Image Sharpening by Flows Based on Triple Well Potentials

Guy Gilboa, Nir Sochen and Yehoshua Y. Zeevi

G. Gilboa and Y.Y. Zeevi are with the Department of Electrical Engineering, Technion - Israel Institute of Technology, Technion city, Haifa 32000, Israel. e-mail: gilboa@tx.technion.ac.il, zeevi@ee.technion.ac.il

N. Sochen is with the Department of Applied Mathematics, University of Tel-Aviv Ramat-Aviv, Tel-Aviv 69978, Israel e-mail: sochen@math.tau.ac.il

Abstract

Image sharpening in the presence of noise is formulated as a non-convex variational problem. The energy functional incorporates a gradient-dependent potential, a convex fidelity criterion and a high order convex regularizing term. The first term attains local minima at zero and some high gradient magnitude, thus forming a triple well-shaped potential (in the one-dimensional case). The energy minimization flow results in sharpening of the dominant edges, while most noisy fluctuations are filtered out.

Keywords: *image filtering, image enhancement, image sharpening, nonlinear diffusion, hyper-diffusion, variational image processing.*

I. INTRODUCTION

We address the issue of sharpening images degraded by blur-type operations and contaminated by additive noise. The approach is based on an evolutionary sharpening process, which in our case is derived from an energy minimization flow of a multi well-shaped energy density function. Somewhat similar type of flows were examined in the analysis of formation of microstructures in crystals [2], [11].

Let us first review the relation between nonlinear diffusion processes and energy minimization flows. We define a potential function (energy density) $\Psi(|\nabla I|)$ and a corresponding energy functional

$$E(I) = \int_\Omega \Psi(|\nabla I|^2) dx. \tag{1}$$

Minimization of this functional, using a gradient descent method, leads to a nonlinear diffusion process:

$$I_t = \text{div}(J(\nabla I)) = \text{div}(c(|\nabla I|^2)\nabla I), \tag{2}$$

where $J(\cdot)$ is the flux function given by

$$J(\nabla I) \doteq c(\cdot)\nabla I = 2\Psi'(|\nabla I|)\nabla I, \tag{3}$$

and $c(\cdot)$ is the diffusion coefficient. The initial condition is $I|_{t=0} = I_0$, where I_0 is in image processing applications the input image. Note that Neumann boundary conditions are assumed. (For more details see [10], [37], [33] and the references therein.)

Typical monotonically-increasing denoising potentials attain their minimum at zero. This type of potentials can be classified as either convex potentials (e.g. linear diffusion, Charbonnier et

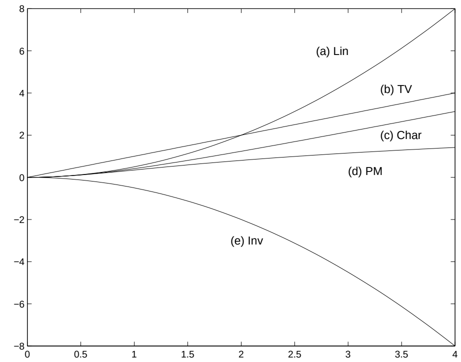

Fig. 1. Potentials $\Psi(s)$ plotted as a function of the gradient magnitude s of some classical processes: (a) Linear forward diffusion ($\Psi(s) = \frac{1}{2}s^2$), (b) TV ($\Psi(s) = s$), (c) Charbonnier et al. ($\Psi(s) = \sqrt{k^4 + k^2 s^2} - k^2$, $k = 1$), (d) Perona-Malik ($\Psi(s) = \frac{1}{2}k^2 \log(1 + (\frac{s}{k})^2)$, $k = 1$), (e) Linear inverse (backward) diffusion ($\Psi(s) = -\frac{1}{2}s^2$).

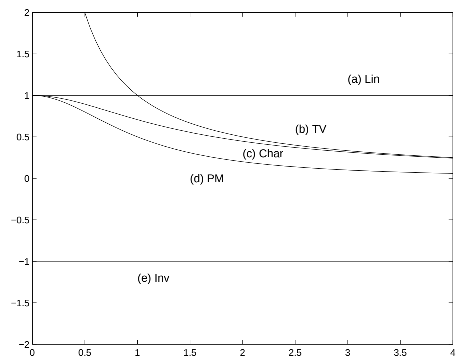

Fig. 2. Diffusion coefficients $c(s)$ plotted as a function of the gradient magnitude s of the above processes: (a) Linear forward diffusion ($c(s) = 1$), (b) TV ($c(s) = \frac{1}{s}$), (c) Charbonnier et al. ($c(s) = \frac{1}{\sqrt{1+s^2/k^2}}$, $k = 1$), (d) Perona-Malik ($c(s) = \frac{1}{1+s^2/k^2}$, $k = 1$), (e) Linear inverse (backward) diffusion ($c(s) = -1$).

