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Image Decomposition into a Bounded Variation Component and an Oscillating Component

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Abstract. We construct an algorithm to split an image into a sum $u + v$ of a bounded variation component and a component containing the textures and the noise. This decomposition is inspired from a recent work of Y. Meyer. We find this decomposition by minimizing a convex functional which depends on the two variables u and v, alternately in each variable. Each minimization is based on a projection algorithm to minimize the total variation. We carry out the mathematical study of our method. We present some numerical results. In particular, we show how the u component can be used in nontextured SAR image restoration.

Keywords: total variation minimization, *BV*, texture, restoration, SAR images, speckle

1. Introduction

1.1. Preliminaries

Image restoration is one of the major goals of image processing. A classical approach consists in considering that an image f can be decomposed into two components $u+v$. The first component u is well-structured, and has a simple geometric description: it models the homogeneous objects which are present in the image. The second component v contains the oscillating patterns (both textures and noise). An ideal model would split an image into three components $u + v + w$, where v should contain the textures of the original image, and w the noise.

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In Section 1, we begin by recalling some models proposed in the literature. Then our model is introduced in Section 2. We give a powerful algorithm to compute the image decomposition we want to get. We carry out the mathematical study of our model in Section 3. We then show some experimental results. We compare our algorithm with the classical total variation minimization method in Section 4. In Section 5, we give an application to SAR images, the u component being a way to carry out efficient restoration.

1.2. Related Works

1.2.1. Rudin-Osher-Fatemi's (ROF) Model. Images are often assumed to be in $BV(\Omega)$, the space of functions with bounded variation (even if it is known that such an assumption is too restrictive [1]). We recall here the definition of $BV(\Omega)$ (we suppose that Ω, the domain of the image, is a bounded Lipschitz open set):

Definition 1.1. $BV(\Omega)$ is the subspace of functions $u \in L^1(\Omega)$ such that the following quantity is finite:

$$J(u) = \sup\left\{ \int_\Omega u(x)\,\mathrm{div}\,(\xi(x))dx / \xi \in C_c^1(\Omega;\mathbb{R}^2), \|\xi\|_{L^\infty(\Omega)} \le 1 \right\} \tag{1.1}$$

$BV(\Omega)$ endowed with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + J(u)$ is a Banach space.

If $u \in BV(\Omega)$, the distributional derivative Du is a bounded Radon measure and (1.1) corresponds to the total variation $|Du|(\Omega)$.

In [11], the authors decompose an image f into a component u belonging to $BV(\Omega)$ and a component v in $L^2(\Omega)$. In this model v is supposed to be the noise. In such an approach, they minimize (see [11]):

$$\inf_{(u,v)\in BV(\Omega)\times L^2(\Omega)/f=u+v} \left(J(u) + \frac{1}{2\lambda}\|v\|^2_{L^2(\Omega)} \right) \tag{1.2}$$

In practice, they compute a numerical solution of the Euler-Lagrange equation associated to (1.2). The mathematical study of (1.2) has been done in [4].

1.2.2. Meyer's Model. In [8], Y. Meyer points out some limitations of the model developed in [11]. He proposes a different decomposition which he believes is more adapted:

$$\inf_{(u,v)\in BV(\mathbb{R}^2)\times G(\mathbb{R}^2)/f=u+v} (J(u) + \alpha\|v\|_G) \tag{1.3}$$

The Banach space $G(\mathbb{R}^2)$ contains signals with large oscillations, and thus in particular textures and noise. We give here the definition of $G(\mathbb{R}^2)$.

Definition 1.2. $G(\mathbb{R}^2)$ is the Banach space composed of the distributions f which can be written

$$f = \partial_1 g_1 + \partial_2 g_2 = \mathrm{div}(g) \tag{1.4}$$

with g_1 and g_2 in $L^\infty(\mathbb{R}^2)$. On G, the following norm is defined:

$$\|v\|_G = \inf\left\{ \|g\|_{L^\infty(\mathbb{R}^2)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^2} |g(x)| / v = \mathrm{div}(g), \; g = (g_1, g_2),\; g_1 \in L^\infty(\mathbb{R}^2), \quad g_2 \in L^\infty(\mathbb{R}^2), \; |g(x)| = \sqrt{|g_1|^2 + |g_2|^2}(x) \right\} \tag{1.5}$$

The justification of the introduction of the space G to model patterns with strong oscillations comes from the next result (see [8]):

Lemma 1.1. *let f_n, $n \ge 1$ a sequence of functions in L^2 (D) with the three following properties (D is a disc centered at 0 with radius R):*

1. *There exists a compact set K such that the supports of the f_n, $n \ge 1$ are embedded in K.*
2. *There exists $q > 2$ and $C > 0$ such that $||f_n||_{L^q(\mathbb{R}^2)} \le C$*
3. *The sequence f_n converges to 0 in the distributional sense.*

Then $\|f_n\|_G$ converges to 0 when n tends towards infinity.

A function belonging to G may have large oscillations and nevertheless have a small norm.

1.2.3. Vese-Osher's Model. Vese and Osher have first proposed an approach for the resolution of Meyer's program. They have studied the problem (see [12]) $(f \in L^2(\Omega))$:

$$\inf_{(u,v)\in BV(\Omega)\times G(\Omega)} \left(\int |Du| + \lambda\|f-u-v\|_2^2 + \mu\|v\|_{G(\Omega)} \right) \tag{1.6}$$

where Ω is a bounded open set. To compute their solution, they replace the term $\|v\|_{G(\Omega)}$ by $||\sqrt{g_1^2+g_2^2}||_p$

(where $v = \text{div}(g_1, g_2)$), which approximates the L^∞-norm when p goes to $+\infty$. For numerical reasons, the authors use the value $p = 1$, and they claim they did not see any visual difference when they used larger values for p. Then they formally derive the Euler-Lagrange equations. They report good numerical results.

These two authors, together with Solé, have proposed another approach to this problem in [9], where they propose a more direct algorithm in the case $\lambda = +\infty$ and $p = 2$.

2. Our Approach

In this section we introduce our model. We first formulate it in the continuous-setting. Then we propose a discretization, and provide a mathematical study and an algorithm for the discretized model.

2.1. Presentation

We propose to solve the following variant of Osher and Vese's functional [12]:

$$\inf_{(u,v)\in BV(\Omega)\times G_\mu(\Omega)} \left(J(u) + \frac{1}{2\lambda}\|f - u - v\|^2_{L^2(\Omega)} \right) \tag{2.7}$$

where

$$G_\mu(\Omega) = \{v \in G(\Omega)/\|v\|_G \leq \mu\} \tag{2.8}$$

We remind that $\|v\|_G$ is defined by (1.5) (where we replace $\mathbb{R}^2$ by Ω). The parameter μ plays the same role as the one in problem (1.6). We will precise the link of our model with Meyer's one later (we will get it by letting $\lambda \to 0$). Let us introduce the following functional defined on $BV(\Omega) \times G(\Omega)$:

$$F_{\lambda,\mu}(u, v) = \begin{cases} J(u) + \dfrac{1}{2\lambda}\|f - u - v\|^2_{L^2(\Omega)} \\ \text{if } v \in G_\mu(\Omega) \\ +\infty \quad \text{if } v \in G(\Omega)\backslash G_\mu(\Omega) \end{cases} \tag{2.9}$$

$F_{\lambda,\mu}(u, v)$, is finite if and only if (u, v) belongs to $BV(\Omega) \times G_\mu(\Omega)$. Problem (2.7) can thus be written:

$$\inf_{(u,v)\in BV(\Omega)\times G(\Omega)} F_{\lambda,\mu}(u, v) \tag{2.10}$$

2.2. Discretization

We are now going to study (2.10) in the discrete case. We take here the same notations as in [3]. The image is a two dimension vector of size $N \times N$. We denote by X the Euclidean space $\mathbb{R}^{N\times N}$, and $Y = X \times X$. The space X will be endowed with the scalar product $(u, v)_X = \sum_{1\leq i,j\leq N} u_{i,j} v_{i,j}$ and the norm $\|u\|_X = \sqrt{(u, u)_X}$ .To define a discrete total variation, we introduce a discrete version of the gradient operator. If $u \in X$, the gradient ∇u is a vector in Y given by: $(\nabla u)_{i,j} = ((\nabla u)^1_{i,j'}, (\nabla u)^2_{i,j})$, with

$$(\nabla u)^1_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{if } i < N \\ 0 & \text{if } i = N \end{cases} \quad \text{and}$$

$$(\nabla u)^2_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{if } j < N \\ 0 & \text{if } j = N \end{cases}$$

The discrete total variation of u is then defined by:

$$J_d(u) = \sum_{1\leq i,j\leq N} |(\nabla u)_{i,j}| \tag{2.11}$$

We also introduce a discrete version of the divergence operator. We define it by analogy with the continuous setting by div $= -\nabla^*$ where ∇^* is the adjoint of ∇: that is, for every $p \in Y$ and $u \in X$, $(-\text{div}\, p, u)_X = (p, \nabla u)_Y$. It is easy to check that:

$$(\text{div}\,(p))_{i,j} = \begin{cases} p^1_{i,j} - p^1_{i-1,j} & \text{if } 1 < i < N \\ p^1_{i,j} & \text{if } i = 1 \\ -p^1_{i-1,j} & \text{if } i = N \end{cases} + \begin{cases} p^2_{i,j} - p^2_{i,j-1} & \text{if } 1 < j < N \\ p^2_{i,j} & \text{if } j = 1 \\ -p^2_{i,j-1} & \text{if } j = N \end{cases} \tag{2.12}$$

From now on, we will use these discrete operators.

