3 - P_1 - P_5)$$

For the case 1 we obtain

$$P_1 \Psi = \frac{1}{2\sqrt{3}} (2P_1 + 2P_2)$$

$$P_2 \Psi = \frac{1}{2\sqrt{3}} (2P_1 + 2P_2)$$

$$P_3 \Psi = \frac{1}{2\sqrt{3}} (-P_1 - P_2)$$

$$P_4 \Psi = \frac{1}{2\sqrt{3}} (-P_1 - P_2)$$

$$P_5 \Psi = \frac{1}{2\sqrt{3}} (-P_2 - P_1)$$

$$P_6 \Psi = \frac{1}{2\sqrt{3}} (-P_2 - P_1)$$

The probability distribution is therefore

$$\omega(1,2,3) = |(\rho_1 + \rho_2)|^2 = |a_3^3|^2 |a_1^1 a_2^2 + a_2^1 a_1^2|^2$$

Integrating over co-ordinate 3 (as our Earth bound observer does not take into consideration the particle on the Moon) then gives the two-particle probability distribution

$$\omega(1,2) = |a_1^1 a_2^2 + a_2^1 a_1^2|^2$$

which is clearly symmetric.

Repeating the whole procedure for $\psi = f_{21}'$ one obtains the probability distribution for case 1

$$\omega(1,2) = |a_1^1 a_2^2 + a_2^1 a_1^2|^2$$

which again is symmetric.

With $\psi = f_{12}''$ and $\psi = f_{22}''$ one gets

$$\omega(1,2) = |a_1^1 a_2^2 - a_2^1 a_1^2|^2 \text{ and } \omega(1,2) = |a_1^1 a_2^2 - a_2^1 a_1^2|^2$$

which are both antisymmetric distributions.

We conclude that for two-particle clusters

f_{11}' and f_{21}' give symmetric distributions whereas f_{12}'' and f_{22}'' both give antisymmetric distributions. Thus measurements of the symmetry type of two-particle clusters distinguishes not between f_{11}' and f_{21}' , as Steinmann claimed, and which must be indistinguishable as they belong to the same irreducible subspace, but between f_{12}'' and f_{22}'' , which is perfectly consistent since these functions belong to different irreducible subspaces.

As Hartle and Taylor said 'The essential point is that the measurement carries the three particle vector from one irreducible subspace to another. There is no question of its distinguishing between vectors in the irreducible subspace ... [such as f_{11}' and f_{21}']. ... which are certainly indistinguishable'. (55)

Similar results can be obtained using the \bar{P} 's as well. These commute with the P 's i.e. $[\bar{P}_i, P_j] = 0 \quad \forall i, j$ which means that if we consider a state where ψ where

$$\psi = (\alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_6 P_6) |a_1^1 a_2^2 a_3^3\rangle$$

then $\bar{P}_2 P_1 \psi = P_1 \bar{P}_2 \psi$. Thus if ψ is an eigenstate of \bar{P}_2 then $P_1 \psi$ is also an eigenstate with the same eigenvalue. Taking $\psi = f_{11}'$ for example then this implies that as f_{11}' is an eigenfunction of \bar{P}_2 with eigenvalue +1 then $P_1 f_{11}'$, $P_2 f_{11}'$, \dots , $P_6 f_{11}'$ will also be eigenfunctions of \bar{P}_2 with eigenvalue +1, i.e.

$$\bar{P}_2 P_1 f_{11}' = + P_1 f_{11}', \quad \bar{P}_2 P_2 f_{11}' = + P_2 f_{11}', \quad \dots, \quad \bar{P}_2 P_6 f_{11}' = + P_6 f_{11}'$$

With our choice of

wave functions a_1^1, a_2^2, a_3^3 such that a_1^1 and a_2^2 are localized on the Earth and a_3^3 is on the Moon, the only terms in $P_1 f_{11}'$, $P_2 f_{11}'$, \dots etc. which need to be considered are those involving $P_1 |a_1^1 a_2^2 a_3^3\rangle$ and $P_2 |a_1^1 a_2^2 a_3^3\rangle$. Furthermore the ^{se} results

above mean that $P_1 f_{11}'$, $P_2 f_{11}'$, \dots etc. must involve $P_1 |a_1^1 a_2^2 a_3^3\rangle$ and $P_2 |a_1^1 a_2^2 a_3^3\rangle$ in the symmetric combination $(P_1 + P_2) |a_1^1 a_2^2 a_3^3\rangle$.

Hence the two-particle probability distribution must be a symmetric one, as we have already shown.

Similar results are obtained for the other three functions $f_{21}', f_{12}'', f_{22}''$: the symmetry of $\omega(1,2)$ is determined by the eigenvalue of \bar{P}_2 . If \bar{P}_2 has eigenvalue +1, as it does for f_{11}' and f_{21}' then $\omega(1,2)$ is a symmetric distribution. If \bar{P}_2 has eigenvalue -1 as it does for f_{12}'' and f_{22}'' then $\omega(1,2)$ is an antisymmetric distribution.

These results support our previous conclusion and reiterate the point that the eigenvalues of \bar{P}_2 serve to distinguish the symmetry type for clusters whereas the eigenvalues of P_2 do not.

One can generalize these conclusion and show that if paraparticle theory is to be consistent with the Cluster Principle then the particles must possess states associated with whole families of different permutation symmetries, according to the following rule: If a given particle has N -particle states associated with a given

Young diagram then it must have $(N-1)$ -, $(N-2)$ -, ..., 2-particle states associated with all Young diagrams which can be obtained from the first by successively removing squares. (56) This implies that all the infinitely many possible types of paraparticle would have both symmetric and antisymmetric two-particle states. In other words they would behave in pairs like bosons and fermions, as we have shown.

3.2.3. The Statistics of Paraparticles

Although the statistical behaviour of paraparticles has been the subject of some research in the past it has obviously not received the same degree of attention as the conventional Bose-Einstein and Fermi-Dirac Statistics (and with good reason as no paraparticle has ever been found to exist in nature). It is our intention here to examine a certain aspect of this behaviour and to consider some of its consequences.

We have already noted the fact that transitions between states corresponding to subspaces of different representations are not allowed, (there is a superselection rule existing between such states), whereas transitions between states corresponding to subspaces of the same irreducible representation are possible. We shall now demonstrate that although there is no superselection rule imposed on these states the latter transitions cannot in fact occur for *some* two-body collisions.

Such transitions may occur for three-body collisions^(56a), however, since the basis vectors we have chosen are not eigenvalues of \bar{P}_3 . These results have important consequences for both the statistics of paraparticles and the question whether a particle of one statistical type could be transformed into one of another type.

56. Hartle and Taylor op cit p. 2050

56a. By three-body collisions we mean collisions among three bodies, not necessarily simultaneously.

It is assumed that all states $a_1^1, a_1^2, a_2^1, a_2^2, a_3^1, a_3^2$ are orthogonal and that $a_1^1, a_1^2, a_2^1, a_2^2$ are wave functions localized on the Earth, whereas a_3^1, a_3^2 are localized on the Moon. 1 and 2 are then the Earth coordinates and 3 the Moon coordinate.

According to the Cluster Principle terms of the form

a_1^1, a_3^1 or a_1^2, a_3^2 are then identically zero, i.e.

there is no overlap between the wave functions localized on the Earth and Moon respectively.

The first step in our demonstration has already been dealt with on p. 145, where we showed that any symmetric function of the observables, Q , can be expanded in terms of the \bar{P} 's. Thus,

$$\langle f' | P_j Q P_k | f \rangle = \sum_i \langle f' | Q P_i | f \rangle \langle f | P_j \bar{P}_i P_k | f \rangle$$

$$\text{i.e. } Q \approx \sum_i c_i \bar{P}_i$$

where $c_i = \langle f' | Q P_i | f \rangle$ and the \approx sign means an equation in the restricted sense.

This if Q is taken to be some interaction energy V we have

$$\langle f' | P_j V P_k | f \rangle = \sum_i \langle f' | V P_i | f \rangle \langle f | P_j \bar{P}_i P_k | f \rangle$$

$$\text{or } V \approx \sum_i c_i \bar{P}_i \quad 44$$

Thus the interaction energy V is equal, in the restricted sense of course, to a linear function of the \bar{P} 's, with coefficients c_i given by $\langle f' | V P_i | f \rangle$

We now assume that there is only a *two-body* force acting between the particles (57) which means that the interaction energy V will consist of a sum of parts each referring to only two particles. This will result in all the matrix elements V_i vanishing, except those for which \bar{P}_i is either the identical permutation or simply an interchange of two places, i.e. $\bar{P}_i = \bar{P}_1$ or \bar{P}_2 .

Thus for two body forces V can be written as

$$V = V_{12} + V_{23} + V_{13}$$

where the subscripts refer to place interchanges. Our expression 44 will then reduce to $V \approx V_1 + \sum_{r < s} V_{rs} \bar{P}_{rs}$ 45

57. Such as the $\frac{u}{\lambda}$ Coulomb force between electrons for example.

where V_{rs} is the matrix element referring to the interchange of places r and s occupied by the particles. (58)

It should be noted that this assumption implies that we are only considering first order perturbation terms and thus the eigenvalues of 45 will give only the first order corrections to the energy levels.

Given this assumption we can now write
our expansion of V as

$$V \approx \sum_i c_i \bar{P}_i$$

$$\Rightarrow V = c_1 \bar{P}_1 + c_2 \bar{P}_2 + c_3 \bar{P}_3 + c_4 \bar{P}_4 + c_5 \bar{P}_5 + c_6 \bar{P}_6$$

where $\bar{P}_1, \bar{P}_2, \dots, \bar{P}_6$ are the place permutation operators defined above. We have to show that certain terms in this expansion are zero. To do this means looking at the coefficients c_i i.e. at the first term in the product

$$\langle f' | V \bar{P}_i | f \rangle = \langle f | P_j \bar{P}_i P_k | f \rangle$$

as it is this that will in general be zero for certain \bar{P}_i . In the second term, for any value of \bar{P}_i that might give us zero we can always find values of P_j and P_k that give 1. In other words, there is no generality to be had from the second term because of the particular nature of P_j and P_k .

We suppose that V is some suitable short range interaction such that it cannot reach to the Moon and thus cannot induce transitions between the eigenstates a^3 and $a^{3'}$ of the Moon bound particle and eigenstates $a^1, a^{1'}, a^2, a^{2'}$ of the two particles located on the Earth. Thus all interaction terms containing either a^3 or $a^{3'}$ of the form $a_1' V_{13} a_3^3$ or $a_3^{3'} V_{13} a_1^1$ are zero. The only terms to survive are those for which both a^3 and $a^{3'}$ are outside the interaction term, which means that the only

58. We are essentially following Dirac at this point, although our demonstration will be more general since he considered only the same state function $|f\rangle$
See Dirac op cit p. 218 and p. 224.

terms in 46 which are not zero are those containing interaction terms of the form V_{12} between a_1', a_2' ; a_1, a_2 only. Thus the only coefficients in our expansion which are not equal to zero are c_1 and c_2 associated with \bar{P}_1 and \bar{P}_2 , these being the only ones with interaction terms V_{12} of the form just described.

As an example we shall consider the coefficient c_3 in 46 associated with \bar{P}_3 :

$$\langle f' | V P_3 | f \rangle = \iiint f'^* V P_3 f d_1 d_2 d_3$$

Expanding $V, |f\rangle$ and $|f'\rangle$ we have

$$\begin{aligned} \langle f' | V P_3 | f \rangle &= \iiint a_1'^* a_2'^* a_3'^* (V_{12} + V_{23} + V_{13}) P_3 a_1, a_2, a_3 d_1 d_2 d_3 \\ &= \iiint a_1'^* a_2'^* a_3'^* (V_{12} + V_{23} + V_{13}) a_1, a_2, a_3 d_1 d_2 d_3 \\ &= \iiint a_3'^* a_3 (a_1'^* a_2'^* V_{12} a_2, a_1) d_1 d_2 d_3 + \\ &\quad \iiint a_1'^* a_3 (a_2'^* a_3'^* V_{23} a_3, a_2) d_1 d_2 d_3 + \\ &\quad \iiint a_2'^* a_2 (a_1'^* a_3'^* V_{13} a_3, a_1) d_1 d_2 d_3 \\ &= \iiint a_3'^* a_3 (a_1'^* a_3, V_{12} a_2'^* a_2) d_1 d_2 d_3 + \\ &\quad \iiint a_1'^* a_3 (a_2'^* a_2, V_{23} a_3'^* a_3) d_1 d_2 d_3 + \\ &\quad \iiint a_2'^* a_2 (a_1'^* a_3, V_{13} a_3'^* a_3) d_1 d_2 d_3 \end{aligned}$$

Now consider each of these terms in turn. In the first one has

$$a_1'^* a_3, V_{12} a_2'^* a_2$$

but V is short range and cannot connect a_3 and a_2 .

Thus V_{12} cannot reach between a_3 and a_2 so this term is zero. In the second term we have

$$a_2'^* a_2, V_{23} a_3'^* a_3$$

Again the short range nature of V means that V_{23} cannot induce transitions between a_2 and a_3 so this term is zero. Finally in the third term we have

$$a_1'^* a_3, V_{13} a_3'^* a_3$$

which again is zero as V_{13} cannot connect a_3 with a_3 .

Thus all three terms involving V_{12} , V_{23} , V_{13} are zero and so

$$\langle f' | V P_3 | f \rangle = 0 \text{ or } C_3 = 0$$

Similarly it can be shown that the coefficients C_4 , C_5 and C_6 are all also zero.

However C_1 and C_2 contain terms for which both a^3 and $a^{3'}$ fall outside the interaction term, thus giving finite non-zero values to these terms. Let us consider for example, the coefficient C_2 :

$$\langle f' | V P_2 | f \rangle = \iiint f'^* V P_2 f d_1 d_2 d_3$$

Expanding gives

$$\begin{aligned} \langle f' | V P_2 | f \rangle &= \iiint a_1'^* a_2'^* a_3'^* (V_{12} + V_{23} + V_{13}) P_2 a_1^1 a_2^2 a_3^3 d_1 d_2 d_3 \\ &= \iiint a_1'^* a_2'^* a_3'^* (V_{12} + V_{23} + V_{13}) a_2^1 a_1^2 a_3^3 d_1 d_2 d_3 \\ &= \iiint a_3'^* a_3^3 (a_1'^* a_2'^* V_{12} a_2^1 a_1^2) d_1 d_2 d_3 + \\ &\quad \iiint a_1'^* a_1^1 (a_2'^* a_3'^* V_{23} a_2^1 a_3^3) d_1 d_2 d_3 + \\ &\quad \iiint a_2'^* a_2^1 (a_1'^* a_3'^* V_{13} a_2^1 a_3^3) d_1 d_2 d_3 \\ &= \iiint a_3'^* a_3^3 (a_1'^* a_2^1 V_{12} a_2^1 a_2'^*) d_1 d_2 d_3 + \\ &\quad \iiint a_1'^* a_2^1 (a_2'^* a_1^1 V_{23} a_3^1 a_3^3) d_1 d_2 d_3 + \\ &\quad \iiint a_2'^* a_2^1 (a_1'^* a_2^1 V_{13} a_3^1 a_3^3) d_1 d_2 d_3 \end{aligned}$$

The first of these three expressions contains the interaction term

$$a_1'^* a_2^1 V_{12} a_2^1 a_2'^*$$

It can immediately be seen that this is non-zero, as the short range nature of V does allow V_{12} to induce transitions between $a_1'^*$ and $a_2'^*$ or a_1^2 and a_2^1 . However, the expression as a whole can only be finite if $a^{3'} = a^3$. This follows from the orthogonality of these functions, i.e.

$$\int a_3'^* a_3^3 d_3 = \begin{cases} 0 & \text{if } a^{3'} \neq a^3 \\ 1 & \text{if } a^{3'} = a^3 \end{cases}$$

Thus $a^{3'}$ and a^3 must not only overlap in order for C_2 to be non-zero, they must in fact be equal.

The second term contains $a_2'^* a_1^1 V_{23} a_3^1 a_3^3$ which is zero as V_{23} cannot connect $a_2'^*$ with a_3^1 or a_1^1 with a_3^3 .

In the third expression we have

$$a_1^{1'} a_1^2 V_{13} a_3^{3'} a_3^3$$

but again the short range nature of V prevents V_{13} from inducing transitions between $a_1^{1'}$ and $a_3^{3'}$ or a_1^2 and a_3^3 , so this term is also zero.

We thus obtain

$$C_2 = \text{finite if } a^{3'} = a^3$$

It can be shown similarly that the only term in C_1 which is finite is that containing V_{12} and then only if $a^{3'} = a^3$.

Thus we have shown that C_1 and C_2 are both non-zero and finite, but only if $a^{3'} = a^3$, and C_3, C_4, C_5, C_6 are all zero. The expansion given by 46 is therefore cut-off above C_2 . We therefore have

$$V \approx C_1 \bar{P}_1 + C_2 \bar{P}_2 \quad 47$$

Thus V commutes with \bar{P}_2 , written as $[V, \bar{P}_2] = 0$ 48 and therefore, for the particular case of ^{the} two-body collisions _{under discussion,} between the particles, there can be no transitions between states which possess opposite eigenvalues of the place permutation operator \bar{P}_2 . Since the two subspaces of the same triangular representation are spanned by two sets of eigenvectors, f_{11}', f_{21}' and f_{12}'', f_{22}'' , respectively, which possess opposite eigenvalues of \bar{P}_2 , we conclude that transitions are forbidden between states corresponding to these subspaces. In other words, given our conditions there can be no transitions between states corresponding to equivalent irreducible representations. Thus our demonstration is complete.

We shall now consider the bearing this result has on the possibility of a particle changing its statistical type. This was first suggested by Pauli in 1927 (59). He considered the collision between an electron and some N -particle system and concluded that such a collision could induce transitions between states of the system corresponding to different irreducible representations of S_N . Thus, for example, a system of bosons

could be transformed into a system of fermions via some mixed symmetry state.

We note, first of all, that the existence of the superselection rule, separating states corresponding to subspaces of different representations of S_N , means that transitions induced by interactions between the N-particles cannot occur between states corresponding to the symmetric and anti-symmetric representations (60), for example, nor between the symmetric and triangular representations. No such rule exists for states corresponding to equivalent representations. As we shall see directly transitions between these states can, in general, occur.

The relevance of this circumstance to the Pauli problem is as follows. If the third particle interacts with the two-particle subsystem, transitions between bosonic and fermionic states can occur, as Pauli suspected. *However*, if the third particle is located on the Moon our discussion of the **C**luster Principle above confirms that its mere existence cannot produce such transitions for two-particle systems on the Earth. We now proceed to show how the general three-body collision can produce transitions between equivalent representations.

60. E Wigner (1927) p.883 and W. Heisenberg (1930)
p.156

The demonstration proceeds as before but now

a_1, a_1', a_3^3 and $a_3^{3'}$ are taken to be the wave-functions localized on the Earth with $a_2^2, a_2^{2'}$ on the Moon. The Cluster Principle then dictates that terms of the form a_1, a_1^2 or a_3^3, a_3^2 are identically zero. Our result from p. 145 is obviously still applicable and so the interaction energy can still be written as

$$V \approx \sum_i c_i \bar{P}_i$$

where the equation is taken in the restricted sense.

If V is again short range then all interaction terms containing either a^2 or $a^{2'}$ i.e. of the form

$$a_1' V_{12} a_2^2 \quad \text{or} \quad a_2^{2'} V_{12} a_1'$$

are zero. Thus the only terms of the expansion to survive are those

containing interaction terms of the form V_{13} between $a_1, a_1', a_3^3, a_3^{3'}$ only. The only coefficients not equal to zero are therefore c_1 and c_3 associated with \bar{P}_1 and \bar{P}_3 .

Consider for example the coefficient c_2 associated with \bar{P}_2 :

$$\langle f' | V \bar{P}_2 | f \rangle = \iiint a_1'^* a_2^{2'*} a_3^{3'*} (V_{12} + V_{23} + V_{13}) \bar{P}_2 a_1 a_2^2 a_3^3 d_1 d_2 d_3$$

Expanding as before we obtain

$$\begin{aligned} \langle f' | V \bar{P}_2 | f \rangle &= \iiint a_1'^* a_2^{2'*} a_3^{3'*} (V_{12} + V_{23} + V_{13}) a_1 a_2^2 a_3^3 d_1 d_2 d_3 \\ &= \iiint a_3^{3'*} a_3^3 (a_1'^* a_1 V_{12} a_2^2 a_2^{2'*}) d_1 d_2 d_3 + \\ &\quad \iiint a_1'^* a_1 (a_2^{2'*} a_2^2 V_{23} a_3^{3'*} a_3^3) d_1 d_2 d_3 + \\ &\quad \iiint a_2^{2'*} a_2^2 (a_1'^* a_1 V_{13} a_3^{3'*} a_3^3) d_1 d_2 d_3 \end{aligned}$$

Considering each term in turn the short range nature of V implies that

$$a_1'^* a_1^2 V_{12} a_2^1 a_2'^* = 0, \quad a_2^{2'}* a_2'^* V_{23} a_3^1 a_3^3 = 0$$

and

$$a_1'^* a_1^2 V_{13} a_3^1 a_3^3 = 0.$$

c_1 and c_3 however contain terms for which both a^2 and $a^{2'}$ fall outside the interaction term, giving finite non-zero values.

Expanding c_3 gives

$$\begin{aligned} \langle f' | V P_3 | f \rangle &= \iiint a_1'^* a_2^{2'}* a_3^3* (V_{12} + V_{23} + V_{13}) a_3^1 a_2^2 a_1^3 d_1 d_2 d_3 \\ &= \iiint a_3^3* a_1^3 (a_1'^* a_2^2 V_{12} a_3^1 a_2'^*) d_1 d_2 d_3 + \\ &\quad \iiint a_1'^* a_3^1 (a_2^{2'}* a_2^2 V_{23} a_3^3* a_1^3) d_1 d_2 d_3 + \\ &\quad \iiint a_2^{2'}* a_2^2 (a_1'^* a_3^1 V_{13} a_3^3* a_1^3) d_1 d_2 d_3 \end{aligned}$$

and now $a_1'^* a_3^1 V_{13} a_3^3* a_1^3$ is non zero. Proceeding as before we therefore obtain

$$V \approx c_1 \bar{P}_1 + c_3 \bar{P}_3 \quad 48$$

Since we chose our wavefunctions to be eigenfunctions of \bar{P}_2 we could conclude from our previous result 47 that transitions could not take place between states possessing opposite eigenvalues of \bar{P}_2 . No such conclusion can be drawn from our result 48 above because our wavefunctions are not eigenfunctions of \bar{P}_3 .

This can be seen by referring back to Table 2 on p. 162. As we have noted f_{11}' can be written in the form

$$f_{11}' = \frac{1}{2\sqrt{3}} (2\bar{P}_1 + 2\bar{P}_2 - \bar{P}_3 - \bar{P}_4 - \bar{P}_5 - \bar{P}_6)$$

Using Table 2 we have

$$\bar{P}_3 f_{11}' = \frac{1}{2\sqrt{3}} (2\bar{P}_3 + 2\bar{P}_5 - \bar{P}_1 - \bar{P}_6 - \bar{P}_2 - \bar{P}_4)$$

Also $f_{21}' = \frac{1}{2} (-\bar{P}_3 + \bar{P}_4 + \bar{P}_5 - \bar{P}_6)$

and $\bar{P}_3 f_{21}' = \frac{1}{2} (-\bar{P}_1 + \bar{P}_6 + \bar{P}_2 - \bar{P}_4)$

Similarly f_{12}'' and f_{22}'' are not eigenfunctions of \bar{P}_3 either.

Thus our two particle cluster on the Earth is in this case not bosonic or fermionic. Of course, we could make it so, as it was before, through a unitary transformation which would make the wavefunctions eigenfunctions of \bar{P}_3 . However, they then could not simultaneously be eigenfunctions of \bar{P}_2 since \bar{P}_3 and \bar{P}_2 do not commute.

We therefore conclude that, with three-body collisions, transitions can occur between states corresponding to subspaces spanned by the vectors f_{11}' and f_{21}' and f_{12}'' and f_{22}'' through the collision of particle 1 with particle 3. ^(60a) In this case states corresponding to

\mathcal{E}_t' and \mathcal{E}_t'' can be connected. Thus in a three-body collision involving particle 3 colliding with particle 1 followed by 2, the first collision can take the particles from a state supported by one subspace to a state supported by another whereas the second can induce transformations between states supported by the same latter subspace.

These results obviously have important consequences for the weighting assignments in the statistics of paraparticles. The statistical weight of an arrangement of particles arises from the number of ways in which that arrangement can be realised. Thus the arrangement represented by

o	o
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 is given weight 2 in Maxwell-Boltzmann statistics, as it corresponds to the two possible microstates ^a

1o	o2
----	----

^b and ^a

2o	o1
----	----

^b but only weight 1 in Bose-Einstein and Fermi-Dirac statistics since in both the latter there is only one possible microstate as particle permutations are not observable. The sum of the weights for a particular kind of statistics gives the total number of possible arrangements of the particles for that kind of statistics.

60a. 'Particle 1' means the wave packet associated with the state a, and so on.

For two particles distributed among two states the statistics are no different from ordinary quantum statistics, (parabosons and parafermions of order 2 exhibit the same statistical behaviour as bosons and fermions respectively). Thus to bring out the extraordinary nature of parastatistics we shall consider three particles distributed over three states.

Our procedure is as follows. The central question is whether or not transitions are possible between our initial arrangement and some final arrangement. The statistical weight given to this final arrangement will then depend upon which of these transitions are allowed. Although any given arrangement corresponds to two possible states represented by different subspaces of the same representation (e.g. \mathcal{E}_ϵ' and \mathcal{E}_ϵ''), it is not necessary to consider transitions from both initial states since we would obviously obtain the same results whether the initial states are represented by subspaces spanned by f_{11}' and f_{21}' or by f_{12}'' and f_{22}'' . Also we need only consider one vector out of either of the above two pairs since there are no transitions within a subspace.

Thus we shall represent our initial state by f_{11}' , and the final state will be represented by the vector obtained from this by setting certain wavefunctions equal to one another depending on the final arrangement considered. (Of course the subspace representing the final state is actually spanned by two vectors obtained from f_{11}' and f_{21}'). We recall that f_{11}' is given by

$$f_{11}' = \frac{1}{\sqrt{3}} \{ 2|a_1^1 a_2^2 a_3^3\rangle + 2|a_2^1 a_1^2 a_3^3\rangle - |a_1^1 a_3^2 a_2^3\rangle - |a_3^1 a_2^2 a_1^3\rangle - |a_2^1 a_3^2 a_1^3\rangle - |a_3^1 a_1^2 a_2^3\rangle \}$$

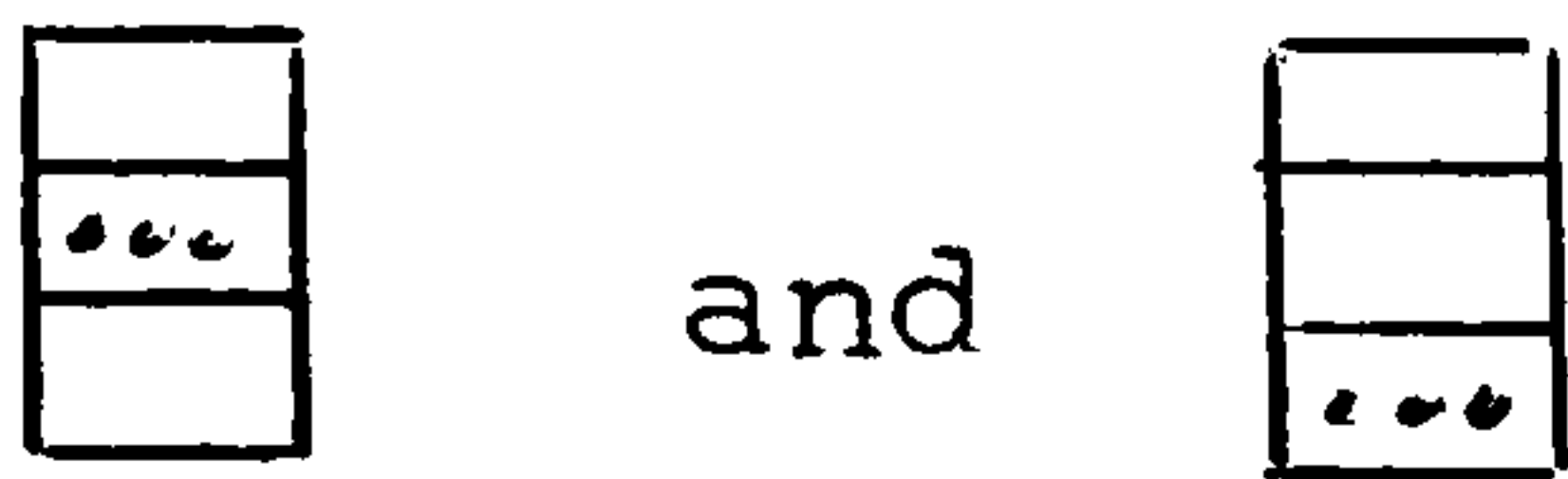
The first states to be considered are those corresponding to the arrangement $\begin{array}{|c|} \hline a^1 \\ \hline a^2 \\ \hline a^3 \\ \hline \end{array}$ i.e. one particle in each state. The wavefunctions a^1 , a^2 and a^3 are then all different and the derived function is simply the same as f_{11}' . In this case, with three-body collisions as we have seen, there is the possibility of transitions occurring between states corresponding to subspaces of equivalent representations, i.e. between states represented by f_{11}' and f_{12}' giving a

doubling of the allowed states and hence also the weight. Thus in this case a weight of 2, rather than 1, is assigned to the arrangement.

With the arrangement given by $\begin{array}{|c|} \hline \dots \\ \hline \\ \hline \end{array}$, i.e. all three particles in state a^1 . the three wave-functions are all equal to one another, thus $a^3 = a^2 = a^1$. The derived function is then

$$f_{11}' = \frac{1}{2\sqrt{3}} \{ 2|a_1^1 a_2^1 a_3^1\rangle + 2|a_1^1 a_2^1 a_3^1\rangle - |a_1^1 a_2^1 a_3^1\rangle - |a_1^1 a_2^1 a_3^1\rangle - |a_1^1 a_2^1 a_3^1\rangle - |a_1^1 a_2^1 a_3^1\rangle \}$$

Transitions to the final state represented by this function are therefore not allowed and weight 0 is assigned in this case. It can easily be shown that a similar result holds for the arrangements given by



With the arrangement $\begin{array}{|c|} \hline \dots \\ \hline \cdot \\ \hline \end{array}$ we have $a^3 = a^1$ and

the state function for the final state is given by

$$f_{11}'' = \frac{1}{2\sqrt{3}} \{ |a_1^1 a_2^2 a_3^1\rangle + |a_2^1 a_1^2 a_3^1\rangle - 2|a_1^1 a_3^2 a_2^1\rangle \}$$

Similarly:

$$f_{12}'' = \frac{1}{2} \{ -|a_1^1 a_2^2 a_3^1\rangle + 2|a_1^1 a_3^2 a_2^1\rangle - |a_2^1 a_1^2 a_3^1\rangle \}$$

In this case there is no doubling of states, and thus there is only one set of transitions from the initial to the final state. Accordingly weight 1 is assigned to this arrangement for three-body collisions.

By continuing in this manner with the other possible arrangements of three particles over three states we obtain the table of weights shown in Table 5. We have included for comparison the weights for particles obeying Gentile's 'paragas' statistics (61) In this theory the average number of particles in a group of states is dependent upon a parameter d giving the maximum number of particles which can occupy any given state. There is therefore simply a form of generalised quantum statistics with F-D and B-E statistics as special cases, obtained when $d = 1$ and $d = \infty$ respectively. In particular there is no doubling of weights due to transitions between

subspaces supporting equivalent representations as in our case above. Thus when $d = 2$ the arrangement

a
b
c

is given weight 1 along with all the others, except those with all three particles in one state which are assigned weight 0, and so we obtain the set of weights shown.

Thus we can see immediately that the statistical behaviour of paraparticles is not the same as that exhibited by a 'paragas'. Table 5 demonstrates that the total number of allowed states for the former is greater than for the latter. From this table we obtain Table 6 which gives the weights for the various distribution numbers Λ_{a^1} , Λ_{a^2} , Λ_{a^3} , giving the number of particles in the states a^1 , a^2 and a^3 . Again we see that for one particle in state a^1 , a^2 , and a^3 respectively the paraparticle weight is different from that of the paragas. This table can in turn be used to give the probability of finding a certain number of particles in a given state by simply dividing the above weights by the total i.e. by the total number of accessible states.

This gives the set of figures displayed in Tables 7 and 8, which clearly illustrate the differences between both paraparticle and paragas statistics and 'ordinary' quantum statistics and between paraparticle and paragas statistics themselves. Thus comparing paragas and 'ordinary' quantum statistics we clearly see the so-called 'intermediate' nature of the former. For example for paragas statistics the probability of finding one particle in any state is greater than for B-E statistics but less, of course, than the corresponding probability for F-D statistics, (and also less than the probability for classical M-B statistics). To give another illustrative example the probability of finding two particles in any state is greater for paragas statistics than for any of the others, including paraparticle statistics. Thus in this limited sense particles obeying the former show a greater tendency to

Final States	M.B. Statistics	F.D. Statistics	B.E. Statistics	Paraparticle Statistics	Paragas Statistics
	6	1	1	2	1
	3	0	1	1	1
	3	0	1	1	1
	3	0	1	1	1
	3	0	1	1	1
	3	0	1	1	1
	3	0	1	1	1
	1	0	1	0	0
	1	0	1	0	0
	1	0	1	0	0
Total no. of allowed States (= total wts)	27	1	10	8	7.

TABLE 5 : Table of Weights

Distribution Numbers	MB	FD	BE	Paraparticle	Paragas
$n_1 = \begin{cases} 1 \\ 2 \\ 3 \end{cases}$	12	1	3	4	3
	6	0	2	2	2
	1	0	1	0	0
$n_2 = \begin{cases} 1 \\ 2 \\ 3 \end{cases}$	12	1	3	4	3
	6	0	2	2	2
	1	0	1	0	0
$n_3 = \begin{cases} 1 \\ 2 \\ 3 \end{cases}$	12	1	3	4	3
	6	0	2	2	2
	1	0	1	0	0

TABLE 6 : Table of Distribution Numbers

Probability	MB	FD	BE	Paraparticle	Paragas
$P(n_{a^1}=1)$	$4/9$	1	$3/10$	$1/2$	$3/7$
$P(n_{a^1}=2)$	$2/9$	0	$2/10$	$1/4$	$2/7$
$P(n_{a^1}=3)$	$1/27$	0	$1/10$	0	0
$P(n_{a^2}=1)$	$4/9$	1	$3/10$	$1/2$	$3/7$
$P(n_{a^2}=2)$	$2/9$	0	$2/10$	$1/4$	$2/7$
$P(n_{a^2}=3)$	$1/27$	0	$1/10$	0	0
$P(n_{a^3}=1)$	$4/9$	1	$3/10$	$1/2$	$3/7$
$P(n_{a^3}=2)$	$2/9$	0	$2/10$	$1/4$	$2/7$
$P(n_{a^3}=3)$	$1/27$	0	$1/10$	0	0

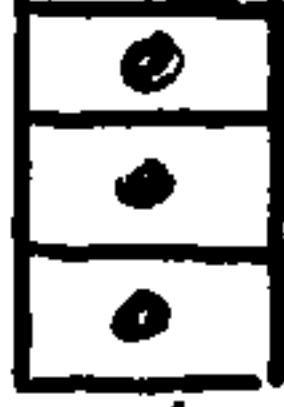
TABLE 7: Table of Probabilities

Probabilities	MB	FD	BE	Paraparticles	Paragas
1 particle in each state	$2/9$	1	$1/10$	$1/4$	$1/7$
2 particles in any state	$2/3$	0	$6/10$	$3/4$	$6/7$
3 particles in any state	$1/9$	0	$3/10$	0	0
at least 1 particle in state a^1	$19/27$	1	$6/10$	$3/4$	$5/7$
at least 1 particle in state a^2	$19/27$	1	$6/10$	$3/4$	$5/7$
at least 1 particle in state a^3	$19/27$	1	$6/10$	$3/4$	$5/7$

TABLE 8: Table of Probabilities

to cluster together than bosons do, although this is not true of course for three particles in any state.

