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ILL-POSEDNESS FOR THE DERIVATIVE SCHRÖDINGER AND GENERALIZED BENJAMIN-ONO EQUATIONS

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Abstract. Ill-posedness is established for the initial value problem (IVP) associated to the derivative nonlinear Schrödinger equation for data in $H^s(\mathbb{R})$, $s < 1/2$. This result implies that best result concerning local well-posedness for the IVP is in $H^s(\mathbb{R})$, $s \geq 1/2$. It is also shown that the (IVP) associated to the generalized Benjamin-Ono equation for data below the scaling is in fact ill-posed.

1. Introduction

In this paper we are concerned with the ill-posedness of the initial value problems (IVP) associated to the derivative nonlinear Schrödinger equation and the generalized Benjamin-Ono equation, hereafter (DNLS) and (GBO) equation respectively.

The notion of well-posedness we will use in this work is the same as in [7], that is, the existence, uniqueness, persistence property and continuous dependence of the solution upon the data.

To describe our results, we begin by considering the IVP associated to the DNLS equation, that is,

$$\begin{cases} \partial_t u = i\partial_x^2 u + \partial_x(|u|^2 u), & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

This equation appears as a model of the Alfvén solitons in plasma physics (see [13],[22]). From the point of view of partial differential equations it has been extensively studied (see [6], [15], [16], [19], [20] and references therein).

Recently, Takaoka in [19] showed that the IVP (1.1) is locally well-posed in $H^s(\mathbb{R})$, $s \geq 1/2$. To prove this result he used the techniques introduced by Bourgain ([5]) and Kenig, Ponce, Vega ([8], [10]) plus a gauge transformation. He also showed by means of an example that the best possible result using the key estimate in his proof was indeed $H^{1/2}$.

On the other hand, a scaling argument ([10]) suggests that the best possible value to obtain local well-posedness in H^s is $s = 0$. Indeed, if u is a solution of (1.1), then

$$u_\lambda(x,t) = \lambda^{1/2}\, u(\lambda x, \lambda^2 t), \tag{1.2}$$

for any $\lambda \in \mathbb{R}$, is also a solution of DNLS with data

$$u_\lambda(x,0) = \lambda^{1/2}\, u_0(\lambda x).$$

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A straightforward calculation gives

$$\|D_x^s u_\lambda(0)\| = \lambda^s \, \|D_x^s u_0\|$$

which implies that the highest derivative term in the H^s norm is invariant under the scaling transform (1.2) for the value $s = 0$.

Our purpose here is to show that the IVP associated to the DNLS equation is ill-posed in $H^s,\ s < 1/2$, which will imply that the best possible local well-posedness result is the one in [19].

To establish this result we will follow the recent method introduced by Kenig, Ponce, Vega [11] (see also [2], [3]), to establish ill-posedness for the IVP associated to the cubic Schrödinger equation, KdV and mKdV equations. We will describe briefly their method: the idea is to show that the solution does not depend continuously (or uniformly continuously) on its data in H^s, by constructing a sequence converging to the data in H^s while the corresponding sequence of solutions does not converge in H^s. The sequence consists of solitary wave solutions. The extra difficulty we have in our case is the lack of Galilean invariance for solutions of the derivative Schrödinger equation, which is a key point in the treatment of ill-posedness for the focusing cubic Schrödinger equation. To replace the Galilean invariance we still have a two-parameter family of solitary wave solutions that allows us to obtain the desired result.

We also consider the IVP associated to the GBO, that is,

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + u^k \partial_x u = 0, \quad x \in \mathbb{R}, t > 0,\ k = 1, 2, \dots, \\ u(x,0) = u_0(x), \end{cases} \tag{1.3}$$

where $\mathcal{H}$ denotes the Hilbert transform. For $k = 1$ we have the well known Benjamin-Ono equation which was deduced by Benjamin [1] and Ono [14] as a model in internal-wave propagation. The best result regarding local well-posedness in H^s is due to Ponce [18], (see also [12]). He showed that the IVP associated to the Benjamin-Ono equation is locally (globally) well-posed in $H^s,\ s \geq 3/2$. For the GBO equation with $k > 1$, local well-posedness is known in $H^s,\ s > 3/2$, for any data. For small data, Kenig, Ponce and Vega [9] proved that (1.3) is locally well-posed in $H^s(\mathbb{R})$, with

$$\begin{aligned} s &> 1 \quad \text{if} \quad k = 2, \\ s &> 5/6 \quad \text{if} \quad k = 3, \\ s &\geq 3/4 \quad \text{if} \quad k \geq 4. \end{aligned} \tag{1.4}$$

These are the best results known to date.