We are now in position to introduce the discrete version of the space G.

Definition 2.3.

$$G^d = \{v \in X/\exists g \in Y \text{ such that } v = \text{div}\,(g)\} \tag{2.13}$$

and if $v \in G^d$:

$$\|v\|_{G^d} = \inf\left\{\|g\|_\infty/v = \text{div}\,(g),\, g = (g^1, g^2) \in Y,\ |g_{i,j}| = \sqrt{\left(g^1_{i,j}\right)^2 + \left(g^2_{i,j}\right)^2}\right\} \tag{2.14}$$

where $\|g\|_\infty = \max_{i,j} |g_{i,j}|$.

Moreover, we will denote:

$$G_\mu^d = \{v \in G^d / ||v||_{G^d} \leq \mu\} \tag{2.15}$$

$\|\cdot\|_{G^d}$ is closely linked with J_d, as stated by the following proposition.

Proposition 2.1.

$$J_d(u) = \sup_{v \in G_1^d} (u, v)_X \tag{2.16}$$

and

$$||v||_{G^d} = \sup_{J_d(u) \leq 1} (u, v)_X \tag{2.17}$$

We already know (2.16). To prove (2.17), we need the following lemma (which is stated in [8]).

Lemma 2.2. *Let $u \in X$ and $v \in G^d$. Then*:

$$(u, v)_X \leq J_d(u)||v||_{G^d} \tag{2.18}$$

Proof: Let $g \in Y$ such that $v = \mathrm{div}\,(g)$.

$$\begin{aligned}(u, v)_x &= (u, \mathrm{div}(g))_X \\ &= -(\nabla u, g)_Y \leq J_d(u)||v||_{G^d}\end{aligned} \tag{2.19}$$

And we deduce (2.18) from it. □

We also need the next result:

Lemma 2.3. *The functions $u \mapsto \frac{J_d(u)^2}{2}$ and $v \mapsto \frac{||v||^2_{G^d}}{2}$ are dual in the sense of the Legendre-Fenchel duality.*

Proof: We recall here (see [5, 6, 10]) the definition of the Legendre-Fenchel transform of H:

$$H^*(v) = \sup_{u \in X}((u, v)_X - H(u)) \tag{2.20}$$

We want to show that $u \mapsto \frac{J_d(u)^2}{2}$ and $v \mapsto \frac{||v||^2_{G^d}}{2}$ are dual with respect to this definition. Let us denote by ϕ the function $\phi(t) = \frac{t^2}{2}$. It is well known that $\phi^* = \phi$.

$$\begin{aligned}\left(\frac{\|\cdot\|^2_{G^d}}{2}\right)^*(u) &= \sup_{v \in X}((u, v)_X - \phi(||v||_{G^d}))) \\ &= \sup_{t \geq 0} \sup_{||v||_{G^d=t}} ((u, v)_X - \phi(||v||_{G^d})) \\ &= \sup_{t \geq 0} \sup_{||v||_{G^d=1}} ((u, tv)_X - \phi(t||v||_{G^d})) \\ &= \sup_{||v||_{G^d=t}} \sup_{t \geq 0}(t(u, v)_X - \phi(t))\end{aligned}$$

And since ϕ is even we get:

$$\begin{aligned}\left(\frac{\|\cdot\|^2_{G^d}}{2}\right)^*(u) &= \sup_{||v||_{G^d=t}} \phi^*((u, v)_X) \\ &= \frac{1}{2}\left(\sup_{||v||_{G^d=t}} (u, v)_X\right)^2\end{aligned} \tag{2.21}$$

But $\sup_{||v||_{G^d=1}}(u, v)_X = \sup_{||v||_{G^d \leq 1}}(u, v)_X$, and we conclude with (2.16). □

Proof of Proposition 2.1: We want to prove (2.17). Lemma 2.2 gives an inequality. Let us show the reverse inequality. We denote by ∂H the subdifferential of H(see [6, 10]), and we recall that

$$w \in \partial H(u) \Longleftrightarrow H(v) \geq H(u) + (w, v - u)_X, \quad \text{for all } v \text{ in } X \tag{2.22}$$

Let $v \in G^d$. We recall that (see [5]), if H is convex: $H(u) + H^*(v) = (u, v)_X$ if and only if $u \in \partial H^*(v)$. We apply this result with $H^*(v) = \left(\frac{J_d(\cdot)^2}{2}\right)^*(v)$. Since $\left(\frac{J_d(\cdot)^2}{2}\right)^*$ is convex continuous, we know that $\partial H^*(v)$ is not empty. Let $u \in \partial H^*(v)$. From Lemma 2.3, we get: $\frac{J_d(u)^2}{2} + \frac{||v||^2_{G^d}}{2} = (u, v)_X$, i.e:

$$\underbrace{(J_d(u) - \|v\|_{G^d})^2}_{\geq 0} = 2((u, v)_X - J_d(u)||v||_{G^d}) \tag{2.23}$$

Hence $(u, v)_X \geq J_d(u)\|v\|_{G^d}$. And this concludes the proof thanks to Lemma 2.2. □

Proposition 2.2. *The space G^d identifies with the following subspace:*

$$X_0 = \left\{v \in X \middle/ \sum_{i,j} v_{i,j} = 0\right\} \tag{2.24}$$

Proof: We split our proof into two steps.

Step 1: Let us assume that $v \in G^d$. Therefore, there exists $g \in Y$ such that: $v = \mathrm{div}(g)$. But $\sum_{i,j}(\mathrm{div}\ \mathrm{g})_{i,j} = (-\nabla^* g, 1)_Y = (g, \nabla 1)_X = 0$ i.e. $v \in X_0$. Hence $G^d \subset X_0$.

Step 2: Conversely, let $v \in X_0$. Since the kernel of ∇ is the constant images, i.e. the vectors $x \in X$ such that $x_{i,j} = x_{i',j'}$ for all i, j, i', j', it is clear that a discrete Poincaré inequality holds: $||x - \frac{1}{N^2}\sum_{i,j} x_{i,j}||_X \leq c||\nabla x||_Y$. Hence one shows easily that the problem $\min_{x \in X} A(x)$, with $A(x) = \|\nabla x\|^2 + 2(x, v)$, has a solution. This solution satisfies $A'(x) = 0$, that is, $-2\text{div}\,(\nabla x) + 2v = 0$. Hence $v = \text{div}\,(\nabla x) \in G^d$, and we conclude that $X_0 \subset G^d$.

□

The discretized functional associated to (2.9), defined on $X \times X$, is given by:

$$F_{\lambda,\mu}(u, v) = \begin{cases} J_d(u) + \frac{1}{2\lambda}||f - u - v||_X^2 \\ \text{if } v \in G_\mu^d \\ +\infty \quad \text{if } v \in X \backslash G_\mu^d \end{cases} \tag{2.25}$$

The problem we want to solve is:

$$\inf_{(u,v) \in X \times X} F_{\lambda,\mu}(u, v) \tag{2.26}$$

2.3. Total Variation Minimization as a Projection

Introduction: Since J_d defined by (2.11) is homogeneous of degree one (i.e. $J_d(\lambda u) = \lambda J_d(u) \forall u$ and $\lambda > 0$), it is then standard (see [5]) that J_d^* (see (2.20)) is the indicator function of some closed convex set, which turns out to be the set G_1^d defined by (2.15):

$$J_d^*(v) = \chi_{G_1^d}(v) = \begin{cases} 0 & \text{if } v \in G_1^d \\ +\infty & \text{otherwise} \end{cases} \tag{2.27}$$

This can be checked out easily (see [3] for details). In [3], Chambolle proposes a nonlinear projection algorithm to minimize the ROF model. The problem is:

$$\inf_{u \in X} \left(J_d(u) + \frac{1}{2\lambda}||f - u||_X^2 \right) \tag{2.28}$$

The following result is shown:

Proposition 2.3. *The solution of* (2.28) *is given by*:

$$u = f - P_{G_\lambda^d}(f) \tag{2.29}$$

where P is the orthogonal projector on G_λ^d *(defined by* (2.15)).

Algorithm. [3] *gives an algorithm to compute* $P_{G_\lambda^d}(f)$. *It indeed amounts to finding*:

$$\min \left\{ ||\lambda \,\text{div}\,(p) - f||_X^2 : p/|p_{i,j}| \leq 1 \forall_{i,j} = 1, \ldots, N \right\} \tag{2.30}$$

This problem can be solved by a fixed point method:

$$p^0 = 0 \tag{2.31}$$

and

$$p_{i,j}^{n+1} = \frac{p_{i,j}^n + \tau(\nabla(\text{div}(p^n) - f/\lambda))_{i,j}}{1 + \tau|(\nabla(\text{div}(p^n) - f/\lambda))_{i,j}|} \tag{2.32}$$

In [3] *is given a sufficient condition ensuring the convergence of the algorithm*:

Theorem 2.1. *Assume that the parameter* τ *in* (2.32) *verifies* $\tau \leq 1/8$. *Then* $\lambda\text{div}(p^n)$ *converges to* $P_{G_\lambda^d}(f)$ *as* $n \to +\infty$.