If we compare our paraparticle statistics with classical and ordinary quantum statistics we arrive at similar conclusions, not surprisingly. Thus the probability of finding one particle in any state is greater for the former than for both B-E and M-B statistics, but less of course than the corresponding probability in F-D statistics. Again, the tendency for there to be two particles in any state rather than one particle in each state is greater for paraparticle statistics than for M-B or B-E statistics.

The most interesting comparison, however, is between paraparticle and paragas statistics themselves. We notice, first of all, that the probability of finding two particles in any state is greater for the latter than for the former. On the other hand the probability of finding one particle in any of a^1, a^2 or a^3 and also the probability of finding one particle in each state, is greater for paraparticles than for paragas particles. Furthermore the probability of finding at least one particle in any of the three states is thus also greater for the former than for the latter. We conclude from these results, which arise as a consequence of the doubling of the weight assigned to  noted above, that paragas particles exhibit a greater tendency to cluster together than paraparticles do, or, conversely, that the latter exhibit a statistical repulsion as compared to the former. Thus as we have said, the statistics of paraparticles is not the same as that for a paragas.

Finally, to complete this section, we shall turn our attention to the question of whether a theory of paraparticles can be embedded in λ^a theory of fermions or bosons.

A trivial answer can be given by taking the Kronecker, or inner product of a representation with its conjugate

and using the Clebsch-Gordan coefficients to give fermionic and bosonic representations. For our purposes this can be done entirely in terms of the characters of the representations involved.

The Kronecker product of the representations $D^{(\mu)}$ and $D^{(\nu)}$ is given by (62):

$$D_{ik,jl}^{(\mu \times \nu)}(R) = (D^{(\mu)}(R) \times D^{(\nu)}(R))_{ik,jl} = D_{ij}^{(\mu)}(R) \cdot D_{kl}^{(\nu)}(R) \quad 49.$$

The reduction of this product representation is then performed by

$$\text{using, first of all } \chi^{(\mu \times \nu)}(R) = \sum_{i,j} D_{ii}^{(\mu)}(R) D_{jj}^{(\nu)}(R) = \chi^{(\mu)}(R) \cdot \chi^{(\nu)}(R) \quad 50$$

(where the χ 's are the group characters), to compute the character of each element in the product representation and then using

$$a_{\mu} = \frac{1}{h} \sum_i N_R \chi_i^{(\mu)*} \chi_i \quad (63) \quad 51$$

to find the number of times each irreducible representation is contained in the product representation. These two steps provide the coefficients in the Clebsch-Gordon series

$$D^{(\mu)} \times D^{(\nu)} = \sum_{\sigma} a_{\sigma} D^{(\sigma)} \quad (64) \quad 52$$

Alternatively the two steps can be combined and

$$a_{\sigma} = \frac{1}{h} \sum_a \chi^{(\mu)}(R) \chi^{(\nu)}(R) \overline{\chi^{(\sigma)}(R)} \quad (65)$$

used.

Let us now apply the procedure as outlined above to the symmetric group S_3 . The character table for this group has already been given on p. 164. Using 50

62. Hammermesh op cit p. 128
 63. Ibid p. 105; Appendix p. 344
 64. Ibid p. 148
 65. Ibid p. 147

we obtain the following characters of all the product representations

	$(1^3) \times (1^3)$	$(1^3) \times (3)$	$(1^3) \times (2,1)$	$(3) \times (3)$	$(3) \times (2,1)$	$(2,1) \times (2,1)$
K_1	1	1	2	1	2	4
K_2	1	-1	0	1	0	0
K_3	1	1	-1	1	-1	1
	 (1^3)	 (3)	 $(2,1)$	 (1^3)	 $(2,1)$	 $(1^3) + (3) + (2,1)$

TABLE 9

Most of the results can be obtained by inspection. It should be noted that the product of two one-dimensional representations is a one-dimensional representation and must be irreducible, and also that the product of the representations $(2,1)$ with any other gives $(2,1)$ again.

We are interested in the inner product of two triangular representations i.e. $(2,1) \times (2,1)$, which is the only reducible product in the table. This is a four-dimensional representation and the sum of the absolute squares of its characters is $(4)^2 + 2(1)^2 = 18$. From

$$\sum_i \chi_i \chi_i^* N_i = h \sum_{\mu} a_{\mu}^2$$

we obtain $18 = h \sum_{\mu} a_{\mu}^2 = 6 \sum_{\mu} a_{\mu}^2$

$$\therefore \sum_{\mu} a_{\mu}^2 = 3$$

This has the unique solution

$$a_{(1^3)} = a_{(3)} = a_{(2,1)} = 1$$

so

$$(2,1) \times (2,1) = (1^3) + (3) + (2,1).$$

Alternatively using 51 we have

$$a_{(1^3)} = \frac{1}{6} [(1)(4)(1) + (2)(1)(1) + (3)(0)(1)] = 1$$

$$a_{(3)} = \frac{1}{6} [(1)(4)(1) + (2)(1)(1) + (3)(0)(1)] = 1$$

$$a_{(2,1)} = \frac{1}{6} [(1)(4)(2) + (2)(1)(-1) + (3)(0)(0)] = 1$$

which gives the number of times each irreducible

representation is contained in the product representation. This gives the same result as before, namely

$$(2,1) \times (2,1) = (1^3) + (3) + (2,1) \quad \text{or, in terms}$$

of Young tableaux:

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}$$

This result arises from the fact that $(2,1)$ is self conjugate, i.e.

$$(2,1) \times (2,1) = (\underline{2},1) \times (\widetilde{2},1) = (1^3) + (3) + (2,1) \quad 54$$

Thus we conclude that the inner product of the triangular representation with itself gives both symmetric, (3) and the anti-symmetric (1^3) representations.

Although this method is straightforward, it does not lead to general formulae. It is known that the Clebsch-Gordon series gives a completely general result for the rotation group (66), and it is interesting to see whether there exist similar general formulae for the symmetric group. These will be more complicated because of this group's more complex structure. A straightforward procedure for the general reduction of inner products has been given by Murnaghan and Gamba (67) and the formulae useful for our purposes is

$$(n-1, 1) \times (n-1, 1) = (n) + (n-1, 1) + (n-2, 2) + (n-2, 1) \quad 55$$

For $n = 3$ this gives $(2,1) \times (2,1) = (3) + (2,1) + (1,2) + (1^3)$

Again it can be seen that we obtain the symmetric and anti symmetric representations (68).

Now we have shown that the inner product of the triangular representation with itself contains both symmetric and anti-symmetric representations. Applying Murnaghan and Gamba's method leads to the general result that if (μ) and (ν) denote representations then the product $(\mu) \times (\nu)$ contains the identity representation once if and only if $(\mu) = (\nu)$ and $(\mu) \times (\nu)$ contains the alternating representation (1^n) once if and only if

66. *Ibid* p. 367 ff

67. Hammermesh op cit p. 253-256

68. These results can also be obtained using graphical methods. See Hammermesh *Ibid* p. 257-259.

$(\mu) = (\tilde{\nu})$ (69) where $(\tilde{\nu})$ is the conjugate of (ν) .

In other words, the inner product of any representation (μ, n) say with itself always contains the symmetric representation and the inner product of any (μ, n) with its conjugate always contains the anti-symmetric representation.

We conclude that symmetric (bosonic) representations and anti symmetric (fermionic) representations may be obtained from the inner product of representations (corresponding to paraparticle states with themselves or with their conjugates respectively. In this sense a paraparticle theory can be embedded in a bosonic or fermionic theory. The converse, however, is not true. We cannot ^{in general,} generate paraparticle states within a bosonic or fermionic theory which incorporates extra degrees of freedom, such as spin, because of the consequent restriction to symmetric and anti-symmetric representations respectively.

Thus, let us consider, as an example, an assembly of N electrons, neglecting, for the moment, their spins. (70) At first sight it would seem that the statistical behaviour of such an assembly is identical to that of an assembly of paraparticles of order 2. However, in the former case one obtains certain representations, for example those corresponding to the situation where all spins are parallel, which do not occur in the latter, owing to the restriction that the wave-function for the assembly must be symmetric on interchange of 2 particles and ^{anti-}symmetric on interchange of more than 2. Thus the statistics of parafermions of order 2 corresponds to the case in ordinary Fermi statistics where the existence of the spin parallel state obtained by neglecting the extra degrees of freedom represented by the spin, is forbidden.

69. Ibid p. 256

70. T. Okayama (1952) p. 523

Essentially the same point was also made by Wigner, who, noting that the number of different eigenvalues for the assembly corresponds to the number of different irreducible representations but that only certain of these correspond to allowed energy states, wrote 'Eigenvalues of other representations do not correspond to actually existing stationary states but are forbidden by a principle independent of the eigenvalue equation, the Pauli exclusion principle' (71). Thus he chose a set of basis functions narrowly defined as linear combinations of 2 linearly independent functions of N variables each of which can have two values. He then demonstrated that with such a set one can only generate all the representations of the symmetric group of degree N for $N \leq 4$, but that for $N > 4$ an ever increasing number cannot be produced. (72) However, these latter are precisely the ones whose eigen values do not obey ^{the} exclusion principle and so those representations which can be generated using these basis functions are the only ones necessary for a treatment of electrons taking into account the spin.

This result, that one can generate all the representations from a consideration of the spin part of the wave-function only up to $N = 4$ finds its analogue in the theory of quarks. These can be regarded either as particles obeying 'ordinary' statistics but possessing an extra degree of freedom known as 'colour' or as para-Fermions of order 3. (73) By considering the problem of forming bound states from a para-Fermi field of order p Ohnuki and Kamefuchi demonstrated that only for $p \leq 3$ could all bound states be described by parafield theory (74). Furthermore for $p = 2$ no fermion bound states are possible and so para-fields of order 1 and 3 are accorded privileged status in the theory. The former corresponds to the three triplet 'colour' model and the latter to the paraquark model. As regards the formal relationship between these two models, the state

71. A.P. Wigner (1959) p. 128

72. Ibid p. 139-140

73. We shall consider this paraquark model in the context of the history of quantum statistics in section 3.4.

vector space of the latter is a subspace of the space corresponding to the general space of the former. The introduction of the colour index then allows a correspondence to be established between the states of the space of the paraquark model and certain states of the space appropriate to the Fermion description. These states are those which form singlets with respect to the SU (3) colour group in the latter, which restriction implies that the colour index is unobservable. In general every bound state in the latter theory corresponds to one bound state in the former, but the converse correspondence does not hold; certain bound states in the paraquark model have counterparts in the colour theory which are degenerate.

Thus we conclude that any type of parastatistics together with some sort of 'hidden variable machinery', such as SU (3) will give any representations desired, but that the converse is not true. In particular the statistics of any given paraparticle theory can be reproduced by an ordinary bosonic or fermionic theory in which hidden variables are introduced (75).

Having given the most general quantum mechanical theory of indistinguishable particles we shall now explicate its consequences regarding the individuality of these particles.

3.3 Identity and Individuality in Quantum Physics

Perhaps the most striking and fundamental difference between classical and quantum statistics is that in the former a permutation of indistinguishable (in the sense

74. Y. Ohnuki and S. Kamefuchi (1968) p. 1290. See also A.N. Mitra and S.A. Moszkowski (1968) p. 1474-1475.
75. Y. Ohnuki and S. Kamefuchi (1982) p. 353-355. Landshoff and Stapp make a similar suggestion and give an example in which a particular para-Fermi model is reduced to a theory containing only 'ordinary' fermions. P.N. Landshoff and H.P. Stapp (1967) p.74-77.

of possessing all intrinsic, non-spatio-temporal properties in common) particles leads to a new state whereas in the latter it does not. Particle permutations are not regarded as observable in Q.M. as we have seen. Since the ^{counting as distinct} permuted and unpermuted arrangements in classical physics is what led to the attribution of individuality to the particles, it can obviously be expected that the quantal counting will have serious consequences as regards a similar attribution in quantum physics.

Thus the arrangement

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 is given weight 6 in Maxwell-Boltzmann statistics, corresponding to the 6 possible permutations of 3 particles, between 3 states, where it is only given weight 1 in Fermi-Dirac, Bose-Einstein and Gentile's paragas statistics, and weight 2 in paraparticle statistics as we have seen. Similarly if we consider the possible states available to two indistinguishable particles distributed over two distinct one-particle states, then the arrangement represented by

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 is given weight 2 in classical statistics but only weight 1 in the quantum form. In Q.M. the two classical arrangements

1•	2•
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^a _b and

2•	1•
----	----

^a _b obtained through a permutation of the particles, are regarded as one and the same state of the two-particle system and counted as such. In quantum statistics, as we have said, a particle permutation does not lead to an observably different, separately countable arrangement.

Historically, there were two responses to this peculiar statistical behaviour. One was to argue that the loss of the particles' statistical independence implied that they were subject to rather strange acausal forces. The alternative rejected such abnormal forces and discussed this behaviour in terms of the particles having lost, in some way, their classical individuality. We shall discuss both of these responses before presenting our own view which argues that there are two positions, one of which is the second response above, which can equally well be adopted, at least

if we keep within the domain of first-quantized theory. However, as we shall see, the implications of Quantum Field Theory (Q.F.T.) and also metaphysical prejudice, may lend support to one position over the other.

3.3.1 Individual particles + acausal forces

This view was first suggested by Ehrenfest early in the history of quantum statistics (76). He argued that Planck's quantum theory involved a choice of those states assigned equal a priori probabilities which was different from that made in classical statistical mechanics. Ehrenfest returned to this argument several times, notably in 1911 when he constructed a generalization of the a priori probability, or weight function, as part of an attempt to rationalize Planck's work. He had also long been puzzled by Planck's combinatorial formula for the distribution of quanta over oscillators. In 1914 he and Kamerlingh Onnes published their famous intuitive proof of this formula (as we shall see in the next section), and Ehrenfest noted that if it were correct then the quanta could not be statistically independent but must exhibit some kind of non-classical correlation.

Some years later this point was taken up by Reichenbach. (77) As we discussed in Ch.2 Reichenbach held that classical particles possess 'material genidentity' whereas quantal particles do not and are said to possess functional genidentity. He noted that this latter interpretation assumes a weighting assignment (78) which gives equal a priori probabilities to all distinguishable arrangements. He then asked if this was the only possible assignment which could be given and answered that it was not (79). Reichenbach's conclusion was that there were

76. P. Ehrenfest (1905) p. 1301 and (1911) p.91

77. H. Reichenbach (1956) p. 224-249

78. He called it an 'extension rule'.

79. c.f. Ehrenfest's work.

two ways of looking at quantum statistics, neither of which is any more correct than the other. Roughly equating material genidentity with individuality and functional genidentity with non-individuality (of a sort to be discussed shortly), this choice is as follows.

The particles can either be regarded as non-individuals with an associated weighting assignment of equal a priori probabilities, or they can be taken to be individuals, in which case the a priori probabilities are not equal. This second case implies that the statistics are not independent and hence that there exists causal anomalies in the behaviour of the particles. For bosons these anomalies consist in a mutual dependence in the motions of the particles (the famous Bose-Einstein condensation), which could effectively be characterized as action at a distance since the particles could be far apart (80). For fermions, on the other hand, the causal anomaly expresses itself in the Exclusion Principle, if this is interpreted in terms of a force acting between the individual particles (81).

The choice is therefore between regarding the particles as non-individuals which behave 'normally' or as individuals whose behaviour displays causal anomalies. Both are equivalent descriptions. However, as only the first of these descriptions supplies a 'normal' system, in the sense of one free from causal anomalies, Reichenbach rejected the second on the grounds that acausal interactions are inadmissible (82).

Thus the Ehrenfest-Reichenbach claim is that if the particles are regarded as individuals then there must exist acausal interactions between them. This is perfectly valid as one could always 'mimic' the statistics by postulating some peculiar ad hoc non-local (action at a distance) acausal forces acting between the

80. Ibid p.234

81. Ibid p.235. This force has been accorded a quasi-realistic status in some discussions of the so called 'exchange interaction'

82. Ibid p. 235-236. Elsewhere he suggested that acausal anomalies might be suppressed through the adoption of a three-valued logic in quantum theory. See H. Reichenbach (1965 reprint) p.160-166.

particles. However this view can be discarded, as Reichenbach himself indicated, on the grounds that its implication of acausality simply does too much violence to our view of reality. This is therefore another good example of the underdetermination of theories by experimental data, and the way in which, given such underdetermination, other factors come into play. Thus one description is rejected on the metaphysical grounds that its implications of non-locality bring it into conflict with Special Relativity, regarded in this context as a set of regulative maxims. By metaphysics we mean here some sort of justification of the conceptual rationale of a theory and we shall discuss the way such considerations can be used to decide between two theories, or interpretations, in more detail in the next chapter.

We shall now consider the alternative response to the statistical behaviour of quantal particles.

3.3.2 Non-individual particles

The non-observability of particle permutations in Q.M. and the consequent assignment of weight 1 to the arrangement

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 has been taken to imply that the particles cannot, as in C.M., be labelled or at least that such labels are meaningless, because they are all 'mixed up' by the particle permutations. Therefore, it is argued, the particles cannot be regarded as individuals but must be, in some sense 'non-individuals'. This 'non-individuality' is not, as Post has pointed out (83), mere indistinguishability taken to the limit. In both classical and quantum statistics particles of the same species, or natural kind, are indistinguishable, as we have argued. Non-individuality obviously means more than this. Something which could be attributed to the particles in classical statistics cannot now be attributed to them in the quantum theory. Different opinions have been held as to what this 'something' is.

Some authors have argued that the non-individuality arises through a denial of the existence of well-defined distinct spatio-temporal trajectories for quantal particles. Thus Landau and Lifshitz stated that the difference between classical and quantum statistics '...follows at once from the uncertainty principle'. (84) They argued that '... by virtue of the uncertainty principle, the concept of the path of an electron (for example) ceases to have any meaning. If the position of an electron is exactly known at a given instant, its coordinates have no definite value even at an infinitely close subsequent instant. Hence by localising and numbering the electrons at some instant, we make no progress towards identifying them at subsequent instants; if we localise one of the electrons, at some other instant, we cannot say which of the electrons has arrived at this point.

Thus, in quantum mechanics, there is in principle no possibility of separately following each of a number of similar particles and thereby distinguishing them. We may say that, in Q.M. identical particles entirely lose their 'individuality'. The identity of the particles with respect to their properties is here very far reaching: it results in the complete indistinguishability of the particles.' (85).

This form of argument can be given the following gloss:

- 1). A necessary condition for an object, such as an elementary particle to be an individual is that it must possess a well defined continuous trajectory in space-time. (This, we recall, is the second criterion of classical space-time individuality).

- 2). If the distance between two particles of the same species becomes so small that there is an appreciable overlap in their ranges of possible positions then it cannot be said that the particles possess well defined trajectories, and hence an unambiguous assignment of individuality can no longer be made. (Re-identification after such a close encounter also becomes

84. Landau and Lifshitz (1965) p.209

85. Ibid p. 209; Also D. ter Haar (1958) p.72 and J.M. Jauch (1966) p.276

problematic.)

3) Heisenberg's Uncertainty Principle implies that particles in Q.M. are not in general localized and that their wave functions and hence ranges of possible location, do overlap. Therefore such particles cannot be said to be individuals but must be regarded as 'non-individuals'.

According to this argument then, particles in Q.M. are non-individuals because their space-time trajectories are not well defined and thus the second criterion of classical space-time individuality is violated.

Post has criticised this argument on the grounds that the Uncertainty Principle does not abolish localizability, it only limits it (86). The spread in position can be reduced in many ways, by taking a more massive body for example. Or one could imagine two indistinguishable particles far enough apart to make any overlap of their wavefunctions negligible. Space-time individuality could thus be attributed to these particles and they could be identified and individuated through a permutation, as long as the distance between them always remains large. This would result in the particles changing positions and since they are regarded as individuals the final configuration can be said to be different from the initial one. Yet particle permutations are not observable in Q.M. and the final configuration would not, in fact, be counted as distinct from the initial one. This apparent paradox leads to the conclusion that non-individuality must be due to something else other than the supposed loss of well defined trajectories through the Uncertainty Principle.

As Post has indicated then the no-well-defined-trajectories argument turns on the interpretation of *the* Uncertainty Principle. McMullin has argued that the principle has been interpreted in at least four different ways. (87)

86. H. Post (1963) p.19

87. E. McMullin (1954)

- 1). It is impossible to measure simultaneously conjugate variables;
- 2). It represents a limitation on the precision of measurements in the sense that the accuracy of knowledge of one variable decreases by measuring its conjugate,
- 3). It is a statistical principle which relates the scatter of one sequence of measurements with that of another.
- 4). It is a mathematical expression of the fundamental complementarity of phenomena in Q.M.

In terms of this classification Landau and Lifshitz have adopted interpretation 1 whereas Post appears to hold interpretation 2; thus they are arguing from different ideological viewpoints. An even more extreme position is to adopt interpretation 3. Proponents of the 'statistical' (or more properly 'stochastic') approach to Q.M. (88) regard particles as classical individuals possessing simultaneous position and momentum and traversing a classical spatio-temporal trajectory. The state vector is regarded as a description not of an individual system but of an ensemble of similarly prepared systems and thus quantum phenomena is considered to be the manifestation of the statistical behaviour of a collection of basically classical individuals. The difficulties associated with this approach are well known and we shall not consider them here (89). However, we will remark that classical space-time individuality is only partially reinstated in this interpretation,^{as} it is still not possible to predict uniquely the particular trajectory which would result from a particular causal disturbance. (90)

If one accepts the inherent indeterminism of Q.M., embodied in the Uncertainty Principle, then it is clearly not possible to ascribe classical space-time

88. K. Popper (1967) p.7.; H. Margenau (1963), p.469
A. Lande (1973)
89. See M. Jammer (1974) p.440-469
90. M. Audi (1973).

individuality to the particles, even granted Post's argument. If the particle is localized at one instant then it is not possible to predict its position at subsequent future instants. Thus if one were an advocate of space-time individuality for classical particles one would have to relinquish this view for quantal particles and adopt some alternative.

As we have tried to indicate these arguments are heavily influenced by ideological prejudice and to the extent that each interpretation possesses certain difficulties the question of which one should be adopted over the others is not one which can be decided by purely rational argument alone.

Another possible reason for rejecting classical space-time individuality in Q.M. is that the Superposition Principle implies the inapplicability of the Impenetrability Assumption for quantal particles. Thus the general ket for an assembly of particles corresponds to a state for the assembly for which one cannot say that each particle is in its own state, but only that each particle is partly in several states, in a way which is correlated (91) with the other particles being partly in several states (92). In other words, each particle partakes of all the states of all the other particles. There is a sense, then, in which one could say that two, or more, such particles can be in the same place at the same time - in fact they can be in several places at the same time.

The oft-quoted example of two bosons both in the same one-particle state is therefore unnecessarily restrictive. Two fermions distributed among two distinct one-particle states will also partake with equal probabilistic weight of each state in the superposition demanded by anti-symmetrization (93). So in this sense, two particles, whether bosons or fermions, are always in the same state.

91. Compare with Ehrenfest's position.

92. Dirac op cit p. 207-208.

93. This is not true of the higher paraparticles but then there is no way of telling which particle is in which state because of the restriction to symmetric observables.

As Post says 'Any one labelled particle occurs simultaneously in all positions occupied by like particles'. (94) and '... the location of an electron at a given moment is as numerous as the number of electrons in the universe '. (95)

This is a well known and fundamental feature of Q.M. embodied in the Superposition Principle which requires the existence of strange (i.e. non-classical) relationships between the states of a dynamical system such that whenever the system is definitely in one state it can be considered to be partly in each of two or more other states. The non-classical nature of this principle is further revealed through the probability of a certain result for an observation being intermediate between the corresponding probabilities for the original states, rather than the result itself being intermediate between the results for the original states, as it would be classically. Thus when an observation is made on any system in a given state the result will not in general be determinate. The Superposition Principle therefore demands indeterminacy in the results of observations and philosophical arguments based upon it are obviously closely related to those using considerations of indeterminacy, as above.

In this, non-classical, sense then, particles in Q.M. do not obey the Impenetrability Assumption, which, we recall, is the first criterion of classical space-time individuality. Thus, if two, or more, particles can occupy the same state at the same time. albeit in this nonclassical and perhaps restrictive sense this form of individuality cannot be attributed to them. Such particles must therefore be regarded as non-individuals.

94. Post op cit p.19

95. Post Ibid p.20

However, it may be objected that this argument derives its conclusion from an aspect of Q.M. which is by no means unproblematic, namely the status of so-called 'entangled states' within the theory. These lie at the root of many of the fundamental conceptual problems in Q.M., such as the EPR argument and the 'Schrodinger's Cat Paradox' (96) for example, and represent what Schrodinger has called the characteristic trait of Q.M. (97)

Thus let us consider the situation mentioned above, that of two fermions distributed over two one-particle states. The anti-symmetrised wave function for the assembly is then given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \{ |a_1^1\rangle |a_2^2\rangle - |a_1^2\rangle |a_2^1\rangle \}$$

Now the statistical operator (98), or density matrix, ρ_1 , of particle 1 can be written in terms of projection operators thus:

$$\rho_1 = \frac{1}{2} [|a_1^1\rangle \langle a_1^1| + |a_1^2\rangle \langle a_1^2|]$$

or

$$\rho_1 = \frac{1}{2} \Gamma_{a_1^1} + \frac{1}{2} \Gamma_{a_1^2}$$

Similarly, for particle 2 we have :

$$\rho_2 = \frac{1}{2} [|a_2^2\rangle \langle a_2^2| + |a_2^1\rangle \langle a_2^1|]$$

or

$$\rho_2 = \frac{1}{2} \Gamma_{a_2^1} + \frac{1}{2} \Gamma_{a_2^2}$$

Thus the two particles have the same statistical operators. Furthermore, it can easily be shown that they possess the same one-particle probability distribution. Thus the two-particle probability distribution $\omega^{(2)}$ for finding one particle in state a^1 and the other in state a^2 is given by

$$\omega^{(2)}(1, 2) = \langle \Psi | \rho_{12} | \Psi \rangle$$

where

$$\rho_{12} = |1\rangle \langle 1| \otimes |2\rangle \langle 2|$$

Thus one obtains :

$$\omega^{(2)}(1, 2) = \frac{1}{2} \{ |a_1^1, a_2^2|^2 + |a_1^2, a_2^1|^2 - 2 \operatorname{Re}. a_1^1 a_2^2 a_2^1 a_1^2 \} \quad (98a)$$

where $a_i^j = \langle 1 | a_i^j \rangle$ etc.

The one-particle probability distribution giving the

96. See *Summer op cit* ch.6

97. E. Schrodinger (1935) p.557

98. See B. D'Espagnat (1976) Ch.6 for the definition of this operator and a discussion of its properties.

98a. The factor '1/2' in this formula, as compared with the results on p.168, arises since we are concerned here with the probabilities for labelled particles to be in states 1 and 2.

probability that particle 1 occupy state a^1 regardless of, the state of particle 2 is then obtained by integrating the above over the space of particle 2.

Thus we have

$$\omega^{(1)}(1) = \frac{1}{2} \{ |a_1^1|^2 + |a_1^2|^2 \}$$

Similarly the one-particle distribution for particle 2 is given by

$$\omega^{(1)}(2) = \frac{1}{2} \{ |a_2^1|^2 + |a_2^2|^2 \}$$

Hence we conclude that the individual particles have the same statistical operator and the same one-particle probability distribution. It is this which underlies our earlier argument that the two fermions partake with equal probabilistic weight of each state in the superposition and thus can be regarded as being 'in' the same state. They are in the same state precisely insofar as they have the same n_{λ}^e -particle probability distribution and thus cannot be distinguished by any one particle position measurement.

However, expressed in these terms one can easily discern how the argument threatens to come undone. First of all the following well-known counter argument can be given. (99) Referring again to the two-particle distribution one can easily show that the probability of finding both fermions in the same one-particle state is zero, i.e.

$$\omega^{(2)}(1,1) = 0$$

This, of course, is what lies behind the Pauli Exclusion Principle which can be naively expressed in the form that no two fermions can exist in the same one-particle state, or that a quantum state in which two fermions possess identical sets of one-particle quantum numbers cannot exist. (100) Thus it can be argued that in this sense two fermions cannot be in the same state and the I.A. is satisfied. Indeed the Exclusion Principle may be regarded as a generalisation of the I.A. in the

99. See Y Shadmi (1978) p.844; and H. Margenau (1950) Ch. 20, (1944) p. 187.

100. Shadmi op cit p.847

sense that it applies not simply to the spatio-temporal coordinates of the particles as in the classical case, but to the one-particle quantum numbers (101).

Thus on the one hand, regarding the one-particle distributions, it can be argued that the particles can be in the same 'state', whereas on the other, a consideration of the two-particle distribution leads to the conclusion that they cannot. A resolution of this apparent conflict can be effected through a consideration of the nature of these 'entangled states' in general and of the statistical operators in particular. ρ_1 and ρ_2 are not in fact state ascriptions since they describe improper mixtures (102) rather than pure states. As these operators effectively 'encode' all the measurements that can be obtained by operating on the particle concerned these measurements will not be maximal, in the sense of giving the maximum amount of information about the system. In other words since the state function of the system is no longer the product of the separate state functions of the particles, one cannot, from a knowledge of $|\Psi\rangle$ ascribe to each of the particles an individual state function. This is a manifestation of the peculiar, non-classical 'holism' inherent in Q.M. in which complete knowledge of the whole does not generally entail complete knowledge of the parts.

Thus one cannot attribute pure states to each particle and so the question of whether the particles can be described as being somehow in the 'same' state simply does not arise. This consideration of the formalism of the theory therefore leads to the conclusion that the question of whether the I.A. is violated or not is in fact obviated in Q.M. This is an important point and as it has an obvious bearing on the status of Leibniz's Principle of Identity of Indiscernibles in quantum

101. This is effectively what Shadmi argues. Ibid
p. 847

102. The terminology is due to d'Espagnat op cit pp.
pp. 58-62.

physics we shall return to it in our discussion of this Principle in Chapter 4.

Finally it should be noted that we are assuming here that Q.M. is complete. If the positions of particles are introduced as hidden parameters then the above conclusions no longer follow and a form of space-time individuality could be reinstated. However, as the question of the viability of the hidden variables programme is still the subject of much discussion and has been extensively dealt with elsewhere (103), we shall not consider the possible consequences of these theories here.

It has also been argued that particles in Q.M. should be regarded as non-individuals because T.I. cannot be attributed to them. Thus Post argues that individuality or non-individuality must be assigned at the very beginning (104). Quantum Field Theory does this 'automatically' and does not refer to individual particles at all, treating all particles as field excitations and hence as 'non-individuals'. Only the first quantised theory, he claims, and in particular Schrodinger's formulation starts off 'heavily on the wrong foot' (105) by initially assigning a label and hence T.I. to each particle. The individuality is then immediately 'wiped out' by systematically permuting these labels and constructing a wave-function for the assembly out of all the wave functions obtained by permuting the particles, subject to certain symmetry restrictions.

This position therefore asserts the denial of classical T.I. which, in first quantised Q.M. is lost in the particle permutations and in Q.F.T. is not exhibited to begin with. Thus it is claimed 'quantal particles are non-individuals'.

103. See Jammer op cit p. 252-339. For more recent discussions and references see M. Redhead in R. Swinburne (1983) p.151 and (1981) p.1., and d'Espagnat op cit Ch.11.

104. Post op cit p.19

105. Ibid p. 19

'Non-individuality' is, for Post, a primary concept which cannot be explicated in terms of other, more fundamental, notions (106). Non-individual particles are not only indistinguishable they are also devoid of T.I. and thus are regarded as 'identical' in a strong sense. They can differ only as regards location, or, more generally, the state they are in. He concludes that one should not refer to individual particles in Q.M. at all but should speak instead of a state being occupied by 0,1,2,..., etc. particles. Material particles are thus regarded as merely states of disturbance in an unidentifiable, substanceless medium. They are without secondary qualities, without individuality, without substance; 'There is no substance left in physics, only form'. (107)

However Post's argument cannot be conclusive because it is perfectly possible to give a consistent account of quantum statistics in which labels or proper names are ascribed to the particles, as we shall demonstrate.

Thus the apparent loss of particle individuality evident in the Q.M. contraction in the number of states has been accounted for in three different ways, each involving the negation of some aspect of the two forms of classical individuality. We shall now argue that there are in fact two alternative positions which can be consistently adopted, although metaphysical considerations may lead to one being preferred over the other.

3.3.3. Non-Individual Particles + Individuated States or Individual Particles + State Accessibility Restrictions

Our view is that there are two ways of looking at particle individuality in Q.M. One considers particles to be

106. Schrodinger held a similar view. See E. Schrodinger (1957) p. 194.

107. Post Ibid p.20.

non-individuals, but regards the states as individuated. Alternatively the particles can be regarded as possessing T.I. but then restrictions must be imposed upon the set of possible states, still regarded as individuated, such that certain of them are no longer accessible to the particles. The first position can also be arrived at through a consideration of Q.F.T. The second is obviously open to the philosophical criticism which has been directed at the notion of T.I. As we shall discuss later this can be taken to mitigate against the particle approach in favour of the field-theoretic description.

We recall that in the ket for an assembly of indistinguishable particles both the states and the particles were labelled. The most general ket for the assembly is then constructed out of those kets obtained by permuting the particle labels and such a particle permutation has been regarded as rendering these labels meaningless. Thus it has been concluded, as we have seen, that quantal particles are 'non-individuals'.

However, particle permutations are, as we have seen, only one of two kinds of permutation which can be defined in Q.M. The other kind are the place permutation operators, which permute the place labels in the ket. When applied to the purely symmetric or anti-symmetric state vectors these operators produce the same results, and so the practical distinction between them, although not of course the conceptual one, disappears for 'ordinary' fermions and bosons. It is only when we move to, the multi-dimensional subspaces of paraparticle theory that this difference becomes apparent. By restricting its analysis to ordinary quantum statistics only the account of particle 'non-individuality' discussed in the previous section fails to take into consideration the effect and possible implications of the \bar{P} 's.

We recall that the effect of the P 's when operating on a ket is to produce another, indistinguishable from the first one. This is enshrined in the Indistinguishability Postulate expressed as the commutation relation, $[Q, P] = 0$

which can be interpreted as stating that it is not possible to distinguish states differing only in a permutation of the particle labels. It is this which actually underlies the arguments to the effect that particles cannot be regarded as individuals in Q.M.

In other words, since the P's do not commute with one another the I.P. implies that they are not observables of the system. Any vector on which they operate is left in the same subspace, which reflects the fact that two state vectors differing only by a permutation of the particles must represent the same state.

The \bar{P} 's, on the other hand, can take a state vector from one subspace to another as a place permutation can change the physical state of a system. Since they commute with the P's the \bar{P} 's do satisfy I.P. and therefore any self-adjoint function of them can be regarded as an observable. Indeed, I.P. implies that any observable Q must be a symmetric function of the particle labels and we then demonstrated that any such function can be expanded in terms of the \bar{P} 's thus

$$Q \approx \sum_i c_i \bar{P}_i$$

where the equality is taken in the limited sense as explained on p. 145.

It is worth noting however that, contrary to what is often written (108) the \bar{P} 's themselves are not self-adjoint and therefore are not real dynamical variables (109). Thus the \bar{P} 's themselves should not be regarded as observables, although combinations, or rather (self-adjoint) functions of them are.

As we have emphasised in classical physics a particle permutation is regarded as observable and giving rise to a new micro-state, This is taken as justification for the particles being regarded as labelled and for these labels to have meaning in the sense that they designate that which confers individuality upon the particles. In quantum physics, as we now see,

108. Landshoff and Stapp op cit. Y. Ohnuki and S. Kamefuchi (1971) p.20 and p. 27-31

109. See the discussion in Dirac op cit p. 212

the P's are not observable and do not produce new, distinct complexions. Thus the particle labels may be regarded as meaningless and quantal particles as 'non-individuals'. However the place permutations do give rise to new, countably distinct micro-states as they project the state vector from one irreducible subspace to another. Thus the state, or place, labels are not meaningless and the states themselves can be regarded as individuated. Underlying this view, and hence also of the requirement that all observables that distinguish between states differing only in the order of the particle labels, are functions of the \bar{P} 's, is the fact that the wave-functions are completely distinct entities and the particles are experimentally identified by these, as given by the state labels, and not through the positions of the particle labels.

Thus the P's are not observable since the particles are indistinguishable, whereas functions of the \bar{P} 's are observable because the states can be distinguished. If the substantialist approach is adopted then this could be interpreted as implying that the underlying substratum is truly 'unknowable' in quantum physics, in the sense that it cannot even be designated by a meaningful label. Alternatively the above statement could be taken to support the view that the particles should be regarded as nothing more than 'bundles' of qualities, corresponding to the intrinsic properties, which can be distinguished one from another by the states they happen to be in (110).