al. [7], Beltrami diffusion [31]), or nonconvex potentials (e.g. Perona-Malik [24]). Processes derived from convex potentials are well-posed, and their evolution approaches the minimum global energy (zero gradient magnitude everywhere, that is a constant function). Nonconvex potentials retain sturdier edge-preserving properties, but their flux is not monotonic and the theory of proper energy minimization is much more complex in this case. Höllig [13] showed the existence of an infinite number of solutions of a one-dimensional diffusion process with nonmonotonic flux (non-convex potential). Yuo et al. [37] analyzed two-dimensional nonlinear diffusion and proved that processes based on a nonmonotonic flux, with the condition

$$J(|\nabla I| \to \inf) = 0, \tag{4}$$

can have an infinite number of stationary points of the energy functional (and therefore are ill-posed). Both studies were restricted to the case of positive diffusion coefficients. Fortunately, it was discovered that regularizing the process by convolving the gradient with a Gaussian [5], or even by simple discretization [35], causes the evolutionary process to converge onto a constant trivial steady state unique solution. The only apparent instabilities are the staircasing effects [35], [33].

A different, and powerful, approach has become known as the total variation (TV) [28]. This approach, based on a l_1 norm, is a special case in the context of our classification in that it is a non-strictly convex potential. To avoid numerical problems at low gradients, a small constant is usually added in the calculation of the gradient magnitude (i.e $|\nabla I|$ is substituted by $\sqrt{|\nabla I|^2 + \epsilon^2}$), turning the process into a convex one.

(See Figs. 1,2 for examples of potential of some classical processes and of the corresponding diffusion coefficients.)

In cases of monotonically increasing potentials, the diffusion coefficients are positive. Thus the minimum-maximum principal is satisfied (the minimum and maximum of $I(t)$ are bounded by the initial condition I_0, for all $t > 0$ in any dimension) and no real sharpening can occur. Note that this is not the case for numerical schemes of systems with co-dimension > 1 [8]. A classical ill-posed sharpening diffusion process is the linear backward (inverse) diffusion, where the diffusion coefficient $c = const < 0$ and, consequently, the potential is strictly concave. This process attains its minimum energy at infinite gradient magnitudes, causing an explosion of the signal and severe noise amplification. We propose a nonconvex non-monotonic potential that overcomes most of the inverse diffusion instabilities, and yet is still powerful enough to sharpen, and increase contrast of, important features. We address some issues of regularization, and illustrate by numerical examples in one and two dimensions how this process is being implemented. This extends our previous study [12], where we proposed a forward-and-backward (FAB) diffusion process that shifts between denoising and sharpening, according to the local gradient feature. In [29] the authors presented another study involving nonconvex potential using multiple wells. Their work is fundamentally different from ours in that their potential is based on the signal and not on its gradient, and its purpose is image classification. In [17] some interesting bounds on the norm of the solution to a gradient dependent inverse-diffusion problem in one dimension are given. The diffusion coefficient, though, is negative for small gradient magnitudes and the solution, tends, therefore, to create microstructures.

II. The Double Well Potential

Well-shaped potentials have been investigated recently in material science and structural mechanics [11], [2], [19]. In this section we review some of the mathematical and numerical aspects that are relevant to our case.

A mathematical model for the formation of microstructures in certain alloys was presented by Ball and James [2]. The theory is based on an energy minimization process of a double-well potential. The gradient-dependent potential attains its minimum value at symmetry-related deformation gradients [11], [2], [19]. In the one-dimensional case, a typical example of such potential is

$$\Psi_{dw}(I_x) = (I_x^2 - k^2)^2. \tag{5}$$

Although it was not referred to as a diffusion process, and the outcome of this energy minimization flow does not resemble classical diffusion, it can clearly be viewed as a nonlinear diffusion process, with the following diffusion coefficient:

$$c_{dw}(|I_x|) = 4(I_x^2 - k^2). \tag{6}$$

Plots of the potential and of the corresponding diffusion coefficient are depicted in Fig. 3. This is indeed a FAB-type process–for small gradients $|I_x| < k$ it is a backward diffusion process, and for large gradients $|I_x| > k$ it is a forward one. This leads to the sharpening of low gradients and the smoothing of large gradients where both approach a magnitude of k ($I_x = \pm k$).