2.4. Application to Problem (2.26)

Since J_d^* is the indicator function of G_1^d (see (2.16, 2.27)), we can rewrite (2.25) as

$$F(u, v) = \frac{1}{2\lambda}||f - u - v||_X^2 + J_d(u) + J_d^*\left(\frac{v}{\mu}\right) \tag{2.33}$$

With this formulation, we see the symmetric roles played by u and v. And the problem we want to solve is:

$$\inf_{(u,v) \in X \times X} F(u, v) \tag{2.34}$$

To solve (2.34), we consider the two following problems:

- v being fixed, we search for u as a solution of:

$$\inf_{v \in X} \left(J_d(u) + \frac{1}{2\lambda}||f - u - v||_X^2 \right) \tag{2.35}$$

- u being fixed, we search for v as a solution of:

$$\inf_{v \in G_\mu^d} ||f - u - v||_X^2 \tag{2.36}$$

From Proposition 2.3, we know that the solution of (2.35) is given by: $\hat{u} = f - v - P_{G^d_\lambda}(f - v)$. And the solution of (2.36) is simply given by: $\hat{v} = P_{G^d_\mu}(f - u)$.

2.5. *Algorithm*

1. Initialization:

$$u_0 = v_0 = 0 \tag{2.37}$$

2. Iterations:

$$v_{n+1} = P_{G^d_\mu}(f - u_n) \tag{2.38}$$

$$u_{n+1} = f - v_{n+1} - P_{G^d_\lambda}(f - v_{n+1}) \tag{2.39}$$

3. Stopping test: we stop if

$$\max(|u_{n+1} - u_n|, |v_{n+1} - v_n|) \leq \epsilon \tag{2.40}$$

3. Mathematical Results

In this section we carry out the mathematical study of the algorithm (2.37)–(2.40). We first show its convergence when λ is fixed. We then precise the link of the limit of our model (when λ goes to 0) with Meyer's one.

3.1. *Existence and Uniqueness of a Solution for (2.26)*

Lemma 3.4. *There exists a unique couple $(\hat{u}, \hat{v}) \in X \times G^d_\mu$ minimizing $F_{\lambda,\mu}$ on $X \times X$.*

Proof: We split the proof into two steps.

Step 1. Existence

1. We first remark that the set $X \times G^d_\mu$ is convex, and then that $F_{\lambda,\mu}$ is convex on $X \times G^d_\mu$. We thus deduce that $F_{\lambda,\mu}$ is convex on $X \times X$.
2. It is immediate to see that $F_{\lambda,\mu}$ is continuous on $X \times G^d_\mu$. We then deduce that $F_{\lambda,\mu}$ is lower semi-continuous on $X \times X$.
3. Let $(u, v) \in X \times G^d_\mu$. We have $\|v\|_{G^d} \leq \mu$. Moreover, since X is of finite dimension, there exists $g \in X$ such that $u = \operatorname{div}(g)$ and $\|g\|_{L^\infty} = \|v\|_{G^d} \leq \mu$. We deduce from (2.12) that (N^2 is the size of the image):

$$\|v\|_X = \leq 4\,\mu N^2 \tag{3.41}$$

We recall that $X \times X$ is endowed with the Euclidean norm.

$$\|(u, v)\|_{X\times X} = \sqrt{\|u\|^2_X + \|v\|^2_X} \tag{3.42}$$

Thus, if $\|(u, v)\|_{X\times X} \longrightarrow +\infty$, then we get from (3.41) that $\|u\|_X \longrightarrow +\infty$. We therefore deduce, since f is fixed, and since (3.41) holds, that $\|f - u - v\|^2_X \longrightarrow +\infty$. And since $F_{\lambda,\mu}(u, v) \geq \frac{1}{2\lambda}\|f - u - v\|^2_2$, we get $F_{\lambda,\mu}(u, v) \to +\infty$. Hence we deduce that $F_{\lambda,\mu}$ is coercive on $X \times G^d_\mu$. We therefore conclude that $F_{\lambda,\mu}$ is coercive on $X \times X$.

We deduce the existence of a minimizer $(\hat{u}, \hat{v})$.

Step 2. Uniqueness

To get the uniqueness, we first remark that $F_{\lambda,\mu}$ is strictly convex on $X \times G^d_u$, as the sum of a convex function and of a strictly convex function, except in the direction $(u, -u)$. Hence it suffices to check that if $(\hat{u}, \hat{v})$ is a minimizer of $F_{\lambda,\mu}$. Then for $t \neq 0$, $(\hat{u} + t\hat{u}, \hat{v} - t\hat{u})$ is not a minimizer of $F_{\lambda,\mu}$. The result is obvious if $\hat{v} - t\hat{u} \in X\backslash G^d_\mu$. Let us show that if $\hat{v} - t\hat{u} \in G^d_\mu$ then the result is still true. Indeed, if $\hat{v} - t\hat{u} \in G^d_\mu$, we have:

$$\begin{aligned} &F_{\lambda,\mu}(\hat{u} + t\hat{u}, \hat{v} - t\hat{u}) \\ &\quad = F_{\lambda,\mu}(\hat{u}, \hat{v}) + (|1 + t| - 1)J_d(\hat{u}) \end{aligned} \tag{3.43}$$

By contradiction, let us assume that there exists $\hat{t} \neq \{-2, 0\}$ such that $\hat{v} - \hat{t}\hat{u} \in G^d_\mu$ and

$$F_{\lambda,\mu}(\hat{u} + \hat{t}\hat{u}, \hat{v} - \hat{t}\hat{u}) \leq F_{\lambda,\mu}(\hat{u}, \hat{v}) \tag{3.44}$$

As $(\hat{u}, \hat{v})$ minimizes $F_{\lambda,\mu}$, (3.44) is an equality. From (3.43), we deduce that $(|1 + \hat{t}| - 1)J_d(\hat{u}) = 0$. And as $\hat{t} \neq \{-2, 0\}$, we get that $J_d(\hat{u}) = 0$. There exists therefore $\gamma \in \mathbb{R}$ such that for all (i, j), $\hat{u}_{i,j} = \gamma$.

1. If $\gamma = 0$, then $\hat{u} = 0$. Thus $(\hat{u} + \hat{t}\hat{u}, \hat{v} - \hat{t}\hat{u}) = (\hat{u}, \hat{v})$.
2. If $\gamma \neq 0$, then $\hat{v} - \hat{t}\hat{u}$ cannot belong to G^d_μ since its mean is not 0 (see Proposition 2.2). This contradicts our assumption.

There remains to check what happens in the case when $\hat{t} = -2$. In this case, by convexity, we get that if $t \in (-2, 0)$, then $\hat{v} - t\hat{u} \in G^d_\mu$ and

$$F_{\lambda,\mu}(\hat{u} + t\hat{u}, \hat{v} - t\hat{u}) \leq F_{\lambda,\mu}(\hat{u}, \hat{v}) \quad (3.45)$$

Thus we get (3.44), and we can conclude. □

3.2. Convergence of the Algorithm

We show here that our algorithm gives asymptotically the solution of the discrete problem associated to (2.34).

Proposition 3.4. *The sequence $F_{\lambda,\mu}(u_n, v_n)$ built in Section 2.5 converges to the minimum of $F_{\lambda,\mu}$ on $X \times X$.*

Proof: We first remark that, as we solve successive minimization problems, we have:

$$F_{\lambda,\mu}(u_n, v_n) \geq F_{\lambda,\mu}(u_n, v_{n+1}) \geq F_{\lambda,\mu}(u_{n+1}, v_{n+1}) \quad (3.46)$$

In particular, the sequence $F_{\lambda,\mu}(u_n, v_n)$ is nonincreasing. As it is bounded from below by 0, it thus converges in $\mathbb{R}$. We denote by m its limit. We want to show that

$$m = \inf_{(u,v)\in X\times X} F_{\lambda,\mu}(u, v) \quad (3.47)$$

Without any restriction, we can assume that, $\forall n, (u_n, v_n) \in X \times G^d_\mu$. As $F_{\lambda,\mu}$ is coercive and as the sequence $F_{\lambda,\mu}(u_n, v_n)$ converges, we deduce that the sequence (u_n, v_n) is bounded in $X \times G^d_\mu$. We can thus extract a subsequence (u_{n_k}, v_{n_k}) which converges to $(\hat{u}, \hat{v})$ as $n_k \to +\infty$ with $(\hat{u}, \hat{v}) \in X \times G^d_\mu$. Moreover, we have, for all $n_k \in \mathbb{N}$ and all v in X :

$$F_{\lambda,\mu}\left(u_{n_k}, v_{nk+1}\right) \leq F_{\lambda,\mu}\left(u_{n_k}, v\right) \quad (3.48)$$

and for all $n_k \in \mathbb{N}$ and all u in X:

$$F_{\lambda,\mu}\left(u_{n_k}, v_{n_k}\right) \leq F_{\lambda,\mu}\left(u, v_{n_k}\right) \quad (3.49)$$

Let us denote by v a cluster point of (v_{n_k+1}).Considering (3.46), we get (since $F_{\lambda,\mu}$ is continuous on $X \times G^d_\mu$):

$$m = F_{\lambda,\mu}(\hat{u}, \hat{v}) = F_{\lambda,\mu}(\hat{u}, \hat{v}) \quad (3.50)$$