Looking for the best possible local well-posedness results we argue as above: we use a scaling argument to find the critical Sobolev indices. If $u(x,t)$ solves (1.3) then $u_\lambda(x,t) = \lambda^{1/k} u(\lambda x, \lambda^2 t),\ \lambda > 0$, also solves (1.3) with initial data $u_\lambda(x,0)$ satisfying

$$\|u_\lambda(\cdot,0)\|_{\dot{H}^s}^2 = \lambda^{2s + \frac{2}{k} - 1} \|u(\cdot,0)\|_{\dot{H}^s}^2 \tag{1.5}$$

($\dot{H}^s$ is the homogeneous Sobolev space), which implies that the highest derivative that leaves the norm invariant is $s_k = 1/2 - 1/k$.

We can observe that the these results are far from those given by the scaling argument. For instance, for the Benjamin-Ono equation the scaling suggests local well-posedness for $s \geq -1/2$.

Our results in this case give ill-posedness of the IVP (1.3) in Sobolev spaces with index below the one given by the scaling argument. Here we follow closely the ideas in [2] and [3].

The paper is organized as follows. In Section 2 we will deal with the derivative Schrödinger equation. The result concerning the generalized Benjamin-Ono (GBO) equation will be proved in Section 3.

2. The derivative Schrödinger equation

In this section we consider the IVP associated to the derivative Schrödinger equation, that is,

$$\begin{cases} \partial_t u = i\partial_x^2 u + \partial_x(|u|^2 u), \quad x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.1}$$

The notion of well-posedness used here is the one given in the introduction but stretched a little bit by requiring the mapping data, $u_0 \to u(t)$ to be uniformly continuous, where $u(t)$ is the solution of (2.1). In case this requirement is not satisfied we will say that the problem is ill-posed. Thus our main result in this section is

Theorem 2.1. *The IVP (2.1) is ill-posed in $H^s(\mathbb{R})$, $s < 1/2$, in the sense that the mapping data-solution, $u_0 \to u(t)$ is not uniformly continuous.*

Proof. It was proved in [21] that there exist solitary waves in the form

$$u_{c,\omega}(x,t) = e^{-i\omega t} e^{i\psi(x-ct)} a(x-ct)$$

with ω, c real numbers and $\psi(\cdot)$, $a(\cdot)$ real functions given by

$$\begin{aligned} a^2(x) &= (d_3 + d_5 \cosh(d_6 x))^{-1}; \\ \psi'(x) &= \frac{c}{2} + \frac{3}{4} a^2(x); \\ d_3 &= \frac{c}{2(-4\omega - c^2)}; \\ d_5^2 &= \frac{-\omega}{(-4\omega - c^2)^2}; \\ d_6^2 &= -4\omega - c^2. \end{aligned} \tag{2.2}$$

Setting $\alpha = \frac{d_3}{d_5}$, it is easy to see that

$$\psi(x) = \frac{c\,x}{2} + 3\arctan\Big(\frac{\exp(d_6 x) + \alpha}{(1-\alpha^2)^{\frac{1}{2}}}\Big).$$

Let

$$\begin{aligned} \varphi_{c,\omega}(x) = u_{c,\omega}(x,0) &= e^{i\frac{c\,x}{2}} e^{ig(x)} a(x) \\ &= e^{i\frac{c\,x}{2}} e^{ig(x)} \frac{d_6}{(-\omega)^{1/4}} \frac{1}{(\alpha + \cosh(d_6\, x))^{1/2}} \end{aligned}$$

where

$$g(x) = 3\arctan\Big(\frac{\exp(d_6\, x) + \alpha}{(1-\alpha^2)^{1/2}}\Big).$$

Setting

$$\begin{gathered}\tilde{g}(x) = 3\arctan\Big(\frac{e^x+\alpha}{(1-\alpha^2)^{\frac{1}{2}}}\Big), \;\; h(x) = \frac{1}{(\alpha+\cosh x)^{1/2}}, \\ F(x) = e^{i\tilde{g}(x)}h(x),\end{gathered} \tag{2.3}$$

we can write

$$\varphi_{c,\omega}(x) = \frac{d_6}{(-\omega)^{1/4}} e^{i\frac{cx}{2}} e^{i\tilde{g}(d_6\,x)} h(d_6\,x) = \frac{d_6}{(-\omega)^{1/4}} e^{i\frac{cx}{2}} F(d_6\,x)$$

and thus

$$\hat{\varphi}_{c,\omega}(\xi) = \frac{1}{(-\omega)^{\frac{1}{4}}}\hat{F}(\frac{\xi}{d_6} - \frac{c}{2d_6}).$$

