Ill-Posed Problems in Probability and Stability of Random Sums

By

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Preface

This is the first of two volumes concerned with the *ill-posed* problems in probability and statistics. Ill-posed problems are usually understood as those results where small changes in the assumptions lead to arbitrarily large changes in the conclusions. Such results are not very useful for practical applications where the presumptions usually hold only approximately (because even a slightest departure from the assumed model may produce an uncontrollable shift in the outcome). Often, the ill-posedness of certain practical problems is due to the lack of their precise mathematical formulation. Consequently, we can deal with such problems by replacing a given ill-posed problem with another, well-posed problem, which in some sense is "close" to the original one.

Our goal is to show that ill-posed problems are not just a mere curiosity in the contemporary theory of mathematical statistics and probability. On the contrary, such problems are quite common, and majority of classical results fall into this class. Our objective is to identify problems of this type, and re-formulate them more correctly. Thus, we propose alternative (more precise in the above sense) versions of numerous classical theorems in the theory of probability and mathematical statistics. In addition, we shall consider some non-standard problems from this point of view.

The following examples illustrate quite prominent ill-posed problems in statistics and probability theory.

- The classical Central Limit Theorem, as well as the corresponding limit theorem for convergence to a stable law, are ill-posed. Indeed, an arbitrarily small change (in the uniform metric) of the tail of the underlying distribution leads to a shift of the domain of attraction: A normal domain of attraction may convert to a stable one, and vice versa. Alternative versions of these theorems were proposed in Klebanov et al. (1999), Nagaev (1997). The main idea was to replace the limiting distribution with an approximation of the *pre-limiting* distribution.

- The second example comes from the extreme value theory. It is well-known that (under certain conditions) the limiting distribution of an appropriately normalized minimum of non-negative i.i.d. random variables is Weibull. The parameters of the limiting Weibull distribution depend on the rate of convergence to zero of the underlying distribution function. An arbitrarily small change (in the uniform metric) of that distribution function may affect this rate quite severely, and thus the problem of finding the exact limiting distribution appears to be ill-posed. The corrected version of this problem appeared in Klebanov et al. (1999) as well.

- The third example of an ill-posed problem is the classical problem of estimating the location parameter of a normal distribution with known standard deviation. If the distribution of the measurement error is Gaussian, then the optimal equivariant estimator of the location parameter is provided by the sample mean. However, if the sample is contaminated with observations from a heavy-tail distribution and we are using the variance of the limiting distribution as the loss function, then the sample mean becomes unacceptable, since its variance may be infinite. This example has led to the theory of robust estimation, see, e.g., Huber (1981) and Hampel et al. (1986). It is also clear that this problem is closely related to the one concerning the limiting distribution for sums of i.i.d. random variables. Following the recommendations of Klebanov et al. (1999), we consider approximations via the pre-limiting distribution. In this case we can not utilize the variance of the limiting distribution as the loss function. Thus, a corrected formulation of the problem of estimating the location parameter is two-folded, involving the pre-limiting approach as well as the issue of choosing an appropriate loss function.

- The forth example concerns estimation of parameters for distributions with discontinuous densities. For example, consider the problem of estimating the scale parameter θ of the uniform distribution on the interval $(0, \theta)$. It is well known that the sample maximum $X_{n:n}$ is a consistent estimator of θ, and the normalized sequence $n(X_{n:n} - \theta)$ has a non-singular limiting distribution as $n \to \infty$. However, when we replace the uniform distribution with another one, which is smooth and arbitrarily close to it (in the uniform metric), then $X_{n:n}$ is no longer consistent, and the normalizing constant n needs to be replaced by $\sqrt{n}$. Thus, we again end up with an ill-posed problem. Its corrected version is based on the replacement of the limiting distribution of the normalized sequence by the pre-limiting distribution, see Klebanov et al. (1999).

- Our last example is related to the problem of specifying a distribution by a finite number of values of certain functionals, such as moments or the Radon transformation. The latter is particularly common in the area of computer tomography. The proof of the ill-posedness here follows from an interesting example, discussed in Guttmann et al. (1991). The corrected versions appeared in Khalfin and Klebanov (1994), Klebanov and Rachev (1995), Khalfin and Klebanov (1996). Some applications to quantum mechanics are discussed in Klebanov and Rachev (1997a), Klebanov and Rachev (1997).

Ill-posed problems in probability theory such as those in the above examples are the subject of this volume, which is organized in two parts (statistical part is treated in a companion volume). In Part I we start with ill- and well-posedness of functional minimization problems. Then, we introduce new classes of probability distances, and present a comprehensive treatment of quantitative convergence criteria. We end Part I with the problem of inverting the Radon transform, which leads to possible solutions of the computer tomography paradox stated in Guttmann et al. (1991). Since the questions about the correctness of certain probabilistic problems may be resolved through appropriate metrics, we include results from the theory of probability metrics throughout the text.

Part II is devoted to ill-posed problems for sums of random variables. The central object of our study is the random summation scheme, which along with the pre-limit theorems, requires a careful definition and characterization of the limiting distributions. We define analogs of stable distributions for random sums, and study their analytical properties. Our main example is the case of geometric summation, leading to the rich class of *geometric stable* distributions. These distributions have the same domains of attraction and tail as the stable laws, but admit a fundamentally different behavior at the mode. The results on geometric stable laws have been scattered in the literature, and are presented here in a monographic format for the first time.

A companion volume will be devoted to statistical applications of the pre-limit theorems. In that volume, we shall present a modified version of the theory of statistical estimation, and show its connection with the problem of the choice of an appropriate loss function. It turns out, that the loss function should not be chosen arbitrarily. As we explain in the present volume, the availability of certain mathematical conveniences, including the correctness of the formulation of the problem of estimation, lead to rigid restrictions on the choice of the loss function.

The authors thank Prof. George Anastassiou for his interest and helpful remarks. Thomas Plum has our grateful thanks for his technical help in preparing the manuscript. Lev Klebanov's work was supported by MSM grant 113200008 and RFFI grants 02-01-00262, 02-01-00483. Tomasz J. Kozubowski is grateful for the (partial) support provided by the NSF grant DMS-0139927. Sveltozar T. Rachev's research was supported by grants from Division of Mathematical, Life, and Physical Sciences, University of California at Santa Barbara, and the Deutschen Forchungsgemeinschaft.

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May 2005

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