Thus we have arrived at the position where we have non-individual particles together with individuated states. Insofar as the particles exist 'in' the states and the latter in turn are embedded in space-time, this may be regarded as a form of 'space-time individuality', albeit a very weak one, not equivalent to the classical version. Thus the particles are distinguished at any one time by the states they are in. As regards reidentification through time it is

110. Thus Post has proposed an ontology of 'states of disturbance in an unidentifiable medium, without substance'. Post op cit p.20.

true that instead of the position x as in C.M. one has $\psi(x)$ but to reidentify the latter one still has to effect a reidentification through the former. Thus one still has the space-time background in Q.M. with all the attendant problems regarding its interpretation. As in the classical case adopting the relational view leads to circularity whereas the alternative absolutist approach suggests that the points of space-time underlying the states should be regarded as individuals themselves.

As we have said, however, the form of S.T. in this case is much weaker than the classical version since there are no well-defined spatio-trajectories in Q.M. and, as we have seen, there is a sense in which it can be said that the I.A. is violated. As we saw at the end of Chapter 2 this is precisely the situation which holds in the case of fields and the above position regarding particle 'individuality' - or rather the supposed lack of it - finds its most coherent expression in Quantum Field Theory, where an account of many-body systems can be given without referring to particle labels at all. (111) In this case the P's are undefined, which is to be expected as the creation operators in Q.F.T. already contain the effects of 'particle' indistinguishability, but the \bar{P} 's are not and allow a correspondence to be established between the first and second quantised theories (112). We shall briefly consider this situation in section 3.5.

In section 3.4 we shall also indicate how the above view can be found running through the history of quantum statistics where one approach was to count the energy states occupied by various numbers of particles rather than the number of particles in various states.

Before discussing these further points however we

111. Thus Redhead has argued that quantal particles also belong to his new category of entities called 'ephemerals' and that the latter can be regarded as bereft of individuality in the sense that a collection of indistinguishable particles is itself an ephemerality. Redhead (1983) p. 38-39
112. Stolt and Taylor op cit.

must consider the alternative conclusion regarding particle individuality in Q.M. which can be arrived at through further examination of the nature and consequences of the I.P.

We noted in section, 3.2 that the I.P. acts as a super-selection rule which partitions the Hilbert space into a number of irreducible subspaces. Thus, for example, in the three particle case the six-dimensional subspace \mathcal{E} of \mathcal{H}_3 , spanned by the vectors $P|f\rangle$ decomposes into irreducible subspaces invariant under the P's as follows :

$$\mathcal{E} = \mathcal{E}_S \oplus (\mathcal{E}_{E'} \oplus \mathcal{E}_{E''}) \oplus \mathcal{E}_A$$

We can represent this schematically thus:

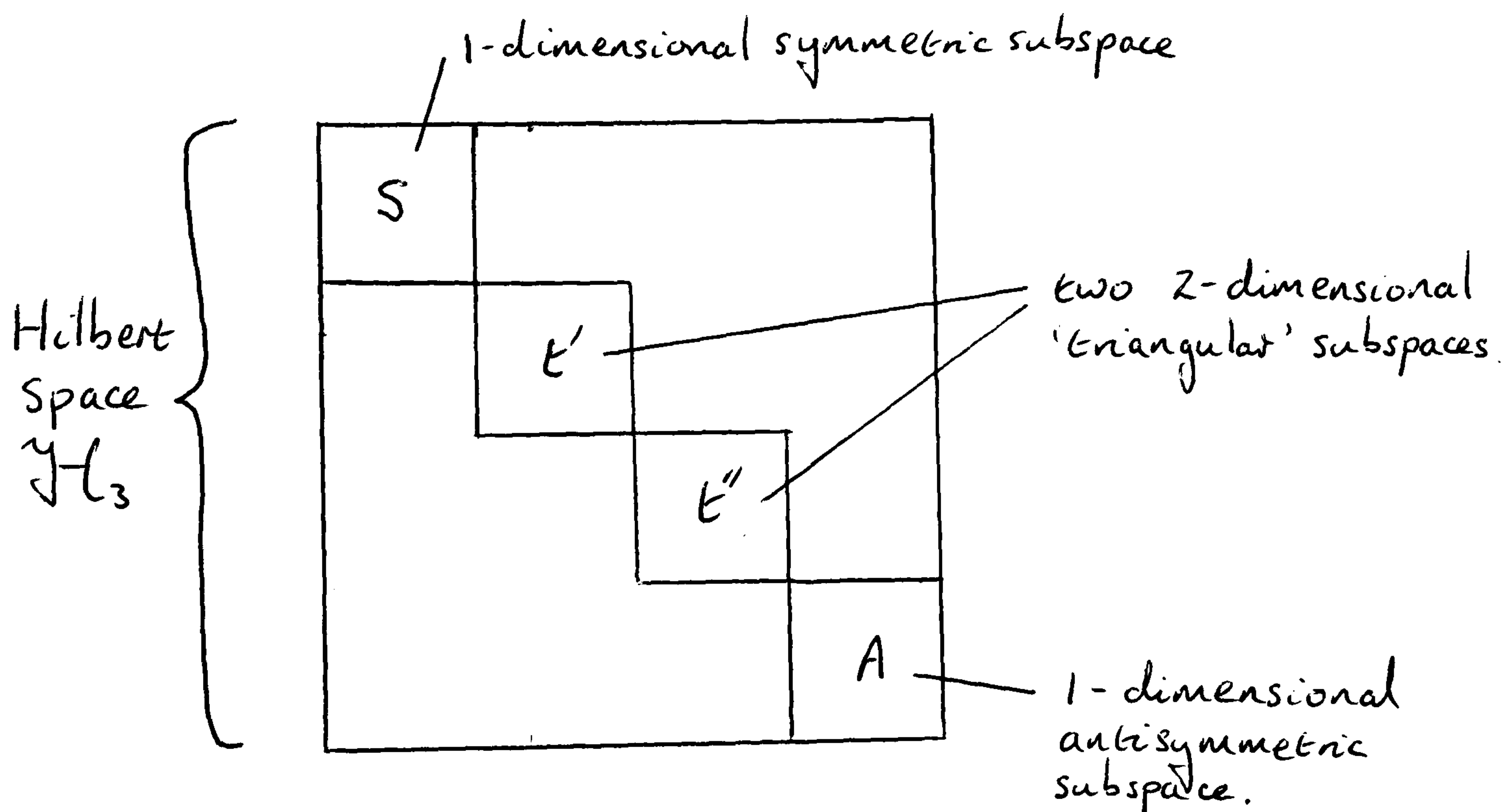


Fig 10

Given I.P. Schur's Lemma implies that all matrix elements of an observable Q connecting different representations are zero. It then follows that if there exist states corresponding to subspaces of different representations then transitions between such states must be forbidden. States corresponding to subspaces of the same representation are not separated in this way and transitions may therefore occur between them as we saw in section 3.2.3.

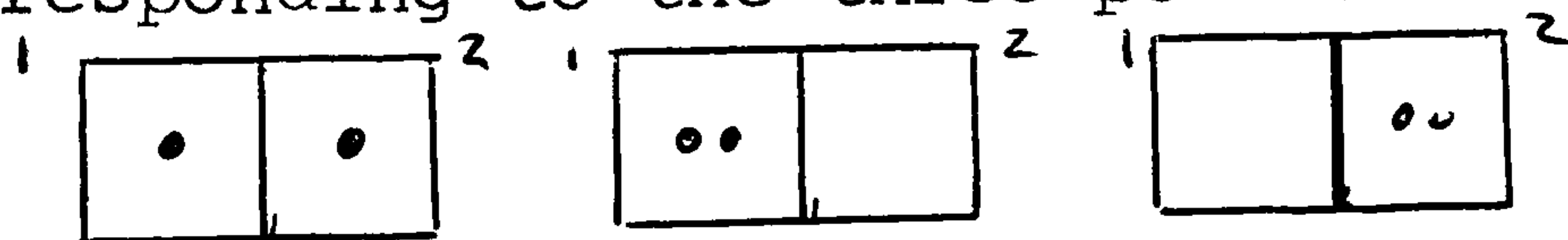
These considerations also hold for the general case of an N-particle system. The Hilbert space then decomposes into a number of subspaces and transitions between states corresponding to certain of them, viz, those of different representations, are forbidden.

Thus I.P. imposes a restriction on the states such that certain of them are rendered inaccessible to the particles. If we consider the time evolution of the system as effected by some Hamiltonian, then we conclude that I.P. implies that if the system starts in one irreducible representation then it will always remain in that representation. Bosons will always be bosons, fermions fermions etc.

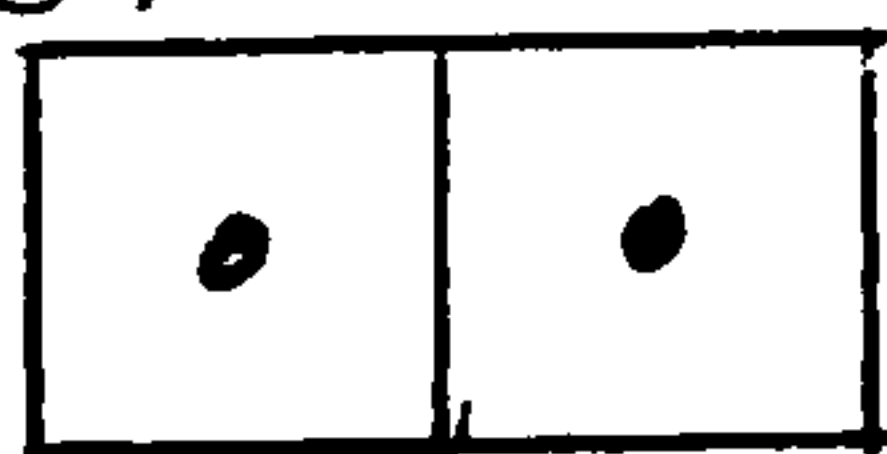
The I.P. can therefore be thought of as an extra postulate of Q.M., an initial condition in the specification of a situation. It divides up the Hilbert space into a number of subspaces and once a system is placed in one of these subspaces the dynamics is such that it can never get out of it.

Thus quantum mechanical particles can be regarded as individuals which are prevented from occupying certain states because of an extra initial condition which has been introduced into the formalism. Once it has been decided which states are inaccessible the dynamics is such that the particles can never get into them.

Thus, for example, let us consider again the distribution of two indistinguishable particles over two one-particle states. If the two states are specified by orthogonal wave functions a^1 and a^2 and the two particles are labelled 1 and 2 (note: we are effectively introducing some form of individuality here) then for B-E statistics the three product wave functions corresponding to the three possible arrangements



are $\frac{1}{\sqrt{2}}\{ |a^1\rangle |a^2\rangle + |a^2\rangle |a^1\rangle \}$ 56 ; $|a^1\rangle |a^1\rangle$; $|a^2\rangle |a^2\rangle$ 56 respectively. For F-D statistics we have

$\frac{1}{\sqrt{2}}\{ |a^1\rangle |a^2\rangle - |a^2\rangle |a^1\rangle \}$ 57
 corresponding to the arrangement  56 and 57 are chosen to retain certain simple symmetry properties under a particle permutation, viz 56 is symmetric

and 57 is anti-symmetric.

We have noted that I.P. can be interpreted as stating that every observable, in particular the Hamiltonian governing the time development of the statevector, must be symmetric under a particle permutation. It then follows that a symmetric state will always remain symmetric and an anti-symmetric state always anti-symmetric. In other words if the initial condition is imposed that the state of the system is either symmetric or anti-symmetric then only one of the states 56 or 57 is ever available to the system and this explains why the statistical weight attaching to the pair of states 56 and 57 is half the classical value.

On this view 56 and 57 are not regarded as the same state any more than the classical states

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \bullet & \bullet \\ \hline \end{array}^2 \quad \text{and} \quad \begin{array}{|c|c|} \hline 2 & 1 \\ \hline \bullet & \bullet \\ \hline \end{array}^2$$

are, just that there exists a dynamical restriction on the accessibility of certain states. This restriction arises from the symmetry of the Hamiltonian under the P's, which in turn can be deduced as a necessary condition for the indistinguishability of the particles, as we have seen, and hence follows from I.P.

On this view then, the non-classical weighting assignment of quantum statistics is 'explained' not in terms of 'non-individuality', but as being due to the accessibility restrictions imposed on the states as an initial condition. In the above example only one of the two possible states is accessible to either kind of particle because the Hamiltonian is such that they cannot get into it - transitions to that state are forbidden. More generally states formed through a particle permutation are not counted not because they do not exist but because they are not available to the particles. We therefore have the rather odd situation in which certain states exist, ontologically speaking, but they cannot be reached by the particles, and this is due, not to the laws of Q.M., but to the restrictions imposed by an initial

condition - the I.P.

So once the symmetry type of the particle is fixed the I.P. ensures that some states are never accessible to the particles. Thus, for example, the state corresponding to the symmetric combination above is inaccessible to fermions and the triangular paraparticle state is not accessible to either bosons or fermions. In general, as we have seen, states carrying different representations are separated by a superselection rule, although states carrying the same representation are not. ^{By} collisions ^{however} these latter states are connected and hence the statistical weight is increased accordingly. These considerations of state accessibility conditions are then a little more complicated for paraparticles than for ordinary particles as we shall see.

Before discussing this, however, there are two further points to note. The first is that what we have given here is a description in which the particles are labelled. The fact that we cannot tell which label attaches to which particle, because they are permuted, does not necessarily compel us to give up such a description. Only an extreme form of positivism would hold that we should. Thus the particles can be regarded as individuals, with the labels designating the underlying substantial substratum. According to this view then the particles possess T.I. exactly as in the classical case, and are therefore simply classical particles which are subject to restrictions on the possible states they can occupy.

The second point is that this notion of state accessibility restrictions can also be found in classical statistical mechanics, although in a much weaker form.

The fundamental problem of statistical mechanics is to discover how to select the correct stationary states of an assembly of a large number of particles. A decision must be made as to whether the assembly, when left to itself, tends to settle down mainly into one or other of a small preferred group of stationary states whose properties control the equilibrium properties of the

assembly, or whether it wanders at random over the whole range of stationary states made accessible by the general conditions of the problem. In the latter case the assembly's equilibrium properties will be determined by the properties of the vast majority of states. These equilibrium properties can only then be calculated by taking a suitable average over all stationary states or by selecting the properties of the most probably occupied type of state, or by some other method which will require justification.

The choice of states can be made in a number of ways all of which, fortunately give the same physical results for large assemblies (113). Each involves some necessary fundamental hypothesis underpinning the choice. Thus, the equilibrium value \bar{Q} of any property Q of the assembly can be obtained by averaging over all the distinct states of the assembly, consistent with a given energy or energy range and any other general conditions, and assigning an equal weight of unity to each state i.e. $\bar{Q} = \frac{\sum_{\text{states}} Q}{\sum_{\text{states}} 1}$ (114). In this case states of a given energy have been selected for the averaging process. The energy is a uniform integral (115) of the equations of motion of the assembly which restricts the assembly's representative point in the phase space to a hypersurface in this space determined by $H(x, p_x) = E$ where $H(x, p_x)$ is the Hamiltonian, x stands for all the positional co-ordinates of the particles, p stands for the corresponding momenta and E is a constant.

Other uniform integrals may exist for classical assemblies in certain circumstances, for example the linear momentum. Any such integral has a profound effect

113. R.C. Tolman (1938) Ch.3. See our discussion of the choice of weighting assignments in Boltzmann's statistical mechanics in Section 2.3.5 of Chapter 2.
114. R.H. Fowler and E.A. Guggenheim (1939) p.7
115. An integral of the equations of motion is a function of the generalised coordinates and momenta and time, whose value remains constant along a path in phase space. A uniform integral is a time independent integral, also called a 'constant of the motion'. See D.A. Lavis (1977) p. 258.

on the trajectory of the representative point in phase space, confining it to a particular surface in this space, or, if a small range of the constant is allowed, to the region enclosed between two such neighbouring surfaces. When such additional uniform integrals exist they must be included in the general condition laid down in the formulation of the fundamental hypothesis. It is then necessary to average over only such regions of phase space, or over such states of the assembly, as conform to the extra requirements.

Such regions or states are called accessible phase space regions or accessible states, respectively.

However, it can be shown that as regards thermodynamic consequences, it makes no difference whether these extra integrals are considered to exist or not. (116). Thus it is generally assumed that it is only the energy integral which imposes any constraints on what phase space regions are accessible to the system (117)

In quantum statistical mechanics, on the other hand, further constraints on state accessibility are introduced over and above those imposed by the energy integral. In this case, as we have seen, not all of the states of an assembly of indistinguishable particles are accessible one from another; they may be analysed into groups of non-combining states by means of their symmetry properties. The symmetry type of any suitably specified set of states is an absolute constant of the motion (118) equivalent to an exact uniform integral in classical terms. The set of accessible states for the purposes of the 'fundamental hypothesis' is that set in the correct energy range, of the appropriate symmetry.

Thus the notion of accessibility can be formulated for both classical and quantum systems, although in the latter case there is the added complication of the symmetry requirements imposed on the wave functions. Corresponding to these there are various accessibility postulates which can be invoked, each giving a

116. *Fowler and Guggenheim op. cit.*

117. P. Penrose (1970) p.124-125.

118. Dirac op cit p. 213

different statistical mechanics. In order to completely specify a quantum system therefore it is necessary to specify not only its Hamiltonian but also which type of statistics - that is, which of the various accessibility postulates - it satisfies (119).

Out of all the non-combining groups of eigenfunctions, defined according to their symmetry properties, the symmetric and antisymmetric groups in particular stand out on account of their simplicity. In the former the eigenfunctions are all symmetric under a particle permutation and the eigenfunctions of the assembly are then restricted to the symmetric subspace of the Hilbert space. Only the states within this subspace are accessible to the assembly and B-E statistics is obtained. The anti-symmetric group consists of eigenfunctions which are anti-symmetric under the P's and in this case the eigenfunctions of the assembly are restricted to the anti-symmetric subspace of the whole Hilbert space. Only the states within this subspace are accessible to the assembly and F-D statistics are obtained (120).

However, as we have emphasised, these two are merely the simplest examples which can be taken from the set of all possible symmetry groups. The other, more complicated, types of symmetry which are possible are obeyed by paraparticles, whose wave functions are symmetric in a certain number of particles and anti-symmetric in the rest (121). The question of what states are accessible to paraparticles is, as we have indicated, more problematic than in the case of ordinary particles.

For particles obeying Gentile's paragas statistics the number of accessible states will simply be

119. If no symmetry requirements at all are imposed then one recovers classical Maxwell-Boltzmann statistics.

120. Penrose op cit p. 930

121. For example a parafermion of order n has a wave-function which is symmetric in n particles but anti-symmetric under the exchange of one of these with all particles other than the n .

intermediate between the number accessible to fermions and bosons. This number will increase as the statistics becomes less fermionic and more bosonic. In the case of paraparticle statistics however, where there is the possibility of transitions occurring between states carrying the same representation, it is a linear combination of states that now becomes accessible to the particles.

Thus after taking into account the above considerations the 'fundamental hypothesis' must be re-written as

$$\bar{Q} = \frac{\sum_{\text{accessible states}} Q}{\sum_{\text{accessible states}} 1} \quad (122)$$

The purpose of this digression was to emphasize that the concept of state accessibility restrictions occurs quite naturally in both C.M. and Q.M. Obviously it plays a larger and more prominent role in the latter because symmetry constraints are imposed over and above the energy constraint laid down in the former. This goes some way towards explaining the greater role played by symmetry in QM compared to CM.

We conclude that there are two alternative positions which can be adopted with regard to particle individuality in Q.M.:

- 1). Non-individual particles + individuated states;
- 2). Individual particles possessing T.I. + state accessibility restrictions.

This conclusion lends support to the argument that one can 'do' quantum statistics with the particles regarded either as individuals or non-individuals so that arguments over individuality become irrelevant. In this context it is interesting to note that Dorling has suggested that counting states and arguing about the individuality of particles is a totally irrelevant exercise as far as the statistics are concerned, whereas

a consideration of the dynamics, specifying the transition probabilities, is absolutely crucial. Thus he has written 'You can't get out the statistics without putting in the dynamics, and it is then quite irrelevant what you count as distinct states and whether you consider the individuals themselves as identical or non-identical'. (123) In terms of this type of approach the statistical behaviour of quantal particles, or their tendency to display certain correlations of the kind exhibited in B-E and F-D statistics, can then be regarded as an 'independent and fundamental property of the particles' (124). Thus this view, with its emphasis on the dynamics, can be regarded as a version of the 'Ehrenfest-Reichenbach description' outlined in Section 3.3.1, in which quantum statistics is interpreted in terms of 'peculiar' interactions, possibly non-local, existing between the particles.

This underdetermination of the philosophical interpretation by the physics implies that one cannot determine any particular distribution of particles over states on the grounds of a priori considerations of particle individuality, or non-individuality, alone. Such a determination can only be obtained by paying due regard to the various transition probabilities involved and hence to the dynamics of the situation.

As in the classical case *our* conclusion regarding particle individuality is supported by historical considerations of the development of quantum statistics. Indeed this development can, to a certain extent, be viewed in terms of successive shifts from one interpretation to another.

123. J. Dorling (1978a) and (1978b) p. 7-9

124. W. De Muynck (1975) p. 334. This work was explicitly based on Mirman's attempt to analyse the experimental meaning of the concepts 'identity', 'individuality' etc. R. Mirman (1973) p. 110.

3.4 The History of Quantum Statistics

The birth of quantum statistics coincides with the birth of quantum theory as a whole, in Planck's famous paper of 1900 (125). In order to change the status of his radiation law from a happy conjecture to a statement of some physical significance Planck was forced to abandon his electro-magnetic H-Theorem Approach and turn to the Combinatorial Approach developed by Boltzmann for gas thermodynamics (126) (thus also changing the status of this approach from a relatively obscure response to a particular criticism to a major component of physics). However, the details of the statistics used by Planck were radically *and* fundamentally different from Boltzmann's.

The statistical step in Planck's derivation of his law lay in a consideration of how a given total energy could be distributed over N linear^a oscillators, all vibrating with frequency ν . He divided this energy into a finite number, P , of energy elements which were then distributed over the resonators. The number of ways of doing this, i.e. the number of 'complexions', is given by

$$\frac{(N+P-1)!}{(N-1)! P!} \quad (127) \quad \text{It is not}$$

immediately clear how Planck arrived at this expression although Rosenfeld has suggested that he discovered it by working backwards from his distribution law. (128)

The rest of the derivation is straightforward and has been extensively discussed and analysed elsewhere. However, it is worth noting Planck's comments at the end. He considered his deduction to be based on only two theorems: the well established relation between the radiation energy density and the average resonator energy and Boltzmann's relationship between the entropy of a system and the logarithm of the total number of

125. M. Planck (1900) p.202, reprinted in D.terHaar (1967) p.80
 126. M. Klein (1970) p. 221; T. Kuhn (1978)p.72 ff;
 A. Pais (1982) p. 368-372; also see Planck in ter Haar op cit p.81 and p.83
 127. Planck Ibid p. 237
 128. L. Rosenfeld (1936) p.149; R. Cranfield (1968) p.69; Kuhn op cit p.100-101; for a discussion of the source of this expression see Kuhn *ibid* p. 282

possible complexions. Planck split the latter into two parts: '(1) The entropy of the system in a given state is proportional to the logarithm of the probability of that state and (2) The probability of any state is proportional to the number of corresponding complexions, or, in other words, any definite complexion is equally as probable as any other complexion (129)'

The first part is just a definition of the probability of the state as far as radiation is concerned. The second was regarded by Planck as the core of his whole theory and its proof must rest ultimately on empirical grounds. He considered it to be a more elaborate and detailed expression of his hypothesis of natural radiation, previously stated in the form that the energy of the radiation is randomly distributed over the various partial vibrations present in the radiation.

Thus the statistical component of Planck's work follows Boltzmann's Combinatorial Approach in broad outline but differs from it in certain profoundly significant respects. First of all, whereas Boltzmann let the size of his energy elements decrease to zero, as we have seen, Planck did not but left them as finite and related to the frequency via $\epsilon = h\nu$ (130). Secondly, and more importantly from the point of view of this thesis, Planck's combinatorial expression above is very different from Boltzmann's. Planck considered, at least implicitly only the number of energy quanta assigned to each oscillator and not which quanta were possessed by which oscillators. Thus the numerator of his expression gives simply the total number of ways of arranging the quanta and oscillators, regarded as $N + P - 1$ distinct elements. (We shall discuss the significance of the -1 when we come to consider Ehrenfest's analysis of this result).

129. Planck in ter Haar op cit p.87

130. It is not clear whether Planck regarded his energy quanta as having real physical significance or as merely convenient mathematical artefacts. See Kuhn op cit and Dorling (1967).

The division by $P!$ implies that permutations of the quanta are not to be regarded as giving rise to new, countably different arrangements. One interpretation of this is that the quanta should be regarded as not only indistinguishable but also unlabeled and hence devoid of individuality (131). Thus whereas Boltzmann considered the distribution of indistinguishable (in the weak sense of possessing all intrinsic properties in common), individual atoms over energy states or cells in phase space, Planck could be regarded as considering the distribution of indistinguishable (in the same sense), non-individual quanta over resonators (132). Alternatively the quanta could be regarded as individuals but with certain arrangements now forbidden.

The non-classical nature of Planck's expression and the questions surrounding its interpretation became a focus of early criticism, particularly from Ehrenfest who undertook to expose the conceptual foundations of the new quantum theory.

Finally it should be noted that Planck's second sub-theorem above, regarding the probability of a state, is simply a restatement of the crucial assumption that equal a priori weights should be assigned to the various complexions. As we shall see it was again Ehrenfest who subject this assumption to critical examination and who first discussed the way in which the weighting assignments of classical statistical mechanics had to be revised in the light of the new physics.

In 1901 Plack presented a more complete but condensed version of his derivation which would be difficult to follow if considered separately from his earlier work.

131. See for example, Pais op cit 0.370 or Kuhn op cit 1.101

132. Cranfield remarks '... in Planck's method, energy elements were the distributed quantities rather than oscillators whose distribution over phase space would have been the analogue of Boltzmann's approach.' Cranfield op cit p. 70.

(133). As a result many of his contemporaries found his work obscure and it was not until he published the 'Lectures on the Theory of Thermal Radiation' in 1906 that comprehension of the nature and implications of his results became more widespread.

However, Lorentz briefly mentioned Planck's work in 1902 and in 1903 remarked that the latter's method of introducing probability was not the only one that could be chosen (134). As Ehrenfest was attending Lorentz's lectures at the time and had read extensively on the problem of black body radiation, it seems reasonable to suggest that his later detailed analysis of the probabilistic assumptions underlying quantum theory, and his investigation of possible alternatives, were greatly influenced by Lorentz's comment. (135)

Ehrenfest's first thoughts on this subject were set down in his 1905 paper in which he expressed his puzzlement over the exact nature of the relationship between Planck's work and Boltzmann's Combinatorial Approach (136). In particular he noted that Planck's choice of those states to be given equal a priori probabilities was different from Boltzmann's. This difference arises from the different choice of events to be counted, and whereas Boltzmann attached equal a priori probabilities to each region of phase space Planck assigned them to each energy distribution (137). Ehrenfest was well aware that a definite hypothesis concerning such assignments was crucial to the Combinatorial Approach and six years later he presented a generalised weight function suitable for the new statistics.

Rayleigh and Jeans were also puzzled by Planck's derivation (138). In particular Jeans argued that

- 133. Kuhn op cit p. 102-103
- 134. H.A. Lorentz (1901) p. 436 and (1903) p. 666
- 135. See Klein op cit p. 231
- 136. P. Ehrenfest (1905) p. 1301; (1959) p.88
- 137. Cranfield op cit p. 70
- 138. Lord Rayleigh (1905) p.54; J. Jeans (1905) p. 293

Planck's combinatorial expression violated the classical probability calculus because he has not specified the population from which such probabilities could be calculated. Such a population could not be introduced with an a priori weighting assignment consistent with Planck's arguments.

1905 also saw the publication of Einstein's famous paper in which he proposed the hypothesis of independent energy quanta in radiation. (139) The central argument of this paper is based, not on Planck's, but on Wien's law which is strictly incorrect. However Einstein was able to obtain the correct quantum conclusion because he treated his quanta as individual Maxwell-Boltzmann particles instead of as bosons (140). It is precisely in the Wien limit that the latter behave like the former. Cranfield has demonstrated that Einstein's result can also be derived using the correct distribution law, Planck's, and the correct statistics, Bose-Einstein, as expected.

Thus it was effectively shown that quanta which are statistically independent will behave in accordance with Wien's law. The fact that Planck's law is actually the correct one therefore suggested that the light quanta were not statistically independent. This point, and the related question of the nature of the interdependence between quanta, was subsequently the subject of much discussion by Ehrenfest and others as we shall see.

The following year Einstein pointed out that Planck's theory pre-supposed the existence of light quanta (141) and in 1907 he noted that if this theory were correct, then energy values other than those given by $h\nu$ are not accessible to the oscillators (142). His analysis of

139. A. Einstein (1905) p.132, English trans. in *Ter Haar op ut* p.91

140. As Cranfield has demonstrated. Cranfield op cit p. 125

141. A. Einstein (1906) p. 199

142. A. Einstein (1907) p.150

energy fluctuations in black body radiation in 1909 produced an expression for the mean square energy fluctuation which was the sum of two terms (143). The classical wave theory of light would give the second term, the wave term, only, and the first, the particle term, corresponded to '... independently moving pointlike quanta with energy $h\nu$ '. (144) This is the first intimation of 'wave-particle duality' in which light, and eventually matter as well, is conceived of as a fusion of wave and particle aspects..

There are two points to note regarding Einstein's statement above concerning light quanta. Firstly, he obviously regarded them as being statistically independent. The non-classical statistical interdependence of quanta, which was to be subsequently emphasized by Ehrenfest, for example, could then be explained by referring to their wave like aspect. Regarded as crests in a system of waves, quanta would become properties of the whole system and there would be no reason to attribute individuality and statistical independence to them. The probability of a quantum existing in a particular state would then be dependent on the other quanta in the system. Thus the separation into particle-like and wave-like aspects allowed the former to be regarded as independent whilst the non-classical statistical interdependence was invested in the latter. As we have said this was the first example of the wave-particle duality which became a way of interpreting the lack of statistical independence between quantal entities. Thus Einstein justified the Bose-Einstein counting procedure by referring to de Broglie's suggestion that particles are analogous to crests in a system of waves; a suggestion that was subsequently taken up by Schrodinger and elaborated into his so-called 'wave mechanics'.

143. See Pais op cit p.402-410 for a good discussion of this analysis.

144. A. Einstein (1909) p. 185-817

Two alternative views of the statistical dependence inherent in quantum theory thus came to be developed. One regarded it as a manifestation of the wave-like nature of all quantal entities, light quanta and material particles. The apparent loss of individuality for quantal particles could then be explained in terms of their complementary wave-like aspects. The other regarded such particles simply as particles which behaved in strange, non-classical ways and could be thought of either as 'non-individuals' in some sense, or as classical individuals restricted as to the states they could occupy.

To a certain extent the difference between these positions corresponds to that between people like Schrodinger, who believed that an electron, for example, is, in some non-visualizable sense, also a wave, and others like Born who argued that the electron is really just a particle whose position and velocity are fundamentally random and which, therefore, exhibits non-classical behaviour. This is, of course, painting the picture with a very broad brush but nevertheless it is true to say that much of the argument over this statistical interdependence was concerned with the question of in what aspect of quantal entities should this non-classical behaviour reside.

The second point to note concerning Einstein's statement is that he referred to the light quanta as point-like. It is clear that by 1909 he was thinking of quanta as particles (145). This view was reinforced by his work on spontaneous and induced radiative transitions in the course of which he derived the result that a light quantum carries momentum $\frac{h\nu}{c}$ (146). Thus emerged the concept of a photon as a particle carrying both energy and momentum.

Planck's 1906 'Lectures ...' (147) contained a rearranged version of his 1900 derivation which gave the combinatorial expression after the law itself. It was

145. See Pais op cit p.403

146. A. Einstein (1916) p.47 and A. Einstein (1917) p.121

147. N. Planck (1906) Ch.4.

emphasized that this law restricted the discussion to the equilibrium case, this distinguishing Planck's problem from Boltzmann's. However the latter counting procedure could also be applied although it would then be necessary to take the sum over all possible Boltzmann distributions compatible with the constraints on the total number of resonators and total energy, to obtain the number of complexions relevant to Planck's problem. Rather than actually go through this summation Planck produced his own combinatorial formula as its result.

He also introduced an alternative approach involving a phase-space description of the equiprobable regions accessible to a resonator. These regions were elliptical rings of area h on the energy hypersurface, within which lay the resonators, and consideration of the distribution of the resonators over these areas *then allowed him to compute directly the number of* complexions corresponding to a given state.

This formed the basis for a quantised statistical treatment of the distribution of entities, particles or resonators, over certain regions of phase space, which was much more akin to the classical combinatorial procedure. Quantal results were ensured, however, by the fixed size given to these regions as determined by h . (148)

Shortly after the publication of Planck's book, Ehrenfest presented a general study of some aspects of radiation theory in the light of Planck's work (149). In particular he derived the distribution law without recourse to resonators and showed that the entropy function obtained was common to Boltzmann and Planck. Ehrenfest therefore concluded that the source of the difference in distribution laws must lie, not in the entropy equations but in the imposed constraints, i.e. in the quantization requirement.

148. Thus Cranfield has shown that whereas the results of classical statistics are invariant with respect to a change in the volume of the cells in phase space, those of quantum statistics are not.

149. P. Ehrenfest (1906) p. 528

It is clear from this work that Ehrenfest regarded Planck's energy quanta as nothing more than a formal device. Einstein, however, believed that the quantization procedure represented a new limitation on possible motions at the molecular level. His consequent application of this procedure to the problem of the specific heats of solids contributed greatly to the general acceptance of the quantum theory (150).

In 1910 Debye gave a derivation of Planck's law, very similar to Ehrenfest's, in which the field vibration modes were quantized directly without using resonators. (151) Lorentz also published a derivation in the line of Planck's 1906 approach, in which the classical Boltzmann combinatorial formula was used and each distribution was specified by a set of integers giving the number of resonators possessing so many energy elements. (152)

Ehrenfest's observations of 1906 were developed in a further paper published in 1911, in which he analysed the restrictions imposed on the weight function of a statistical theory of radiation by the properties of black body radiation. (153) He now realised that Planck's derivation could be brought into a logical form not by introducing additional constraints but by generalising the a priori probability or weight function assigned to the phase space. Thus he abandoned the classical uniform weighting and by considering the restriction imposed upon the generalised weight function by the known properties of black body radiation, showed that this function could depend only on the ratio E/ν and possessed non-zero values only for the points

$$E/\nu = 0, 1, 2, \dots$$

150. Kuhn op cit p. 206

151. P. Debye (1910) p. 1427

152. H.A. Lorentz (1910) p. 1234

153. P. Ehrenfest (1911) p. 91 and (1959) p. 185

He then examined the difference between Planck's quantized resonators and Einstein's light quanta. The prevailing opinion at the time was that ^{the} main difference lay in the separate existence of quanta in empty space. However, Ehrenfest also pointed out that the statistical independence of quanta was another source of contention, with Einstein affirming and Planck denying it. He emphasized that Planck had not assumed this in his derivation and that it led, in fact, to Wien's law which Einstein had used. This problem of interdependent statistical behaviour of Planck's quanta clearly troubled Ehrenfest and he returned to it three years later, as we shall see. It was also discussed by Joffé and Natanson and was the subject of an interesting controversy between Krutkow and Wolfe in 1914.

Natanson was also interested in the differences between Planck's and Einstein's conceptions of quanta. In his 1911 paper he probed the assumptions underlying Planck's consideration of the distribution of energy elements over 'receptacles', and identified three different 'modes of distribution', depending on whether the quanta or receptacles or both were regarded as individuals (154).