Then we have, writing d_{61} and d_{62} as the corresponding constants in (2.2) associated with c_1, ω_1 and c_2, ω_2, respectively, that

$$\begin{aligned}
&\|\varphi_{c_1,\omega_1} - \varphi_{c_2,\omega_2}\|^2_{H^s} = \int (1+|\xi|^2)^s |\hat{\varphi}_{c_1,\omega_1}(\xi) - \hat{\varphi}_{c_2,\omega_2}(\xi)|^2\,d\xi \\
&= \int (1+|\xi|^2)^s \Big|\frac{1}{(-\omega_1)^{1/4}}\widehat{F}(\frac{\xi}{d_{61}} - \frac{c_1}{2d_{61}}) - \frac{1}{(-\omega_2)^{1/4}}\widehat{F}(\frac{\xi}{d_{62}} - \frac{c_2}{2d_{62}})\Big|^2 d\xi \\
&= d_{61}\int (1+|d_{61}\eta|^2)^s \Big|\frac{1}{(-\omega_1)^{1/4}}\widehat{F}(\eta - \frac{c_1}{2d_{61}}) - \frac{1}{(-\omega_2)^{1/4}}\widehat{F}(\eta\frac{d_{61}}{d_{62}} - \frac{c_2}{2d_{62}})\Big|^2 d\eta \\
&\simeq (d_{61})^{2s+1}\Big\{\int (1+|\eta|^2)^s\,\frac{1}{(-\omega_1)^{1/2}}\Big|\widehat{F}(\eta - \frac{c_1}{2d_{61}}) - \widehat{F}(\frac{d_{61}}{d_{62}}\eta - \frac{c_1}{2d_{61}})\Big|^2 d\eta \\
&+ \int (1+|\eta|^2)^s\,\frac{1}{(-\omega_1)^{1/2}}\Big|\widehat{F}(\eta\frac{d_{61}}{d_{62}} - \frac{c_1}{2d_{61}}) - \widehat{F}(\eta\frac{d_{61}}{d_{62}} - \frac{c_2}{2d_{62}})\Big|^2 d\eta \\
&+ \int (1+|\eta|^2)^s\Big|\frac{1}{(-\omega_1)^{1/4}} - \frac{1}{(-\omega_2)^{1/4}}\Big|^2\Big|\widehat{F}(\eta\frac{d_{61}}{d_{62}} - \frac{c_2}{2d_{62}})\Big|^2 d\eta\Big\} \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Let N be a large positive integer to be chosen later. Take

$$c_j = N_j \simeq N, \;\; \omega_j = -(N_j^{4s} + \frac{N_j^2}{4}), \qquad j = 1, 2, \tag{2.4}$$

so that if $N_1 < N_2$,

$$d_{6j} = 2N_j^{2s}, \;\; d_{61} < d_{62}, \;\; |d_{62} - d_{61}| = 2|N_2^{2s} - N_1^{2s}| \simeq |N_2 - N_1| N^{2s-1}.$$

Now, $\widehat{F}$ concentrates in $B_1(0)$, where $B_1(0)$ is the ball of center 0 and radius 1. Thus if $\eta \in B_1(N^{1-2s})$, then $|\eta| \simeq N^{1-2s}$. It follows that, by the mean value theorem and Cauchy-Schwarz's inequality

$$\begin{aligned}
I_1 &= \frac{(d_{61})^{2s+1}}{(-\omega_1)^{1/2}} \int (1+|\eta|^2)^s \left|\widehat{F}(\eta - \frac{c_1}{2d_{61}}) - \widehat{F}(\eta \frac{d_{61}}{d_{62}} - \frac{c_1}{2d_{61}})\right|^2 d\eta \\
&\simeq \frac{N^{2s(2s+1)}}{\sqrt{N^{4s}+N^2}}\, N^{2s(1-2s)} \int \Big| \int_{\eta \frac{d_{61}}{d_{62}} - \frac{c_1}{2d_{61}}}^{\eta - \frac{c_1}{2d_{61}}} \widehat{F}'(\alpha)\, d\alpha \Big|^2 d\eta \\
&\leq \frac{N^{4s}}{\sqrt{N^{4s}+N^2}} \int \Big| \int_{\eta \frac{d_{61}}{d_{62}} - \frac{c_1}{2d_{61}}}^{\eta - \frac{c_1}{2d_{61}}} d\alpha \Big| \Big| \int_{\eta \frac{d_{61}}{d_{62}} - \frac{c_1}{2d_{61}}}^{\eta - \frac{c_1}{2d_{61}}} |\widehat{F}'(\alpha)|^2\, d\alpha \Big|\, d\eta \\
&\simeq C \frac{N^{4s-1}|N_1 - N_2| N^{2s-1}}{N^{2s}} \int |\eta| \Big| \int_{\eta \frac{d_{61}}{d_{62}} - \frac{c_1}{2d_{61}}}^{\eta - \frac{c_1}{2d_{61}}} |\widehat{F}'(\alpha)|^2\, d\alpha \Big|\, d\eta \\
&\simeq C\, N^{4s-2} |N_1 - N_2| (I_{11} - I_{12})
\end{aligned} \tag{2.5}$$