As the potential is non-convex, and along some of its segments decreasing (creating an inverse diffusion flow), this process has stimulated a growing number of studies dealing with both the theoretical and numerical difficulties that it entails. (See for example [2], [4], [14], [19], [20], [22]).

Three main methods for numerical solutions of such problems were proposed [14]:

- Convexification of the potential, wherein the original potential is replaced by its convex hull. In this case there exists a minimizer and it can be easily obtained, but at a cost of changing some of the process characteristics.

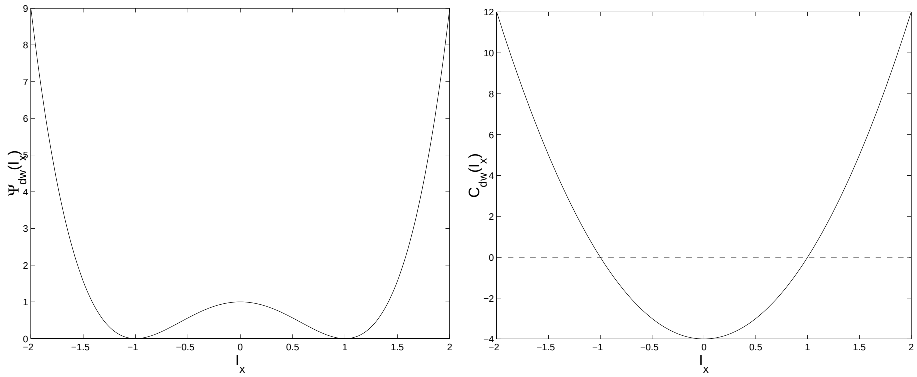

Fig. 3. A double-well potential (left) and the corresponding diffusion coefficient (right). $k = 1$.

- Reformulation of the problem using Young measures (a mathematical tool in the calculus of variations, applying a gradient-generated family of probability measures) [9], [23], [27].
- Direct minimization of the energy functional. In this type of methods, the process may converge towards a fixed point of a local minimum, because of the nonconvex nature of the problem. In some applications, though, such minima are also of interest.

The nature of the double-well and other related problems is quite similar to the formalism of our problem, and numerical techniques in image processing can most likely benefit from the research conducted in the (mathematically and computationally) related field. Yet, we should emphasize the following differences from the problem that we have at hand:

- The potential does not have a ”relaxed” region, where gradients are being smoothed. Specifically, constant functions are unstable.
- The basic solution of the crystalline microstructure intends to have oscillations, which is not desirable in our case.
- The boundary conditions are different (Dirichlet versus Neumann in our case).
- The motivation is different: We are interested in the evolution of the input image, whereas analysis of the double-well model focuses on the final minimal energy state with weak relations to any primary initial evolutionary state.

III. ENERGY WELLS IN IMAGE PROCESSING

A. The model

We assume the following general model of our degraded image I_0:

$$I_0 = B(I) + n, \tag{7}$$

where I_0 is the original image, B is a smoothing (blurring) transformation, not necessarily linear or shift invariant and n is some noise, uncorrelated with the signal (not necessarily white, but not of impulsive nature). We assume that large gradients (i.e. edges) of I are still relatively large in $B(I)$. After some sort of smoothing (or discretization) of I_0 (e.g. $\tilde{I}_0 = I_0 * g_\sigma = B(I)*g_\sigma + n*g_\sigma$) we assume that the gradient magnitude of the noise is less than an upper bound k with a very high probability (e.g. Prob$(|\nabla n * g_\sigma| < k) \approx 1$).

Our objective is to sharpen important edges of the image. That is, edges with a relative large gradient magnitude in a neighborhood and with sufficient support. An imperative requirement is that noise should not be amplified in the process (and preferably even reduced). The noise amplification byproduct is a major drawback of many classical sharpening processes.

B. The Energy Functional

We choose to minimize the following energy functional:

$$E(I) = \int_\Omega (W(|\nabla I|) + \lambda F(I) + \varepsilon R(|\nabla^2 I|))dx. \tag{8}$$

W is a potential generating a selective sharpening flow. Its form is discussed in details below. F is a convex fidelity criterion related to the input image changes

$$F(I) = \rho(|I - I_0|). \tag{9}$$

We choose here $\rho(s) = \frac{1}{2}s^2$ but other choices are possible (e.g. [21]). Note that we assume no a-priori knowledge of the blurring process and avoid therefore the introduction of a blur operator in the fidelity term. For cases of linear and translation invariant blur a blind deconvolution may be a viable option (see [6], [15]). R is a higher order regularizing term. It is a function of the Laplacian and is discussed later.