Thus he used the term 'mode of distribution' for the correlation of energy elements with receptacles in which the distribution is characterised only by the number of receptacles containing a given number of quanta. In this case no account is taken of the possible individuality ('identifiability') of receptacles or energy elements. However if the former are regarded as individuals then every 'mode of distribution' branches out into a number of 'modes of collocation', which specify the number of energy elements in each individual receptacle. This, he argued, corresponded to Planck's microstate. If energy elements are also regarded as individuals then each 'mode of collocation' splits up

into a number of 'modes of association', which associate individual energy elements with individual receptacles. This corresponds to Boltzmann's complexion. (155)

As Natanson noted, the thermodynamic probability of a given mode of distribution depends on whether all modes of collocation or all modes of association are regarded as equally probable. Whereas Einstein took the latter view, Planck took the former and thus, according to Natanson, implicitly regarded his receptacles, the resonators, as individuals but not his energy elements, the quanta. The odd notation is an attempt to distinguish the meaning of the term 'complexion' as used in quantum and classical statistics. In the former a complexion specifies only how many elements are in each receptacle and this is called a mode of collocation. A permutation of two elements between receptacles does not lead to a new mode of collocation. In the latter, a complexion specifies which individual element is in which individual receptacle, and this is termed a mode of association. A permutation of elements between receptacles does lead to a different mode of association.

Joffe also attempted to modify Einstein's idea of quanta so as to reconcile it with Planck's law (156), and three years later a controversy arose between Krutkow and Wolfe^k over just this question.

Krutkow, following Ehrenfest, demonstrated that if classical statistics are used then the assumption of independent quanta leads to Wien's law. Planck's law can only be obtained, he argued, if this assumption is abandoned. Wolfe^k, however, argued that one must distinguish between two meanings of the word 'independent'. If the quanta exist independently then, he claimed

in A. van der Merwe

155. A. Kastler (1983) p.620 This is an interesting although rather broad, account of these developments.

156. A. Joffe (1911) p.534.

Planck's law is obtained, but if, in addition, the quanta are regarded as spatially independent then Wien's law will result (157). Thus he concluded that Einstein had implicitly assumed the quanta to exist independently and to be also spatially independent whereas Planck has assumed only the former, leading to the suggestion that some sort of spatial correlation had to exist between Planck's quanta.

Further light was shed on these questions by Ehrenfest and Onnes' paper of 1914, (158), which gave an intuitive way of understanding Planck's combinatorial formula. Ehrenfest had not been satisfied with the standard derivation given in the text books as they appealed to proof by induction and offered no insight into the peculiar structure of the result. Thus he set out to provide his own.

The problem he considered was to find the number of ways in which P objects could be placed in N containers where only the number of objects in each container is of importance. Two distributions were called identical when corresponding containers, i.e. oscillators, in each distribution possessed the same number of objects, i.e. energy elements. A distribution was represented symbolically thus

$$\text{II } \epsilon \epsilon \epsilon \epsilon \text{ O } \epsilon \epsilon \text{ O O } \epsilon \text{ II}$$

where the II are the 'fixed boundaries', ϵ represents the energy units, and the O separate these units in successive oscillators, numbered from left to right. (159)

157. M. Wolfe^k (1914) p. 133 and 363
 158. P. Ehrenfest and H. Kamerlingh Onnes (1914) p.870 and (1959) p. 353; M. Klein (1970) p. 255-257 and (1959) p. 47-48.
 159. This notation is confusing in that the symbol O has a dual significance of both the separation of quanta in successive oscillators and also that an oscillator possess no quanta. A clearer way of writing it is $\text{|| } \epsilon \epsilon \epsilon \epsilon \text{ | } \epsilon \epsilon \text{ | } \text{| } \epsilon \text{ ||}$ where the dividers separate the energy symbols for different oscillators. Klein (1970) p. 256

With general values of N and P the symbol will contain ϵ P times and O $N-1$ times, and Ehrenfest asked how many different symbols for the distribution could be formed from the given number of ϵ and O . His answer was

$$\frac{(N-1+P)!}{P!(N-1)!} \quad \text{derived as follows.}$$

There are $N-1+P$ quantities ϵ and O which, if regarded as distinguishable, can be arranged in $(N-1+P)!$ ways between the ends \parallel . However the ϵ 's and O 's can be permuted among themselves $P!$ and $(N-1)!$ times respectively and two distributions differing only by such permutations are identical. Therefore as there are $P!(N-1)!$ symbols corresponding to these identical distributions one must divide $(N-1+P)!$ by $P!(N-1)!$ to obtain the number of distinct distributions. The result is $\frac{(N-1+P)!}{P!(N-1)!}$, Planck's combinatorial formula.

This derivation allowed some important conclusions to be drawn regarding the status of Planck's quanta and their relationship with Einstein's. It confirmed Ehrenfest in his old opinion that the energy elements

ϵ and hence also Planck's quanta, were merely a formal device, imbued with no more physical significance than the divider elements. (160) The failure to realise this, he believed, had led to the mistaken interpretation that these quanta were mutually independent and identical to Einstein's. The difference between the two was demonstrated by comparing their different statistical behaviour. Ehrenfest noted that the number of ways of distributing a number P of Einstein's light quanta, regarded as independent, over N_1 and then over N_2 cells in space stand to each other in the ratio $N_1^P : N_2^P$. If Planck's quanta were also regarded as mutually independent then in passing from N_1 to N_2 oscillators, the number of possible distributions would increase in the same ratio. This, Ehrenfest noted, is clearly not true, as Planck's formula gives:

160. Ehrenfest and Kamerlingh Onnes op cit p. 872

$$\frac{(N_1 - 1 + P)!}{(N_1 - 1)! P!} ; \frac{(N_2 - 1 + P)!}{(N_2 - 1)! P!} \quad (161)$$

According to Ehrenfest the explanation was simple, Einstein's quanta could be regarded as existing independently of each other, whereas Planck's could not and were no more than a formal device. 'The real object which is counted remains the number of all the different distributions of N resonators over the energy grades $0, \mathcal{E}, 2\mathcal{E}, \dots$ with a given total P '.
(162)

Finally Ehrenfest gave an example of the quantal reduction in the number of possible arrangements which illustrated the classical nature of Einstein's quanta and the non-classical behaviour of Planck's.

Ehrenfest thus contended that if Planck's quanta were to be considered as statistically independent then the traditional classical counting procedure should be applied rather than Planck's own. (163) Since the number of distributions obtained by Planck differed from that obtained by treating the quanta as independent the entropy change in any appropriately specified process would also have to be different. It is ironic that although Ehrenfest's derivation is now almost universally given as an intuitive underpinning of Planck's procedure, it's original purpose was to argue that this procedure cannot be correct if the quanta are to be independent. This was also the substance of his criticism of Einstein's 1924 paper as we shall see.

It thus follows, in a reciprocal manner, that if Planck's procedure is correct then the quanta cannot be regarded as statistically independent and must exhibit some sort of non-classical correlation. Also, the fact that a permutation of Einstein's quanta gives a new distribution whereas a permutation of Planck's does not implies that the former can be regarded as individuals whereas the latter, nor the oscillators either for the

161. Ibid p. 873; Also see Jammer (1973) p.51-52

162. Ibid p. 873

163. M. Klein (1970a) p. 256

same reason, cannot. Klein^{has} remarked that the blurring of the concept of particle when it comes to light quanta was already implicit in Einstein's results for energy and momentum fluctuations in black body radiation. (164) These results furthered the identification between Einstein's and Planck's quanta because they effectively allowed Einstein to explain quantum phenomena in terms of the wave-like aspect associated with his, individual, quanta, whereas Planck effectively had to appeal to the non-classical correlations between his, non-individual, quanta. The same phenomena was explained by both kinds of quanta and thus they came to be seen as identical.

By the time this 1914 paper had been published, Planck's 1906 method of treating whole regions of phase plane of area h as a single event, had been generalised and by 1913 it had been established that there was a natural unit for phase extension : h for each degree of freedom. This led to the result that each cell in phase space could be given the volume h^3 , thus giving an absolute value for the entropy via the Sackur-Tetrode Equation.

These results implied that the classical assignment of equal a priori weights to equal volumes of phase space had to be abandoned and only certain regions of this space given non-zero weight. Thus for Planck's oscillators the area assigned non-zero weights were those particular ellipses of constant energy whose enclosed areas were integral multiples of h . As we have said, it was Ehrenfest who was virtually alone in recognising that the basis of Boltzmann's proof of the Second Law had been lost in principle as soon as physics had followed Planck in abandoning the classical weighting assignment (165). Thus in 1914 he returned to the problem of obtaining a suitably generalised weight function for quantum statistics and presented a

164. Ibid p. 257

165. Ehrenfest's letter to Bohr, given in Klein op cit p. 283

general discussion which did not rely on the notion of an oscillator (166). He proved that the statistical weights had to be adiabatically invariant, (167), a result which was fundamental in extending the Combinatorial Approach to quantum statistics.

From about 1910 onwards attention shifted increasingly towards the problems of atomic structure and other questions, but the publication of Bose's work in 1924, and the subsequent elaboration of its consequences by Einstein, rekindled many of the old arguments concerning the statistical behaviour of quantal particles.

As is well known Bose asked Einstein to translate his paper and arrange for its publication. In the accompanying letter he emphasized the division of phase space into cells of volume h^3 as a basic assumption of his work, thus clearly indicating its ancestry. Bose presented a derivation of Planck's law on the basis of Einstein's light quantum hypothesis using the traditional combinatorial formula. (168)

He regarded the quanta as particles localizable in space and considered their distribution over the cells of phase space. It was in determining this distribution that Bose deviated from the traditional line and adopted Planck's 1900 approach in specifying the distribution by the numbers of cells containing each possible number of quanta rather than the numbers of quanta in each of the cells. Thus he employed a quantal,

166. P. Ehrenfest (1914) p. 657, (1959) p. 347

167. In terms of adiabatic invariance his 1911 paper can be understood as having established that Boltzmann's relationship between entropy and the number of ways of obtaining the most probable distribution remained valid precisely because Planck had quantized the oscillators' adiabatic invariant, the ratio of its energy to frequency. More generally, he (Ehrenfest) had shown that if, and only if, the weight function was dependent on this adiabatic invariant, then the statistical thermodynamics of the oscillator was secure.

168. S.N. Bose (1924) p. 178 rep. in H.A. Boorse and L. Motz (1966) p. 1013; also see Pais op cit p. 423 ff.

rather than a classical, characterisation of the events to be counted : cells occupied by the quanta.

However, he used a form of the traditional Boltzmann combinatorial formula, giving the number of ways W of distributing A^S cells over N^S quanta as

$$\prod_s \frac{A^s!}{p_s^s! p_i^s! \dots}$$
 where P_i^s is the number of cells containing i quanta. Indeed Bose followed the classical Combinatorial approach very closely, but replaced everywhere 'particles' by 'cells'. (169)

The view (170) that this work represents a significant departure from traditional Boltzmann statistics, as modified by Planck in 1906, is therefore rather simplistic. Bose did depart from the traditional line as regards what was taken as a countable event, although the departure was hardly strikingly original, merely a return to Planck's characterisation of 1900, but the combinatorial formula used and his whole procedure in general, were located entirely within the traditional approach. It is worth remarking that just as Boltzmann's procedure implied that his atoms were statistically independent so Bose's juxtaposition of the classical combinatorial expression with the quantal characterisation of countable events, implied the statistical independence of the cells. The statistical independence of the quanta had vanished. As we shall see, this aspect of Bose and Einstein's work was seized upon by Ehrenfest as a perpetuation of Planck's earlier mistake. Finally we emphasise again that by counting the cells containing the quanta, rather than the quanta themselves, and by asking how many quanta are in a cell rather than which quanta are in a cell, Bose was implicitly regarding the quanta as devoid of individuality in the classical sense. (171)

169. Thus A^S is not the number of standing waves as in Rayleigh and Jeans work, nor the number of particles as in Boltzmann's but the number of cells.

170. L. Wessels (1977) p. 315

171. See M. Klein (1964) p. 29

Shortly after he had arranged for the publication of Bose's paper Einstein presented a paper of his own in which he applied Bose's methods to a gas of material particles. This paper and the two which followed it laid down the foundations of the quantum theory of the ideal gas embodying what is now called Bose-Einstein Statistics.

In his first paper Einstein used exactly the same combinatorial formula and techniques as Bose, suitably modified to take into account the finite mass of the gas atoms and their fixed number, and employed a similar characterisation of countable events, with 'quanta' replaced by 'gas atoms'. (172)

Thus Einstein followed Bose in using the traditional form for the expression for the number of ways of distributing Z^s cells over N particles together with the quantal 'Planck 1900'

$$W = \prod_s \frac{Z^s!}{p_0^s! p_1^s! \dots}$$

characterisation of the events to be counted. (173) The thermodynamic properties of Einstein's gas were consequently more complicated than in the classical case but tended to support his theory. Thus it predicted a value for the entropy at high temperatures which was equal to that given by the Sackur-Tetrode equation, which contained the correct additive constant.

(174) Einstein also showed that at temperatures approaching absolute zero the entropy approached zero for all values of the volume thus demonstrating that his quantum gas also satisfied Nernst's heat theorem. Further support was to come several years later with the experimental verification of the famous low temperature degenerate behaviour of the gas atoms, only hinted at in this first paper.

This behaviour was the subject of one of Einstein's letters to Ehrenfest where he wrote 'From a certain temperature on, the molecules 'condense' without attractive forces, that is, they accumulate at zero

172. A. Einstein (1924) p. 261

173. See Pais op cit p. 428

174. P. Ehrenfest and U. Trkal (1920) p.162, also Ehrenfest (1959) p. 414.

velocity' (175), and he wondered how true his theory was. It is interesting to note that Einstein appears to have ruled out the possibility of the non-classical condensation phenomena being accounted for in terms of peculiar interatomic forces. In a reply, sometime that same year, Ehrenfest criticised this work on the grounds that the gas molecules were not statistically independent (176). This criticism is pivotal to an understanding of the difference between Einstein's 1924 and 1925 papers as we shall shortly see.

Einstein's second paper was published in 1925 and dealt first of all with this condensation phenomena. (177). He then went on to consider Ehrenfest's objection and accepted that it was entirely correct. In the course of the discussion of this question Einstein gave a combinatorial formula and a characterisation of countable events which were completely different from those given in the previous year. Thus he wrote down

$$W = \prod_s \frac{(N^s + A^s - 1)!}{N^s! (A^s - 1)!}$$

where A^s is the number of cells and N^s the number of particles, which clearly is of the same form as Planck's of 1900. (178) However he now took a distribution to be characterised, in a classical manner, by the number of particles in each available cell, and counted the number of ways in which the particles could be distributed over the cells. This is a combination which is the reverse of the one used in 1924.

It is surprising that Ehrenfest directed his criticism not at this work as would perhaps be expected, but at the 1924 paper which explicitly used a version of the traditional combinatorial formula. To what then was Ehrenfest objecting? Clearly it was to the characterisation of countable events used in 1924 which, as we have noted, implied that the cells were statistically independent but the particles were not. Thus Ehrenfest

175. Pais op cit p. 432

176. Klein op cit p. 31

177. A. Einstein (1925) p.3

178. If certain trivial substitutions are made then Einstein's expression is completely identical with the product over all frequencies of Planck's.

criticised Planck for the statistical interdependence inherent in his combinatorial formula of 1900 and criticised Einstein for the same thing, manifested now in the characterisation of what was to be counted.

Einstein admitted the truth of this objection and attributed the statistical dependence to some kind of mutual interaction between the particles. Thus, referring to his expression above, he wrote, 'The formula, therefore, expresses indirectly a certain hypothesis on a mutual influence of the molecules which for the time being is of a quite mysterious nature.'(179).

It was absolutely crucial that this statistical dependence was not eliminated by the 1925 reworking of the theory, despite Ehrenfest's objections, because as Einstein obviously realised, it lay at the very heart of the condensation phenomenon to which he attached so much importance. But if the particles were now regarded as distinguishable (180), in a classical sense, in what did this dependence lay? The answer is that it lay in the wave-like aspect which Einstein now attributed to his gas atoms.

The 1924 paper also represents a return by Einstein to the study of fluctuations about equilibrium but considered now in the context of material gas atoms rather than of radiation (181). He succeeded in obtaining an expression for the mean square fluctuation of the number of particles which was analogous to the one for the mean square energy fluctuation of electromagnetic radiation and which was also the sum of two terms. However, it was now the first term which was familiar

179. Einstein op cit.

180. Thus Brush's brief analysis is both over simplistic and incorrect. S.G. Brush (1983) p.131

181. Einstein op cit. See also Pais op cit p. 437

and which applied to a distribution of distinguishable particles. The second term was associated with waves in the case of radiation and thus Einstein was led to '... interpret it in a corresponding way for the gas, by associating with the gas a radiative phenomenon'. (182)

It was at this point that Einstein turned to deBroglie's ideas and suggested that a de Broglie matter wave-field should be associated with the gas particles thus accounting for their statistical dependence. As we have noted this suggestion was subsequently developed by Schrodinger into his theory of wave mechanics and the wave-like nature of material particles became a way of understanding their lack of statistical independence.

It is interesting to note that Einstein also showed that for a gas of independent particles the entropy must violate either the requirement that it be extensive or Nernst's heat theorem. The entropy expression for the quantum gas satisfied both of these and Einstein regarded this as a good reason for preferring his procedure, even if it could not be shown to be superior on a priori grounds (183).

Following the publication of a third paper (184) tidying up some of the details, Einstein's theory provoked a number of responses (185). Planck's theory of the ideal gas was presented as a conservative reply to Einstein and derived the entropy on the basis of the former's 1906 work, employing the $N!$ division in order to avoid counting 'redundant' complexions formed by the permutation of two atoms (186). This procedure was justified on the grounds that such a permutation produced no change in the state of the gas. Thus Planck was insisting here on the use of what Gibbs called the generic rather than the specific phase (187). Unlike Einstein Planck was deeply sceptical of any strange statistical

182. Pais Ibid p. 438; V.V. Raman and P. Forman (1969) p.291

183. See also A. Einstein (1925) p. 18

184. Ibid

185. See W. Pauli (1927) p.81 and M. Klein (1964) p.31

186. M. Planck (1925) p.49; See P. Hanle (1977) p.177

for a good discussion of these responses.

187. J.W. Gibbs (1902) Ch.15.

interaction between the atoms and thus his theory did not predict any degenerate behaviour at low temperatures.

The procedure of dividing the number of possible arrangements, obtained by implicitly regarding the atoms as individuals and statistically independent, by $N!$ in order to obtain the 'correct' quantal count had been the subject of a dispute between Planck and Ehrenfest.

(188) Planck had vigorously defended the division, on the above grounds, while Ehrenfest had argued that it was ad hoc and simply not cogent. These arguments greatly influenced Schrodinger who believed that one could only divide by $N!$ when the gas was in the condensed state in which the atoms were virtually held fixed, so the permutation number $N!$ becomes physically meaningful and the atoms become 'identifiable' (189).

A year later Schrodinger reconsidered this problem in the light of Bose and Einstein's work and again attacked Planck's justification for the $N!$ division on the grounds that the molecules were either individuals or not and the theory should be constructed accordingly without first assuming they were and then 'correcting away' the resulting multiplicity (190). Thus he adopted a holistic view which attributed quantum states not to individual gas atoms but to the body of the gas as a whole. However, Schrodinger could not find a way to carry out such a programme in a physically plausible way.

This defect forced him to amend his approach and in 1926 he applied Boltzmann's combinatorial procedure to a gas considered, significantly, as a collection of de Broglie matter waves. (191) This gave a theory broadly similar to Einstein's but which ruled out the possibility of the condensation effect predicted by the latter.

188. P. Ehrenfest and V. Trkal (1920), (1921) p.609;
M. Planck (1921) p. 365 (1924) p. 673
189. E. Schrodinger (1924) p. 41-45
190. E. Schrodinger (1925) p. 434
191. E. Schrodinger (1926) p.95; also see Pais op cit
p. 438-439

In 1928 the He I - He II phase transition was discovered (192) and this was subsequently interpreted as an example of B-E condensation, but not until 1938 however. (193) Thus it was not the experimental verification of a crucial prediction which decided between Einstein's gas theory and the rival approaches of Planck and Schrodinger but rather, as we shall now see, the accommodation of this theory within a self-consistent theoretical framework by Heisenberg and Dirac (194).

Before discussing this it is worth noting that the above account serves to illustrate Klein's remark that '... the same impasse which blocked the understanding of the statistics of photons was at least the case of a detour in the statistics of atoms and molecules. In both cases the classical concept of the particle was at fault, since non-interacting classical particles are necessarily independent'. (195)

Following the construction of quantum mechanics as we now know it, during the years 1925-1927 (196), three different authors independently applied the new theory to the statistical mechanics of indistinguishable particles.

Thus Fermi attempted to formulate a theory of the ideal gas which was consistent with both the new Q.M. and the third law of thermodynamics (197). He assumed the

192. See W.H. Keeson (1942)

193. F. London (1938) p. 947; Brush op cit p. 172-203, gives an excellent discussion of the developments surrounding this phenomenon.

194. Planck continued to propound his approach until 1926 but appears to have realised that he ^{had} strayed away from the mainstream of quantum statistical mechanics.

195. M. Klein (1959) p.55

196. M. Jammer (1973)

197. E. Fermi (1926a) p.145 and (1926b) p.902; F. Rasetti's introduction to Fermi's collected papers has a good discussion of the genesis of this work. E. Fermi (1962) p. 178

validity of Pauli's Exclusion Principle (198) for gas atoms and obtained an expression for the number of arrangements of N_s molecules distributed over Q_s states, subject to the constraint that not more than one molecule could be in any one state:

$$P = \frac{Q_s!}{N_s!}$$

The Boltzmann relation and standard thermodynamics then gave the equation of state for such a gas, now known as the Fermi gas formula. From this Fermi derived the Sackur-Tetrode equation thus demonstrating that his gas would exhibit the correct behaviour at low temperatures.

With the publication of this work there were then two very different theories of the ideal gas, each embodying a different form of statistics. Shortly afterwards Heisenberg explicated the connection between these two forms and the symmetry characteristics of states of systems of indistinguishable particles.

In his first paper, published in June 1926, Heisenberg showed that two indistinguishable systems, i.e. particles, which were weakly coupled, always behaved like two oscillators for which there were two sets of non-combining states (199). Thus he demonstrated that the eigenfunctions of one system were symmetric in all the co-ordinates whereas those of the other were anti-symmetric. The fact that the two sets of states were not connected then followed from the symmetry of the Hamiltonian of the system under a particle permutation. An example of such systems, according to Heisenberg, were the two electrons in the helium atom and in July he investigated more fully the theory of such two electron atoms using the Schrodinger approach (200). The conclusion he reached was that only those states whose eigenfunction are anti-symmetric in their electron co-ordinates can arise in nature.

Dirac had read Fermi's paper but claims to have forgotten it (201) and had not seen Heisenberg's at all

198. W. Pauli (1925) p. 765

199. W. Heisenberg (1926) p. 411

200. W. Heisenberg (1926b) p. 499

201. Archive for the History of Quantum Physics.

although it is mentioned in a note added in proof. Thus his August 1926 paper (202) represents a completely independent effort which went further than those above in setting the two forms of statistics and the corresponding theories of the ideal gas within their correct theoretical context.

He began with the fundamental requirement that the theory should not make statements about unobservable quantities and noted that it then followed that two states which differed only by the interchange of two particles and which were therefore physically indistinguishable, must in fact be counted as only one state (203). This in turn implied that out of the set of possible two-particle eigenfunctions there were only two which satisfied the conditions that the eigenfunction should correspond to both of the above states and should be sufficient to give the matrix representing any symmetric function of the particles, these two being the symmetrical and the anti-symmetrical eigenfunctions. Dirac then noted that the theory as it then stood was incapable of deciding which of these two actually applied in nature (204), indicating that it had not yet been realised that each symmetry type had its own domain of applicability.

He then showed how these results could be extended to any number of non-interacting particles, writing the anti-symmetric eigenfunction in determinantal form, from which the Exclusion Principle followed quite naturally. (205) Dirac concluded by remarking that 'The solution with symmetrical eigenfunctions ... allows any number of electrons to be in the same orbit so that this solution cannot be the correct one for the problem of electrons in an atom.' (206). Thus as the theory cannot

202. P.A.M. Dirac (1926) p.661

203. Ibid p. 667

204. Ibid p. 669

205. Ibid p. 669-670

206. Ibid p. 670

tell which solution is correct extra-theoretical considerations had to be appealed to.

Dirac then applied this theory to the ideal gas, noting that the two possible types of eigenfunction would give two different solutions of the problem.

By multiplying together the single particle eigenfunctions he obtained the eigenfunctions for the assembly as a whole, from which the symmetric and anti-symmetric ones could be selected. The former were identified with Bose-Einstein statistics, taken to be applicable to light quanta (207), and the latter with what are now known as Fermi-Dirac statistics, regarded as applying to electrons in atoms and gas molecules.

The equation of state of an ideal gas was then derived on the assumption that the solution with anti-symmetrical eigenfunctions is the correct one, so that not more than one molecule can be associated with each de Broglie matter wave (208). The set of such waves associated with the molecules was then divided into a number of sub-sets such that the waves in each sub-set are associated with molecules of about the same energy. Assuming that equal a priori weights are assigned to all stationary states of the assembly, the probability of a distribution in which N_s molecules are associated with the A_s waves in the s 'th set is given by

$$W = \prod_s \frac{A_s!}{N_s! (A_s - N_s)!} \quad (209)$$

Boltzmann's relation was then used to give the entropy and the equation of state was obtained very straightforwardly.

Dirac's combinatorial formula, above, is simply a suitably amended version of the well-known expression

207. This is a rather odd identification given that Dirac presumably knew of Einstein's 1925 papers. He may perhaps have been influenced by the controversy surrounding this work and the arguments against it put forward by Planck and Schrodinger.

208. Ibid p. 672

209. Ibid p. 673

for the number of ways in which $\binom{n}{m}$ objects can be selected from a set of n objects, written as $\frac{n!}{m!(n-m)!}$

and often referred to as the number of combinations of n things taken m at a time (210). Thus it gives the number of ways in which N molecules can be selected from a set of A molecules associated with A waves, or the number of possible combinations of A waves taken N at a time. We can understand Dirac's formula in the following way: if we have A_s waves and N_s molecules then the number of ways of associating these molecules with the waves, such that no wave is associated with more than one molecule is given by $\frac{A_s!}{(A_s - N_s)!}$.

However a permutation of the molecules does not lead to a new arrangement and so this result must be divided by the total number of arrangements formed by such permutations, $N_s!$, giving $\frac{A_s!}{N_s!(A_s - N_s)!}$. We

repeat this procedure for each set of de Broglie waves and the probability of a particular distribution is then given by the product of the above over all sets

i.e. $\prod_s \frac{A_s!}{N_s!(A_s - N_s)!}$.

In other words Dirac's formula is simply a version of Planck's non-traditional 1900 formula, suitably amended to take account of the Exclusion Principle. It is clear that Dirac was concerned with the distribution of particles over de Broglie waves, or states, and thus he implicitly adopted a classical characterisation of particles distributed over states as the events to be counted.

It is not clear whether Dirac regarded the particles as classical individuals, subject to restrictions on the set of states they can occupy, or as 'non-individuals' in some sense. In the former case the division by $N_s!$ in the combinatorial formula above can be interpreted in terms of giving the required reduction in the statistical weights, whereas on the latter view it is introduced to eliminate the unobservable and hence

meaningless permutations of the 'non-individual' particles. There is certainly nothing in this paper which indicates that Dirac supported the latter view.

The great explanatory power of Dirac's work, particularly as regards electrons in metals, for example, led to its rapid acceptance by the physics community. Out of the very many papers exploring the consequences of this theory we shall just note Heitler and London's analysis of the hydrogen molecule in which they explained the consequences of Fermi-Dirac statistics in terms of saturable, non-dynamic forces - the so-called 'exchange' forces of attraction and repulsion - existing between particles (211). As we have seen this is one way of accounting for the statistical dependence of quantal particles and Heitler and London's suggestion follows very naturally on from Einstein's and Ehrenfest's remarks.

Following the realisation that there were now three kinds of statistics, classical, Bose-Einstein and Fermi-Dirac, several papers were published exploring the relationship between them. Thus Fowler, in 1926, made the first attempt to construct a general form of statistical mechanics embracing all three types. (212) Ehrenfest and Uhlenbeck tackled the question as to whether B-E or F-D statistics are necessarily required by the formalism of Q.M. or whether there were areas in which classical statistics was still valid. (213) They concluded that it is the imposition of symmetry requirements on the set of all solutions of the Schrodinger equation for an assembly of particles, obtained by considering the permutations of all the particles among themselves, which produce the symmetric and anti-symmetric combinations and hence give rise to B-E and F-D statistics respectively.

If no constraints are imposed then Maxwell-Boltzmann statistics are the most appropriate form to use. Thus

211. W. Heitler and F. London (1927) p. 455

212. R.H. Fowler (1926) p. 432

213. P. Ehrenfest and G.E. Uhlenbeck (1927) p. 24

they demonstrated that the Q.M. formalism does not, by itself, necessarily imply one or other of the two forms of quantum statistics. It is the imposition of specific symmetry requirements (214) on the wave-functions which determines which form is obtained.

We have noted that the non-classical statistical dependence of the particles manifested in quantum statistics could be 'explained' by reference to the de Broglie matter waves associated with each particle. Thus Uhlenbeck contended that Schrodinger had shown in 1926 that Einstein's gas theory could be obtained by considering the gas either as an assembly of particles and applying B-E statistics or as a system of standing de Broglie waves and applying classical M-B statistics. (215) He used the recently developed wave mechanics to extend this discussion into a detailed analysis of the various interpretations of the three forms of statistics and their different areas of applicability. The conclusion he reached was that the lack of independence of quantal particles was introduced in a natural way in the appropriate wave interpretation. Thus Ehrenfest commented 'One must distinguish between identity and independence, between the particle and the wave picture'. (216)

It is interesting to note that the development of quantum statistics took place entirely within the Combinatorial Approach and therefore was capable of treating equilibrium situations only. The neglect of Boltzmann's alternative H-Theorem Approach ended in 1928 with Nordheim's attempt to construct a theory of quantum statistical mechanics based on this approach which would give, not only all the results derived previously, but also expressions applicable to phenomena

214. These can be regarded as initial conditions as we have noted.

215. G.E. Uhlenbeck (1927). This is not strictly correct, as we have seen, because Schrodinger's approach, using matter waves, did not give the condensation phenomena to which Einstein attached so much importance.

216. Klein op cit p. 58

associated with non-equilibrium states. (217)

Nordheim began by following the classical H-Theorem approach but noted that the time variation of the distribution function for an assembly of particles could be separated into two parts. One was a flux term due to the particles own motion and external forces, which following Darwin and Kennard's demonstration that a single quantal particle behaved classically under the influence of external forces, (218) remained unaltered by the quantum theory. The other term was a collision term due to interactions between the particles, which Nordheim rewrote to take account of the fact that in Q.M. the probability of a collision depends not only on the number of particles in the initial states, as in C.M. but on the number in the final states also. (219).

Having done this he then substituted for the number of particles undergoing collisions in his version of Boltzmann's transport equation from which followed the well-known distribution functions for quantum statistics. To justify this derivation he established the quantal version of the H-Theorem, obtaining an expression for H and showing that it could only decrease and that its derivative would be zero in the equilibrium case (220)

However, Nordheim's conclusion that he had constructed a theory of quantum statistics based purely on kinetic arguments is not strictly correct as combinatorial initial conditions were introduced in the consideration of the flux term in order to obtain the correct quantal behaviour. Thus he wrote 'The prohibition of certain states of motion, characteristic for quantum statistics, emerges here not out of the law of motion but out of the choice of the initial state ... of a proper wave group. Thus, for instance, it is prohibited in the Fermi-

217. L.W. Nordheim (1928) p. 689

218. Darwin (1927) p. 258 and Kennard (1927) p. 326

219. Nordheim op cit p. 691-693

220. Ibid p. 695

Dirac statistics to choose an initial distribution, the density in the phase space of which is greater than one particle per h^3/m^3 .¹ (221) In other words the results of quantum statistics can only be obtained if certain states are regarded as inaccessible and a particular initial state selected. Once this choice has been made the laws of Q.M. can only ensure that there are no transitions to the forbidden states. Thus, on Nordheim's own admission, the H-Theorem alone cannot give the appropriate statistics, some external, non-kinetic constraint, such as the symmetry restrictions imposed upon the wave-functions, must be imposed.

A similar attempt was made by Ornstein and Kramer to derive F-D statistics from purportedly purely kinetic arguments (222). However, their derivation was based on the Exclusion Principle which, again, can be regarded as a non-kinetic initial restriction on the set of accessible states. Further work was continued along these lines by Halpern and Doermann and others (223).

It was an historical accident that the sufficiency condition for particle indistinguishability embodied in the Symmetrization Postulate namely that the eigenfunction should be either symmetric or anti-symmetric, was first demonstrated within the context of B-E and F-D statistics. The realisation that this was not a necessary condition only came later with the conception of parastatistics. However, even in 1926 Dirac realised that the symmetric and anti symmetric functions were merely the two simplest out of the set of possible eigenfunctions for the assembly. Thus he later wrote 'It appears that all particles occurring in nature are either fermions or bosons, and those only anti-symmetrical or symmetrical states for an assembly of similar particles are met with in practice. Other more complicated kinds of symmetry are possible mathematically

221. Ibid p. 691

222. L.S. Ornstein and H.A. Kramers (1927) p. 481

223. O. Halpern and F.W. Doermann (1939) p. 1077; for an excellent review of these developments and quantum statistical mechanics in general see D. ter Haar (1955) p. 312

but do not apply to any known particles'. (224)

The investigation into these alternative symmetry types was initiated by Gentile in 1940. (225) He used Bose's combinatorial methods to derive an expression for the average number of particles in a group of states which was dependent upon a parameter d giving the maximum number of particles which could occupy any given state. F-D and B-E statistics were special cases of these intermediate statistics, obtained when $d = 1$ or ∞ respectively. From this Gentile arrived at an expression for the entropy of a gas of particles obeying these statistics also dependent on d from which the F-D and B-E entropies could also be obtained as special cases.

ter Haar (226) and Sommerfeld (227) objected on Q.M. grounds that, apart from the purely academic cases considered by Müller (228), the only application of Gentile's statistics was in the case of a system of bosons in which d equals the number of particles in the system N . Several authors had discussed the implications of Gentile's formula when $d = N$ and ter Haar concluded that in this case there would be no difference between intermediate and B-E statistics. From an inspection of integrals in the complex plane Wergeland (229) and Schubert (230) had both concluded that it did not matter whether one took $d=N$ or $d=\infty$.

However ter Haar's demonstration that Gentile's work conflicted with the postulates of Q.M. was based only on S.P. from which, as we have seen, one can only obtain conventional quantum statistics.

In 1952 a work was published in Japan by Okayama which received little attention then or since, but which anticipated several later results (231). Thus he distinguished between particle and place permutations and

224. P.A.M. Dirac (1930), (1958) p. 211

225. G. Gentile (1940) p. 493

226. D. ter Haar (1952a) p.199 and (1952b) Ch. IV.

227. A. Sommerfeld (1942) p. 1988

228. H. Müller (1950) p.199.

229. H. Wergeland (1944) p. 51

230. G. Schubert (1946) p. 113

231. T. Okayama (1952) p. 517

realised that the fact that a wave function operated on by the former represents the same state as the original function only implies the conventional statistics if it is assumed that the two functions must be proportional and must lie in a one dimensional irreducible subspace. If this assumption is dropped and the possibility is allowed that a single state could correspond to some larger collection of vectors spanning a multi-dimensional subspace, then other statistical types will result. Okayama noted that these would be related to various representations of the symmetric group and could thus be investigated using the apparatus of group theory (232).

He also inquired into the relationship between para-statistics and the degeneracy which results when certain degrees of freedom assigned to conventional particles are neglected (233), as we noted on p 189, and demonstrated that paraparticles also obey the Cluster Principle, upon which the practical utility of conventional statistics is dependent. (234). Finally he tried to develop a second quantised version of para-statistics but this was not particularly successful, as his generalised commutation relations allowed only the trivial solution that the Hilbert space consists only of null vectors. (235)

Apparently unaware of Okayama's contribution Green made a similar, but much more successful attempt in 1953, (236) following Wigner's demonstration that the Q.M. commutation relations were not uniquely determined by the equations of motion (237), which suggested that more general forms of such relations were possible. Thus Green's aim was to relax the formal structure of

232. Ibid p. 517-521

233. Ibid p. 523 We considered this relationship at the end of Section 3.2.3.

234. Ibid p. 523-524.

235. Ibid p. 530; see the comments by S. Kamefuchi & Y. Takahashi (1962) p. 200-202

236. H.S. Green (1953) p. 270

237. E.P. Wigner (1950) p. 711

Q.F.T. in order to allow a possible resolution of some of the problems with which it was beset. His basic requirement was that any quantization scheme would be regarded as satisfactory if it ensured the equations of motion. This permitted the introduction not only of the usual fermion and boson commutation relations, but also of whole families of alternative commutation schemes obeyed by paraparticles. (238). Nowadays these schemes are labelled by an integer p and thus if we quantize with the p 'th scheme one talks of a parafermion or paraboson field of order p . The conventional relations are recovered when $p = 1$.