where

$$I_{11} = \int_0^\infty \eta \int_{\eta \frac{d_{61}}{d_{62}} - \frac{c_1}{2d_{61}}}^{\eta - \frac{c_1}{2d_{61}}} |\widehat{F}'(\alpha)|^2\, d\alpha\, d\eta$$

and

$$I_{12} = \int_{-\infty}^0 \eta \int_{\eta \frac{d_{61}}{d_{62}} - \frac{c_1}{2d_{61}}}^{\eta - \frac{c_1}{2d_{61}}} |\widehat{F}'(\alpha)|^2\, d\alpha\, d\eta.$$

We estimate each term on the right hand side of (2.5). Fubini's theorem gives

$$\begin{aligned}
I_{11} &= \int_{\frac{-c_1}{2d_{61}}}^\infty |\widehat{F}'(\alpha)|^2 \int_{\alpha + \frac{c_1}{2d_{61}}}^{\left(\alpha + \frac{c_1}{2d_{61}}\right)\frac{d_{62}}{d_{61}}} \eta\, d\eta\, d\alpha \\
&= \frac{1}{2} \int_{-\frac{c_1}{2d_{61}}}^\infty |\widehat{F}'(\alpha)|^2 \left(\alpha + \frac{c_1}{2d_{61}}\right)^2 \left[\left(\frac{d_{62}}{d_{61}}\right)^2 - 1\right] d\alpha.
\end{aligned} \tag{2.6}$$

A similar argument yields

$$I_{12} = \frac{1}{2} \int_{-\infty}^{-\frac{c_1}{2d_{61}}} |\widehat{F}'(\alpha)|^2 \left(\alpha + \frac{c_1}{2d_{61}}\right)^2 \left[1 - \left(\frac{d_{62}}{d_{61}}\right)^2\right] d\alpha. \tag{2.7}$$

Combining (2.6) and (2.7) it follows that

$$I_{11} - I_{12} = \frac{1}{2} \int_{\mathbb{R}} |\widehat{F}'(\alpha)|^2 \left(\alpha + \frac{c_1}{2d_{61}}\right)^2 \left[1 - \left(\frac{d_{62}}{d_{61}}\right)^2\right] d\alpha.$$

Observe that

$$\frac{(d_{62})^2 - (d_{61})^2}{(d_{61})^2} \simeq \frac{4(N_2^{4s} - N_1^{4s})}{4N^{4s}} \simeq \frac{(N_2 - N_1) N^{4s-1}}{N^{4s}} = \frac{N_2 - N_1}{N}.$$

Returning to (2.5), we have

$$\begin{aligned}
I_1 &= C\, N^{4s-2}|N_1 - N_2| \int_{\mathbb{R}} |\widehat{F}'(\alpha)|^2 \left(\alpha + \frac{c_1}{2d_{61}}\right)^2 \left[1 - \left(\frac{d_{62}}{d_{61}}\right)^2\right] d\alpha \\
&= C\, N^{4s-3}(N_1 - N_2)^2 \int_{\mathbb{R}} |\widehat{F}'(\alpha - \frac{c_1}{2d_{61}})|^2\, \alpha^2\, d\alpha \\
&\simeq C\, N^{4s-3}(N_1 - N_2)^2\, N^{2(1-2s)} \|\widehat{F}'\|_2^2 \\
&= C\, \frac{(N_1 - N_2)^2}{N} \|\widehat{F}'\|_2^2.
\end{aligned} \tag{2.8}$$