By passing to the limit in (2.38), we get: $\tilde{v} = P_{G^d_\mu}(f - \hat{u})$. But from (3.50), we know that: $\|f - \hat{u} - \hat{v}\| = \|f - \hat{u} - \tilde{v}\|$. By uniqueness of the projection, we conclude that $\tilde{v} = \hat{v}$. Hence $v_{n_k+1} \to \hat{v}$. By passing to the limit in (3.48) ($F_{\lambda,\mu}$ is continuous on $X \times G^d_\mu$), we therefore have for all v:

$$F_{\lambda,\mu}(\hat{u}, \hat{v}) \leq F_{\lambda,\mu}(\hat{u}, v) \quad (3.51)$$

And by passing to the limit in (3.49), for all u :

$$F_{\lambda,\mu}(\hat{u}, \hat{v}) \leq F_{\lambda,\mu}(u, \hat{v}) \quad (3.52)$$

(3.51) and (3.52) can respectively be rewritten:

$$F_{\lambda,\mu}(\hat{u}, \hat{v}) = \inf_{v\in X} F_{\lambda,\mu}(\hat{u}, v) \quad (3.53)$$

$$F_{\lambda,\mu}(\hat{u}, \hat{v}) = \inf_{u\in X} F_{\lambda,\mu}(u, \hat{v}) \quad (3.54)$$

But, from the definition of $F_{\lambda,\mu}(u, v)$ (see (2.33)), (3.54) is equivalent to (see [5]):

$$0 \in -f + \hat{u} + \hat{v} + \lambda\partial J_d(\hat{u}) \quad (3.55)$$

and (3.53) to:

$$0 \in -f + \hat{u} + \hat{v} + \lambda\partial J^*_d\left(\frac{\hat{v}}{\mu}\right) \quad (3.56)$$

The subdifferential of $F_{\lambda,\mu}$ at $(\hat{u}, \hat{v})$ is given by:

$$\partial F_{\lambda,\mu}(\hat{u}, \hat{v}) = \frac{1}{\lambda}\begin{pmatrix} -f + \hat{u} + \hat{v} + \lambda\partial J_d(\hat{u}) \\ -f + \hat{u} + \hat{v} + \lambda\partial J^*_d\left(\frac{\hat{v}}{\mu}\right) \end{pmatrix} \quad (3.57)$$

And thus, according to (3.55) and (3.56), we have:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \partial F_{\lambda,\mu}(\hat{u}, \hat{v}) \quad (3.58)$$

which is equivalent to: $F_{\lambda,\mu}(\hat{u}, \hat{v}) = \inf_{(u,v)\in X^2} F_{\lambda,\mu}(u, v) = m$. Hence the whole sequence $F_{\lambda,\mu}(u_n, v_n)$ converges towards m the unique minimum of $F_{\lambda,\mu}$ on $X \times G^d_\mu$. We deduce that the sequence (u_n, v_n) converges to $(\hat{u}, \hat{v})$, the minimizer of $F_{\lambda,\mu}$, when n tends to $+\infty$. □

3.3. Link with Meyer's Model

We examine here the link between the discrete model (2.34) and Meyer's problem. We first recall the discrete version of Meyer's problem:

$$\inf_{(u,v)\in X\times G^d/f=u+v} H_\alpha(u,v) \tag{3.59}$$

with

$$H_\alpha(u,v) = (J_d(u) + \alpha\|v\|_{G^d}) \tag{3.60}$$

The following result is straightforward:

Lemma 3.5. *There exists a solution* $(\hat{u},\hat{v}) \in X \times G^d$ *of problem* (3.59).

Proof: (3.59) is equivalent to $\inf_{v\in G^d} H_\alpha(f-v,v)$. It is immediate to verify that H_α is convex, coercive and continuous on G^d. Hence there exists $\hat{v} \in G^d$ such that

$$H_\alpha(f-\hat{v},\hat{v}) = \inf_{v\in G^d} H_\alpha(f-v,v) \tag{3.61}$$

Let us denote $\hat{u} = f - \hat{v}$. Then $(\hat{u},\hat{v})$ is a solution of (3.59). □

Remark. We do not know if a uniqueness result holds for problem (3.59). We then recall problem (2.34):

$$\inf_{(u,v)\in X\times X} F_{\lambda,\mu}(u,v) \tag{3.62}$$

with

$$F_{\lambda,\mu}(u,v) = \frac{1}{2\lambda}\|f-u-v\|^2 + J_d(u) + J_d^*\left(\frac{v}{\mu}\right) \tag{3.63}$$

Let us consider the problem

$$\inf_{(u,v)\in X\times X/f=u+v} J_d(u) + J_d^*\left(\frac{v}{\mu}\right) \tag{3.64}$$

One easily shows the next result:

Lemma 3.6. *There exists* $(\tilde{u},\tilde{v}) \in X \times X$ *solution of* (3.64).

Proof: (3.64) is equivalent to

$$\inf_{v\in X} J_d(f-v) + J_d^*\left(\frac{v}{\mu}\right) \tag{3.65}$$

It is immediate to see that the function to minimize in (3.65) is convex, coercive and lower semi-continuous on X. Hence there exists $\tilde{v} \in X$ such that

$$J_d(f-\tilde{v}) + J_d^*\left(\frac{\tilde{v}}{\mu}\right) = \inf_{v\in X} J_d(f-v) + J_d^*\left(\frac{v}{\mu}\right) \tag{3.66}$$

Denoting $\tilde{u} = f - \tilde{v}$. Then $(\tilde{u},\tilde{v})$ is a solution of (3.64). □

Proposition 3.5. *Let us fix* $\alpha > 0$ *in problem* (3.59). *Let* $(\hat{u},\hat{v})$ *a solution of problem* (3.59). *We fix* $\mu = \|\hat{v}\|_{G^d}$ *in* (3.64). *Then*:

- $(\hat{u},\hat{v})$ *is also a solution of problem* (3.64).
- *Conversely, any solution* $(\tilde{u},\tilde{v})$ *of* (3.64) (*with* $\mu = \|\hat{v}\|_{G^d}$) *is a solution of* (3.59).

Proof: We split the proof into two steps.

Step 1. We first want to show that $(\hat{u},\hat{v})$ is a solution of (3.64) (with $\mu = \|\hat{v}\|_{G^d}$). As $(\hat{u},\hat{v})$ is a solution of (3.59) (the existence of $(\hat{u},\hat{v})$ is given by Lemma 3.5) and as $\|\hat{v}\|_{G^d} = \mu$, then $\hat{u}$ is solution of

$$\inf_{u\in X/u=f-v,\|v\|_{G^d}=\mu} J_d(u) + \alpha\mu, \tag{3.67}$$

i.e. $\hat{u}$ is solution of

$$\inf_{u\in X/u=f-v,\|v\|_{G^d}=\mu} J_d(u) \tag{3.68}$$

Since the set $\{u \in X/u = f - v, \|v\|_{G^d} = \mu\}$ is contained in $\{u \in X/u = f - v, \|v\|_{G^d} \le \mu\}$, we have:

$$\inf_{u\in X/u=f-v,\|v\|_{G^d}=\mu} J_d(u) \ge \inf_{u\in X/u=f-v,\|v\|_{G^d}\le\mu} J_d(u) \tag{3.69}$$

By contradiction, let us assume that

$$\inf_{u\in X/u=f-v,\|v\|_{G^d}=\mu} J_d(u) > \inf_{u\in X/u=f-v,\|v\|_{G^d}\le\mu} J_d(u) \tag{3.70}$$

Thus, there exists $v' \in X$ such that $\|v'\|_{G^d} < \mu$ and

$$J_d(f-v') < \inf_{u\in X/u=f-v,\|v\|_{G^d}=\mu} J_d(u) \tag{3.71}$$

Denoting by $u' = f - v'$, we have: $J_d(u') + \alpha\|v'\|_{G^d} < J_d(u') + \alpha\mu$. But since $(\hat{u}, \hat{v})$ is a solution of (3.59):

$$J_d(\hat{u}) + \alpha\|\hat{v}\|_{G^d} < J_d(u') + \alpha\|v'\|_{G^d} < J_d(u') + \alpha\mu \tag{3.72}$$

Hence (we recall that $\|\hat{v}\|_{G^d} = \mu$), we get from (3.72) that $J_d(\hat{u}) < J_d(u')$. This contradicts (3.71). We conclude that (3.70) cannot hold. Hence:

$$\inf_{u\in X/u=f-v, \|v\|_{G^d}=\mu} J_d(u) = \inf_{u\in X/u=f-v, \|v\|_{G^d}\leq\mu} J_d(u) \tag{3.73}$$

From (3.68), we see that $\hat{u}$ is a solution of $\inf_{u\in X/u=f-v, \|v\|_{G^d}\leq\mu} J_d(u)$, i.e. $\hat{u}$ is a solution of

$$\inf_{u\in X/u=f-v} J_d(u) + J_d^*\left(\frac{v}{\mu}\right) \tag{3.74}$$

Hence $(\hat{u}, \hat{v})$ is also a solution of (3.64).