Green noted that although this generalized theory implied that a new state would result from the interchange of two particles, the particles would always divide into groups in such a way that such permutations within a group would not lead to a new state, and that furthermore interactions can be devised which prevent the formation of new states by permutations between the groups. These, suspiciously ad hoc, moves allowed him to say that the 'Principle of Indistinguishability of Identical Particles' is retained in parafield theory.

The physical properties of Green's paraparticles were first studied by McCarthy in 1955. (239) He argued that analysis of the spectra of certain bound states could give a means of conclusively identifying paraparticles whereas scattering experiments could not. The problem of distinguishing paraparticles in nature was later taken up by Messiah and Greenberg. McCarthy also concluded that the thermodynamic properties of a parafermion gas were identical to those of Gentile's paragas and those for a paraboson gas corresponded to those of a number of superimposed Bose gases. (240)

During the next few years it was shown that Green's parafield theory was consistent, both internally and with the fundamental principles of physics (relativistic

238. Green op cit p. 273

239. I.E. McCarthy (1955) p.131

240. Ibid p. 139

and non-relativistic), (241) and that its formalism could be extended to allow for interactions between the fields (242). The only problem, recognised by Volkov, for example, as a serious one, was the lack of evidence for the existence of paraparticles in nature (243).

This problem was addressed by Greenberg and Messiah in the course of their construction of a consistent first quantized paraparticle theory (244). By rigorously examining the role of S.P. within Q.M. they concluded that it was neither necessary for a consistent quantal treatment of indistinguishable particles nor strongly supported by experiment. The first point led them to replace S.P. by a form of the I.P. which then permitted the existence of, previously forbidden, mixed symmetry paraparticle states through the reduction of the Q.M. state vector space into irreducible subspaces of dimension greater than one. The set of vectors spanning such a multi-dimensional subspace was termed a 'generalised ray' and Greenberg and Messiah demonstrated that the associated indeterminacy causes no difficulty in the interpretation of the theory because measurable results on a state associated with such a ray do not depend on which state vector in the multi-dimensional subspace is chosen to represent the state. (245)

They then showed that it follows from I.P. and Schur's lemma that there exists a superselection rule between vectors in inequivalent representations. (246) However, their proposal that any paraparticle would have N-

241. Kibble and Polkinghorne (1957) p. 252; D.V.Volkov (1959) p. 1107; S. Kamefuchi and Y. Takahashi op cit; G.F. Dell'Antonio, O.W. Greenberg and E.C.G. Sudarshan (1964) p. 403
 242. D.V. Volkov (1960) p. 375; S. Kamefuchi and J. Strathdee (1963) p. 166
 243. Volkov op cit.
 244. A.M.L. Messiah and O.W. Greenberg (1964) p. 248
 245. Ibid p. 251
 246. Ibid p. 252. We have discussed this in section 3.2.

particle states corresponding to just one representation of the symmetric group was later shown to be incompatible with the Cluster Principle, discussed in Section 3.2.2.

It was also made clear that those arguments which purport to show that state vectors must be either symmetrical or anti-symmetrical can only be carried through if an extra assumption is introduced to the effect that states which cannot be distinguished by any observation are represented by the same vector to within a phase factor (247). Thus Galindo and Yndurain's conclusion that paraparticles cannot be regarded as indistinguishable in the usual sense, is incorrect because their argument assumes the existence of a complete set of commuting observables, which is equivalent to the assumption that the state vectors transform under a one dimensional representation of the permutation group (248). Perhaps the greatest and most fundamental achievement of Greenberg and Messiah's work was to demonstrate that assumptions such as this are not a necessary part of the Q.M. formalism and can be replaced by the less restrictive rule that states ^{can} be represented by any multi-dimensional irreducible subspace invariant under the permutation group.

In the second half of their paper they gave a very detailed discussion of the direct experimental tests of SP and concluded that in many cases such experiments actually tested I.P. instead (249). This prompted them to lay down a set of criteria for valid tests of S.P. and some possible types of such tests were given. Finally, they surveyed the experimental evidence for S.P. for various kinds of particles and concluded that although the statistical character of many types of particles was accepted and well established, there were some for which it was not.

247. Ibid p. 253

248. A. Galindo and F.J. Yndurain (1963) p. 1040

249. Messiah and Greenberg op cit p. 259-267

Although Messiah and Greenberg thus demonstrated that the status of many known particles as regards their symmetry type was unclear, the question whether paraparticles themselves existed remained open. A possible answer was, however, given later that same year by Greenberg when he suggested that the 'statistics problem' in quark theory could be resolved if quarks were regarded as para-fermions of order 3.(250)

It had been suggested, on various grounds, that quarks should be assigned spin $1/2$ (251) and therefore, according to the spin-statistics theorem, they should possess totally anti-symmetric wave functions. This worked well in the cases of mesons but failed in that of baryons where the symmetric quark model, which demanded that the total wavefunction be symmetric, had proven successful in classifying the baryon spectrum. (252)

Greenberg's way out of this dilemma was to propose that quarks be regarded as parafermions of order $p=3$. This allowed them to possess separate wave functions which were symmetric under permutation of any two quarks, as required by the symmetric quark model, and also overall, composite wave functions which were anti-symmetric under permutations with other composite states, as demanded by the spin statistics theorem,

The price paid for this resolution was the introduction of three labels assigned to each quark (253). These labels were subsequently interpreted, in the three triplet quark model (254), as representing a new

250. O.W. Greenberg (1964) p. 600

251. F. Gursey and L.A. Radicati (1964) p. 173; A. Pais (1964) p. 173; Gursey, Pais and Radicati (1964) p. 299

252. F.E. Close (1979) Ch.5.

253. Greenberg op cit p. 602

254. M.Y. Han and Y. Nambu (1965) p. 1006

three-fold degree (255) of freedom, later called colour. The colours possessed by a quark provided the extra degree of freedom which could be used to anti-symmetrise the otherwise symmetric wave-function. Thus there were two alternative models (256) which could solve the statistics problem : one regarded quarks as paraparticles, the other took them to be 'ordinary' particles possessing an extra degree of freedom.

Insofar as all three triplet models have the same consequences with regard to hadron (baryon and meson) spectroscopy the above two models are equivalent experimentally (257). As regards the relationship between their formalisms we shall note the following. There exists a certain transformation (258) which effectively transforms paraquark fields into degenerate Bose or Fermi Fields. Thus it would appear that the paraquark and colour models are formally equivalent. However, the Hilbert space of the transformed para-fields is larger than that of the untransformed fields. It can then be shown that the Hilbert space generated by the degenerate Bose and Fermi fields acting on the vacuum contains all physically different states for sets of local observables invariant under the groups $SO(p)$, $O(p)$, and $U(p)$. However, this is only true for para-fields for the last group, $U(p)$. (259). Furthermore, all the irreducible representations of each group are produced with the degenerate Bose and Fermi fields and so the physical states are represented

255. It is only with three different values of colour, corresponding to a parafield of order $p=3$, that Fermi statistics are allowed for quarks, and unique baryon states are produced. O.W. Greenberg and C.A. Nelson (1967) p. 79-80. Recall Ohnuki and Kamefuchi's point discussed on p.190

256. Various other kinds of models were also proposed. See Greenberg and Nelson op cit, for a taxonomy of the different types.

257. Han and Nambu op cit p. 1010; O.W. Greenberg and M. Resnikoff (1968) p. 1844. Greenberg and Nelson op cit p. 84-88.

258. The 'Klein' transformation. Ibid p. 85ff for details

259. Greenberg and Nelson op cit p. 85-87.

redundantly. In the para-field case each different state is represented 'non-redundantly' by one vector in the Hilbert space generated by the para-field acting on the vacuum (260). Thus it can be concluded that the para-quark and colour models have the same states only in certain cases, which depend on the choice of observables in the former, and so are 'formally equivalent' only to this extent.

If the required choice of observables is made however the two models can be regarded as equivalent, both formally and in terms of their observable implications. The question then arises as to why the colour model was subsequently almost unanimously chosen to be the fundamental interpretation of quark theory? (As is well known, a Yang-Mills gauge theory of Han-Nambu colour became the dominant theory of the strong interactions in the early 1970's and formed the core of the new theory of quantum chromodynamics).

A possible answer is that, first of all, the formalism of the colour model was more familiar to elementary particle physicists and was easier to use, and secondly colour could be easily gauged whereas the parafield model could not (261). Thus although the two descriptions were equivalent formally, under the circumstances noted above, they were not equivalent heuristically, the colour theory being 'more fruitful' (262) in the sense of possessing a greater capacity for generating new and successful lines of development.

There is one further point we wish to make concerning this historical episode. As we shall discuss in the next section the quantum field theoretic description of an N-particle system regards the 'particles' as merely excitations of the field. It is in this sense

260. Recall our discussion on p.

261. O.W. Greenberg, private correspondence 12/6/81.

As regards the second point note the qualifying remarks in Greenberg and Nelson op cit p. 88

262. M. Redhead (1975) p. 77

that one can say that the interpretation of the first quantised theory in which the particles are regarded as 'non-individuals' finds its most coherent expression in Q.F.T. However, the above equivalence between the parafield and colour descriptions suggests that the particle individuality 'lost' in the field theory can be regained, in a sense, by transforming to an alternative model in which the particles are regarded as distinguishable through the introduction of a new quantum number. Thus quarks become distinguishable through the attribution of different colours just as electrons with different spins can be regarded as distinguishable.

Furthermore, it was subsequently shown, as we shall see, that parafield theory was (completely) formally equivalent to the first quantised theory of paraparticles, at least for particles of finite order. Thus following the 'chain' of equivalence we can argue that particles regarded as indistinguishable in one first quantised description may be considered to be distinguishable in another.

If colour is regarded as an extrinsic property of the particles then one could interpret quarks with different colour as merely different states of the same particle, possessing different values of colour. On the other hand if colour is regarded as an intrinsic property then quarks with different colour are different kinds of quarks. Thus the introduction of colour may be regarded as corresponding to a further classification of particles into different natural kinds.

An analogous situation exists in the case of neutrons and protons which may be regarded either as different kinds of particles or as different states of the same kind of particle, which may possess different values of isospin. (263) Thus the distinguishability, as far as the intrinsic properties are concerned, of the particles in the first case is 'taken up' by the states, in the

263. A. Messiah (1962) Ch.14; See also Hamermesh for a group theoretical discussion of isospin. Hamermesh op cit. p. 433-435

second, through the isospin, regarded as an extra degree of freedom.

These examples illustrate the general point that as an intrinsic property can always be reinterpreted as an extrinsic one, and vice versa, two different kinds of particles can be subsumed as different manifestations of the same kind of particle, and conversely, a set of one kind of particle can be further classified into different natural kinds.

The theoretical consequences of parafield theory and its relation to first quantised paraparticle theory, continued to be explored during the mid-'60's. (264) In particular it is worth noting Greenberg and Messiah's conclusion, based on further theoretical work together with the experimental evidence then available, that no particle known at that time could be a paraparticle (265) Furthermore it was not possible for a paraparticle to decay into ordinary particles or vice versa.

We have already dealt with Steinmann's claim that paraparticle theory is not consistent with the Cluster Principle (266) and a similar refutation of this conclusion can be found in Landshoff and Stapp's general discussion of the problems involved in establishing a correspondence between first and second quantised paraparticle theory (267). In particular, as Greenberg had already pointed out (268), the particle permutations are not represented in the Hilbert space of Q.F.T. and so the I.P. is, in general, ill defined in the second quantised approach. Landshoff and Stapp suggested this problem could be overcome if a distinction were made between particle and place permutations. The latter were regarded as observables (269) and as well defined in

264. O.W. Greenberg and A.M.L. Messiah (1965) p. 500,
(1965) p.1155 and O.W. Greenberg (1966) p. 29
265. Greenberg and Messiah op cit p. 1165-1166.
266. O. Steinmann (1966) p. 755
267. P.V. Landshoff and H.P. Stapp (1967) p.72
268. Greenberg op cit p. 35
269. But note our comments on p.207

parafield theory. Indeed, they argued, the I.P. is equivalent to the requirement that all observables that distinguish among states differing only in the order of the variables, are functions of the \bar{P} 's. (See our discussion in 3.2.1). In this form the I.P. is equally applicable in both the first and second quantised approaches and can thus be used to establish the correspondence between the two. As we shall see this suggestion was subsequently taken up and elaborated by Stolt and Taylor.

In 1969 Hartle and Taylor rewrote the formalism of first quantised paraparticle theory in a very lucid and cogent manner. (270) It is this form of the theory which we have used as the basis for the outline of the first quantised theory of indistinguishable particles given in Section 3.2. As we have already discussed its most significant characteristics we shall not consider them again here. However it is worth repeating that it was shown how Greenberg and Messiah's generalised ray could be eliminated and the usual connection between states and rays in Q.M. restored, by moving to a subspace of lower dimension and using the one dimensional ray belonging to such a subspace to label the mixed symmetry states (271). Hartle and Taylor also showed that the theory was consistent with the Cluster Principle and argued that it then followed that '... for a given kind of paraparticle there is a whole family of allowed symmetry types, with the property that whenever the family contains an $(N+1)$ -particle symmetry type $D^{(N+1)\mu}$ it contains all N -, $N-1$, ..., 2-particle symmetries whose Young diagrams can be obtained from that of $D^{(N+1)\mu}$ by removal of successive blocks.' (272).

270. J.B. Hartle and J.R. Taylor (1969) p. 2043

271. Ibid p. 2046

272. Ibid p. 2050

Finally as regards the occurrence in nature of such particles they conclude '... although there is no theoretical reason to exclude para-particles, their properties are sufficiently disagreeable for one to hope sincerely that there will continue to be no evidence in their favour'. (273)

Further work by Stolt and Taylor demonstrated that paraparticles could be classified into those of finite and those of infinite order (274). A particle is said to have finite column (row) order p if it has states associated with some Young diagram of p columns (rows), but no states associated with any diagram of more than p columns (rows). If a particle has neither finite row nor column order then it must have states with diagrams of arbitrarily many rows and columns. Such a particle is said to be of infinite order. Finite order paraparticles can be further divided into parabosons and parafermions of order p , a division which closely resembles that of parafield theory. (275). However, another result needed to be established before an identification could be made between the first and second quantise approaches.

Kamefuchi and Ohnuki had been pursuing their own programme since 1967 prior to which Ohnuki has published a study of the statistical properties of two-body bound states of paraparticles (276). Their 1967 work continued in this vein, (277) but in 1969 they embarked upon a general group theoretical investigation of the wave functions of indistinguishable particles (278). Their

273. Ibid p. 2051

274. R.H. Stolt and J.R. Taylor (1970) p. 2226

275. Ibid p. 2228. A similar classification can be established for infinite order paraparticles except those with states corresponding to all Young diagrams.

Hartle Stolt and Taylor (1970) p. 1759

276. Y. Ohnuki (1966) p. 285

277. S. Kamefuchi and Y. Ohnuki (1968) p. 1279

278. S. Kamefuchi and Y. Ohnuki (1969) p. 337

main conclusion was that a necessary and sufficient condition for particles to obey parastatistics is that there exists one and only one irreducible invariant sub-space which uniquely corresponds to each Young diagram whose first row or column is shorter than or equal to p . (279) In other words the N -particle wave functions associated with a para-Fermi (Bose) field of order p support a representation of the permutation group containing every irreducible representation with p or fewer columns (rows). This was exactly the result which Stolt and Taylor needed.

Following this work Kamefuchi and Ohnuki went on to consider the necessary and sufficient condition for particles to be indistinguishable. Arguing that the premises used to establish I.P. were too restrictive they adopted the following field theoretic condition : those particles are regarded as indistinguishable which result from the second quantisation of a given field, since they share the same intrinsic properties (280). (C.f our distinction in the Introduction to Ch.1). The Q.M. of such particles then follows from the corresponding field theory. They then obtained a general theory of indistinguishable particles by simply translating the field theoretic results of many particle systems into the Q.M. language. This carried with it the surprising discovery that this formalism could be generalised even further to give a theory of particles not obeying either ordinary or para-statistics. (281) These results are collected together and discussed in more detail in their recent book. (282).

The complete equivalence between parafield theory and first quantised paraparticle theory was finally established by Stolt and Taylor in 1970 (283), although

279. Ibid p. 345-346

280. S. Kamefuchi and Y. Ohnuki (1970) p. 543

281. Ibid p. 569 ff; A similar generalisation was considered by Carpenter the same year, following Landshoff and Stapp's work. See K.M. Carpenter (1970)

282. Y. Ohnuki and S. Kamefuchi (1971) p.19 and (1982)

283. R.H. Stolt and J.R. Taylor (1970. p.1.

for paraparticles of finite order only (284). Such an equivalence required two things : first, a 1-1 correspondence between the states of the two theories must be established by constructing a 1-1 isomorphism between the rays of the corresponding Hilbert spaces. Clearly this is not possible with Messiah and Greenberg's generalised rays so Stolt and Taylor used the alternative formalism of Hartle and Taylor in which the conventional correspondence between states and rays is preserved. Secondly, a correspondence must be established between the permutation operators in the two theories. This cannot be done with the particle permutation operators since these are undefined in the second quantised formalism. However, place permutation operators can be defined in both theories and as Landshoff and Stapp demonstrated, these do correspond to one another.

Thus Stolt and Taylor began their proof by showing that the physical significance of the \bar{P}_i 's in the second quantised theory is the same as in the first quantised. Ohnuki and Kamefuchi's result, above, allowed them to demonstrate that the vectors in each theory transform under the \bar{P}_i 's, and their second quantised analogues, in exactly the same way. (285). It was then shown that the Fock and Hilbert spaces of the second and first quantised formalisms respectively decompose into a sum of subspaces in exactly the same way also, and that the sums in these decompositions run over the same set of irreducible subspaces. Thus if the basis vectors in the two theories are identified with one another then this defines an isomorphism between the two state spaces. (286). Furthermore it is clear from the definitions of the \bar{P} 's and their analogues, that they possess the same matrix elements between corresponding vectors and hence are corresponding operators. Thus Stolt and Taylor

284. Paraparticles of infinite order have no second quantised analogues although Stolt and Taylor conjectured that the Q.F.T. formalism could be generalised to include them.

285. Ibid p. 13-14

286. Ibid p. 15

concluded 'Our construction of the desired isomorphism is complete; Every finite order first-quantised paraparticle has a second quantised analogue; and vice versa'. (287).

Thus the Q.M. of indistinguishable particles of any statistical type (288) can be given a formulation in terms of field excitations, in which the particles are not labelled from the outset and the creation operators in the field theory contain the effects of particle non-individuality. We shall briefly return to parastatistics and Q.F.T. in the next section.

Apart from the odd piece of work here and there, such as Green's suggestion that neutrinos could be parafermions of order 2 (289) and Govorkov's study of the connection between parastatistics and simple Lie algebras (290), there has been little further discussion of paraparticle theory. Presumably, one factor which could account for the sudden loss of interest, apart from the simple absence of anything new to add to the theory, was the realisation that the theory was, in a sense, unnecessary because, as the development of quantum chromodynamics indicated, one could always devise an equivalent formalism, which was simpler and more accessible, by assigning extra degrees of freedom to the particles and using ordinary statistics.

This concludes our history of quantum statistics. We particularly wish to emphasise the following points. First, questions of individuality, 'identity', statistical independence etc., played an important and explicit role in shaping the development of 'ordinary' statistics. Secondly, the philosophical alternatives which we discussed in Section 3.3 find their analogues, or rather origins, in the development of the formalism of the theory and its interpretation. Thus, the use of a classical characterisation of the events to be counted

287. Ibid p. 15

288. Assuming, of course, that Stolt and Taylor's conjecture (fn 284) is correct and a similar equivalence can be established for paraparticles of infinite order.

289. H.S. Green (1972) p. 1400

290. A.B. Govorkov (1980) p. 1673

of, for example, particles in cells, implies that the former were being implicitly regarded as classical individuals. However, a non-traditional combinatorial formula was used, concomitant with this characterisation, which could be interpreted in terms of a restriction being imposed on the set of possible states accessible to the particles. Alternatively, a traditional formula could be employed but with a characterisation of countable events in which one counted cells containing particles, thus implicitly regarding the latter as 'non-individuals' and the cells, or states, as individuated. Another approach was to maintain that the particles were, in fact, classical individuals, but subject to peculiar interactions. Finally, the non-classical statistical behaviour of the 'particles' was also attributed to their 'wave-like' natures and thus they could be regarded as 'non-individuals' precisely in this sense, that they were 'also' waves.

All four positions can be drawn from the history of the development of quantum statistics and thus, as in the classical case our philosophical conclusions are supported by historical considerations. Indeed, the development of quantum statistics can be characterised to a large extent, in terms of shifts from one position to another.

Thirdly, the development of parastatistics demonstrates that the Q.M. formalism is not as rigid as is often thought and can be reformulated to admit entities, such as paraparticles and parafields, which were previously excluded. Thus formal restrictions as regards ontology should not be regarded as absolute. Philosophical positions also are not uniquely determined by considerations of the formalism of the theory. This point, already made in Section 3.3 above, is reiterated by the equivalence between the paraquark and colour theories which demonstrated that the individuality lost by introducing a field theoretic description can be 'taken up' by the introduction of a new parameter.

Finally, we note that although the first and second quantised approaches are formally equivalent, they differ heuristically, and, more importantly from the

point of view of this thesis, there may be strong arguments against the philosophical implications of one which undermine it in favour of the other.

We shall now further discuss Q.F.T. albeit rather briefly.

3.5. Individuality and Quantum Field Theory

As is well known we can arrive at a quantised field description of an N-particle system via two routes :

1). The N-particle Schrodinger equation is taken and subjected to the Fock space formulation i.e. to the formalisation of second quantisation.

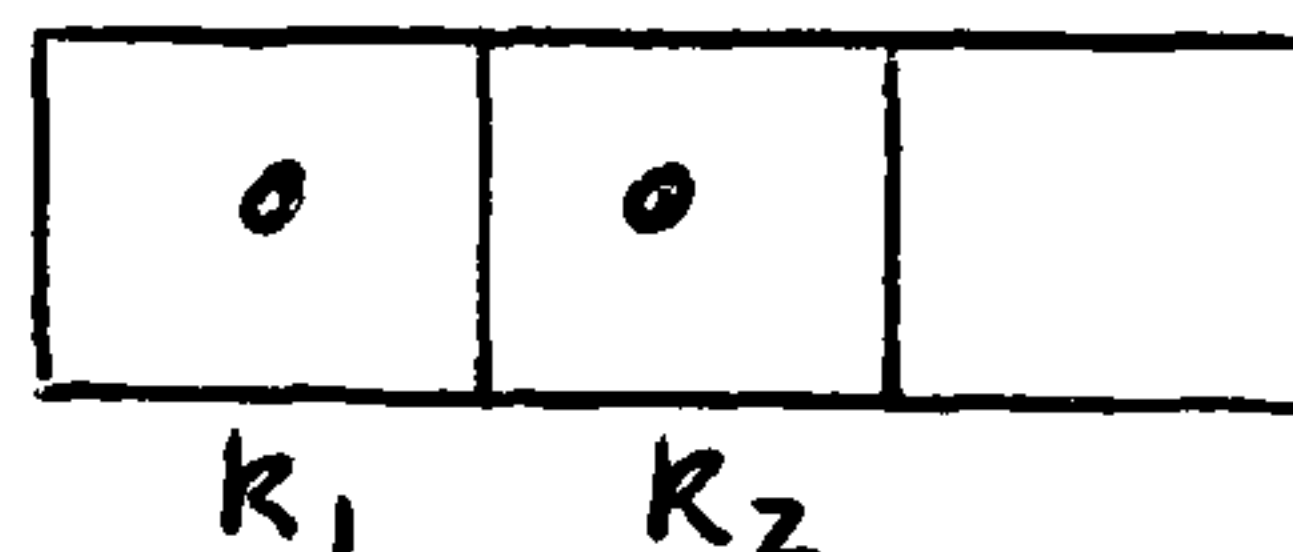
2). The 1-particle Schrodinger equation is taken, regarded as describing a mechanical field (i.e. resolved into normal modes) and the techniques of field quantisation applied (291)

Formally speaking these are completely equivalent, but the quantised field which results can be given two meanings according to the route taken. On the first one begins with particles which have T.I. attributed to them, both particles and states are labelled and thus both the P_i 's and the \bar{P}_i 's play an important role. On the second route modes of a field are considered and thus one begins with non-individuals, only the states are labelled and thus only the \bar{P}_i 's have a role to play.

Now when a many particle system is regarded as a quantised field one no longer talks of which particles are in which states but only one how many particles are occupying a particular state, as given by the occupation numbers. Thus the system is represented in occupation number space in which only the states are specified and the number of particles or rather field excitations, occupying the states.

The role of the \bar{P}_i 's in Q.F.T. can then be understood as follows. We begin with the vacuum state which contains

no particles and which we represent by the ket $|0\rangle$. The situation with one particle in the k 'th state is represented by $\alpha_k |0\rangle$, where α_k is a creation operator, and one particle in the k_1 'th state, one in the k_2 'th state is given by $\alpha_{k_1} \alpha_{k_2} |0\rangle$ and so on. Using our ball and box analogy what we are giving here is a method of describing the distribution of balls among boxes i.e.



If we consider a system of bosons then we can construct the general ket $\alpha_1 \alpha_2 \dots |0\rangle$. Applying a place permutation operator gives :

$$\bar{P}_i \alpha_1 \alpha_2 \dots |0\rangle = \alpha_{r_1} \alpha_{r_2} \dots |0\rangle$$

but the creation operators commute i.e.

$$[\alpha_1, \alpha_2] = 0$$

$$\therefore \alpha_{r_1} \alpha_{r_2} \dots |0\rangle = \alpha_1 \alpha_2 \dots |0\rangle$$

$$\therefore \bar{P}_i \alpha_1 \alpha_2 \dots |0\rangle = \alpha_1 \alpha_2 \dots |0\rangle$$

So the effect of a place permutation on a state is simply to reproduce that state. For bosons the states are symmetric under the \bar{P} 's and this translates into a certain type of commutation relation between the creation and annihilation operators in Q.F.T. In the case of fermions, the states are anti-symmetric under the \bar{P} 's and this translates into anti-commutators in the field theory. Thus in general there is a 1-1 correspondence between the symmetry types in the 'ordinary' particle description and the commutators in the field theory.

The situation is a little more complex in the case of paraparticles because one must give not only the occupation numbers but also the irreducible representation under which the state transforms, and the symmetry type of the state under the \bar{P} 's, in order to distinguish states transforming under the same representation. (292). This is why the \bar{P} 's play a more important role in Q.F.T. than in the first quantised theory.

292. cf Y. Ohnuki and S. Kamefuchi (1971) p. 19 especially pp. 21-23.

This is all translated into a field theory of paraparticles via Green's ansatz which supplies a generalization of the commutators and anti commutators mixed up in such a way as to characterise the symmetry type in the correct way. Simple commutation or anti-commutation of the creation operators would not capture the symmetry type of the paraparticles. If the anti-commutators of ordinary Q.F.T. are somewhat arbitrary then Green's ansatz is even more artificial in the way it is designed to reproduce the correct symmetry type. This supports the view that field quantisation is really just a formal device. (293)

Thus Q.F.T. is the most 'natural' framework in which to regard the particles as 'non-individuals' because they are represented as merely field excitations and are not labelled at the very outset. Thus there are no shuffling of particles as in the Schrodinger formulation and the P's remain undefined. The \bar{P} 's are well defined however, and the states are labelled, leading one to conclude that the states can be regarded as individuated.

Insofar as the states are embedded in space-time this is a form of space-time individuality. However, as in the first quantised case, it is a very weak form because the Impenetrability Assumption is clearly violated in the case of quantum fields. When we discussed a possible field description of C.M. we noted that T.I. could not be attributed to the field configurations, but that they could be individuated through their trajectories. This cannot be done in the case of Q.F.T. because of the overlapping wave functions of the 1-particle states. Together with our remarks above this supports the view, already outlined in the classical case, that quantum fields are merely properties of the points of space-time, to which individuality is attributed.

In the first quantised approach we noted that the statistical dependence of the particles could be accounted for in terms of the wave-like aspect attributed to them. As waves they become, in a sense, properties of

the entire system, a description which clearly invites a holistic approach. However, we must be careful here. The problems concerning the interpretation of de Broglie's 'matter waves' and the wave function in general, usually centering on the question 'what is it that's doing the waving?', are well-known. Furthermore as Dirac had pointed out, this notion of 'wave-like aspects' and the term 'wave mechanics' in general arises, both historically as we have seen, and formally, out of nothing more than an analogy between the systems of Q.M. for which the quantal Superposition Principle holds, and systems in C.M., such as vibrating ropes, for which also a superposition principle holds, accounting for the vibration in terms of the superposition of a number of waves (294). However, the nature of the superposition relationships between the states in Q.M. is very different from that in the classical theory because they require us to accept that whenever a system is definitely in one state it is also partly in each of two or more other states. Thus it is the non-classical Superposition Principle of Q.M. which lies behind this idea of the particles having some sort of wave-like aspect, and it is this which must be addressed, as we have done, briefly, on p. 199 ff, in any discussion of particle non-individuality and statistical dependence.

It might be thought that these problems could be circumvented by turning to Q.F.T. which, it is said, offers a 'natural' description of particles as field excitations and which therefore resolves the wave-particle duality problem. (295) Unfortunately, however, this is not correct, as Redhead has demonstrated (296).

294. The source of the analogy is that in both cases the superposition principle leads to a mathematical formalism in which the equations governing the behaviour of the system are linear in the unknowns.

295. P.A.M. Dirac (1927) p. 243

296. Redhead op cit p. 26-28.

The representation in which the field amplitude is diagonal can be called the field representation and likewise the representation in which the number operators are all diagonal can be termed the particle representation. However, in this latter representation the field amplitude is not sharp because the number operators do not commute with the creation and annihilation operators. Thus one cannot simultaneously ascribe sharp values to both the particle number and the field amplitude and so wave-particle duality is manifested again, albeit in a different form.

Thus the creation and annihilation operators can be interpreted either as creating or destroying particles or as increasing and decreasing, in a discrete way, the excitation level of the oscillators comprising the field. The latter view allows, again, a very 'natural' description of the situation where the number of 'particles' is not conserved. Particles are then regarded as simply quantised excitations of particular modes of the field and the field configurations can come into and go out of being without the kinds of conceptual and philosophical problems associated with creating and destroying individuals which occur with the particle interpretation.

If a particle is annihilated and ceases to exist then clearly, existence not being a predicate, any individuality that was attributed to it is destroyed along with the particle. The annihilation of a particle followed by the creation of another, indistinguishable from the first, is not a process involving the same, i.e. identical in our sense, individual.

One possible way of retaining individuality in this situation is to talk not of the creation and annihilation of particles but of particles undergoing transitions from 'active' to 'frozen' states in which they cannot be observed through any physical interaction. Thus Dirac originally interpreted photon radiation as such a transition from an unobserved to an observable state and similarly regarded electron-positron pair annihilation and creation in terms of transitions to

and from unoccupied states in an ^{un}observable negative energy 'sea'. (297) Reichenbach has also demonstrated that a consistent description of quantum statistics could be given in these terms (298).

With this interpretation the particles can obviously still be regarded as having T.I. attributed to them although they cannot always be reidentified because there are periods in their histories during which they are completely unobservable and '... exist in the form of nothingness'. (299) There is an obvious analogy here with the case of 'inaccessible states' which we discussed on p. 210ff. Likewise the particles still exist, ontologically speaking, when they are 'frozen' they just cannot be observed in any way. They are there but we cannot see them. Of course it might be objected that physics should not be concerned with unobservable quantities and this gives another reason for preferring the field description over the particle representation.

It might be argued that given all this one should drop all reference to 'non-individual' particles and refer instead to field excitations. This is perfectly permissible of course, but it must be remembered that we are not compelled to give up a description in which the particles are regarded as individuals. On the other hand if we do insist on adopting the particle representation then we must be prepared to face the philosophical objections to the notion of T.I. and to the idea of 'inaccessible' states, which will come as a result. As we have already said, although the representations are formally equivalent, objections such as these may mitigate against one in favour of the other.

3.6 The Gibbs Paradox

The Gibbs paradox has two aspects. The first is that the

297. P.A.M. Dirac op cit.

298. H. Reichenbach op cit p. 259

299. Ibid p. 262

expression for the entropy obtained from classical Boltzmann counting is not extensive and thus when a sample of gas is mixed with a sample of the same kind of gas there is an entropy increase. This is unsatisfactory on two grounds. First it is not in agreement with the results of classical phenomenological thermodynamics. Secondly, it can be argued that insofar as thermodynamics is a phenomenological theory, in which no reference to 'particles' need be made, nothing physical happens when two gases of the same kind are mixed. Thermodynamically speaking 'mixing' is a concept which is only applicable to different kinds of gases and only becomes meaningful when applied to the same kind when kinetic-atomic considerations are introduced, as in the 'reduction' of the theory to statistical mechanics. Thus from a purely thermodynamic point of view the mixing of a gas with itself is not a 'real', or at least not a physical, event and so the entropy change should be zero.

A resolution of this form of the paradox can be obtained if the classical expression for the number of possible complexions is divided by $N!$. This gives an expression for the entropy - the Sackur Tetrode equation - which is extensive as required (300). The justification for this $N!$ division is that it eliminates redundant complexions obtained through a permutation of the particles. This suggests a result which, as we have emphasised, is absolutely fundamental to Q.M. namely that particle permutations are not regarded as observable and the quantal combinatorial formula takes account of this fact. It is in this sense that Q.M. has been said to 'resolve' Gibbs paradox (301).

300. W. Yourgrau, A. van der Merwe and G. Raw (1966) p. 236-237.

301. T.H. Boyer (1970) p. 849. See also D. Hestenes (1970) p. 840 and R. Rosen (1963) p. 232 who sweep the problem under the carpet by adopting an operationalist approach.

However, even if the $N!$ division is carried out and an extensive entropy obtained there is still the second aspect of the paradox to contend with, which is the form in which it was originally given by Gibbs.

Let us consider a box divided into two halves of equal volume by a partition. Suppose that one half contains a mole of gas A and the other a mole of gas B, both at a sufficiently low pressure that deviations from ideal behaviour are negligible. If the partition is removed the gases mix at constant temperature and pressure and there is an entropy change of $2R \ln 2$, where R is the gas constant per mole. However, if the two halves initially contained the same gas, A say, then upon removing the partition the entropy change would be zero. Now it would seem reasonable to assume that we can make the two gases as much alike as we want; in other words there is no limit to the degree of resemblance which they might have. Yet we would still get this abrupt decrease in the entropy as the two gases became identical. As Gibbs said '... the increase of entropy due to the mixing of given volumes of the gases at a given temperature and pressure would be independent of the degree of similarity or dissimilarity between them'. (302)

It is this mismatch between the continuous change in the entropy and the apparently continuous change in the distinguishability of the gases which constitutes the essence of this aspect of the paradox.

It is worth noting that Gibbs himself does not appear to have regarded this as a paradox, but rather gave it as a plausibility argument for choosing generic phases over specific phases. The latter are phases which are altered by a particle permutation the former are phases which are not. One can define the entropy with respect to either but Gibbs argued that we should choose the former because we would not regard the diffusion of a gas into itself as producing any change in the entropy. Thus, after giving a

version of the 'paradox' he wrote, 'It is evident therefore that it is equilibrium with respect to generic phases and not with respect to specific, with which we have to do in the evaluation of entropy...' (303).