Now we estimate I_2

$$\begin{aligned}
I_2 &= \frac{(d_{61})^{2s+1}}{(-\omega_1)^{1/2}} \int (1 + |\eta|^2)^s \left|\widehat{F}(\eta\frac{d_{61}}{d_{62}} - \frac{c_1}{2d_{61}}) - \widehat{F}(\eta\frac{d_{61}}{d_{62}} - \frac{c_2}{2d_{62}})\right|^2 d\eta \\
&= \frac{(d_{61})^{2s+1}}{(-\omega_1)^{1/2}} \left(\frac{d_{61}}{d_{62}}\right)^{2s+1} \int |\eta|^{2s} \left|\widehat{F}(\eta - \frac{c_1}{2d_{61}}) - \widehat{F}(\eta - \frac{c_2}{2d_{62}})\right|^2 d\eta \\
&\simeq \frac{(d_{61})^{2s+1}}{(-\omega_1)^{1/2}} N^{2s(1-2s)} \int \left|\int_{\eta - \frac{c_2}{2d_{62}}}^{\eta - \frac{c_1}{2d_{61}}} \widehat{F}'(\alpha)\, d\alpha\right|^2 d\eta \\
&\leq \frac{(d_{61})^{2s+1}}{(-\omega_1)^{1/2}} N^{2s(1-2s)} \int \left|\int_{\eta - \frac{c_2}{2d_{62}}}^{\eta - \frac{c_1}{2d_{61}}} d\alpha\right| \left|\int_{\eta - \frac{c_2}{2d_{62}}}^{\eta - \frac{c_1}{2d_{61}}} |\widehat{F}'(\alpha)|^2\, d\alpha\right| d\eta \\
&\simeq \frac{(d_{61})^{2s+1}}{(-\omega_1)^{1/2}} N^{2s(1-2s)} \left|\frac{c_1}{2d_{61}} - \frac{c_2}{2d_{62}}\right| \int \int_{\eta - \frac{c_2}{2d_{62}}}^{\eta - \frac{c_1}{2d_{61}}} |\widehat{F}'(\alpha)|^2\, d\alpha d\eta \\
&\simeq N^{4s-1}\left|\frac{c_1}{2d_{61}} - \frac{c_2}{2d_{62}}\right|^2 \|\widehat{F}'\|_2^2 \\
&\simeq \frac{|N_2 - N_1|^2}{N} \|\widehat{F}'\|_2^2.
\end{aligned} \tag{2.9}$$

Finally,

$$\begin{aligned}
I_3 &= (d_{61})^{2s+1} \int |\eta|^{2s} \left(\frac{1}{\sqrt[4]{-\omega_1} - \sqrt[4]{-\omega_2}}\right)^2 |\hat{F}(\eta\frac{d_{61}}{d_{62}} - \frac{c_2}{2d_{62}})|^2 d\eta \\
&= (d_{61})^{2s+1} \frac{(\sqrt[4]{-\omega_2} - \sqrt[4]{-\omega_1})^2}{\sqrt{\omega_1\omega_2}} \int |\eta|^{2s} |\hat{F}(\eta\frac{d_{61}}{d_{62}} - \frac{c_2}{2d_{62}})|^2 d\eta \\
&= (d_{62})^{2s+1} \left(\frac{\omega_2 - \omega_1}{4\sqrt[4]{(-\omega_o)^3}}\right)^2 \frac{1}{\sqrt{\omega_1\omega_2}} \int |\eta|^{2s} |\hat{F}(\eta - \frac{c_2}{2d_{62}})|^2 d\eta \\
&\simeq N^{2s(2s+1)-3} \frac{|N_1^2 - N_2^2|^2}{N^2} \left(\frac{c_2}{2d_{62}}\right)^{2s} \|\widehat{F}\|^2 \\
&\simeq N^{4s-3}(N_1 - N_2)^2 \|F\|^2,
\end{aligned} \tag{2.10}$$

where we have used the mean value theorem and $\omega_o \in]\omega_2, \omega_1[$.

Set

$$\alpha = \frac{d_3}{d_5} = \frac{c}{2\sqrt{-\omega}} < 1, \quad \beta^2 = 1 - \alpha^2 = \frac{-4\omega - c^2}{-4\omega};$$

this implies

$$\frac{\alpha}{\beta} = \frac{c}{\sqrt{-4\omega - c^2}} \text{ (and } \frac{1}{\beta}) \simeq N^{1-2s}.$$

We can take N large enough so that, setting $\overline{\theta} = \arctan \frac{\alpha}{\beta}$,

$$\frac{\pi}{2} - \overline{\theta} \simeq \frac{\pi}{2} - \arctan N^{1-2s} \simeq N^{-2s}, \tag{2.11}$$

since $2s < 1$.