Step 2. Let us now consider $(\tilde{u}, \tilde{v})$ a solution of (3.64) (the existence of $(\tilde{u}, \tilde{v})$ is given by Lemma 3.6). We can repeat the computations we made in Step 1. We get that $\tilde{u}$ is a solution of:

$$\inf_{u\in X/u=f-v, \|v\|_{G^d}=\mu} J_d(u) + \alpha\mu \tag{3.75}$$

We therefore have: $J_d(\tilde{u}) - \alpha\mu = J_d(\hat{u}) + \alpha\|\hat{v}\|_{G^d}$. But as $(\tilde{u}, \tilde{v})$ is a solution of (3.64), we have $\|\tilde{v}\|_{G^d} \leq \mu$. Hence $J_d(\tilde{u}) + \alpha\|\tilde{v}\|_{G^d} \leq J_d(\hat{u}) + \alpha\|\hat{v}\|_{G^d}$. And since $(\hat{u}, \hat{v})$ is a solution of (3.59), we get that:

$$J_d(\tilde{u}) + \alpha\|\tilde{v}\|_{G^d} = J_d(\hat{u}) + \alpha\|\hat{v}\|_{G^d} \tag{3.76}$$

We thus conclude that $(\tilde{u}, \tilde{v})$ is a solution of (3.59). □

In fact, we can say more about the link between Meyer's problem (3.59) and our limit problem (3.64). α being fixed, let us denote by

$$Z_\alpha = \left\{ v_\alpha,\, v_\alpha \text{ is a solution of the problem } \inf_{v\in G^d} H_\alpha(f-v, v) \text{ (see (3.59))} \right\} \tag{3.77}$$

$$S_\alpha = \left\{ v_\alpha\|_{G^d},\, v_\alpha \text{ is a solution of the problem } \inf_{v\in G^d} H_\alpha(f-v, v) \text{ (see (3.59))} \right\} \tag{3.78}$$

We know that Z_α and S_α are not empty thanks to Lemma 3.5. We consider the two multi-applications:

$$\begin{aligned} Y : \mathbb{R}_+ &\to \mathbb{P}(G^d) \\ \alpha &\mapsto Z_\alpha \end{aligned}$$

and

$$\begin{aligned} T : \mathbb{R}_+ &\to \mathbb{P}(\mathbb{R}_+) \\ \alpha &\mapsto S_\alpha \end{aligned}$$

where $\mathbb{P}(G^d)$ (resp. $\mathbb{P}(\mathbb{R}_+)$) stands for the set of subsets of G^d (resp. $\mathbb{R}_+$).

We want to show a kind of reciprocal result to Proposition 3.5, i.e. that, for a certain range of μ, there exists α such that $\mu \epsilon T(\alpha)$.

The following result holds:

Proposition 3.6.

1. *T is a nonincreasing multi- application.*
2. $Y(0) = \{f - \bar{f}\}$ *and* $T(0) = \|f - \bar{f}\|_{G^d}$ (*where* $\bar{f}$ *stands for the mean value of f over* Ω).
3. *If* α *goes to* $+\infty$, *then* $Y(v_\alpha)$ (*resp.* $T(v_\alpha)$) *goes to* $\{0\}$ (*resp.* $\{0\}$) (*with respect to the Hausdorff metric*).

Proof: We successively show the three points of the proposition. If we pick v_α in Z_α, we denote by $u_\alpha = f - v_\alpha$.

1. Let $\alpha_2 > \alpha_1 > 0$. Let us pick v_{α_1} in Z_{α_1} and v_{α_2} in Z_{α_2}. Let us denote by $u_{\alpha_1} = f - v_{\alpha_1}$ and $u_{\alpha_2} = f - v_{\alpha_2}$. Then, as v_{α_1} in Z_{α_1}, we have in particular:

$$J_d(u_{\alpha_1}) + \alpha_1\|v_{\alpha_1}\|_{G^d} \leq J_d(u_{\alpha_2}) + \alpha_1\|v_{\alpha_2}\|_{G^d} \tag{3.79}$$

And as v_{α_2} in Z_{α_2}, we also have:

$$J_d(u_{\alpha_2}) + \alpha_2\|v_{\alpha_2}\|_{G^d} \leq J_d(u_{\alpha_1}) + \alpha_2\|v_{\alpha_1}\|_{G^d} \tag{3.80}$$

Adding the two last inequalities, we get:

$$\alpha_1\|v_{\alpha_1}\|_{G^d} + \alpha_2\|v_{\alpha_2}\|_{G^d} \leq \alpha_1\|v_{\alpha_2}\|_{G^d} + \alpha_2\|v_{\alpha_1}\|_{G^d} \tag{3.81}$$

And then

$$\underbrace{(\alpha_2 - \alpha_1)}_{>0 \text{ by hypothesis}} \left(\|v_{\alpha_2}\|_{G^d} - \|v_{\alpha_1}\|_{G^d} \right) \geq 0 \quad (3.82)$$

Hence $\|v_{\alpha_2}\|_{G^d} \geq \|v_{\alpha_1}\|_{G^d}$, which proves the first point of the proposition.

2. Let us now prove the second point of the proposition. We have (see (3.60)) $H_0(f-v, v) = J(f-v) \geq 0$ for all $v \in G^d$. Choosing $v_0 = f - \bar{f}(v_0 \in G^d$ since $\bar{v}_0 = 0)$, we get $H_0(f - v_0, v_0) = J(f - v_0) = J(\bar{f}) = 0$. Hence $0 = \min_{v \in G^d} H_0(f - v, v)$. We deduce that $v_0 \in Z_0$. Moreover, $J(u) = 0$ if and only if $u = \bar{u}$. Let v_1 be a solution of $\min_{v \in G^d} H_0(f - v, v)$. We thus have $\overline{f - v_1} = f - v_1$. And as $v_1 \in G^d$, we also have $\bar{v}_1 = 0$. Then $f - v_1 = \overline{f - v_1} = \bar{f} - \bar{v}_1 = \bar{f}$, i.e. $v_1 = v_0$. We conclude that $\{v_0\} = Z_0$. This shows the second point of the proposition.
3. Let us now prove the third point of the proposition. Let us pick v_α in Z_α, and let us denote by $u_\alpha = f - v_\alpha$. By definition of Z_α, we have for all $(u, v) \in X \times G^d$ such that $f = u + v$:

$$J_d(u_\alpha) + \alpha\|v_\alpha\|_{G^d} \leq J_d(u) + \alpha\|v\|_{G^d} \quad (3.83)$$

We choose $u = f$, and $v = 0$. We get:

$$J_d(u_\alpha) + \alpha\|v_\alpha\|_{G^d} \leq J_d(f) \quad (3.84)$$

- First case: if f is constant (i.e $f = \bar{f}$), then $J_d(f) = 0$. Hence (3.84) implies that $J_d(u_\alpha) = \|v_\alpha\|_{G^d} = 0$. We conclude that $v_\alpha = 0$, and $u_\alpha = \bar{f} = f$.
- Second case: if now f is not constant (i.e $f \neq \bar{f}$), then $J_d(f) > 0$ Hence (3.84) implies that $\|v_\alpha\|_{G^d} \leq \frac{J_d(f)}{\alpha}$. Thus $Y(\alpha) \to \{0\}$ as α goes to $+\infty$ (with respect to the Hausdorff metric).

□

Remark. In fact, we have shown that $T : \mathbb{R}_+ \to [0, \|f - \bar{f}\|_{G^d}]$. In particular, T has uniformly bounded values:

1. $T(0) = \{f - \bar{f}\}$
2. If $\alpha > 0$, then if $v_\alpha \in T(\alpha)$, we have

$$\|v_\alpha\|_{G^d} \leq \|f - \bar{f}\|_{G^d} \quad (3.85)$$

Proposition 3.7. *T is u.s.c.* (*upper semi continuous*) (*i.e. T has a closed graph and convex compact values*).

Proof: We split the proof into two steps:

Step 1. Let us set $\alpha \in \mathbb{R}_+$. By definition of S_α, one easily checks that $T(\alpha)$ is convex and closed in $\mathbb{R}$. Moreover we have shown that $T(\alpha)$ is uniformly bounded (see (3.85)). Therefore, $T(\alpha)$ is compact in $\mathbb{R}$.

Step 2. Let us now consider a sequence (α_n, v_{α_n}) where $\alpha_n \in \mathbb{R}_+$ and $v_{\alpha_n} \in Z_{\alpha_n}$. Assume that there exists (α_0, v_0) in $\mathbb{R}_+ \times G^d$ such that $(\alpha_n, v_{\alpha_n}) \to (\alpha_0, v_0)$ as n goes to $+\infty$. As v_{α_n} in Z_{α_n}, we have for all $(u, v) \in X \times G^d$ such that $f = u + v$:

$$J_d(f - v_{\alpha_n}) + \alpha_n \|v_{\alpha_n}\|_{G^d} \leq J_d(u) + \alpha_0\|v\|_{G^d} \quad (3.86)$$

By passing to the limit as n goes to $+\infty$, we get:

$$J_d(f - v_0) + \alpha_0\|v_0\|_{G^d} \leq J_d(u) + \alpha_0\|v\|_{G^d} \quad (3.87)$$

Hence v_0 belongs to Z_{α_0}, and therefore $\|v_0\|_{G^d}$ is in S_{α_0}. This shows that T has a closed graph.

□

Corollary 3.1. *For all μ in $(0, \|f - \bar{f}\|_{G^d})$, there exists α in $\mathbb{R}_+$ such that there exists (u, v) in $X \times G^d$ with $\|v\|_{G^d} = \mu$ and solving Meyer's problem* (3.59).