Attempts at resolving the paradox typically fall into one or other of two groups. Some authors have argued that the indistinguishability of the particles is a simple Yes/No concept and thus eliminate the 'mismatch' and resolve the paradox, by denying that the gases can pass continuously from distinguishable to indistinguishable. A typical example is Mandl who writes '... there is no sense in which we can gradually let the properties of two kinds of molecules tend to each other and hence the entropy difference ... tend to zero ... The molecules either are identical or they are different. In the latter case their difference in properties can be used to actually effect a separation by means of semipermeable membranes ... and similar devices.' (304) This view is justified by an appeal to the quantal character of elementary particles whose intrinsic properties can possess only discrete values. Substances composed of such particles cannot therefore be infinitesimally distinguishable. However, this does not account for the fact that the entropy change is the same whether the two substances are very similar, i.e. the particles possessing the same values for all intrinsic properties except one, say, or very different. It seems implausible that the constituent particles can differ as regards only one, and any one, intrinsic property and the entropy change remains the same, whatever the property that is different, but that as soon as they possess the same value for this property the entropy change falls to zero. It also seems implausible that the change in the value of the entropy change should be the same whatever the change in the value of the intrinsic property taking the

303. Gibbs (1960) p.206-207.

304. F. Mandl (1971) p. 131

particles from being distinguishable to indistinguishable. In other words two kinds of particles can possess the same values for all intrinsic properties except one. Then the entropy change falls from $2R \ln 2$ to 0 whether this property is the spin which becomes the same, or the rest mass. Furthermore the change in the value of the entropy change is the same whether the difference in the value of the one remaining non-equal intrinsic property changes from x to 0 or y to 0.

Thus this view cannot account for the observations that the change in the value of the entropy change is independent of the degree of similarity of the two kinds of particles, of which intrinsic property the two kinds of particles happen to differ with regard to, and also independent of the change in value of this property which makes the particles indistinguishable and of the same kind.

However although this 'Yes/No' position is incorrect with regard to indistinguishability, it is true for 'non-individuality'. Thus Post has written that '... there is an important difference between mere indistinguishability and identity in the strong sense, that is, being exactly alike and devoid of individuality. Identity is not simply extreme similarity pushed to the limit ... You do not approach identity gradually.' (305) One of the most fundamental points which we have tried to emphasise has been the difference between indistinguishability and non-individuality. The former applies to both classical and quantal particles of the same kind, and is concerned with the intrinsic properties possessed by the particles, whereas the latter only applies to quantal particles and is concerned with the loss of the classical individuator, whether it be the underlying substratum or the particle's spatio-temporal trajectory. Thus non-individuality cannot be continuously approached from indistinguishability

and in this sense non-individuality is a Yes/No concept, particles of the same kind are either non-individuals or they are not. As we shall see, consideration of non-individuals, rather than indistinguishability, can in fact lead to a resolution of the paradox.

An alternative to the 'Yes/No' approach is to use the overlap of the particle wavefunctions as a continuous measure of indistinguishability and then show that the entropy change varies continuously from 0 to $2R \ln 2$ as the overlap decreases from 1 to 0. Thus in this case the 'mismatch' is removed by denying the discontinuous change in entropy.

This view has been propounded by von Neumann (306) and Schrodinger (307), who suggested that the overlap of wave-functions representing two quantum states could be used to give a continuous measure of indistinguishability. Thus von Neumann demonstrated that the entropy change could be made to decrease to zero continuously if the two gases differed only in their quantum states and if these states could be continuously varied from orthogonality to proportionality. Lande used this argument in reverse and argued that the thermodynamic assumption that any two gases are either strictly separable by a filter or strictly inseparable, is false for two gases which are alike but whose molecules are in two different quantum states, as this leads to states lying between complete separability and inseparability. Thus he wrote 'Actually there must be some scale of equality degrees, observable as a gradual scale of diffusion entropy values from $Nk \ln 2$ to zero, changing from total separability to lack of separability'. (308) Lande's argument hinges on a demonstration that the expression for the diffusion entropy can be made dependent upon a continuous function expressing the degree of distinguishability of the gases.

306. J. von Neumann (1955, 1943) p. 370 ff.

307. E. Schrodinger (1952) p. 58 ff.

308. A. Lande (1973) p. 83

Klein has developed this approach further by analysing the mixing of two isomers containing nuclei in different energy states. (309) He calculated the value of the entropy of mixing as a function of the overlap of the states of the particles that are mixed and showed explicitly that the entropy change varies continuously from 0 to $2R \ln 2$ as the overlap decreases from 1 to 0.

As we have remarked one can regard the non-individuality of quantal particles as a manifestation of the wave-like aspect of the particles and explain it in terms of the overlap of the wavefunctions. Thus this approach above resolves the paradox by turning to the non-individual character of quantal particles. However, one can raise the objection that if the wave functions are overlapping then it is difficult to see how one can talk about distinct things at all, indistinguishable or not. In the limit there is just some sort of smear and it seems nonsensical to talk of indistinguishability. The concept simply ceases to apply.

Despite this the above work does establish an important point by demonstrating that the entropy change of $2R \ln 2$ is obtained not only when two particles of different kinds are mixed but also when two gases composed of particles of the same kind but in different states are mixed, for example a monatomic gas A diffusing into a gas B consisting of the same substance as A but with all its nuclei in an excited metastable state. This emphasis on the states of the constituent particles was seized upon by Lesk who argued that 'There is an element of convention in regarding particles as distinguishable or indistinguishable - what in these cases is indistinguishable is not the particles themselves but the ^{ir} set of allowed states' (310) Furthermore, in order to decide whether a continuous measure of indistinguishability is possible, and therefore whether some sort of partial indistinguishability

309. M. Klein (1948) p. 80 and (1959) p. 73

310. A.M. Lesk (1980) p. 112

is really feasible, '... one must examine not the particles themselves but the restrictions on transitions between sets of allowed energy levels'. (311)

A useful example which he gives is that of conformational isomers, where the molecules may share the high energy levels, so that there is no distinction between the isomers, but not the low energy ones. When samples of such isomers mix the shared energy states constitute a partial indistinguishability which does decrease the change in entropy upon mixing although not to an observable extent. However, as Lesk notes, it is unlikely that such partial indistinguishability could ever be realised in practice. If two species of molecules did share populated sets of energy levels in this way then transitions would occur between these sets and the two species could not be regarded as distinct. If, on the other hand, the rate of inter-conversion was so slow that the species could be regarded as distinguishable then the shared energy states would be insufficiently populated to cause any observable decrease in the entropy change.

This approach to the paradox can also be obtained very straightforwardly from our discussions of particle individuality in Q.M. Thus in terms of the view in which quantal particles are attributed with individuality in the form of T.I. but subject to state accessibility restrictions, the difference in the entropy change can be said to arise from the fact that many states of a system of distinguishable particles correspond to a single state of a system of indistinguishable ones. There are very many more states accessible to the particles of the mixed system of distinguishable gases than there are to the particles of the mixture of indistinguishable gases. Thus it is not distinguishability of the particles which is important here but the accessibility of possible energy states. In both cases the particles are individuals, in the classical sense,

but in the second there is a reduction in the number of accessible states, which is a consequence of the symmetry restrictions imposed upon the observables of the system by I.P., corresponding to the reduction in the entropy change.

On the other hand, regarding the particles as non-individuals cannot alter the entropy change of $2R \ln 2$ in the case of the mixing of distinguishable gases because the particles of one gas are distinguishable from the particles of the other; they possess different values of at least one intrinsic property. More significantly the states of the mixed system are distinguishable because it is they which, on this view, 'carry' the individuality. In the case of the mixing of two gases of the same species the particles are then all both non-individual and indistinguishable, and there is then an equivalence of states for the gases, so questions of indistinguishability do not arise. The entropy change is still zero.

To illustrate this choice let us take Lesk's example of the mixing of two isomers. (312) Any state of either molecule can then be regarded as a state of the system consisting of the constituent atoms, which of course will be the same in both cases. We may choose to regard these systems either as individual particles with independent sets of quantum states or as non-individual particles occupying different regions of phase space separated by activation energy barriers. Adopting the latter alternative cannot alter the entropy change, as Lesk realized, because the states of the mixed system are distinguishable. The difference in the internal states of the system then allows the two states differing only by the permutation of one isomer with the other to be different.

Thus we agree with Lesk that it is through a consideration of the states of the particles, rather than the particles themselves, that we arrive at an understanding of this aspect of Gibbs paradox. The sudden reduction in the entropy change is to be expected because the number of available states decreases. However, this is

not to be taken to imply that indistinguishability is a straightforward Yes or No affair. It is theoretically possible for there to be a form of partial indistinguishability in the sense that particles may share certain sets of energy states, but it is difficult to see how this could be realised in practice without either destroying the distinction between the particles or reducing the effect on the entropy change to negligible proportions.

3.7 Conclusion

By considering in detail the formalism of Q.M. with particular regard to the role of particle and place permutations and to the consequences of the Indistinguishability Postulate, we have concluded that, as in C.M., there are two positions which can be consistently adopted concerning particle individuality. One regards the particles as non-individuals, in some sense, but with the states as distinguishable. The other attributes T.I. to the particles, but then non-classical state accessibility restrictions must also be imposed. It must be emphasised that both views are consistent with the I.P. and, as we have tried to indicate, manifestations of each can be found in the history of quantum statistics.

It has been argued that quantum physics has so degraded the concept of a 'particle' that it can no longer be meaningfully applied to the entities concerned (313). To a certain extent this is obviously true. By virtue of the profound differences between C.M. and Q.M. quantal 'particles' possess very different characteristics from classical 'particles', and these differences may be held to be so great that it is impossible to establish a correspondence between the two terms. Thus if it is taken to be an important, or 'essential', characteristic of particles that they be individuals then it is clearly difficult, even self-contradictory, to conceive of 'non-individual' particles.

As we have remarked, the most 'natural' framework for this view is Q.F.T. where all reference to particles can be dropped from the very beginning. The above contradiction only arises because the view of particles as non-individuals is an imperfect translation of the field theoretic approach into the language of first quantised Q.M.

It can therefore be argued that the only way to preserve a particle approach, in the above sense, within Q.M. is to adopt the alternative view of particle individuality and introduce T.I. In this case the particles can be regarded as classical individuals but only at the expense of accepting that non-classical correlations must exist between them such that certain states, although existing, are rendered inaccessible. Of course, we are implicitly assuming that the characteristic of being an individual has a higher priority than that of being statistically independent so that we can still refer to 'particles' in the absence of the latter as long as the former still applies.

In other words if our first position is adopted then conceptual difficulties are encountered regarding 'non-individual particles'. These can be avoided by choosing the alternative view in which one can refer to the particles as individuals, but then some other classical characteristic must be abandoned. Q.M. itself only dictates that there should be some differences in the characteristics of quantal and classical particles, but we still have a certain degree of choice as to where these differences lie, corresponding to the choice between the two views above.

As would be expected, and as we have shown, this choice between two metaphysical credenda can be found running through the history of the development of quantum theory. Thus for example the statistical dependence of quantal 'particles' was explained, by Einstein and others, by appealing to their wave-like aspect, which was a predecessor of Q.F.T. The generalisation of this to material particles as well as photons and, in particular, Schrodinger's mathematical formulation of de Broglie's

'matter wave' idea, seriously threatened the particle nature of entities such as the electron for example, On the other hand their corpuscular aspect was strongly implied by collision experiments which led Born to suggest that Schrodinger's wave function does not actually represent, in some way, the system, but determines the probability that the particle is in a certain region of space (314). As Jammer remarks, Born's original probabilistic interpretation of the wave-function Ψ can be summarised as saying that

'... $|\Psi|^2 d\tau$ measures the probability of finding the particle within the elementary volume $d\tau$, the particle being conceived in the classical sense as a point mass possessing at each instant both a definite position and a definite momentum. Contrary to Schrodinger's view, Ψ does not represent the physical system nor any of its physical attributes but only our knowledge concerning the latter'. (315) Thus Born maintained that quantal entities, like the electron, could be regarded as individual particles, but only at the expense of accepting that the statement 'the system is at a given time and place in a certain state'

314. M. Born (1926a) p. 803 and (1943) p. 23. This interpretation was also influenced by Einstein's conception of wave-particle duality for light quanta (Interview 18/10/62, Archive for the History of Quantum Physics). However, although Born claimed that Einstein regarded the wave field as a 'ghost field' whose waves guided the photons along their trajectories there is little evidence for this view in Einstein's own work. For a more detailed and comprehensive discussion of Born's interpretation see Jammer op cit p. 38 ff.

315 Jammer op cit p. 42-43

no longer has any meaning. (316)

Furthermore, although T.I. is the only way of making sense of the particle approach in Q.M. it has attracted a great deal of philosophical criticism, as we shall see in the next Chapter. Thus, as we have said, although the field and particle approaches are completely equivalent, formally speaking, they differ both as regards their heuristic value and their underlying metaphysical principles. The adoption of one approach rather than the other will therefore depend to a certain extent, on one's own philosophical disposition.

316. M. Born (1926b) p. 129. As is well known, Born's original interpretation runs into severe difficulties in accounting for the electron diffraction phenomena revealed by the two slit experiment, which suggested that the wave function must be something physically real rather than just a representation of our knowledge. In an attempt to achieve some sort of resolution of this conflict Heisenberg suggested that Born's probability waves could be conceived as a quantitative formulation of the Aristotelian concept of *potentia*, which determined the *probability* for an event to take place.
See Jammer op cit p. 44

CHAPTER FOUR

A PHILOSOPHICAL EPILOGUE

4.1 Philosophical Criticism of T.I.

The concept of T.I. involves, as we have seen, the notion of the substantial substratum, a notion which has been regarded with suspicion by many philosophers, especially those within the empiricist tradition, ever since Berkeley's celebrated criticism of Locke. The grounds for this suspicion rest, at their most basic level, on the supposedly unknowable character of this underlying substance. If it cannot possibly be known then it is no more than a philosophical myth, which is unnecessary and can therefore be dispensed with. A 'thing' is no more than a set, or bundle of co-existing qualities with nothing existing separately over and above this set.

There are two points which can immediately be made about this gloss. The first is that the empiricist tradition finds a natural ally in the positivist-operationalist-instrumentalist attitude (1) which permeated physics at the beginning of the century and which played such a fundamental role in the development of Q.M. and its orthodox interpretation (2). The empiricist's treatment of 'unknowables', like substance, finds its analogy in the positivists' rejection as metaphysical and scientifically useless anything which is not experimentally vindicable. Given the initial difficulty of constructing a suitable theoretical framework capable of consistently accounting for and accommodating the phenomena of quantum physics, positivism, with its implicit admonition to be content with the partial knowledge that could be gained as a result of observation and experiment only, clearly seemed a very

1. We realise that by grouping them together like this we are ignoring important differences between these positions.
2. See Jammer (1974)

attractive position to adopt (3). However, it can also be characterised, less nobly, as a retreat from the fundamental aspiration to obtain knowledge of everything and, more importantly, may unduly restrict or impede the progress of science. The most well-known example of this is Mach's opposition to Boltzmann's atomicist ideas, and the recent debate about hidden variable theories in Q.M. testifies to the live nature of the issues involved(4).

Secondly, the empiricist argument is by no means conclusive. It can be argued that although admittedly substance cannot be presented naked, without the accompanying properties, this does not imply that they cannot be distinguished. Substance and properties are logically, but not practically, distinct, in the sense of the subject predicate distinction. There can be no properties unless there can be substances for them to be properties of and to which they can belong. The relationship between substance and properties is akin to that between an owner and his/her possessions, as we have said. Thus the defence of 'substance' can be anchored to the subject-predicate distinction in logic or the noun-adjective distinction in language.

The sanctuary of this anchorage has been threatened by the empiricist analysis of logic and language. Thus Russell's theory of descriptions has been used to replace singular terms by variables and purely predicative general terms (5). Quine has suggested that language should be tightened up so that vague sentences of ordinary speech can be replaced by eternal

3. The influence of the opponents to the mechanist approach in the 19th century, such as Mach, also played an important role of course.
4. Jammer op cit p. 253-339. The connection between the anti-atomic stance of the late 19th century and recent no hidden variable proofs is provided by Mach and the logical positivists who, as we have said, greatly influences the development of the 'orthodox' interpretation of Q.M. with its rejection of hidden variables.
5. B. Russell (1972) p. 56. Russell himself did not apply his theory to all singular terms, 'this', for example, escapes.

sentences using definite descriptions to identify the subjects of discourse. (6) These descriptions would then contain only general predicative terms. Likewise Ayer, and others, have tried to show that language is, or at least need only be, purely predicative and that one can do without the proper names which are used in T.I. to designate the underlying substance (7).

However, the question of whether a thing can, or should be, regarded as anything more than the set of its properties is quite separate from that of whether we can talk about that thing entirely in terms of this set. In fact it is not at all clear whether we can, in practice, limit ourselves in this way and still 'do' everything we want to. In particular the empiricist 'bundle' theory runs into severe difficulties with regard to individuality, which can be resolved^d very simply by invoking the notion of substance.

Furthermore, one must be clear as to how the properties in the 'bundle' are to be regarded. The view that they are particulars has been advanced as a solution to the problem concerning the multiple location of a property (8), but this approach has been rejected as incoherent (9). It is more common to regard properties as universals but then the bundle theory requires that the Principle of Identity of Indiscernibles (P.I.I.) be true. This can easily be seen as follows. If things are merely bundles of properties which are universals, then it is necessarily true that if two such things possess exactly the same set of properties then they must, in fact be one and the same thing, i.e. in the absence of any individuator besides the properties,

6. W. Quine (1960) p.191, and (1950) p. 220

7. A.J. Ayer (1954) p.16, Strawson has argued that there are no logically proper names nor any Russellian definite descriptions. P.F. Strawson (1960) p. 320

8. G.F. Stout (1930)

9. D. Armstrong (1978) Vol. I p. 77-79. Armstrong presents a whole range of arguments against both the bundle theory, Ch.9 especially p. 98-101, and the Lockean account of particulars, Ch.10 esp. p.104-107, before advancing his own 'objective theory of universals' in Vol. II.

the bundle theory must require that indistinguishability implies identity and hence the necessary truth of P.I.I. However, there are, as we shall see in a later section, grounds for rejecting P.I.I., in physics, which clearly weakens the empiricist position.

On the other hand the relationship between the substratum and the properties stands in need of explication. It is not sufficient to say that substance stands in the relation of support^P to the properties because, as Berkeley showed, the term 'support' in this sense is an empty metaphor. Substance clearly does not lie beneath the properties supporting them as the walls of a house lie beneath and support the roof. To say that substance, and the relation of support, are 'sui generis' and can be acquainted with by anybody who grasps that things 'have' properties, is very weak and amounts to no more than saying that the substratum is something unique unto itself standing in a relation to the properties which is also unique to itself. The absence of any satisfactory explication of this relation represents one of the most serious drawbacks of the substantivalist position.

Despite this the introduction of the notion of 'substance' does provide a very coherent and plausible solution to the problem of individuality. In particular it can account for the individuality of two indistinguishable things, such as two classical electrons, whereas the bundle theory must either appeal to P.I.I. in a form in which spatial location is included as one of the properties of a thing, or turn to what we have called space-time individuality. Both putative solutions presuppose the truth of the Impenetrability Assumption, which is violated, in a sense by the 'particles' of Q.M. and therefore has only a limited validity. Armstrong has argued that it is logically possible for there to exist particulars which are^{not} spatio-temporal at all (10) which casts further doubt on the role of spatio-temporal location as individuator.

10. Armstrong op cit p. 119 & Vol.II Ch.18. The Scholastic angels would be one example of such entities. See T. Aquinas (1947-8) Part 1, Q.50, Art.4.

Furthermore, if it is claimed that the complete set of a thing's properties must include its positional characteristics, which necessarily distinguish that thing from other, indistinguishable, things, then it must be acknowledged that the status of position is different from that of other properties in the set. To ascribe a positional predicate to a thing involves anⁱⁿeliminable reference to another individual or position, unless of course some sort of absolute view of space is adopted. The invocation of some coordinate system cannot provide an answer since the origin of such a system must still be specified. As Quinton says 'Individuals cannot be decomposed into sets of properties that are sufficient to individuate them unless positional properties are included in the set. But if positional properties are included the decomposition into properties is incomplete. Further individuals remain to be decomposed, inviting us into an infinite regress (11).

There are various ways of escaping this regress. One is to take the position where 'I' am at the present moment as the absolute point of origin of all one's positional characteristics of things. This is the one position which does not have to be picked out by its relation to anything else, and by their relation to which everything else is, in the final analysis, related. Thus position as individuator connects a thing to the 'here-and-now'. (12)

However, it is logically possible, as we noted in Chapter 1, to imagine a universe containing, for example, only two indistinguishable spheres and nothing else, not even an observer. The above account cannot then answer the question of how individuality can be attributed to the spheres since there is no fundamental 'I' to which their spatial positions can be referred - unless of course, some sort of idealist view is also

11. Quinton op cit p. 18

12. Ibid p. 19-20

adopted (13). A relational view of space clearly cannot help as the spatial relation of each sphere to the other is the same and thus cannot be used to individuate them. The only choice left is to take up the absolutist position and say that the spheres are individuated via their relations to the points of space regarded as some sort of absolute, underlying matrix or substratum.

We shall return to this example in Section 4.2, for the moment it is sufficient to note the difficulties it poses for the bundle account.

Thus we conclude that spatial position must be regarded, not as one of the properties of a thing, but as a substratum underlying these properties. The notion of substance is therefore resurrected, in the form of absolute space and in the sense of the individuator. Thus it would seem that the bundle theorists, in rejecting material substance as individuator and turning to space-time individuality, are forced to accept spatial substance in the same role. A concrete individual can then be thought of as a set of properties manifested at a certain point in space-time. (14) The label or name attached to a thing then designates the underlying substratum of space-time which acts as the individuator. In this way space-time individuality degenerates into a form of T.I.

This absolutist view of space-time is not, as we have seen, conclusively precluded by either the Special or General Theories of Relativity; indeed certain interpretations of the latter, to the effect that space-time would possess a structure even in the total absence of matter would obviously strongly support this position.

Having argued that individuality can be conferred upon things through the points of an absolute space-time, the question is how are these points themselves individuated? One possible answer is to relate the points to some macroscopic reference frame (15), identified by some unique set of predicates, such as the human

13. This is rejected by Quinton himself.

14. Quinton op cit p. 28-29

15. Strawson op cit p. 37

body. However, this is obviously just another version of the 'here-and-now' argument and suffers from the same defects. Furthermore, it is clearly open to accusations of circularity. Macroscopic objects, ^{are} on this view, individuated through their relations to the points of an underlying absolute space-time, and these points are then individuated through their relations to the macroscopic objects acting as some sort of reference frame.

Alternatively one could answer that the points themselves possess T.I. Each space-time point can, in principle, be labelled and these labels designate the substantial substratum which acts as individuator. Now, however, there is the epistemological difficulty of individuating these points. With material entities, such as particles, this is achieved via their spatial positions; T.I. is revealed through space-time individuality. This option is clearly not available to the points of space-time themselves. For one thing, it would invite an infinite regress, for another one cannot conceive how a position could move in time !

This difficulty serves to illustrate the mutual dependence of material objects and space-time objects as regards individuation. This interdependence could be broken by those interpretations of General Relativity, mentioned above, which suggest that space-time would possess some curvature in the absence of matter. This internal structure might then allow the points to be individuated. Thus the answer to the question whether the points of space-time can be individuated by the internal structure of space-time itself would seem to depend upon developments in the theory of General Relativity.

Strawson's example of a chess board provides a useful illustration of the points at issue here (16). This would appear to possess an internal structure which allows every square to be individuated (c.f. the

procedure used in writing down the moves in chess). However, the board is finite and bounded and it is by reference to this boundary that the squares are actually individuated. Thus in writing down the moves the convention is introduced that rows are numbered beginning with the nearest to the white player and columns are labelled by the piece occupying the square at the foot of the column. This provides a two-dimensional macroscopic reference frame which allows each square to be uniquely individuated. Thus unless the universe is similarly taken to be bounded in some way then some sort of external reference frame must be introduced.

Attributing space-time individuality to an object requires that a relation be established 'downwards' between the object and the underlying space-time point. For elementary particles, regarded as point like, this relation is one of coincidence between unextended point-like entities. Every particle is coincident with some point of space-time. The Impenetrability Assumption can then be regarded as stating that no two material point-like objects can be coincident, or no space-time point can be coincident with more than one material object. In other words the I.A. required there to be a 1-1 correspondence only between point-like particles and the points of space-time.

This relation, whether it be one-one as I.A. requires, or many-one, is passive in the sense that it does not determine in any way the relations between the points of space-time themselves. In other words the structure of space-time is not affected or altered by the existence of material objects. A similar independence does not obtain, however, in the Riemannian generalization of the four-dimensional pseudo-Euclidean Minkowski space-time appropriate to General Relativity. In this case the structure of space-time is connected in lawlike manner with the distribution of mass 'in' it (although it is not clear, as we have seen, whether one can go further and say that the mass distribution

causes changes in space-time), and the relation between the material and space-time points can be described as active. This, therefore, strengthens the interdependence regarding individuality.

The other aspect of space-time individuality, that of reidentifying things through time via their continuous, well defined spatio-temporal trajectories, also requires that a relation exist between the points of space-time themselves, 'side-ways' along the trajectory between successive points of the path. This relation clearly does not exist independently of the material object travelling along the path as it is established by the various requirements of continuity, causality etc., discussed in Chapter 2. As we remarked there, there is an obvious connection between the possibility of reidentifying, or 'tracing', a thing and the Impenetrability Assumption. As Robinson has demonstrated it is, at the very least, difficult to argue for such a possibility and against this assumption. (17) Penetrability implies no possibility of reidentification.

This situation occurs most obviously in the case of fields, as we have seen, but in the case of Q.M. particles also, it is difficult to see what meaning can be given to the notion of the space-time trajectory of a single particle, when the particles manifest the sort of holism which is a consequence of the Superposition Principle and which is exemplified by the so called 'entangled states'.

Also to reidentify some thing through time it must be established that the places occupied by that thing at successive times are either identical or connected by a continuous path. However this requires that the places themselves be individuated and again the question arises whether this can be done internally, through some inherent structure, or externally, through some kind of reference frame, Taking the second option

requires the selection of some privileged continuant in order to avoid any circularity involving the mutual dependence of material things and space-time points. The identity of this privileged continuant must be established independently of any consideration of spatio-temporal trajectories. Eliminating 'myself' and the 'here-and-now' as possible candidates, for the reasons given above, those things which are continuously observed throughout the period of time during which other things begin and cease to be observed, may fulfill this role (18). However, although such things can form a sufficiently complex positional scheme which could precisely specify spatio-temporal locations, there is still the problem of how these primary continuants are themselves individuated and again the possibility of circularity arises. It would seem that a combination of T.I. together with some sort of internal structure, still offers the best solution to this dilemma.

Thus although T.I. must face empiricist criticism, directed principally at the notion of an unknowable substance, it does at least give a credible account of individuality. Advocates of the view that things are mere bundles of qualities must therefore appeal to some other principle of individuation. One possibility is the Principle of Identity of Indiscernibles, but this, as we shall see in the next section, has a dubious validity. The alternative is to invoke some form of space-time individuality which involves a circularity from which it is difficult to escape. Insofar as the notion of a particle as a labelled individual can only be retained in Q.M. if one is prepared to admit T.I. any arguments against this concept will serve to undermine the particle approach as a whole. Similar conclusions apply to the field approach and space-time individuality. Thus although they are formally equivalent each approach carries with it its own metaphysical problems and any weighing of the two must place these in the balance also.

18. D. Wiggins op cit

4.2 Physics and P.I.I.

As we saw in Chapter 1 Leibniz gave several different versions of P.I.I. as his position shifted in response to Clarke's criticism and derived it both from the Complete Notion of Individual and the Principle of Sufficient Reason. In particular it is worth noting the difference between '... it is not possible for there to be two individuals entirely alike or differing in number only.' and '... there are not in nature two indiscernible real absolute beings'. In the first statement Leibniz is saying that the principle is necessary (19), in the second that it is contingent. This is a distinction of great importance as we shall now see.

In terms of second order logic with equality P.I.I. can be written thus:

$$\forall \phi \{ \phi(a) \longleftrightarrow \phi(b) \} \rightarrow a = b$$

where \underline{a} and \underline{b} are any two individual constants and ϕ is a variable ranging over the possible attributes of these individuals. The question arises, what sort of attributes should be included in the range of ϕ ?

If the attribute 'being identical with \underline{a} ', which is certainly true of \underline{a} , is included then P.I.I. becomes simply a theorem of second order logic. If such trivializations are ignored however, a weak and strong form of the principle can still be distinguished (20):

1) Weak form of P.I.I.: The range of ϕ includes properties of spatio-temporal location.

The principle can then be read as saying that it is not possible for there to exist two things possessing all properties in common.

19. Leibniz actually distinguished at least two senses of 'necessary', a principle could be absolutely necessary in the sense that its denial was self-contradictory, or it could be morally necessary in the sense that God wills it to be so and it could not possibly be otherwise. See his fifth letter to Clarke in H.G. Alexander op cit.
20. Quinton op cit p. 24-25.

2) Strong form of P.I.I.: The range of \emptyset excludes properties of spatio-temporal location.

The principle can then be taken as stating that it is not possible for there to exist two things possessing all properties, excluding spatial ones, in common.

The weak form of P.I.I. has been described as trivial (Wiggins) (21), and necessary (Quinton) (22). In fact it is neither. It is not trivial because as we shall see, interesting consequences follow from it when placed in a physical context. It is necessary only if the I.A. is accepted, and, as we have seen, there may be grounds for rejecting this assumption. Obviously, if I.A. were not true, so that two numerically distinct particles could possess exactly the same spatio-temporal properties, i.e. occupy the same place at the same time, and be qualitatively indiscernible, then weak P.I.I. would also not be true. (23)

The strong form does not presuppose I.A. and is always only contingently true. Leibniz himself emphasised the importance of non-spatio-temporal properties in distinguishing two separately indiscernible things: 'It is always necessary that beside the difference of time and place there be internal principles of distinction... thus although time and place (external relations, that is) serve in distinguishing things, we do not easily distinguish them by themselves ... The essence of identity and diversity consists ... not in time and place '. (24)

However, as Wittgenstein notes, (25) strange results follow from the strong form of P.I.I. because it denies the possibility of a universe consisting only of two qualitatively indistinguishable spheres, a point which was subsequently taken up by Black and others.

21. D. Wiggins (1967). R. Adams (1979) p.11, makes this claim also.

22. Quinton op cit p. 25

23. Thus the weak form of P.I.I. is not logically necessary, as is sometimes claimed, because it is possible to conceive of two things occupying the same space-time point and using some other principle of individuation, apart from spatial location, such as T.I. to individuate them & count them as separate.

24. Leibniz (1916) II 27.1, 27.3

25. L. Wittgenstein op cit 5.5302

(26). We shall consider this in more detail below.

We shall now examine the status of these two forms of the Principle in both classical and quantum physics.

4.2.1. P.I.I. and Classical Physics

Classical particles obey the I.A. and particles of the same kind, or species, are considered to be qualitatively indistinguishable as we have seen. Clearly then such particles are examples of things which violate the strong form of P.I.I. Insofar as the latter seeks to reduce the individual to a bundle of properties or qualities, this implies that an alternative principle of individuation must be sought for (27). There are two possibilities: space-time individuality or T.I.

If space-time individuality is attributed to classical particles then strong P.I.I. cannot be true, as we have seen, because such particles possess all qualitative properties in common and only differ as regards spatial location, which is inadmissible. Weak P.I.I. is not violated because of this difference. If the particles are attributed with this form of individuality then it follows from I.A. that they cannot also possess the same spatial properties and therefore cannot have all properties in common.

However, this last conclusion suggests that an absolute view of space must be held as we noted in section 1.3 of chapter 1.

We recall that the argument runs as follows.

We assume that it is possible for there to exist a universe consisting of only two distinct indistinguishable particles and no observer or any other material thing. On a relational view of space there is then no way of distinguishing the particles since there is

26. M. Black (1952)

27. In other words it implies that things cannot be individuated by their properties only. See our discussion in the preceding section and Chapter 1.

no external reference frame to which they can be related differently, and the spatial relation of each particle to the other is exactly the same. Thus on this view of space, the particles possess all properties and relations in common, including spatial ones, and therefore violate the weak form of P.I.I.

The only alternative would appear to be the adoption of an absolute view of space in which it is regarded as some sort of substratum. This would provide the necessary frame of reference which would allow the spatial positions of the particles to be differentiated. They would not then possess all properties in common, the spatial ones being excluded from the shared set, and weak P.I.I. is satisfied.

Although this latter alternative is ruled out by Special Relativity (which may be regarded, as we saw in Section 2.2.4 of Chapter 2, as having merely replaced absolute space by an absolute 'space-time'), either possibility would be anathema to a Leibnizian. Weak P.I.I. together with a relational account of space implies that the two particle situation above is not possible. Therefore if our assumption, that it is possible, is accepted then we have the dilemma, particularly acute for a Leibnizian, that either weak P.I.I. is false or an absolute view of space is adopted.

Black has presented similar arguments in which a possible universe is imagined containing two iron spheres and nothing else, and he concludes that P.I.I. is then violated. More elaborate universes were the¹_λ devised in which the same violation could occur. These arguments have failed to convince many philosophers, including, for example, Ayer who wrote '... such examples seem intelligible to us only because we tacitly introduce into them some further feature by reference to which we do in fact discern between the objects which we are supposing to be indiscernible' (28). In other words our

28. A.J. Ayer (1954) p. 33.

initial assumption is rejected and the dilemma evaporates. Presumably Ayer would accept that this 'extra feature' need not be 'the observer' but could be the substratum of absolute space, for example. However, someone who accepts Berkeley's dictum that the statement 'physical objects exist unperceived' is self-contradictory would argue that it is not possible to even conceive of a state of affairs in which 'the observer' is absent. The very act of imagining a situation involves some kind of observer albeit a mental one.

In similar vein Cassullo has argued that the dispute between Black and those who would defend the truth of P.I.I. can be resolved through an investigation of how the possibility of a state of affairs is determined (29). Thus he suggests that the clearest sense of conceiving of a state of affairs, as well as the sense relevant to determining its possibility, is to visualize it. Whenever one visualizes a group of objects one visualizes them in some spatial configuration and consequently whenever more than one object is visualized each occupies a different position in the visual field. However, if one accepts Russell's account of the spatial structure of the visual field then objects in this field occupy different positions by virtue of a difference in their positional qualities. Thus, in order to visualize or imagine two particles one must visualize them as occupying different positions in the visual field. But then they must differ in their positional qualities and hence do not possess all properties in common. P.I.I. is therefore true.

There are a number of places at which this argument can be attacked, such as the correctness of Russell's theory of the visual field, but we shall content ourselves with remarks which also apply to the Idealist

position in general.

Thus one can criticise Casullo for failing to distinguish between the statement 'one cannot think of two objects as existing, without a mind which is doing the thinking' and the statement 'one cannot think of two objects as existing apart from, i.e. outside of, a mind'. The first is obviously true, the second may not be and, through the attitude it generates, characterises the difference between the Realist and Idealist positions. A clear distinction must be drawn between what is involved in the process of thinking about, or visualizing a state of affairs, and what is entailed by the state of affairs itself. The fact that the process of *visualization* of two objects involves a difference in their positional qualities is a function of one's thought processes only and does not entitle one to infer that such a difference must actually exist in the external state of affairs (30). Of course it is impossible for me to imagine two spheres without the 'me' which is doing the imagining, but it is possible for me to imagine a situation in which the spheres existed ~~but~~ the 'me' did not, and in which there is no way of ascribing a difference in spatial predicates to them on a relational account of space.

This is a well-known argument against the Idealist account in general (31). Although it is true that thoughts cannot exist without minds thinking them, this does not prove that objects cannot exist separate from or in the absence of such minds. The question of whether such objects 'really' exist may be open but it is not a logical impossibility, as Berkeley claimed. As with thinking so with perception also. Perception cannot occur without a mind doing the perceiving but it is at least logically possible that that which is perceived can.

30. The same mistake is made by the Indian mystic Krishnamurti, who argues that since we cannot think, or conceive, of death, in the sense of absolute non-existence, so true death, in this sense, cannot actually occur.

31. There are others, of course.

It is not our intention here to give a detailed analysis of the Idealist arguments in general, we merely wish to point out that such arguments cannot be conclusive in resolving the dilemma regarding P.I.I.

As we suggested in Chapter 2, space-time individuality could be taken to imply that the points of space-time themselves could possess a form of T.I. Regarding these points as individuals then implies, as they are indiscernible, obviously, that they must also violate P.I.I. However they only violate the strong form as they cannot, by definition possess the same spatial properties! (32) Leibniz himself would have argued that the points of space-time lie beyond the domain of applicability of the Principle as *he* believed that it applied to substances only, (although, as we have seen, space-time could be regarded as substantial in a certain sense, perhaps even in the Leibnizian sense).