Now we evaluate the L^2-norm of $\hat{F}$ and $(\hat{F})'$:

$$\begin{aligned} \|F\|^2 &= \int_{-\infty}^{\infty} \frac{dx}{\alpha + \cosh x} = \int_{-\infty}^{\infty} \frac{2e^x dx}{2\alpha e^x + e^{2x} + 1} \\ &= 2\int_0^{\infty} \frac{du}{(u+\alpha)^2 + \beta^2} = \frac{2}{\beta} \arctan \frac{u+\alpha}{\beta}\Big|_0^{\infty} \\ &= \frac{2}{\beta}(\frac{\pi}{2} - \overline{\theta}) \simeq N^{1-4s}. \end{aligned} \tag{2.12}$$

On the other hand, we have that

$$\|(\hat{F})'\|^2 = \|xF\| = 2\int_{-\infty}^{\infty} \frac{x^2\, e^x}{2\alpha e^x + e^{2x} + 1}\, dx \le C. \tag{2.13}$$

Replacing (2.12) in (2.10) and (2.13) in (2.8) and (2.9), we get, respectively

$$I_1,\ I_2 \le \frac{C\,(N_1 - N_2)^2}{N}; \quad I_3 \le \frac{C\,(N_1 - N_2)^2}{N^2}. \tag{2.14}$$

Then (2) can be estimated by

$$\|\varphi_{c_1,\omega_1} - \varphi_{c_2,\omega_2}\|_{H^s}^2 \le \frac{C\,(N_1 - N_2)^2}{N}. \tag{2.15}$$

Since $2s < 1$ we can choose

$$N_1 = N \ \text{ and } \ N_2 = N + \delta N^s \tag{2.16}$$

to have

$$\|\varphi_{c_1,\omega_1} - \varphi_{c_2,\omega_2}\|_{H^s}^2 \le C\,\delta^2 N^{2s-1} \le C\,\delta^2.$$

Next we consider the corresponding solutions $u_{c_1,\omega_1}(x,t)$ and $u_{c_2,\omega_2}(x,t)$ at time $t = T$. As in (2), we get, from (2.12),

$$\begin{aligned} \|\varphi_{c,\omega}\|_{H^s}^2 &\simeq \frac{d_6^{2s+1}}{(-\omega)^{\frac{1}{2}}} \int |\eta|^{2s} |\hat{F}(\eta - \frac{c}{2d_6})|^2 d\eta \\ &\simeq \frac{N^{2s(2s+1)}}{N} N^{(1-2s)2s} \|F\|^2 \simeq C. \end{aligned} \tag{2.17}$$

We shall compute $\|u_{c_1,\omega_1}(\cdot,T) - u_{c_2,\omega_2}(\cdot,T)\|_{H^s}$, using the fact that

$$\|u_{c_j,\omega_j}(\cdot,T)\|_{H^s} = \|\varphi_{c_j,\omega_j}\|_{H^s} \simeq C,$$

$j = 1, 2$ (by the invariance of the solitary wave solutions), and

$$\|u_{c_1,\omega_1}(\cdot,T) - u_{c_2,\omega_2}(\cdot,T)\|_{H^s}^2 \ge N^{2s} \|u_{c_1,\omega_1}(\cdot,T) - u_{c_2,\omega_2}(\cdot,T)\|^2;$$

on the other hand, we have that

$$u_{c_j,\omega_j}(x,T) = e^{-i\omega_j T}\ e^{i\psi(x - c_j\,T)} \frac{d_{6j}}{(-\omega_j)^{1/4}}\, h(d_{6j}(x - c_j\,T)), \qquad j = 1, 2.$$

The support of $u_{c_j,\omega_j}(T)$ is concentrated in $B_{(d_{6j})^{-1}}(Tc_j),\ j=1,2$. Thus, given $\delta>0$ and $T>0$ if c_1,c_2 are chosen such that

$$T(c_2-c_1)\gg \max(\frac{1}{d_{61}},\frac{1}{d_{62}})\simeq N^{-2s}, \tag{2.18}$$

there is no interaction and

$$\|u_{c_1,\omega_1}(\cdot,T)-u_{c_2,\omega_2}(\cdot,T)\|^2\simeq\|u_{c_1,\omega_1}(\cdot,T)\|^2+\|u_{c_2,\omega_2}(\cdot,T)\|^2\simeq N^{-2s}.$$

Combining the above estimates we get

$$\|u_{c_1,\omega_1}(\cdot,T)-u_{c_2,\omega_2}(\cdot,T)\|^2_{H^s}\geq C.$$

So, if we choose N such that $N^{3s}\gg\frac{1}{T\delta}$ then, from the choice of c_1,c_2 in (2.16)

$$T(c_2-c_1)=T\delta N^s>>N^{-2s}$$

and we get (2.18).