Proof: This a consequence of Proposition 3.6, Proposition 3.7 and the next theorem (applied to the multi-application $T_\mu = T - \mu$) which we state without proof. □

Theorem 3.2. *Let us consider a multi- application L*:

$$\mathbb{R} \to \mathbb{P}(\mathbb{R})$$
$$\alpha \mapsto [L_{\min}(\alpha), L_{\max}(\alpha)]$$

Let us assume that L is such that:

1. *L is u.s.c* (*upper semi-continuous*).
2. *There exists $a \in \mathbb{R}$* (*resp. $b \in \mathbb{R}$*) *such that $L_{\min}(a) \leq 0$* (*resp. $L_{\max}(b) > 0$*).

Then there exists $c \in [a, b]$ such that $0 \in L(c)$.

Figure 1. Original synthetic image.

Remark. This corollary completes the result of Proposition 3.5. It completely closes the link between Meyer's problem (3.59) and our limit problem (3.64).

3.4. Role of λ

We show here that problem (3.64) is obtained by passing to the limit $\lambda \to 0^+$ in (3.62).

Proposition 3.8. *Let us fix* $\alpha > 0$ *in* (3.59). *Let us assume that problem* (3.59) *has a unique solution* $(\hat{u}, \hat{v})$. *Set* $\mu = \|\hat{v}\|_{G^d}$ *in* (3.62) *and* (3.64). *Let us denote* (u_λ, v_λ) *the solution of problem* (3.62). *Then* (u_λ, v_λ) *converges to* $(u_0, v_0) \in X \times X$ *as* λ *goes to* 0. *Moreover,* $(u_0, v_0) = (\hat{u}, \hat{v})$ *is the solution of problem* (3.64).

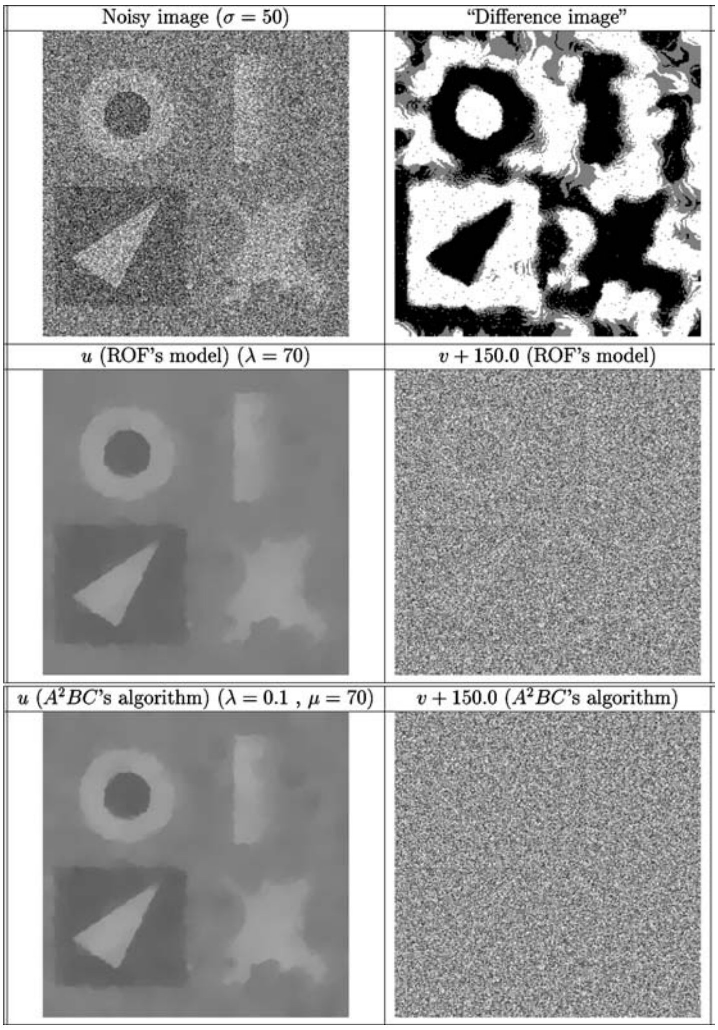

Figure 2. Comparison for $\sigma = 50$ (SNR = 8.36).

Remark. In the case when the solution of problem (3.59) is not unique, the result of Proposition 3.8 does not hold. We can just show that any cluster point of $(u_{\lambda_n}, v_{\lambda_n})$ is a solution of problem (3.64) and thus of (3.59).

Proof of Proposition 3.8: The existence of $(\hat{u}, \hat{v})$ is given by Lemma 3.6. The existence and uniqueness of (u_λ, v_λ) is given by Lemma 3.4.

Since (u_λ, v_λ) is the solution of problem (3.62), we have $v_\lambda \in G_\mu^d$, i.e. $\|v_\lambda\|_{G^d} \leq \mu$. As we saw in the proof of Lemma 3.4, this inequality implies:

$$\|v_\lambda\|_X \leq 4\,\mu N^2 \tag{3.88}$$

Since (u_λ, v_λ) is the solution of problem (3.62), we have:

$$F_{\lambda,\mu}(u_\lambda, v_\lambda) \leq F_{\lambda,\mu}(f, 0) \tag{3.89}$$

which means

$$F_{\lambda,\mu}(u_\lambda, v_\lambda) \leq J_d(f) \tag{3.90}$$

And the left hand-side of (3.90) is given by:

$$\begin{aligned} &F_{\lambda,\mu},(u_\lambda, v_\lambda) \\ &\quad = J_d(u_\lambda) + \frac{1}{2\lambda}\|f - u_\lambda - v_\lambda\|_X^2 + J_d^*\left(\frac{v_\lambda}{\mu}\right) \\ &\quad = J_d(u_\lambda) + \frac{1}{2\lambda}\|f - u_\lambda - v_\lambda\|_X^2 \end{aligned} \tag{3.91}$$

Hence $J_d(u_\lambda) + \frac{1}{2\lambda}\|f - u_\lambda - v_\lambda\|_X^2 \leq J_d(f)$, and

$$\|f - u_\lambda - v_\lambda\|^2 \leq 2\lambda J_d(f) \tag{3.92}$$

As $\|v_\lambda\|_X$ is bounded (from (3.88)), we conclude that if $\lambda \in [0; 1]$, u_λ is bounded by a constant $C > 0$ which does not depend on λ.

Consider a sequence (λ_n) which goes to 0 as $n \to +\infty$. Then, up to an extraction (since $(u_{\lambda_n}, v_{\lambda_n})$ is bounded in $X \times X$), there exists $(u_0, v_0) \in X \times X$ such that $(u_{\lambda_n}, v_{\lambda_n})$ converges to (u_0, v_0). By passing to the limit in (3.92), we get: $\|f - u_0 - v_0\|_X = 0$, i.e. $f = u_0 + v_0$.

To conclude the proof of the proposition, there remains to show that (u_0, v_0) is a solution of problem (3.64). We first notice that as $\forall\lambda > 0$, and since $\|v_\lambda\|_{G^d} \leq \mu$, we get: $\|v_0\|_{G^d} \leq \mu$. Let $(u, v) \in X \times X$ such that $f = u + v$. We have:

$$\begin{aligned} &J_d(u) + J_d^*\left(\frac{v}{\mu}\right) + \frac{1}{2\lambda}\underbrace{\|f - u - v\|^2}_{=0} \\ &\geq J_d\big(u_{\lambda_n}\big) + J_d^*\left(\frac{v_{\lambda_n}}{\mu}\right) + \frac{1}{2\lambda_n}\big\|f - u_{\lambda_n} - v_{\lambda_n}\big\|^2 \\ &\geq \underbrace{J_d\big(u_{\lambda_n}\big) + J_d^*\left(\frac{v_{\lambda_n}}{\mu}\right)}_{\to J_d(u_0) + J_d^*(\frac{v_0}{\mu})} \end{aligned}$$

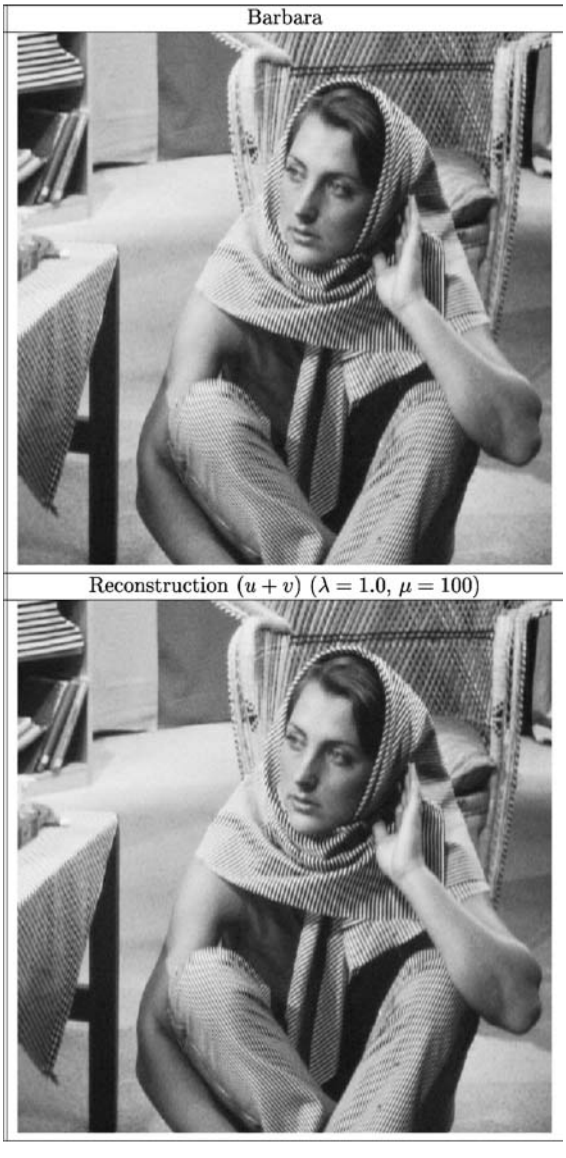

Figure 3. Barbara image.