Classical particles can also have T.I. attributed to them. In this case it is possible for them to violate both forms of P.I.I. since they are indistinguishable and yet also regarded as distinct, and T.I. says nothing about their spatio-temporal properties. If in addition it is assumed that the I.A. is obeyed then only the strong form is contingently false. However, as we have seen, it ^{maybe} possible for there to exist entities, fields for example, to which T.I. can be attributed but which also violate I.A. In such cases the weak form of the Principle would also be false. Even though it is not entailed by T.I. there are good grounds for assuming that the I.A. is obeyed by classical particles. In the quantal case there is a sense in which it is not and thus both weak and strong P.I.I. ^{maybe} violated, as we shall see, although a case could be made for saying that things which violate the I.A. cannot be regarded as 'particles'.

In both kinds of individuality the particles are labelled, the label designating that which confers

32. The I.A. does not apply to the points of space-time since they obviously cannot move.

individuality - the underlying spatial location or substantial substratum respectively. If these labels, or proper names could be regarded as properties of the particles, in some sense, then the particles could be regarded as distinct through a qualitative difference and both forms of P.I.I. would be obeyed. Of course, if this were so then there would be no need for either T.I. or space-time individuality since the particles would no longer be indistinguishable.

In Chapter 1 we noted that, broadly speaking, there were two views of proper names. The 'sense theory' regards proper names as no more than a sort of disguised definite description whereas the 'no sense' theory holds that proper names simply stand for objects and have no meaning other than standing for objects. The difference between the two is that the latter view holds that naming is prior to describing whereas the former contends that describing is prior to naming for a name only names by describing the object it names. As we have said, although it seems implausible to claim as the sense theory does, that proper names are just shorthand for definite descriptions, this view does at least give an account of identity and existential statements which present problems for the no sense position.

On the sense theory then proper names are only a kind of shorthand for a description of an object. But since a description is simply a list of the object's properties, proper names, on this view, are no more than shorthand summaries of the properties of objects. Therefore a difference in proper names must signify a difference in properties - somewhere on the list there must be a difference. Thus if the proper names used to label the particles are taken to fall under this view then the particles must be discernible and P.I.I. is obeyed.

This line has recently been followed by De Munyck who advocates the view that indices, or labels, should be regarded as intrinsic properties of the particles (3³).

33. W.M. De Mynck (1975) p. 327

He acknowledges that this implies that particles become distinguishable in C.M. and Q.M. , but then argues that this says more about the theories than about the experimental meaning of the labels. The languages of both C.M. and Q.M. are those of distinguishable particles and are therefore inadequate for dealing with particles which are experimentally indistinguishable. Thus it is argued that two particles are indistinguishable if they agree on all their dynamically relevant intrinsic properties and that they can therefore be indistinguishable with respect to one set of experiments but distinguishable with respect to another. Accordingly a label can only serve to distinguish a particle when it is possible to extend the experiments under which the particle is regarded as indistinguishable, in such a way that the index becomes dynamically relevant with respect to some new property.

This is certainly a rather odd way of looking at elementary particles. It is difficult to see how the particle labels introduced into classical statistical mechanics can be regarded as properties on a par with charge or mass. Unlike the latter these labels are not the subject of any theory nor are they invoked to account for the behaviour of the particles, their *raison d'être* being metaphysical: to stand for that which confers individuality upon the particles. If they were to be regarded as properties then they are rather artificial ones being not possessed by the particle itself but imposed somehow from without. They are not essential, in the sense that a particle could not possess them and still be a particle of that kind, and neither are they extrinsic in Jauch's sense of being dependent upon the state of the system. Particle labels could perhaps be regarded as extra-dynamical intrinsic properties of the particles (34) in the same way that the colour of a billiard ball is, (at least as regards collision phenomena). However, this is a false and misleading analogy. The colour of a billiard

34. Sudarshan and Mehra op cit. p.245

ball is a secondary property related, by some body of theory, to the primary properties of the particles of which the ball is composed. By definition these particles cannot themselves possess such secondary properties, and it is difficult to see what meaning could be given to a primary property that was both intrinsic and extra-dynamical. Even the hidden variables of Q.M. have some bearing on the dynamical behaviour of the particles - indeed that is exactly why they are posited. Without this saving grace particle labels fall victim to Ockham's razor and must be excluded from the set of properties possessed by the particles.

Furthermore, De Muynck's approach does not correspond to the way in which particle labels are regarded in C.M. itself. As we have emphasised, classical particles are considered to be indistinguishable individuals, with their individuality designated by the particle labels. If space-time individuality is attributed then the label stands for the spatio-temporal location at which the properties predicated of the particle are instantiated. In the case of T.I. the label denotes the substantial substratum which bears or possesses the properties. This is the important point : whichever form of individuality is attributed to the particles, the labels designate that which underlies the properties not the properties themselves.

Thus the labels or proper names used in C.M. denote something other than the set, or any element of the set, of properties predicated of the particles and are therefore used, or regarded, in a manner consistent with the 'no sense' view. They do not, and cannot, plausibly, generate the qualitative difference necessary to preserve P.I.I. We therefore conclude that classical particles certainly violate the strong form of the principle, and may also render the weak form contingently false if the requirement that they satisfy the I.A. is dropped. This would be rather bizarre, however, and if space-time individuality is attributed or T.I. plus

I.A. then the weak form remains true; classical particles of the same kind do not possess all properties in common, including spatio-temporal ones.

5.2.2. P.I.I. and Quantum Physics

The status of P.I.I. within Q.M. has been the subject of some debate among philosophers of science, particularly recently. We shall examine each of the various arguments in turn, indicating, where we can, their deficiencies, before setting down our own view.

Most authors have argued that the Principle is refuted by Q.M. and the debate has then shifted to the question of the manner in which it fails to hold. Barbour, however, has argued that Q.M. definitely supports P.I.I. (35) a claim which is based on certain passages from the Leibniz-Clarke correspondence such as the following "'Tis a thing indifferent, to place three bodies equal and perfectly alike, in any order whatsoever; and consequently they will never be placed in any order by him who does nothing without wisdom'. (36). According to Barbour this is exactly the situation which pertains to Q.M. where only one out of the six possible permutations of three electrons is actually counted.

This argument contains two mistakes. First in the passage cited Leibniz is arguing, from the Principle of Sufficient Reason, not that there must be only one arrangement of three particles but that there will be no such arrangement. Secondly, Barbour fails to make the distinction between P.I.I. as applied to the arrangements of particles and the Principle as applied to the particle themselves (37). It is clear from the passages given that Leibniz was arguing that one cannot distinguish arrangements differing in the order of the particles i.e. differing as regards a particle permutation. As we have seen, this is false in C.M.

³⁵. J.B. Barbour (1980) p. 34

³⁶. Leibniz in H. Alexander (1956), *Leibniz Fourth Letter, para I-IV*

³⁷. Lucas makes the same mistake. See J. Lucas (1973) para. 25.

but true in Q.M. However, the fact that indistinguishable arrangements are identical in the latter does not imply the same result for particles. Powers has recently made the same point, arguing that the reduction in the number of arrangements in quantum statistics may be taken to support P.I.I., as applied to arrangements obviously, whereas the particles themselves refute it. (38).

Post has contended that quantal particles, being non-individuals, violate P.I.I. for the following reason (39). Such particles are, in his terms 'strongly identical' (40), in the sense of being both indistinguishable and devoid of individuality, and yet there is more than one of them. Thus they are numerically distinct and completely indiscernible and therefore Leibniz's Principle cannot hold.

This argument immediately invites the question, how, exactly are the particles numerically distinct? In what sense can it be said that there is more than one of them? Post's answer is that they differ as regards location, or more generally, bearing in mind the well-known problems concerning the non-localizability of the photon, that they differ with regard to their state. Indeed Post has proposed an ontology of states in which all reference to individual particles is eliminated.

However, this cannot be correct because first of all it is possible for two particles, two photons say, to exist in the same one particle state, and secondly as we have noted, the Superposition Principle can be taken to imply that there is a sense in which each particle partakes of all the states of all the other particles. Post himself contradicts his own answer by writing '... the location of an electron at a given moment is

38. J. Powers (1982) p. 156ff

39. Post op cit p. 19

40. The particles are not identical in our sense of the term since they are regarded as numerically distinct,

as numerous as the number of electrons in the universe'. (41) Clearly, some other principle by which the particles can be said to be numerically, or countably distinct must be sought for. As we shall see this is provided by a consideration of the particles energy and momenta. Thus on the two photon example we conclude that there are two particles in the state, rather than one, because the total energy and momentum are twice that of one particle.

Cortes has initiated a more recent debate by arguing that the logical form of P.I.I. is restricted to individuals and is therefore violated by the 'non-individual' particles of Q.M. which it cannot encompass (42). His argument can be divided into two claims. The first is that the logical proof of Leibniz's Principle presupposes that all entities in the domain of discourse are individuals and that the question regarding the possible existence of non-individuals remains open. All that can be established by an analysis of the way names and variables function is that '... the quantificational logic as normally used is not capable of expressing anything whatsoever about non-individuals'. (43)

Cortes adopts a form of space-time individuality and defines an individual to be a spatio-temporal thing which differs in some respect from other spatio-temporal things (44). Thus particles which do not differ in any respect, including their spatio-temporal properties, are, on these terms, non-individuals (45). Rather confusingly Cortes also defines a non-individual to be anything which violates P.I.I. so that this Principle is limited to the domain of individuals by definition. This allows the possibility that quantal particles do not violate the Principle, but merely fall outside its range

41. Post op cit p. 20

42. A. Cortes (1976) p. 491

43. Ibid p. 499

44. Ibid p. 493

45. See our discussion on p.

of application. In particular it does not apply to waves and, as we have pointed out, the supposed 'non-individuality' of quantal particles can be interpreted as being due to their wave-like aspect. This point, regarding the applicability of P.I.I. was brought up by Teller, as we shall see, and constitutes the essence of our own position.

The second step in Cortes's argument is to demonstrate that non-individual entities actually exist and that P.I.I. is therefore violated in fact. To do this he takes the example of photons¹ in a mirror lined box or laser beam, which being bosons can simultaneously occupy exactly the same state. According to the Copenhagen Interpretation (46) such photons have all physical properties (47) in common and are therefore indistinguishable. Furthermore they are spatio-temporally related to every other thing in the universe in exactly the same way and are therefore non-individuals, in Cortes's terms. Clearly their state cannot determine whether the photons are numerically distinct so Cortes must turn to some other justification for saying that, for example, there are two photons in the box. This is provided by the observation that, in this example, the total energy of frequency ν in the box is $2h\nu$, together with the theory of Q.M. which says that the energy of one photon in such a situation is $h\nu$. Thus we surmise from these additional considerations, that there are two photons present. (48).

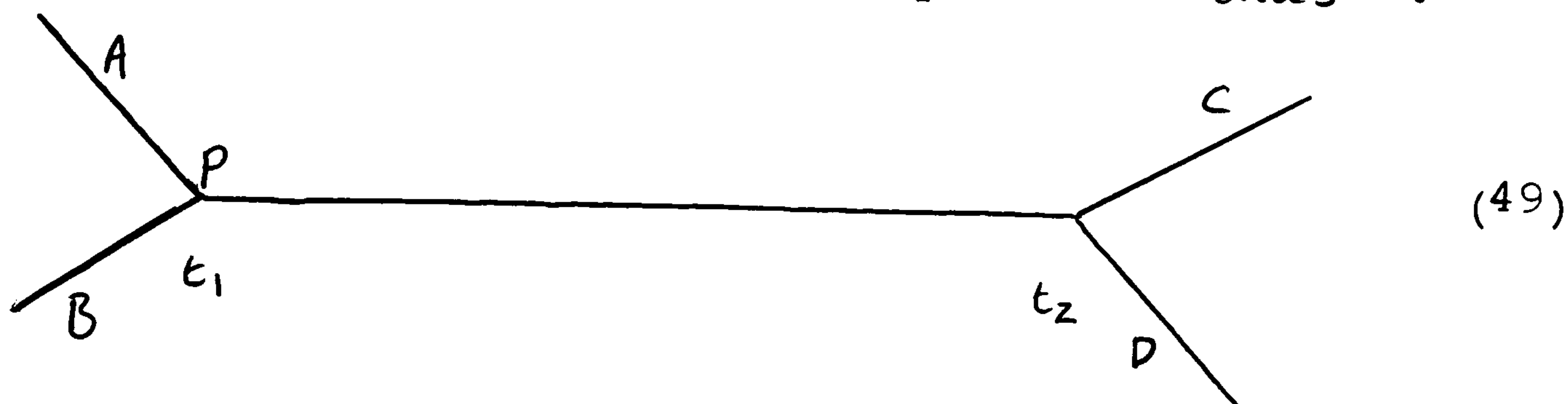
However, it can be argued that two objects possessing all properties in common, even spatio-temporal ones, may be individuated by their distinct spatio-temporal trajectories. To counter this Cortes considers the example of two indistinguishable objects with different historical^c origins but which occupy the same spatial location for a certain time before separating to different

46. This states that the wave function for a given system of particles gives all the physically possible information about that system. i.e. hidden variable theories are excluded.

47. 'Physical' properties are those which lead to observable consequences.

48. Ibid p. 500

historical ends. This can be represented thus :



Cortes's argument is that if their history really does individuate the objects then it must individuate them at P and it should therefore be logically possible to decide whether the object that arrived at C came from A or B or partially from both. This is impossible however, and he concludes that history cannot always serve to individuate objects, in particular it cannot do so for photons. This in fact is a general point about the failure of spatio-temporal trajectories to individuate in a situation where I.A. fails to hold.

Thus Cortes concludes that photons are indistinguishable non-individuals which can be counted as distinct and which therefore violate P.I.I. Further more he shows that a proof of the logical form of P.I.I. suitably modified to take account of such entities, is impossible to carry through successfully.

Barnette attacked the above argument, regarding the inability of spatio-temporal trajectories to individuate when I.A. breaks down, on the grounds that it confuses epistemological arguments with metaphysical ones. (50) The fact that there is no way of telling between t_1 and t_2 which object came from A and which from B does not compel us to give up the metaphysical claim that the predicate 'object which is identical with the object having history A ' is satisfied by one and only one of the two objects for all times between t_1 and t_2 . Given that this predicate is satisfied by one and only one of the objects at all times, it follows that for all times, including t_1 through to t_2 there is some

49. Ibid p. 503

50. R.L. Barnett (1978) p. 466

description which holds for one object and not the other. Therefore P.I.I. is not violated.

However, as Teller has correctly pointed out, it is not the case that in practice we cannot distinguish the two photons, there is in principle nothing which can serve to individuate them. (51) This certainly makes a difference because metaphysical questions may depend on what characteristics are available which could, at least, in principle, serve to individuate the particles. Given this, and the absence of any such characteristics for the photons when they occupy the same state, it is begging the question to say that they are individuated by differences in some historical property.

Ginsberg responded to Barnette's paper by turning to Q.F.T. and the concepts of creation and annihilation of particles to break the spatio-temporal connection between the photons at P and those at A,B,C and D in Cortes's example (52). Thus he characterised this example as follows. The two-photon system can be represented by a two-particle state of the quantised electro-magnetic field with each particle regarded as an excitation of one of the field modes. Such a mode may contain any number of excitations which in particle language means that any number of photons can occupy a field mode simultaneously. Thus a two-particle state of the field can be understood either as one in which one mode has two excitations or as one in which two different modes each have one excitation. This point was subsequently taken up by Teller as we shall see.

Thus before t_1 the field is in a two-particle state with one photon, or field quantum, in mode A and one in mode B. From t_1 to t_2 the field is in a two-particle state with both photons in mode P. After t_2 the field is still in a two-particle state with one photon in mode C and the other in mode D.

In the Q.F.T. description two photons are destroyed at t_1 and two 'new' ones created, these are then destroyed in turn at t_2 and two others created. Given this

51. P. Teller (1983) p.314

52. A. Ginsberg (1981) p. 487

Ginsberg argues that it is false to claim that the predicate 'object identical to the object having history A' is possessed by either one of the two photons existing between t_1 and t_2 since neither of these are identical to any of the photons existing before t_1 or after t_2 . There is also no asymmetry in the relationship between the new and old photons so the former do not differ in their historical connections to other objects.

Thus, granted certain assumptions such as completeness, it can be concluded that the photons existing before t_1 and after t_2 are excitations of different field modes and do not possess all properties in common. The photons existing between t_1 and t_2 are not identical to any of these, are excitations of the same field mode and have all physical properties in common. Thus Ginsberg concludes that these latter photons have identical histories and since they are indistinguishable yet numerically distinct they violate P.I.I. (53)

The first point to be made regarding this argument concerns the creation and annihilation operators which it involves. As we have noted these play a central role in Q.F.T. where a process beginning with a particle in some initial state and ending with a particle of the same kind in some final state, is not regarded as involving one and the same thing undergoing a change in properties, but as the annihilation of the particle in the initial state and the creation of a different particle in the final state. Thus, as Ginsberg noted, this description implies that particles are not identical, in the sense of being the same particle, through changes and therefore, precludes the possibility of reidentification through such changes (54). In Q.F.T. there is no enduring substance underlying the accidents.

This description could be applied to all temporal stages leading to an interpretation of motion in terms of a sequence of creations and annihilations. As we have

53. Ibid p. 491

54. Ibid p. 489

noted this was the view adopted by the Mutakallimun.

On the other hand, as Teller notes, the objects of Q.F.T., separated as they are by the creation and annihilation processes, can be grouped together to compose enduring individuals. Indeed this is exactly what we will have to do if we are to describe a world in which things undergo change. However, we do not have a free hand in this matter, as certain groupings will be ruled out on grounds of inconsistency with theory or objective fact. We shall return to this point shortly.

The second point about Ginsberg's approach concerns the assertion that the two photons in the same state are numerically distinct. They are if the particle interpretation is adopted in which the total energy $2h\nu$ is regarded as being due to two particle-like photons each of energy $h\nu$. However, in the field, or wave, representation the above situation is described in terms of the level of excitation of a field oscillator being raised by 2 units. In this latter case one clearly does not have two distinct indiscernible individuals and P.I.I. remains intact.

This is the line adopted by Teller who takes the general view that we can slice reality as finely or as coarsely as we like, and how sensible such a slicing is depends on the context (55). Thus the entities of Q.M. can be regarded in certain circumstances as waves and in others as particles. In particular Cortes's two-photon example only works because of the wave-like aspect of photon behaviour. Thus Teller writes 'We get a two particle state with both particles in exactly the same state only by virtue of a description of state by a wave-function which is analysable as two one-particle waves superimposed.' (56). But then, he

55. Teller op cit

56. Ibid p. 314

continues, there is nothing to prevent us from saying that we have one two-particle wave in the state, i.e. slicing reality more coarsely. Instead of the two indiscernibles things there is now only one and P.I.I. remains true. Thus he concludes that whether or not Leibniz's Principle is violated depends upon an arbitrary decision as to how the two-particle state should be regarded.

This conclusion is then applied to Ginsberg's quantum field theoretic analysis. As Ginsberg himself noted the creation and annihilation operators can be interpreted either in terms of the creation and annihilation of particles or in terms of increasing or decreasing the level of excitation of the field oscillators by one unit. These units, or field quanta, are particle-like in certain respects, such as their discrete nature, but wave-like in others. Thus, as we have noted, Q.F.T. indulges just as much in wave-particle duality as first quantized Q.M. and the two interpretations above correspond to the two ways of arriving at a quantised field briefly discussed on p. 266.

Therefore, in the Cortes example, between t_1 and t_2 one could say either that there are two photons, or field quanta, in the same state, or that the level of excitation of a field oscillator has been increased by two units. The former case represents a violation of P.I.I. the latter does not. According to Teller it is arbitrary how the state is regarded. There is one total oscillatory state which can be regarded either as having no parts or as being analysable in an infinite number of ways into various components possessing certain individuating characteristics. The analogy he gives is that of two waves travelling in opposite directions. Where they cross we can say that we either have two waves in the same state or one partless wave of amplitude equal to the sum of the amplitudes of the waves before the crossing.

The fact that the level of excitation of the field oscillators can only be changed in discrete units of $h\nu$ does not compel us to adopt the particle interpreta-

tion any more than the fact that a certain bank may accept deposits or withdrawals in units of £1,000 compels us to regard a deposit of £2,000 as two £1,000's. In other words the mere fact that the total energy in the two-photon state is $2h\nu$ does not compel us to say that there are two distinct entities existing in the same state.

Finally, for the photons to violate P.I.I., they cannot be observed as distinct, for the 'obvious' reason that any spatio-temporal difference would serve to distinguish them. This example must therefore be an unobserved state, and since such states enter the theory as superpositions of an infinite number of other states, one is faced with the question of whether this state actually occurs.

Thus Teller's overall conclusion is that the question of whether Q.M. refutes P.I.I. is one whose answer depends upon an arbitrary decision as to how certain states of the theory are to be regarded.

Although this approach is, we believe, broadly correct, it contains several confused or dubious points. Firstly, Teller's analogy between the two photon example and two classical waves on a rope should not be pushed too far. As he indicates, but fails to make explicit, we get a two-particle state with both particles in the same state only by virtue of the Superposition Principle which requires us to regard any state as the superposition of two, or more, other states and conversely any two, or more, states may be superposed to give a new state. The fact that a similar principle holds for classical wave systems invites an obvious analogy but the difficulties involved in regarding the wave-function in a classical manner are well known (58) and as Dirac has said 'The analogies are ... liable to be misleading'. (59) It is the Superposition Principle which should be considered in any discussion of Cortes's example, rather

58. See Jammer op cit p. 33 and p. 43-44

59. Dirac (1978) p. 14

than the somewhat confusing notion of wave-like behaviour, and as we shall see when we come to our own account, this allows us to generalize the discussion to include fermions.

Secondly, any metaphysical slicing of reality should be undertaken with caution. Teller himself has noted Kripke's argument that the answers to the question of which groups of temporal parts into larger individuals are sensible may be objectively determined and are not just a matter of convention (60). Nevertheless Teller appears to believe in the doctrine of temporal parts which holds that for every period of time that an individual thing exists there is a temporal part of that thing, which is unique to that period of time. Thus he writes '... we can string together distinct parts in various ways to make wholes. This holds for temporal parts of objects conventionally conceived. We can consider temporal stages as independently existing things which can be grouped together in larger individuals in various ways, which ones counting as sensible depending on the circumstances'. (61)

This doctrine has been attacked by Chisholm on the grounds that it is not adequate to the experiences we have of the unity of certain things, such as ourselves, that it multiplies things beyond necessity and thus violates requirements of economy and simplicity and finally that the case for the doctrine is based on a false analogy between space and time (62). This analogy states that just as an object that is extended through space at a given time has, for each portion of space that it occupies, a spatial part that is unique to that portion of space at that time, so any object that persists through a period of time has, for each sub-period of time that it exists, a temporal part that is unique to that sub-period of time. However, Chisholm argues, we cannot say about time everything we

60. Teller, preprint of (1983) p.7

61. Ibid p.7.

62. R.M. Chisholm in M.K. Munitz (1971) p. 14-15

can say about space because although one and the same thing can be at two different times in one and the same place, one and the same thing cannot be in two different places at one and the same time.

Chisholm is wrong on this last point. The disanalogy is not between space and time considered in themselves, but exists only as regards the relations between objects in space and time since the Superposition Principle implies that there is a sense in which it can be said that quantal particles can be in two different places at the same time. Nevertheless the general thrust of Chisholm's argument is sound; the ill-considered slicing of reality and unrestrained grouping of temporal parts can lead to bizarre results. (63)

Teller also fails to emphasise that it is not just the discrete nature of their energy which leads us to regard the field quanta as particle like; they also possess momentum and satisfy the dispersion law characteristic of a particle of zero rest mass. His analogy with integral financial transitions is therefore misleading and fails to support the view that the particle and field interpretations can be given equal weight, at least formally.

Finally, as regards the argument concerning unobserved states, such states are required to violate the weak form of P.I.I. only; the strong form can, of course, still be refuted if there is a spatio-temporal difference between the photons.

This brings us on to our own view in which we argue that the question as to whether weak P.I.I. is violated or not by Q.M. is dependent upon the way in which 'mixed states' are interpreted in the theory.

63. Thus the doctrine of temporal parts has been used to defend the theological doctrine of original sin by arguing that God could regard you, I or anyone else together with Adam as merely temporal parts of one 'moral individual' and could therefore punish you, I or anyone else for the sins committed by Adam. Ibid p. 13.

First of all quantal particles, like their counterparts in C.M. clearly violate the strong form of the Principle since, at the very least, they possess all intrinsic properties in common and are therefore indistinguishable, in the weak sense, yet countably distinct.

As regards weak P.I.I. we can say the following. We noted in section 3.3 of Chapter 3 that the general ket for an assembly of particles can be interpreted as corresponding to a state for the assembly for which one cannot say that each particle is in its own state but only that it is partly in several states. In other words each particle 'partakes' of all the states of all the other particles, and this result is true not only for bosons, such as photons, but for fermions also.

Thus, for example, we have noted that two bosons or two fermions distributed over two distinct one-particle states partake with equal probabilistic weight in the superposition demanded by symmetrization or anti-symmetrization. In the boson case the wave function for the assembly is

$$|\Psi_S\rangle = \frac{1}{\sqrt{2}} \{ |a'_1\rangle |a'_2\rangle + |a'_2\rangle |a'_1\rangle \}$$

and the statistical operators for particles 1 and 2 respectively are

$$\rho_1 = \frac{1}{2} \Gamma_{a'_1} + \frac{1}{2} \Gamma_{a'_2} \quad ; \quad \rho_2 = \frac{1}{2} \Gamma_{a'_1} + \frac{1}{2} \Gamma_{a'_2}$$

With two fermions we have

$$|\Psi_A\rangle = \frac{1}{\sqrt{2}} \{ |a'_1\rangle |a'_2\rangle - |a'_2\rangle |a'_1\rangle \}$$

and statistical operators,

$$\rho_1 = \frac{1}{2} \Gamma_{a'_1} + \frac{1}{2} \Gamma_{a'_2} \quad ; \quad \rho_2 = \frac{1}{2} \Gamma_{a'_1} + \frac{1}{2} \Gamma_{a'_2}$$

In each case the two particles, bosons or fermions, respectively, have the same statistical operator. Thus as we also noted in section 3.3, they possess the same one-particle probability distribution. In the fermion case, as we have seen, this is given by

$$\omega^{(1)}(1) = \frac{1}{2} \{ |a'_1|^2 + |a'_2|^2 \}$$

for particle 1 and

$$\omega^{(1)}(z) = \frac{1}{2} \{ |a_2^z|^2 + |a_2^1|^2 \}$$

for particle 2. In other words the observable distribution of the two particles is identical and hence they cannot be distinguished by any one particle measurement. (64) For example, the probability of finding particle 1 at position $\underline{1}$, and particle 2 at position $\underline{2}$, is the same as the probability of finding 1 at $\underline{2}$ and 2 at $\underline{1}$. This is true also for other observables such as energy and momentum and thus one can conclude that the two particles cannot be distinguished by a measurement of their one-particle properties (65).

It is in this sense that two particles, bosons or fermions, distributed over two one-particle states might be said to be in the 'same' state, namely in the sense that they possess the same statistical operators, the same one-particle probability distribution and thus cannot be distinguished by measurements made on one particle only. If this is accepted then, as we noted in Chapter 3, the Impenetrability Assumption cannot be taken to hold for quantal particles and since the validity of weak P.I.I. rests on this assumption this cannot be taken to hold either.

If this were correct then the argument between Cortes et al over the two-photon example would be redundant since the particles are according to the above account, 'in' each others states before

64. Such measurements involve operators of the form $Q_i \otimes I$ on the product space $\mathcal{H}_1 \otimes \mathcal{H}_2$, where \mathcal{H}_1 and \mathcal{H}_2 are the one particle Hilbert spaces, Q_i is the operator pertaining to one particle whose states are described by vectors in one Hilbert space and I is the unit vector in the other Hilbert space.

65. See H. Margenau (1950) p. 441

t_1 (66). It could then be concluded that P.I.I. is violated by the photons, not only between t_1 and t_2 but also prior to t_1 and after t_2 , in fact at all times. Thus the situation depicted by Cortes would be regarded as both unrealizable and unnecessary.

On the other hand, however, it has been argued that Q.M. supports P.I.I. at least for fermions, since the Pauli Exclusion Principle implies that no two fermions can exist in the same one-particle state, or a quantum state in which two fermions have identical sets of one-particle quantum numbers cannot exist (67). In other words for two fermions in the same one-particle state the two particle probability distribution is zero i.e. $\omega^{(2)}(1,1) = 0$ since the anti-symmetric wave-function vanishes when both fermions are in the same one-particle state. Thus the probability of finding particle 1 and particle 2 at the same position, Γ_1 say is zero.

Clearly the Exclusion Principle is here regarded as a generalisation of I.A. for Q.M. and it can therefore be concluded that weak P.I.I. is not violated for fermions precisely because the results above are interpreted as saying that fermions cannot possess all properties, in particular spatio-temporal ones, in common. A consideration of the statistical correlation (68) between fermions thus appears to lead to the conclusion that weak P.I.I. is, in this sense, supported by Q.M.

66. One can also question the attribution of distinct, well defined trajectories to the photons, given their supposed 'non-localisability'. The difficulties of constructing a position operator in the first quantised theory which satisfies the Newton-Wigner-Wightman conditions are well known. In the second quantised theory localisation involves the creation of additional particles and so all particles, not just photons, are in this sense 'non-localisable'. See M. Redhead (1977) p. 72 for references and further discussion.

67. Y. Shadmi op cit

68. The operators then relate to two-particle observables and are of the form $Q_1 \otimes R_2$ i.e. the tensor product of one particle observables pertaining to each particle.

Thus, on the one hand, a consideration of the one particle distribution seems to imply that weak P.I.I. is violated, whereas on the other, a consideration of the two particle distribution suggests that it is not, that it is in fact supported by Q.M. This apparent conflict is in fact a result of a play, of sorts, on the word 'state' (69) and it can be resolved as we saw in the last chapter, through an examination of the nature of the 'states' referred to in each account.

We noted that the statistical operators ρ_1 and ρ_2 in either case, represent in fact 'mixed states' rather than pure states and the first argument only works if the former are regarded as 'states' on a par with the latter. However ρ_1 and ρ_2 cannot be regarded as state ascriptions because they represent 'improper mixtures' rather than pure states (70) and so one cannot attribute pure states to each individual particle in this case. Therefore, the question of the particles being somehow in 'the same ^{state}' simply does not arise.

It is important to realise that the whole crux of the matter lies in the fact that one cannot distinguish pure states and mixed states by means of observations made on one of the particles alone. The expectation value for any observable O on the pure state contains, in general, 'interference' or 'cross product' terms which do not occur in the expression for the expectation value of O in the mixed state. However these terms vanish when O is of the form $Q_i \otimes I$,

where Q_i is any
observable ^(70a) referring to one particle only. Observables

69. Thus D'Espagnat has discussed some aspects of this 'play on words' noting that for some authors a statistical operator describes a state whereas for others it describes at best a mixture of states.
D'Espagnat op cit p. 100

70. Ibid p. 61, ρ_1 and ρ_2 are not idempotent i.e. $\rho^2 \neq \rho$

70a. The discussion here does not assume that the two particles are indistinguishable, so the class of observables is not restricted to $\sqrt{}$ symmetric functions of the particle labels.

such as this, which are unable to distinguish pure and mixed states are called 'insensitive'. Furthermore, it can be shown that the expectation value of $Q_i \otimes I$ when the assembly is in a mixed state is equal to the expectation value of Q_i in the state represented by ρ_1 or ρ_2 . In other words pure states behave like mixtures with respect to measurements performed on one particle only (71).

Therefore in order to distinguish between pure states and mixed states one must perform a correlation measurement between observables relating to particle 1 and observables relating to particle 2 i.e. one must measure the expectation value of an observable of the form $Q_1 \otimes R_2$ (72). The joint position measurements in the two fermion system, relating to the probability of finding both fermions at the same position, whose results are encapsulated in the Exclusion Principle, are precisely measurements of this sort. As we have noted, it is through a consideration of the statistical correlation exhibited by two fermions that the conclusion is drawn that P.I.I. is supported by Q.M.

However no such conclusion can in fact be drawn since the relational properties involved in these correlations between the particles lie outside the domain of applicability of P.I.I. In other words, the fact that no two fermions can exist in the same one-particle state expresses a relational property of the particles, to the effect that if one fermion is in a particular one-particle state all other fermions are excluded from that state. We noted in Chapter 1 that Leibniz attempted to reduce all relations, in particular position, to predicates of the individual and it is these latter properties, such as the one-particle properties discussed above, to which P.I.I. applies. It simply cannot encompass entities or collections of entities which manifest the irreducibly relational properties

71. Note the comment in Jammer op cit p. 479 fn 14

72. For such observables to be 'sensitive' in this sense neither Q nor R must act on an eigenstate.

considered here, and so the question ^{regarding} its validity or otherwise cannot be well formed.

Thus our overall conclusion regarding the status of P.I.I. in Q.M., given the existence of these 'entangled states', is as follows.

First of all, although the particles have the same statistical operators, with regard to such states, and the same one-particle probability distribution, the question as to whether they are in 'the same' state, thereby violating P.I.I., is not a meaningful one to ask since the particles cannot be attributed with pure states. This is not true, of course, for states of the assembly which are simple products of one-particle states. In the case of two bosons in the same one-particle state, for example, the wave function is given by $|\Psi\rangle = |a'_1\rangle |a'_2\rangle$ and one can attribute definite states to the individual particles. However the statistical ^{el} operators representing the entangled states given ^{by} for example,

$$|\Psi_S\rangle \text{ and } |\Psi_A\rangle$$

represent 'improper mixtures' and so one cannot associate pure states with the individual particles when in such entangled states.

Secondly, if we consider the two-particle probability distribution and those measurements relating to correlations between the particles then we are still not entitled to come to a definite conclusion regarding the validity of P.I.I. because such considerations involve the sorts of relational properties which are outside the scope of the Principle. These properties may be called 'inherently relational' since they cannot be reduced to non-relational properties (73)

73. This can be expressed in the notion of 'supervenience' in the sense that inherently relational properties do not supervene upon the non-relational properties of the relata. The view that there are such properties has been called the 'doctrine of relational holism' by Paul Teller, who recently argued, at the 1984 Annual Conference of the British Society for the Philosophy of Science, that Q.M. incorporates this doctrine in an all pervasive way. It should be noted that if such inherently relational properties were admitted in the discussion of P.I.I., then it would support a violation of the principle since each particle enters symmetrically into the relation.

and the fact that they are incorporated in Q.M. is a consequence of the theory's non-classical holism, as exemplified by the 'entangled states'.

Therefore the question as to whether weak P.I.I. is valid or not is simply obviated by Q.M., in the sense that one cannot say that it is either true or false since it does not apply to those particles which are described by the theory.

As we discussed in Chapter 3 one interpretation of the non-classical statistical correlations which quantal particles display is that they should be regarded as 'non-individuals', in the sense that they are part of some 'grand cosmological' collective (74). Clearly P.I.I. is inapplicable in this case since it does not apply to parts of wholes. Indeed one could now try to argue again that Q.M. supports P.I.I. since the indistinguishable particles have become identical in the sense of being parts of some global 'thing'. This is exactly the sort of reasoning which underpins Teller's argument regarding the wave-like behaviour of particles and its implication of 'partless wholes' which are not counter examples of P.I.I.

One could in fact go even further and employ the Leibnizian argument that as each particle includes among its set of predicates all the states of all the other particles of the same kind, then from the viewpoint of the Complete Notion of an Individual, there is simply one global individual - the collective of all particles of the same kind, and the set of predicates of this individual, contained in its complete notion, includes the states of all the particles.

However this argument does not take account of the fact that the energy and momentum associated with the assembly are integer multiples of the energy and momentum of one particle. Thus to take the example of two particles in a box, we can say that there are

two particles in the box because the total energy and momentum are twice what they would be if there were only one particle in the box. It is on this basis that we say that the particles are countably distinct and yet 'entangled' and thus P.I.I. is not applicable. (75)

There are two further points to note. Firstly in the case of paraparticles the statistical weights are not the same so they do not partake equally of each others states. However, in this case the restriction to symmetric observables, embodied in the I.P., implies that there is no way of telling which particle is in which state.

Secondly our account above does not hold if a hidden variable theory is introduced since in such cases the particles may be regarded as having definite properties of their own, and thus as being in definite states, even when the assembly is in an entangled state (76). However, we shall not consider such possibilities here (77).

Thus, to repeat our conclusion, Q.M. effectively sidesteps the question of the validity of P.I.I. since, as a consequence of their holistic behaviour, quantal particles lie outside the Principle's domain of applicability. In this respect we are in broad agreement with Teller and Cortes although we believe that our account illuminates more clearly the underlying reasons for this conclusion.