3. The generalized Benjamin-Ono equation

The generalized Benjamin-Ono equation has, for $k=1$, an explicit solitary wave

$$u_c(x,t)=\frac{4c}{(cx-c^2t)^2+1}, \tag{3.1}$$

and for $k>1$, Weinstein in [23] proved that there exists a solitary wave $u_c(x,t)=\varphi_c(x-ct)$ of (1.3), that is, φ_c is a solution of

$$-c\varphi'+\varphi^k\varphi'+(\mathcal{H}\varphi)''=0 \tag{3.2}$$

where $\varphi_c\in C^\infty(\mathbb{R})\cap H^{1/2}(\mathbb{R})$ is positive, symmetric and decreasing in $|x|$. $\square$

Theorem 3.1. *The initial value problem (1.3) with $k=1$ is ill-posed in $H^s(\mathbb{R})$, $s<-1/2$ and, with $k>1$, it is ill-posed in $\dot{H}^{s_k}(\mathbb{R})$ with $s_k=1/2-1/k$ in the sense that the mapping data-solution, $u_0\to u(t)$ is not uniformly continuous.*

Proof. For $k=1$ and $s>1/2,\ c=1/\varepsilon$ we have, from (3.1),

$$\|u_\varepsilon(\cdot,0)\|^2_{H^{-s}}=\|(1-\Delta)^{-s/2}u_\varepsilon(\cdot,0)\|^2=16\pi^2\int_{-\infty}^{\infty}\frac{e^{-4\pi\varepsilon|\xi|}}{(1+|\xi|^2)^s}d\xi \tag{3.3}$$

which converges uniformly as $\varepsilon\to0$ to $16\pi^2\|\delta\|^2_{H^{-s}}$ (since $\hat{\delta}=1$). Now we prove that $u_\varepsilon(\cdot,0)$ converges also weakly to $4\pi\delta$: let $\varphi\in C_o^\infty(\mathbb{R})$, then

$$\begin{aligned}\lim_{\varepsilon\to0}\langle u_\varepsilon,\varphi\rangle&=\lim_{\varepsilon\to0}\int\frac{4/\varepsilon}{x^2/\varepsilon^2+1}\varphi(x)dx=4\lim_{\varepsilon\to0}\frac{\varphi(\varepsilon y)}{y^2+1}dy\\&=4\pi\varphi(0)=4\pi\langle\delta,\varphi\rangle,\end{aligned}$$

thus implying the convergence in H^{-s}. For $t>0$ the invariance in t implies the convergence of $\|u_\varepsilon(\cdot,t)\|_{H^{-s}}$ to $4\phi\|\delta\|_{H^{-s}}$ but $u_\varepsilon(\cdot,t)\to0$ weakly, since, as φ has compact support,

$$\begin{aligned}\lim_{\varepsilon\to0}\langle u_\varepsilon,\varphi\rangle&=\int\frac{4/\varepsilon}{(x/\varepsilon-t/\varepsilon^2)^2+1}\varphi(x)dx\\&=4\lim_{\varepsilon\to0}\int\frac{\varphi(t/\varepsilon+\varepsilon y)}{y^2+1}dy=0.\end{aligned}$$

This proves that the initial value problem for the Benjamin-Ono equation is locally ill-posed in $H^s(\mathbb{R})$ for $s<-1/2$.

Now, for $k > 1$, let

$$\varphi_{k,c}(x) = c^{1/k}\varphi_1(cx),$$

where φ_1 solves (3.2) with $c = 1$ ($\varphi_{k,c}$ solves (3.2) with $c > 0$), and

$$u_{k,c}(x,t) = \varphi_{k,c}(x - ct) = c^{1/k}\varphi_1(cx - c^2t)$$

which is a solitary wave solution for (1.3) with speed of propagation c.

Let us evaluate the $\dot{H}^{s_k}$-norm of the difference of two solitary wave solutions with speeds c_1, c_2 at $t = 0$ and $t > 0$:

$$\begin{aligned} \|(u_{k,c_1} - u_{k,c_2})(\cdot,0)\|^2_{\dot{H}^{s_k}} &= \|D^{s_k}(\varphi_{k,c_1} - \varphi_{k,c_2})\|^2 \\ &= \|D^{s_k}\varphi_{k,c_1}\|^2 + \|D^{s_k}\varphi_{k,c_2}\|^2 - 2\langle\varphi_{k,c_1}, \varphi_{k,c_2}\rangle_{s_k}. \end{aligned} \tag{3.4}$$