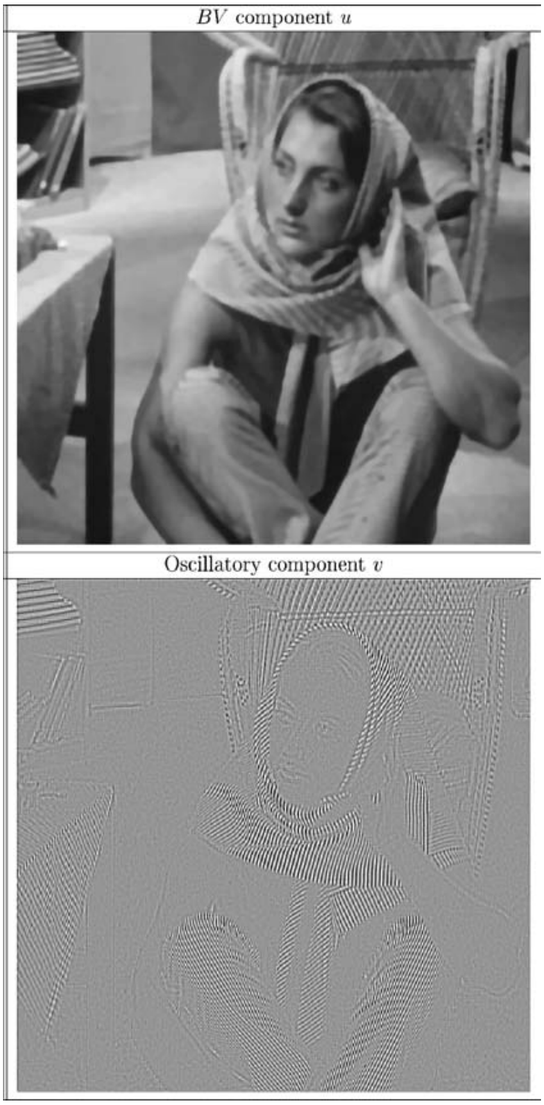

Figure 4. Decomposition with our model for $\lambda = 1.0$ and $\mu = 100$. $\|v_{A^2BC}\|_{L^2}$ here is equivalent to $\|v_{\mathrm{ROF}}\|_{L^2}$ in Fig. 5.

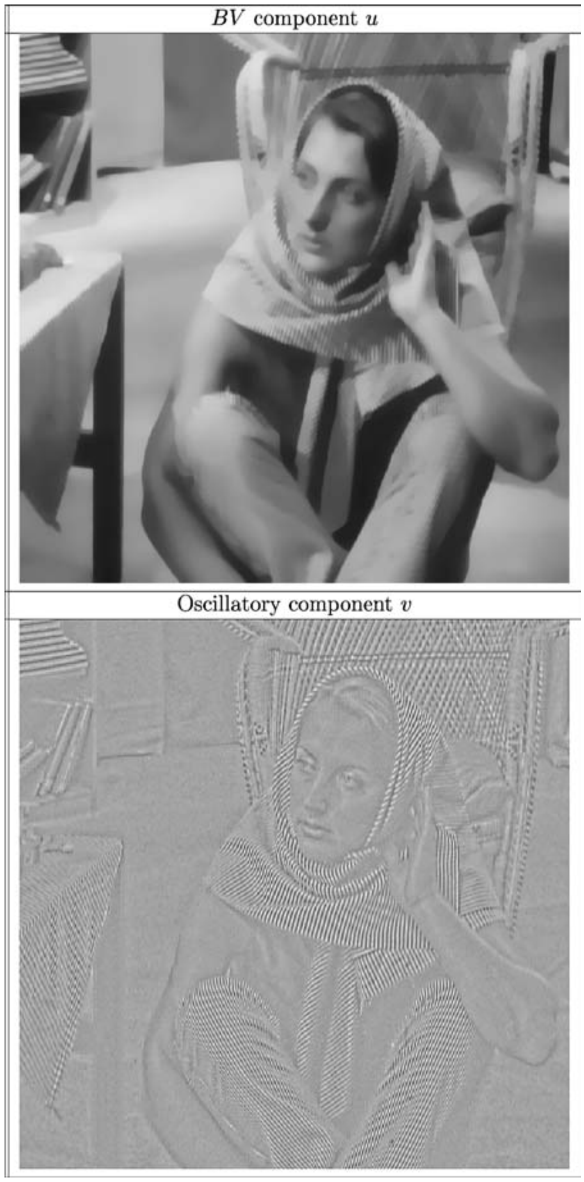

Figure 5. Decomposition with the ROF model for $\lambda = 43$. $\|v_{\mathrm{ROF}}\|_{L^2}$ here is equivalent to $\|v_{A^2BC}\|_{L^2}$ in Fig. 4.

Hence (u_0, v_0) is a solution of problem (3.64). And as we have assumed that problem (3.64) has a unique solution, we deduce that $(u_0, v_0) = (\hat{u}, \hat{v})$, i.e. (u_0, v_0) is the solution of problem (3.64). $\square$

4. A Comparison

4.1. Introduction

In this section, we intend to compare Rudin-Osher-Fatemi (ROF) problem (1.2) with Meyer's one (1.3). We put some noise (a gaussian noise of variance σ^2) on an image provided by the GdR-PRC ISIS (http://www-isis.enst.fr/) (see Fig. 1), and we perform both a total variation algorithm and our algorithm (2.37)–(2.40). We have chosen to use Chambolle's algorithm to minimize the total variation (Section 2.3).

We display the results on Fig. 2. The "difference image" is obtained from the v components of both algorithms. We denote by V_{A^2BC} the v component given by our algorithm, and v_{ROF} the one given by the total

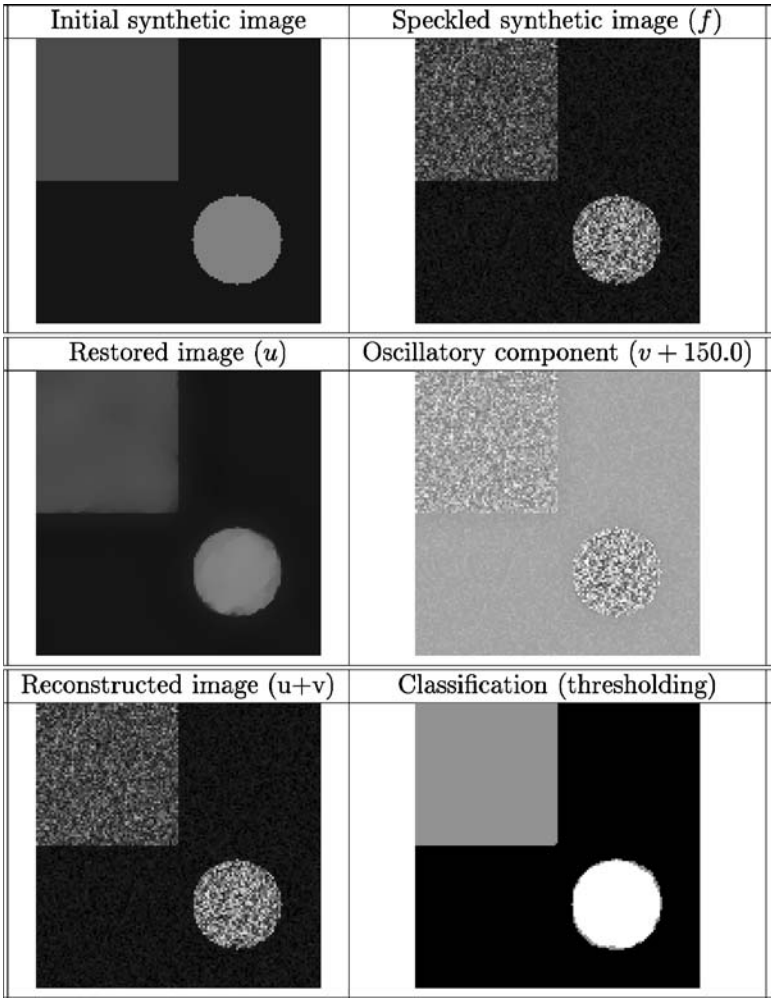

Figure 6. Simple synthetic image ($\lambda = 0.01$ and $\mu = 80$).

variation minimization algorithm. The value of a pixel in position (i, j) is 255.0 (i.e. white) if $v_{A^2BC}(i, j) > v_{\mathrm{ROF}}(i, j)$, 127.0 (i.e. gray) if $v_{A^2BC}(i, j) = v_{\mathrm{ROF}}(i, j)$ and 0.0 (i.e. black) if $v_{A^2BC}(i, j) < v_{\mathrm{ROF}}(i, j)$.