4.3 Essentialism and Natural Kinds

We have, at several places in this thesis, referred to particles 'of the same kind' without fully explaining what we mean by this phrase. It is our intention in this section to attempt just such an

75. cf Margenau op cit p. 441

76. D'Espagnat op cit p. 79

77. Ibid Ch11 and Jammer op cit Ch. 7

explication and to examine briefly the doctrine of essentialism as it applies to elementary particles.

The central idea of recent essentialist accounts (78) is that not all of the members of the set of properties predicated of an object are logically on a par with respect to that object's identity, but that there exists a subset of properties which are necessary to the object's identity in the sense that if any member of the subset were absent the object would not be an object of that kind. The properties in this subset are called essential properties and their possession uniquely characterises a particular natural kind. Those properties predicated of the object which are not members of this subset can be called 'accidental' properties.

Such accounts find a particularly useful role in characterising the conditions necessary for the re-identification of an object through time. Thus Hirsch has argued that '... the core idea of the identity of an articulated object is simply the idea of a continuous space-time path that is traced under an articulative E term'. (79), where 'E term broadly means 'essential term'. Likewise Brody has recently said that a necessary condition for an object o existent at t' to be identical with the object o existent at t is that '... o must have retained some of its properties that it had at t , the ones that o cannot lose without going out of existence' (80). In other words for two objects at different times to be identical, in the sense of being actually the same object, the object at the later time must possess the same subset of essential properties as the object at the earlier time. This distinction between essential and accidental properties then allows us to distinguish between substantial and accidental change. The former

78. P.T. Geach (1962) p. 44 and Ch.2 in general; D. Wiggins (1967) op cit; E. Hirsch in Munitz (1971)p.31

79. Hirsch op cit p. 49

80. B. Brody (1980) p. 83

occurs through the loss of essential properties, the latter through the loss of accidental ones.

This doctrine and its associated notion of natural kinds, receives a great deal of plausibility from the classification of elementary particles into different species characterised by differences in the values of a certain subset of properties (81), which we have, following Jauch, called 'intrinsic', such as charge, rest-mass, spin etc. Indeed it may be that these are the only secure examples of natural kinds as we shall see.

So, a typical, and plausible, instance of a natural kind is that of 'being an electron'. Bearing in mind Putnam's distinction between the ontology of natural kinds and the semantics of natural kind words (82), an account of natural kinds can be given in terms of their properties thus: It is assumed that out of all the possible properties which can be predicated of an electron, there exists a certain subset which makes an electron an 'electron'. Then if these are identified, in conjunction, the predicate 'electron' can be applied. These, intrinsic, properties are, or will be, comparatively few in number and are fixed by the relevant physical theory pertaining to the electron. This last point is an important one to which we shall return shortly.

Thus the essential properties of a particle are those which we have, up until now, been calling its 'intrinsic' properties, and this subset corresponds to the Lockean real essence of the natural kind concerned.

In the case of elementary particles the real and nominal essences are identical, although this may not, of course, be true for other putative examples of natural kinds. Gold for example possesses a number of

81. P.E. Hodgson (1980)

81. H. Putnam in N.E. Kiefer and M.K. Munitz (1970)

properties, such as colour, ductility etc, which indicate to an 'expert' whether or not the object they are observing is gold. These properties constitute the nominal essence of gold. However, underlying these there are others, concerned with the atomic structure of the element which correspond to its real essence. The reductive programme in science then ensures that the number of members of the subset corresponding to the real essence is less than the number in the subset corresponding to the nominal essence. Thus the nominal essence 'springs from' the real essence. As we shall now see it is the contraction of their nominal to their real essences which allows elementary particles to be regarded as perhaps the only true examples of natural kinds.

It has been argued that the members of the subset of 'essential' properties are no more important or necessary than any other in the general set of properties which can be predicated of an object. Thus Mellor has written, 'So-called 'essential' properties are ... really no more essential than any other shared properties of a kind. They are just properties ascribed by the primitive predicates in a comprehensive deductive theory of the kind'. (83) If, as Mellor, claims, the reductive programme in science required deducibility then the doctrine of natural kinds is caught on the horns of a dilemma. If deducibility does not hold then some of the macroscopic properties of a kind are not deducible from its microscopic structure and reference to things of that kind is still required. But then, of course, the whole reductive programme, taken to be characteristic of modern science collapses. If, on the other hand, such properties are deducible then they occur in any possible world that the microscopic properties occur in. Therefore, if the microscopic properties are 'essential' in this sense of being true of the object in all possible worlds then so too are

all the macroscopic properties which they explain and the term 'essential' loses its force. In other words Mellor's argument is that the deducibility required by the reductive programme carries the essential nature of the properties from the subset to the set as a whole.

This analysis is correct in so far as the properties constituting the nominal essence are deducible from those constituting the real essence. However, the former set is not identical *with* the set composed of all possible properties which can be predicated of the object. The size and shape, for example, cannot be deduced from the microscopic properties comprising the real essence alone -the interaction of the object with others may have to be considered for example. There may also be microscopic properties which do not belong to the latter subset. In other words there are properties, both macro- and microscopic which are not included in the subsets comprising the nominal and real essences respectively, and which are not necessary for the object to be a thing of that kind. We are thus entitled to regard the properties in these subsets as essential. Of course Mellor is correct in saying that the nominal essence properties are deducible from those constituting the real essence and that therefore both sets should be termed essential. This is exactly what an essentialist does, with the addition that the two sets are distinguished by the prefixes 'nominal' and 'real'.

Furthermore, it is not clear whether Mellor's attack carries any weight in cases where the nominal and real essences are the same and there is therefore no chain of deducibility running between them. Examples of these sorts of natural kinds are elementary particles whose intrinsic properties constitute the real essence but which do not appear to have nominal essences in the way gold, for example, does.

Elementary particles can be regarded as fundamental, if the reductive programme in elementary particle

physics is accepted (84) in the sense of lying on the bottom rung of some explanatory reductive ladder. They can be thought of as 'irreducible' in the theory, or conjunction of theories, in terms of which the explanation is expressed. Of course the irreducible nature of these fundamental particles is not absolute, since it may be revealed that a supposedly fundamental particle is not fundamental at all but composed of some other, more basic, particles. A good example is the proton which was for many years regarded as fundamental but is now considered to be composed of three quarks. In this case the chain of deducibility is restored and the proton's real essence becomes the nominal essence of the quarks to be explained or deduced in terms of their intrinsic or essential properties. Thus the reductive programme can create new natural kinds and what counts as a natural kind is contingent upon the state of physics at the time. The absence of any guarantee that our present theories are complete or 'true' implies that the notion of a natural kind can only, at best, be a relative notion, relative, that is, to a given physical theory.

This idea is strongly rejected by Mellor, although with no good reason so far as I can see (85). Fales, however, has argued that '... what natural kinds we believe there to be at any time will be relative to whatever most fundamental theory is at that time believed to be true; and what natural kinds there are is determined by whatever such theory actually is true'. (86) In order to generate a natural kind classification which avoids the Lockean problems of border-

84. There is of course the 'bootstrap approach' to elementary particle theory, although this has decreased in popularity with the advent and development of quark theory. See Redhead (1980) p.298.

85. Mellor op cit p. 311

86. E. Fales (1979), p. 350.

line cases and gradual substantial change we must look to those theories lying at the lower levels of some reductive hierarchy. The clearest examples of these are the elementary particle theories of physics and, as Fales shows, such theories do indeed determine a problem free classification for the entities in their domain. They do this precisely in the way we have indicated above, by distinguishing, through their laws and symmetry principles, between the intrinsic and extrinsic properties of the particles and constructing a natural kind taxonomy based around the former. Thus Fales contends that at the most fundamental level elementary particles are classified into types '... in just such a way as to precisely reflect the systematically discriminable interaction-governing properties which they display'. (87)

A commitment to a particular fundamental theory therefore implies a commitment to certain restrictions which determine how the fundamental entities belonging to the domain of the theory may be classified. It is the theories of physics which provide the sense in which it can be said that particular fundamental entities possess essential properties by distinguishing between the necessary and contingent truths about these entities. Thus '... putative natural-kind classifications are parasitic upon putatively true theories and are, like the theories themselves, subject to revision under pressure from empirical information about the world '. (88)

However, although certain of the possible natural kind classifications of the fundamental entities of a particular theory may be seen to be infelicitous when viewed against the background of the theory, this does not mean that the theory uniquely determines which classification is applicable. For example, certain gauge theories in elementary particle physics apparently

87. Ibid p. 364

88. Ibid p. 350

'unify' different entities by reducing them to one, more fundamental entity, thus reducing two, or more natural kinds to one. Different kinds of particles are then regarded as merely different states of one kind of more basic particle these states transforming into one another under the gauge transformations. A good example of this procedure is the way in which neutrons and protons are regarded as merely two states of a single particle of a different kind, called the nucleon, with these states transforming under isospin rotations. However, as Redhead has pointed out, although this is a convenient way of talking, it is not necessarily a reduction in our ontology: 'All we can say is that if the gauge symmetry were exact then physics could not decide for us which particle was a proton and which a neutron, but this is not the same thing as saying that neutrons are protons, nor does it compel us to the view that neutrons and protons are just two states of a single entity (the nucleon).' (89) One can conclude from this that arguments based on gauge symmetries do not help to settle the question of whether there are many natural kinds of elementary particles.

Another example is that of the coloured and para-quark descriptions of hadron behaviour as we noted in *Chapter 3*. These descriptions may be regarded as 'roughly' (90) equivalent in the sense that in certain cases, depending on the choice of observables in the latter, they have the same states, but they differ as we have noted on p. 258 as regards the natural kind classification they suggest. These examples support the general conclusion that although a theory may impose certain constraints upon the classification into natural kinds of the entities in its domain, and although this classification may have to be revised in the light of changes to the theory, the relationship between the two is not such that the theory uniquely determines the classification - metaphysical factors may

89. Redhead (1983) p. 31

90. Greenberg and Nelson op cit p. 79

also play a role in deciding which particular classification is to be adopted.

4.4 The Undetermination of Theories

We have emphasised that although there is an ontological difference between the particle and field representations of elementary particles there is no observable difference between them in the sense that no experiment could ever decide which is correct. This is therefore an example of the underdetermination of theories by empirical data. By this we mean that two theories may both be compatible with all actual and possible observations and yet be incompatible with one another as regards their ontological foundations. In such a situation the question as to which of the theories is 'true' is empirically undecidable in the sense that there is no difference in the distribution of truth values over those propositions of the theories deemed observational.

Quine has put forward the strong thesis that all theories are underdetermined by the data (91) whereas Newton-Smith argues in favour of the weaker position that there can be cases of underdetermination (92). The response to this situation can swing between two extremes: the Ignorance response concludes that there are facts which are inaccessible and can simply never be decided by observation. The Arrogance response on the other hand, holds that if we cannot know about something then there is nothing to know about, i.e. there is no matter of fact at stake. Adherence to the latter implies a restriction of the Law of the Excluded Middle to exclude empirically undecidable propositions (93). Given the number of cases of underdetermination this could be a rather severe constraint.

91. W.O. Quine (1970) p. 179

92. Newton Smith W. (1980) p. 68-73 and p. 231-242

93. Ibid p. 232

Furthermore adopting the Arrogance response has the consequence that empirically undecidable propositions cannot be regarded as expressing hypotheses about the facts. Some account of the role these propositions play in the theories containing them is then called for.

Unless such an extreme positivist approach is taken, there are, of course, other factors, of a metaphysical or heuristic nature, which may be relevant in deciding between observationally indistinguishable theories. Thus the particle approach to quantum physics involves an ontological commitment to T.I. and any philosophical arguments against this notion may serve to adjudicate against this approach in favour of the field representation. Two theories may also differ as regards their heuristic value (94). Thus as we have noted, the coloured quark description, in leading to the development of quantum chromodynamics was more heuristically fruitful than the formally equivalent para-quark model. Redhead has emphasised the fruitfulness of the field as compared to the particle, representation in Q.F.T. (95). Other factors, like simplicity, may also play some role, although the difficulties involved in producing effective criteria of simplicity and economy are well known.

We can give several other examples of underdetermination, taken from the history of physics. Thus the electrical theories of Maxwell and the Continental Physicists, although involving different formalisms and representations of the phenomena, were completely equivalent empirically. As Hesse has remarked 'At any given point of development it was generally possible to say that the empirical facts were explained as well by this continental action-at-distance theory as by the British fields'.(96) . However, the two theories were not equivalent heuristically or as regards the ease with which they expressed certain phenomena; the field approach, involving

94. Post has emphasised the role of heuristic guidelines in the construction of theories ^{and} as regards inter-theory relations H.R. Post (1971) p. 213

95. M. Redhead (1983)

96. M. Hesse in E. McMullin (1978) p. 129

continuous action, was obviously more suited to the representation of continuous phenomena, rather than discrete which found easier expression in the continental theory. Thus '... the two theories were not equivalent for all scientific purposes. Theories in science are sometimes rejected, not on grounds of inconsistency with empirical facts, but on the grounds of complexity, incoherence and lack of predictive power' (97)

Another example of a pair of metaphysically incompatible yet empirically equivalent theories are the classical atomist and Boscovichian theories of matter. In simple terms, the former regarded matter as ultimately consisting of impenetrable corpuscles. This was rejected by Boscovich who adopted the view that matter was ultimately composed of point centres of force and should be regarded as active rather than inert. (98) By a suitable adjustment of their periodic properties these centres of force could be mathematically defined so as to produce exactly the same physical effects as the extended impenetrable atoms. The difference between these two approaches was therefore not one which could be decided by observation alone. As Faraday realised (99), it was all a matter of choosing a particular ontology which would satisfy the demands of consistency and also provide the most effective heuristic aid to the future development of the subject.

As Newton-Smith has pointed out (100) such cases of underdetermination clearly present problems to a realist who is going to want to say that theories are either true or false as determined by virtue of how the world 'really' is, that if the theory is true then its theoretical terms are causally connected to certain observable (in the widest sense) phenomena which constitute the evidence of the theory, and, finally that we can 'know' whether it is true or not. Thus a

97. Ibid p. 129

98. R.J. Boscovich (1763) see P.M. Harman (1982)p.82

99. M. Faraday (1844) p. 136

100. Newton-Smith op cit p. 230

realist is led to hold that there is something in virtue of which either the particle or the field approach is true. However, there is no empirical (the reason for our emphasis will become apparent shortly) evidence for thinking that either description is more likely to be true than the other, which is incompatible with the last component in the characterisation of the realist position. To resolve this dilemma the realist must, according to Newton-Smith, either weaken this component and adopt the Ignorance response stated above, or weaken the first component and adopt the Arrogance response. The former maintains that those propositions responsible for underdetermination are either true or false but we cannot know which they are. The latter response involves dropping the assumption that there are things about the world in virtue of which empirically undecidable propositions are either true or false. As we have noted this may involve restricting ^{the} scope of the Law of Excluded Middle to exclude such propositions, and it can conceivably be asked of the ignorant realist what are the grounds for asserting that there exist the inaccessible facts required by this approach.

However, these undesirable consequences may follow only because Newton-Smith is being unduly restrictive in claiming that '... nothing is going to count in favour of ...' (101) one or other of the pair of empirically equivalent theories. As we have seen there are considerations, other than empirical ones, which may promote one theory over another. Thus the realist may weaken or stretch the first component above by requiring that a theory should be approximately true in the sense that its empirical, theoretical, and metaphysical claims represent, with a certain degree of faithfulness, the way the world really is. A theory may then be true in certain respects but false in

101. Ibid p. 232

others. Thus a realist might regard a theory which involves a commitment to a philosophical position which he finds untenable as false or at least a worse approximation to the truth than one whose metaphysical commitments he agrees with. Of course, this may be as much a relative appraisal as those based on simplicity criteria are.

A claim which is less dubious and open to criticism would be to say that a theory which is more heuristically fruitful than another represents a closer approximation to the truth. Thus, for example, it seems plausible to claim that the coloured quark theory is 'more true', or a closer approximation to the truth, than the paraquark theory because, unlike the latter it generated interesting consequences and novel predictions which were subsequently confirmed experimentally. A similar claim might be made for the field, as compared to the particle, approach in quantum physics. Thus by retreating from the requirement of truth to that of approximate truth a realist can introduce non-empirical considerations which may tip the balance one way or the other in the weighing up of two underdetermined theories.

As well as playing an important role in theory appraisal metaphysics also has a crucial role to play in the generation of new theories and the advancement of science in general, as Watkins has emphasised. This role is '... to prepare the way for the most important scientific advances of all'. (102) Thus he argued that if one theory is to effect a 'revolutionary' reduction of another then their metaphysical components must conflict. Harman has also emphasised the role of metaphysics in the articulation of theories, through a detailed analysis of historical case studies (103).

Our conclusion thus gives support to this 'counter revolution' against logical empiricism, which argues that metaphysical considerations can and do play a

102. J.W.N. Watkins (1975) p. 115

103. Harman op cit.

fundamental role in the articulation, appraisal and advancement of scientific theories.

4.5 Conclusion

The primary concern of this thesis has been the inter-relationship between physics and that part of philosophy, or perhaps more properly, metaphysics, which deals with the individuality of things in the world of external perceptible phenomena. We have argued that either one of two broad views of individuality, which we have labelled 'transcendental' and 'space-time' individuality, can be consistently maintained in both classical and quantum physics, and that this corresponds to the observation that either of two interpretations, particle or field respectively, can be placed on the formalisms of these theories. Thus any conflict between philosophical positions involving, as fundamental principles, these two views, cannot be resolved by appealing to physics. For example, someone who advocates the 'bundle theory' of things, with its reliance upon either P.I.I. or space-time individuality, cannot legitimately claim support from quantum theory on the grounds that it disallows the view of particles as 'transcendental', or 'substantial', individuals, because as we have seen, this view can be maintained within Q.M. so long as the notion of 'inaccessible' states is also accepted.

Conversely a physicist cannot claim support for a particular interpretation of the formalism of some theory on the grounds that the rival interpretation lacks any coherent principle of individuality for the entities in its domain, since some such principle is possible for either interpretation. However, this symmetry regarding the reciprocal roles of physics and philosophy - or rather the lack of roles, at least as far as the above discussion is concerned - is broken when it comes to adopting a particular physical interpretation on

grounds other than the above. Thus given the noted underdetermination, in both classical and quantum physics, of the field and particle interpretations by the data, a consideration of empirically testable consequences obviously cannot be used to support a choice of one interpretation over the other. Other considerations will then come into play such as metaphysical connections giving ontological reference to certain terms in the theory and the, related, notion of heuristic fruitfulness. In so far as the former necessarily involves the adoption of a particular philosophical viewpoint, for example regarding individuality, any criticism of or arguments against this viewpoint on philosophical grounds, will serve to mitigate against the interpretation which it is entailed by. Thus, for example, it can be argued that a coherent particle interpretation can only be maintained within quantum physics if the 'transcendental individuality' view is adopted. However as we have indicated, there are strong philosophical arguments which can be levelled against this view, especially by those within the empiricist camp, and these may be regarded as 'tipping the balance' in favour of the alternative field, interpretation.

Thus our overall conclusion is that more than one 'metaphysical package' is generally consistent with a given physical theory and therefore there exists a fundamental underdetermination regarding the 'picture' of reality which physics commits us to. In relating physics to philosophy, the best we can hope for is to rule out those 'packages' or descriptions of reality, which just do not 'fit', in the sense of being inconsistent with a particular interpretation.

Although this may be regarded as a somewhat negative conclusion we feel, nevertheless, that it is significant that such results can be achieved in the philosophy of physics.

APPENDIX : GROUP THEORY

Detailed expositions of the definitions and theorems presented below can be found in M.Hamermesh (1962) or E.P.Wigner (1959).

A.1 Abstract Group Theory

An abstract group G is a set of elements A, B, C, \dots such that a law of composition or 'group multiplication' may be defined which associates a third element with any ordered pair and which satisfies the following requirements :

1. If A and B are in the set then so is AB ; that is, the set is closed under 'group multiplication'.
2. 'Group multiplication' is associative; that is $A(BC) = (AB)C$.
3. The set contains an element E called the identity, or unit element, such that $AE = EA = A$ for every element A of the set.
4. Every element A in the set has an inverse B , denoted by $B = A^{-1}$, such that $AB = BA = E$.

If, in addition, all the elements of the group commute with one another, so that $AB = BA$ for all A and B , then the group is said to be 'commutative' or 'Abelian'.

The number of elements in a group is called the order of the group.

An element B of the group G is said to be conjugate to an element A if $XAX^{-1} = B$ or $X^{-1}BX = A$. If B is conjugate to A and C is conjugate to B then C is also conjugate to A , since if $C = YBY^{-1}$, $B = XAX^{-1}$, then $C = YXAX^{-1}Y^{-1} = (YX)A(YX)^{-1}$. This relation between elements can therefore be used to separate the group into classes of elements which are conjugate to one another. A class is determined by stating one of its elements A_i ; the entire class is then found by forming

all products of the form

$$E A_i E^{-1} = A_i, A_2 A_i A_2^{-1}, \dots, A_n A_i A_n^{-1}$$

Thus all the elements of the group can be divided among the various classes and every element appears in one and only one class.

With regard to our outline of the theory of representations below, it is worth noting that if the group elements are represented by matrices then the traces of all matrices (i.e. the sum of their diagonal elements) in a class must be the same. This is because the operation then becomes that of making a similarity transformation which leaves the trace invariant.

Two groups G and G' are said to be isomorphic if a unique one-to-one correspondence exists between their elements which is preserved under group combination. If the correspondence is not one-to-one, but is such that one and only one element of G' corresponds to every element of G and at least one element of G corresponds to every element of G' , and this correspondence is still preserved under combination, then G and G' are said to be homomorphic.

A.2 The Theory of Group Representations.

A representation of an abstract group G is, in general, any group composed of mathematical entities which is homomorphic to G . In this thesis we have restricted our attention to linear representations in terms of a group of linear operators $D(G)$ in an n -dimensional vector space, the basis for which is provided by a set of n linearly independent basis vectors $\underline{u}_1, \dots, \underline{u}_n$. The representation is then said to have degree n , or is said to be an n -dimensional representation. If R and S are elements of G then the corresponding operators can be denoted by $D(R)$ and $D(S)$ respectively. We then have

$$D(RS) = D(R)D(S) \text{ , } D(R^{-1}) = (D(R))^{-1} \text{ , and } D(E) = 1.$$

In this case, where the vector space is spanned by some chosen basis, the linear operators of the representation can be described by their matrix representatives. We then obtain a homomorphic mapping of the group G onto a group of n -by- n square matrices $D(G)$, with matrix multiplication as the group combination operation. This is then called a matrix representation of the group G and we have

$$D_{ij}(E) = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \text{ , } i, j = 1, \dots, n.$$

$$D_{ij}(RS) = \sum_k D_{ik}(R)D_{kj}(S) \equiv D_{ik}(R)D_{kj}(S) .$$

Different representations are distinguished from one another through the use of superscripts, e.g. $D_{ij}^{(\mu)}(R)$.

If the basis is changed in the vector space then the matrices $D(R)$ will be replaced by their transforms by some matrix C . The matrices $D'(R) = CD(R)C^{-1}$, also provide a representation of G which is equivalent to the representation given by $D(R)$. Equivalent representations have the same group structure. Any given representation can be characterised by the trace $\sum_i D_{ii}(R)$, known as the character of R in the representation D , which is invariant under such transformations. Thus we define the character of the μ th representation as being the set of h numbers (where h is the order of the group), $\chi^{(\mu)}(E)$, $\chi^{(\mu)}(A_2)$, \dots , $\chi^{(\mu)}(A_h)$, where

$$\chi^{(\mu)}(R) = \sum_{i=1}^h D_{ii}^{(\mu)}(R).$$

Thus equivalent representations have the same set of characters. Also, since the matrix representations of all elements in the same class are related by similarity transformations, (see our note above), the invariance of matrix traces demonstrates that all elements in the same class have the same character. Thus we can specify the character of any representation by giving the trace of one matrix from each class of group elements, denoted by $\chi^{(\mu)}(K_k)$, where the classes of G are labelled by $K_1, K_2, \dots, K_k, \dots$

Given any two, or more representations $D^{(1)}(R)$ and $D^{(2)}(R)$, then we can construct a larger representation from them by simply combining the matrices into larger matrices. In other words we could form

$$D(A) = \begin{pmatrix} D^{(1)}(A) & 0 \\ 0 & D^{(2)}(A) \end{pmatrix}$$

where $D^{(1)}(A)$ and $D^{(2)}(A)$ are the matrices representing element A in the original representation and $D(A)$ represents A in the derived representation. If it is possible to reduce the matrices representing all the elements in the group to this form by the same similarity transformation, then the representation is said to be reducible. If this is not possible then the representation is irreducible and cannot be expressed in terms of other representations of lower dimensionality.

In terms of our n -dimensional vector space a representation is said to be fully reducible if this space can be decomposed into the direct sum of subspaces of dimension less than n which are invariant under all transformations of the group. A basis for the entire space can then be formed from the sets of basis vectors spanning each subspace and in this basis the matrices of the representation will assume the reduced form above. If our vector space cannot be decomposed into subspaces of representations with lower dimensionality then the representation is irreducible.

Thus a fully reducible representation can be decomposed into the sum of irreducible representations of lower dimensionality. If this sum contains several irreducible representations which are equivalent to one another then, since these latter are not regarded as distinct, we can say that the sum contains the given irreducible representation a_ν times, where the a_ν are positive integers. We can then write the sum in the form

$$D = \sum_{\nu} a_{\nu} D^{(\nu)}.$$

Furthermore it can be shown⁽¹⁾ that for groups of finite order reducible representations can always be decomposed into a sum of representations.

One can, however, obtain simpler criteria for irreducibility and also show that certain restrictions exist on the number of non-equivalent representations, through what are known as the orthogonality theorems. We consider first the following lemmas.

Lemma I. If D and D' are two irreducible representations of a group G , having dimensions n_1 and n_2 respectively, and if a rectangular matrix A exists which satisfies

$$D(R)A = AD'(R)$$

for all R in G then it follows that, i) if $n_1 \neq n_2$, then $A = 0$, or ii) if $n_1 = n_2$, then either D and D' are equivalent or $A = 0$. (2).

Lemma II (Schur's Lemma). If the matrices $D(R)$ are an irreducible representation of a group G and if

$$AD(R) = D(R)A$$

for all R in G , then $A = \text{constant} \cdot E$. (3)

In other words if a matrix commutes with all the matrices of an irreducible representation then that matrix must be a multiple of the unit matrix. Therefore, if a non-constant commuting matrix exists, i.e. one which satisfies the above relation but is not a multiple of E , then the representation is reducible. On the other hand if no such matrix exists, i.e. if we can find an A which satisfies the above relation but we can show that A is a multiple of E , then the representation D is irreducible.

These lemmas can then be used to prove⁽⁴⁾ the following: The Orthogonality Theorem. If we consider all the non-equivalent irreducible representations of a group G then

1. Hamermesh op cit p.92-93 and p.98.
2. Ibid p.100; Wigner op cit p.76-77.
3. Hamermesh op cit p.100-101; Wigner op cit p.75.
4. Hamermesh op cit p.101-103; Wigner op cit p.79-81.

$$\sum_R D_{il}^{(\mu)}(R) D_{mj}^{(\nu)}(R^{-1}) = \frac{h}{n_\mu} \delta_{\mu\nu} \delta_{ij} \delta_{lm}$$

or, if the representations are unitary,

$$\sum_R D_{il}^{(\mu)}(R) D_{jm}^{(\nu)*}(R) = \frac{h}{n_\mu} \delta_{\mu\nu} \delta_{ij} \delta_{lm}$$

In other words for two non-equivalent irreducible representations $D^{(1)}$ and $D^{(2)}$ we obtain

$$\sum_R D_{il}^{(2)}(R) D_{mj}^{(1)}(R^{-1}) = 0, \text{ for all } i, j, l, m, \text{ or if the representations are unitary, } \sum_R D_{il}^{(2)}(R) D_{jm}^{(1)*}(R) = 0$$

whereas if $\mu = \nu$ then, $\sum_R D_{il}(R) D_{mj}(R^{-1}) = \delta_{lm} \delta_{ij}$
or $\sum_R D_{il}(R) D_{jm}^*(R) = \frac{h}{n} \delta_{lm} \delta_{ij}$ if D is unitary.

One can then show that the number of non-equivalent irreducible representations of a finite group is finite, ⁽⁵⁾ as expressed by the dimensionality theorem

$$\sum_\mu n_\mu^2 = h$$

which is essential for actually obtaining the irreducible representations of any group.

From the above one can derive orthogonality relations for the characters, ⁽⁶⁾ thus

$$\sum_R \chi^{(\mu)}(R) \chi^{(\nu)*}(R) = h \delta_{\mu\nu}$$

Collecting the group elements according to classes, within which the $\chi^{(\mu)}(R)$ are the same, this can be rewritten as

$$\sum_K \chi^{(\mu)}(K) \chi^{(\nu)*}(K) N_K = h \delta_{\mu\nu}$$

where N_K is the number of elements in the class K and the sum now runs over classes. For a given μ the numbers $\chi^{(\mu)}(K) N_K^{1/2}$ form a vector in an k -dimensional space. The vectors obtained in this way from non-equivalent irreducible representations are orthogonal, and it can then be shown that the number of non-equivalent irreducible representations of a group must equal the number of classes in the group.

It can then be shown that the character of a reducible representation $D(R)$ is the sum of the characters of its

5. Hamermesh p.103 and p.106-107; Wigner op cit p.83

6. Hamermesh op cit p.103-104; Wigner op cit p.83-84.

component irreducible representations, i.e.

$$\chi(K_k) = \sum_{\nu} a_{\nu} \chi^{(\nu)}(K_k)$$

One can then obtain $a_{\mu} = \frac{1}{h} \sum_{k} N_k \chi^{(\mu)*}(K_k) \chi(K_k)$ giving the number of times a given irreducible representation is contained in D, and also the following simple criterion for irreducibility: ⁽⁷⁾

$$\sum_{k} N_k |\chi(K_k)|^2 = h$$

These results allow one to draw up the character table for a group directly, without first having to explicitly work out the representation matrices. In such a table the columns are labelled by the various classes preceded by the number N_k of elements in the class, the rows are labelled by the irreducible representations, and the entries in the table are the $\chi^{(\mu)}(K_k)$. Thus we obtain

	$N_k K_1$	\dots	$N_k K_i$	\dots	$N_k K_k$
$D^{(1)}$	$\chi_1^{(1)}$	\dots	$\chi_i^{(1)}$	\dots	$\chi_k^{(1)}$
\vdots	\vdots		\vdots		\vdots
$D^{(\mu)}$	$\chi_1^{(\mu)}$	\dots	$\chi_i^{(\mu)}$	\dots	$\chi_k^{(\mu)}$
\vdots	\vdots		\vdots		\vdots
$D^{(k)}$	$\chi_1^{(k)}$	\dots	$\chi_i^{(k)}$	\dots	$\chi_k^{(k)}$

A.3 The Symmetric Group.

The elements of the symmetric group of degree N, denoted by S_N , are the permutations of N objects, denoted by

$$\begin{pmatrix} 1 & 2 & \dots & N \\ r_1 & r_2 & \dots & r_N \end{pmatrix}$$

The order of S_N is clearly $N!$. A permutation may also be denoted in terms of 'cycles'. A cycle $(r_1 r_2 \dots r_{\lambda})$ is a permutation which replaces every element r_k by the element following it, r_{k+1} , except for the last element, r_{λ} , which is replaced by the first r_1 . Thus the cycle $(r_1 r_2 \dots r_{\lambda})$ is just another way of writing the permutation denoted by $\begin{pmatrix} r_1 & r_2 & \dots & r_{\lambda} \\ r_2 & r_3 & \dots & r_1 \end{pmatrix}$

Cycles which have no elements in common commute and any permutation can be decomposed into a product of

7. Hamermesh op cit p.105

8. Ibid p.111

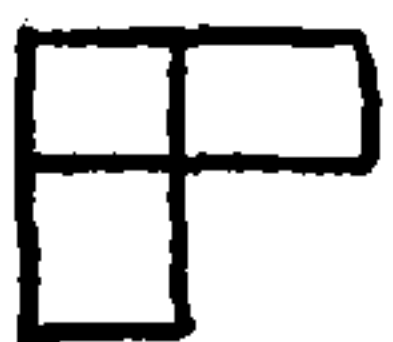
cycles which commute. Thus, for example, the permutation denoted by $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 2 & 5 & 1 \end{pmatrix}$ can be resolved into $(136)(24)(5)$ and this is equivalent to $(361)(24)(5)$ and $(24)(5)(136)$ since the order of the cycles as well as the initial element of each cycle, is of no significance.

The permutations of S_N can of course be divided into different classes and two permutations which have the same number of cycles and whose cycles are of equal length, i.e. contain the same number of elements, are contained in the same class. The number of classes is therefore equal to the number of different possible lengths of cycles. This is equal to the number of ways of dividing N into positive integral summands without regard to order and is called the partition number of N . Since the number of different irreducible representations is equal to the number of classes, as we noted above, it follows that this is also given by the partition number of N . Thus the symmetric group of degree 3 has 3 classes, $(1)(2)(3)$, $(12)(3)$, (123) corresponding to 3 irreducible representation.

Permutations which merely interchange two elements are called transpositions, written, in terms of cycles, thus (ij) . Every permutation can be decomposed into the product of either an even or an odd number of transpositions. The even permutations form a subgroup of S_N known as the alternating group.

A permutation which when resolved into cycles has ν_1 cycles containing 1 element, ν_2 cycles containing 2 elements, ..., ν_N cycles containing N elements is said to have the cycle structure $(1^{\nu_1}, 2^{\nu_2}, \dots, N^{\nu_N})$. As we have noted all permutations which have the same cycle structure form a class in S_N . For every partition of N there exists a corresponding *cycle* structure and so, given the point noted above, every different cycle structure corresponds to a different irreducible representation of S_N . The characters, matrices and dimensions of the irreducible representations of S_N

can be obtained by a variety of different methods (9). Given the results of Chapter 3 of this thesis we shall mention only one, the graphical method involving Young Tableaux (10).

Thus we associate with each partition a graph of the form , which is obtained by placing objects one at a time on the diagram so that at each stage the number of objects in each line is greater than the number in succeeding lines, until all the squares of the graph are filled. This is called a regular application of one square at a time. The number of different ways of building these graphs by regular applications of squares, for each partition then gives the dimension of the corresponding permutation. Thus for example we can find the dimensions of all the irreducible representations of S_3 as follows:

(3)	1 2 3	$N = 1$
(2,1)	1 2 3	$N = 2$
(1 ³)	1 2 3	$N = 1$

A graph of a given shape, as determined by the partition, in which the numbers 1, ..., N are placed in the boxes is called a Young Tableau. When the graph is built up through regular application of numbers the tableau is said to be a standard tableau and the dimension of the representation corresponding to a given partition is equal to the number of standard tableau which can be constructed for that partition. This in turn gives the number of equivalent irreducible representations corresponding to that partition.

Young tableaux can also be used to obtain the characters (11) and basis functions of the irreducible representations. Thus the irreducible representations of

9. See Hamermesh op cit. Ch.7.

10. Ibid p.198-201.

11. Ibid p.201-208.

S_N can be found as follows. One first draws the Young pattern for any partition of N and then put numbers $1, \dots, N$ into the pattern in any order to give a Young tableau. Then one constructs the following quantities:

$$P = \sum_p p, \quad Q = \sum_q \delta_q q$$

where the p 's and q 's are permutations in the rows and columns respectively, δ_q is the parity of q and the sums are over all horizontal or vertical permutations

respectively, of the given tableau. Application of the Young operator $Y = QP$ then gives the

irreducible representations of $S_N^{(12)}$. For different patterns the representations obtained in this way are inequivalent whereas equivalent irreducible representations are obtained from different tableaux with the same pattern. Thus for the group S_3 the partitions (3) and (1^3) have corresponding representations denoted by $\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$ and $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$ respectively, whereas there are two equivalent representations $\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$ and $\begin{array}{|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$ corresponding to $(2,1)$.

With regard to the subject matter of this thesis we note that given a system of N indistinguishable particles one can construct basis functions for the various irreducible representations of S_N by applying to the eigenfunction of the system the Young operators corresponding to all the standard tableaux for a given pattern, thus giving the required basis functions for the corresponding irreducible representation⁽¹³⁾. We conclude by noting that there is a complete correspondence between the symmetry types of the wave functions and these Young patterns.

12. Ibid p.244.

13. Compare with the procedure of section 3.2.1. of Chapter 3.

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