We have

$$\begin{aligned} \langle\varphi_{k,c_1}, \varphi_{k,c_2}\rangle_{s_k} &= \int D^{s_k}\varphi_{k,c_1}(x)\overline{D^{s_k}\varphi_{k,c_2}(x)}dx \\ &= \int \hat{\varphi}_{k,c_1}(\xi)\overline{\hat{\varphi}_{k,c_2}(\xi)}|\xi|^{2s_k}d\xi \\ &= (c_1c_2)^{1/k-1}\int \hat{\varphi}_1(\xi/c_1)\overline{\hat{\varphi}_1(\xi/c_2)}|\xi|^{2s_k}d\xi \\ &= (c_1/c_2)^{1-1/k}\int \hat{\varphi}_1(\eta)\overline{\hat{\varphi}_1(c_1/c_2\eta)}|\eta|^{2s_k}d\eta. \end{aligned}$$

As $\theta := c_1/c_2 \to 1$ we get

$$\lim_{\theta\to 1}\langle\varphi_{k,c_1}, \varphi_{k,c_2}\rangle_{s_k} = \|D^{s_k}\varphi_1\|^2 = \|\varphi_1\|^2_{\dot{H}^{s_k}}. \tag{3.5}$$

Analogously, for $i = 1, 2$,

$$\begin{aligned} \|\varphi_{k,c_i}\|^2_{\dot{H}^{s_k}} &= \int |\xi|^{2s_k}|\hat{\varphi}_{k,c_i}(\xi)|^2 d\xi \\ &= c_i^{2/k-2}\int |\xi|^{2s_k}|\hat{\varphi}_1(\xi/c_i)|^2 d\xi \\ &= c_i^{2s_k+1+2/k-2}\int |\eta|^{2s_k}|\hat{\varphi}_1(\eta)|^2 d\eta = \|\varphi_1\|^2_{\dot{H}^{s_k}}. \end{aligned} \tag{3.6}$$

Replacing (3.5) and (3.6) in (3.4) and taking limits as $\theta \to 1$ we get

$$\lim_{\theta\to 1}\|(u_{k,c_1} - u_{k,c_2})(\cdot,0)\|^2_{\dot{H}^{s_k}} = 0. \tag{3.7}$$

Now for $t > 0$ we have similarly

$$\begin{aligned} \langle u_{k,c_1}(\cdot,t), u_{k,c_2}(\cdot,t)\rangle_{s_k} &= \int D^{s_k}\varphi_{k,c_1}(x - c_1t)\overline{D^{s_k}\varphi_{k,c_2}(x - c_2t)}dx \\ &= \int e^{-2\pi i\xi t(c_1-c_2)}\hat{\varphi}_{k,c_1}(\xi)\overline{\hat{\varphi}_{k,c_2}(\xi)}|\xi|^{2s_k}d\xi \\ &= \int e^{-2\pi i\xi t(c_1-c_2)}c_1^{\frac{1}{k}-1}\hat{\varphi}_1(\xi/c_1)c_2^{1/k-1}\overline{\hat{\varphi}_1(\xi/c_2)}|\xi|^{2s_k}d\xi \\ &= (c_1c_2)^{\frac{1}{k}-1}\int e^{-2\pi itc_1\eta(c_1-c_2)}\hat{\varphi}_1(\eta)\overline{\hat{\varphi}_1(\frac{c_1}{c_2}\eta)}c_1^{2s_k+1}|\eta|^{2s_k}d\eta \\ &= (\frac{c_1}{c_2})^{1-\frac{1}{k}}\int e^{-2\pi itc_1\eta(c_1-c_2)}\hat{\varphi}_1(\eta)\overline{\hat{\varphi}_1(\frac{c_1}{c_2}\eta)}|\eta|^{2s_k}d\eta. \end{aligned}$$

Taking $c_1 = n+1$ and $c_2 = n$ and $n \to \infty$ we have, by the Riemann-Lebesgue lemma,

$$\lim_{n\to\infty} \langle u_{k,n+1}(\cdot,t), u_{k,n}(\cdot,t)\rangle_{s_k} = 0; \tag{3.8}$$

since

$$\|u_{k,c_1}(\cdot,t)\|_{\dot{H}^{s_k}} = \|u_{k,c_2}(\cdot,t)\|_{\dot{H}^{s_k}} = \|\varphi_1\|_{\dot{H}^{s_k}}$$

we have

$$\|u_{k,c_1}(\cdot,t) - u_{k,c_2}(\cdot,t)\|_{\dot{H}^{s_k}} \to 2^{1/2}\|\varphi_1\|_{\dot{H}^{s_k}} \neq 0.$$

□

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