4.2. *Commentaries*

We compare the v component given by our algorithm with the one given by the ROF model. Their mean values are both very close to zero. For instance, in the case of Fig. 2 ($\sigma = 50$), v_{A^2BC} and v_{ROF} have almost the same mean value: -0.7. In the case of the ROF problem, the parameter λ corresponds to the one in (1.2), and in the case of Meyer's problem, the parameters λ and μ correspond to the ones in (2.7). For a given noisy image, we tune these parameters so that $\|v_{A^2BC}\|_{L^2} \simeq \|v_{\mathrm{ROF}}\|_{L^2}$: the v components both contain the same quantity of information. We want to compare the information they contain.

One sees on the "difference image" that $v_{A^2BC}(i, j) > v_{\mathrm{ROF}}(i, j)$ in the darkest regions of the original image (Fig. 1), and that $v_{A^2BC}(i, j) < v_{\mathrm{ROF}}(i, j)$ in the lightest regions. This means that the v component in the ROF model depends more on the mean gray level value of the original image than in the case of Meyer's one. For instance, let us have a look at the dark circle on top left of Fig. 1. In the case of Fig. 2 ($a = 50$), the mean value of the pixels corresponding to this circle is -1.0 in v_{A^2BC} and -4.2 in v_{ROF}. Both components v tend to have a negative mean because in Fig. 1 the circle is a dark component.

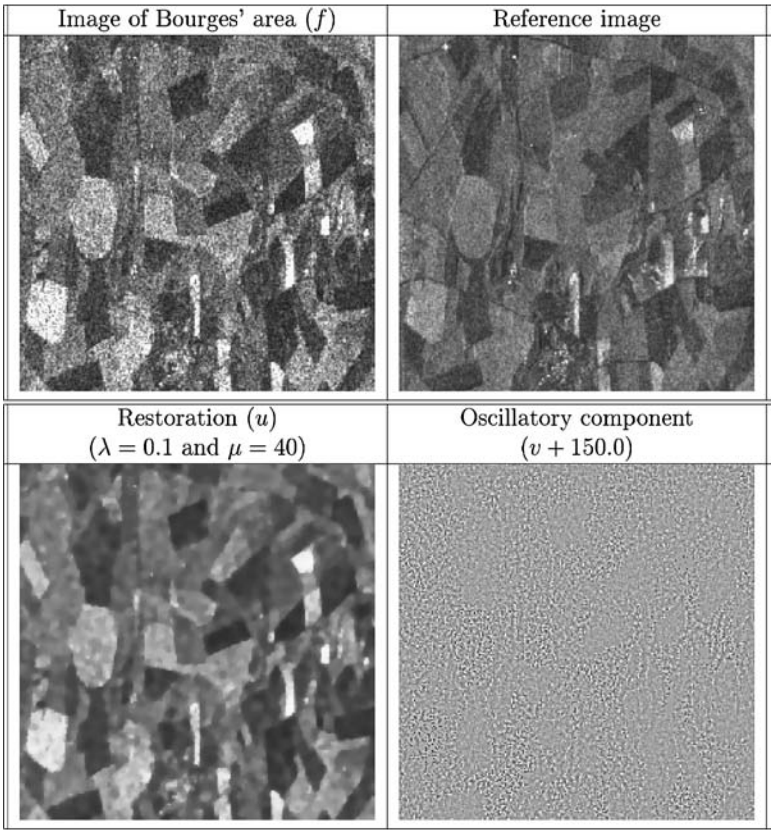

Figure 7. Image of Bourges' area (1).

In homogeneous regions (such as the dark circle we considered just before), we would expect the v component of both models to have a zero mean (the mean of the white gaussian noise we add to the original image). According to the remarks we made before, Meyer's model appears to loose less information than the ROF model. This confirms the assertions by Meyer in [8]. The decomposition he proposes seems to be more adapted to image restoration. Nevertheless, the difference between both methods appears not to be visually very important.

4.3. *Barbara Image*

We have also performed our algorithm on the Barbara image (Fig. 3).

On Figs. 4 and 5, one sees that v_{A^2BC} corresponds more to the texture part of the Barabara image than v_{ROF}. One can also see that Barbara's face appears much more in v_{ROF} than in v_{A^2BC}. This confirms the analysis of Meyer [8]. Moreover, the leg of the table appears much more in v_{ROF} than in v_{A^2BC}, and this is not a textured component of the Barbara image.

For all this reasons, our model (inspired by Meyer's model) gives a better decomposition of an image into a BV component and an oscillatory component than the ROF model.

5. SAR Images Restoration

5.1. *Introduction*

Synthetic Aperture Radar (SAR) images are strongly corrupted by a noise called speckle. A radar sends a coherent wave which is reflected on the ground, and then registered by the radar sensor [7]. When one cares with the reflection of a coherent wave on a coarse surface, then one can see that the observed image is degraded by a noise of large amplitude. This gives a speckled aspect to the image. That is why such a noise is called speckle.

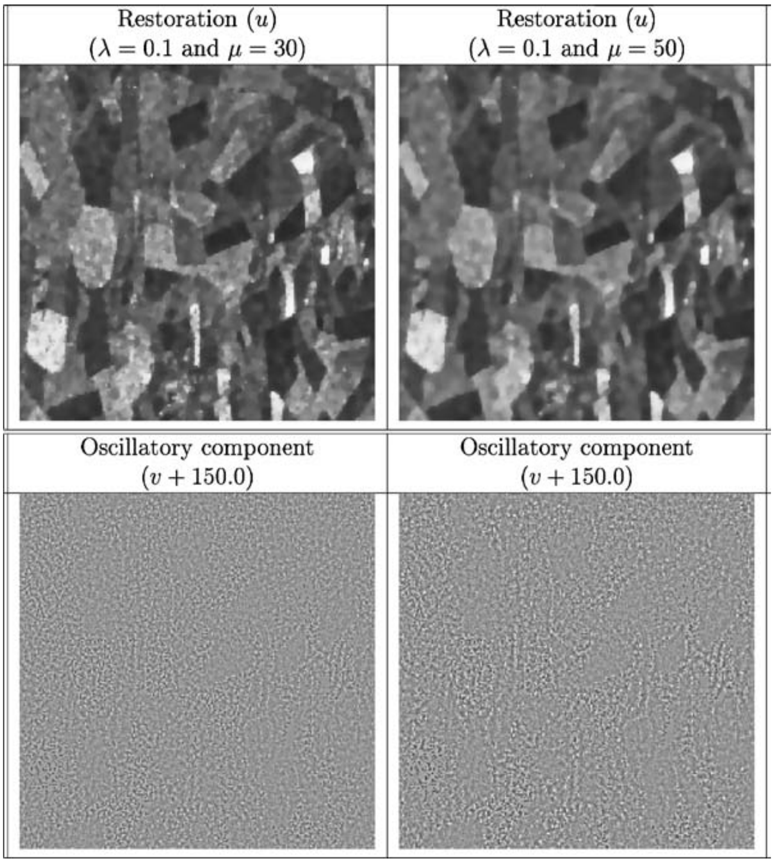

Figure 8. Image of Bourges' area (2).

Link With Our Approach. Contrary to the usual modelization in SAR, the noise in our model is considered to be additive: the image f is decomposed into a component u belonging to BV, and a component v in G. But it is to be noticed that our model is completely different from the classical additive models: in these ones, v is often considered to be a Gaussian white noise, and therefore has a constant variance all over the image. Here, v belongs to G, a space in which signals can have large oscillations but small norm. Moreover the variance of the oscillations of v may not be uniform on the whole image. Remark that by considering u as the restored image (without speckle) we assume that there is no texture in the SAR image.

5.2. *Results on a Synthetic Image*

Figure 6 show why for a SAR image the decomposition proposed by Meyer is very interesting. Indeed, one checks that the v component contains the speckle, and the u component can be regarded as a restoration of the original image (if it does not contain textures). It is difficult to make comparisons with other methods, since the main criterion remains the visual interpretation. Nevertheless, the results we get appear good with respect to existing methods. And above all, our approach being a variational one, computation time are very short. With a processor at 800 MHz and 128 kilo of RAM, it takes less than one minute to deal with an image of size $256 * 256$.

5.3. *Results on a Real Image*

We use a SAR image of Bourges' area provided by the CNES (French Space Agency: http://www.cnes.fr/index_v3.htm). The reference image (also furnished by the CNES) has been obtained by amplitude summation. Figures 7 and 8 show the effect of parameter μ on the restoration process. The larger μ is, the more v contains information, and therefore the more u is averaged. According to the value of μ, we can thus get a more or less restored image, and also more or less smooth.

6. Conclusion

In this article, we present a new algorithm to decompose a given image f into a component u belonging to BV and a component v containing the noise and the textures of the initial image. Our algorithm performs Meyer's program [8]. We use the space G and its norm, and not an approximation as done in [9, 12]. Moreover, we carry out the mathematical study of our model.

We present some numerical results to show the relevance of our algorithm. We also show how the u component can be used for SAR images restoration. More experimental results can be found in [2]: in particular, we show how the v component can be used in textured images classification.

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