C. The Triple Well Potential

We begin by discussing the shape of the potential W derived from our objectives. The blurring process smears edges, thus gradients of large magnitude decrease. We would like to reverse this process and increase medium gradients back to their original state. Therefore high gradients should retain a lower energy state (”cost less energy”) and the energy minimization process would thus be rewarded on edge sharpening.

However, two restrictions must be made: a saturation of the sharpening should be defined so very high gradients would not continue to be sharpened and cause the explosion of the signal. As we do not want to fall in the category of the ill-posed problems of condition (4), very large gradients should be even smoothed slowly, to reduce staircasing.

Secondly, low gradients should not be enhanced in order to avoid as much as possible noise amplification. Specifically, the zero gradient should not contribute any energy (be of zero potential).

From this discussion it follows that a potential intended for sharpening should be constructed of three basic attractors (low energy states) in one dimension: Two for high gradients (of positive and negative values) and one for the zero gradient. In two dimensions the potential is rotationally symmetric. This leads to a triple well-shaped potential.

Formally we set the following requirements:

$$\begin{array}{ll} (a) & W(0) = 0 \\ (b) & W(-s) = W(s), \quad \forall s \\ (c) & W(s) \geq 0, \quad \forall s \\ (d) & \exists\, 0 < a < b < \infty : W'(s \in (a,b)) < 0 \\ (e) & W'(s \to \infty) > 0. \end{array} \tag{10}$$

We suggest the following formula for the potential:

$$W(s) = \sqrt{k_f^4 + k_f^2 s^2} - k_f^2 - \frac{\alpha}{2} k_b^2 \log(1 + (\frac{s}{k_b})^2), \tag{11}$$

where k_f, k_b are parameters determining the lower-gradients *forward* diffusion region and the higher-gradients *backward* diffusion region, respectively ($k_f < k_b$), and α is a weight parameter. In order to fulfill (10.c) a proper bound on α should be set.

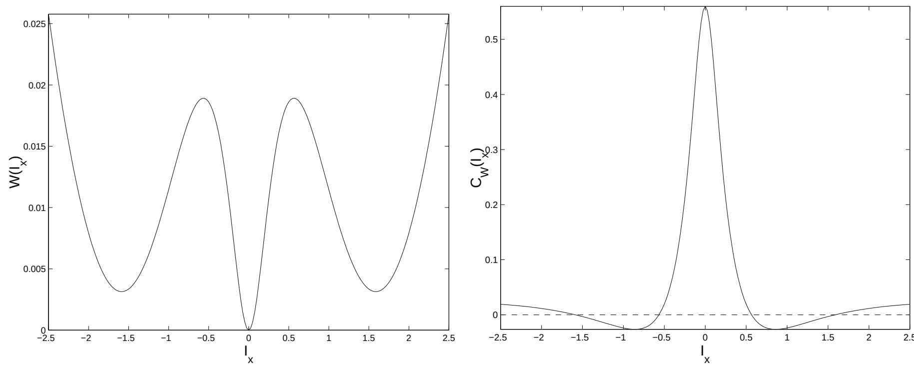

Fig. 4. A triple well potential (left) and the corresponding one-dimensional diffusion coefficient (right). $k_f = 0.2$, $k_b = 1$, $\alpha = 2.2\frac{k_f}{k_b}$.

The corresponding diffusion coefficient is

$$c_W(s) = \frac{1}{\sqrt{1+(s/k_f)^2}} - \frac{\alpha}{1+(s/k_b)^2}. \tag{12}$$

The potential is ’designed’ such that the resultant diffusion coefficient is as simple as possible; After all, we use the diffusion coefficient to compute the flow in the numerical implementation. (See Fig. 4 for plots of W and c_W.) Other, more sophisticated formulas, with more parameters controlling the shape of the potential, can be used. In [12] we proposed a different formula for a forward-backward diffusion coefficient. However, in that study the process was not formulated as a variational problem. As a consequence some of the stabilizing elements introduced here where not included in the earlier study, namely, the restriction to positive potentials, the positive diffusivity at very large gradients and the addition of higher order regularization.

D. Higher Order Regularization

We wish to have the ’smoothest’ possible energy minimizer in order to reduce oscillations between the three low energy states. (The reasoning is similar to what is given in cases of viscosity solutions). For this purpose we add the following high order convex regularization term to the total energy density function:

$$R(|\nabla^2 I|) = \frac{1}{2}|\nabla^2 I|^2. \tag{13}$$

This adds a linear fourth order term $-\nabla^4 I$ to the gradient descent flow, where ∇^4 is the *biharmonic operator* (or *bi-Laplacian*). In the one-dimensional case, $\nabla^4 I = I_{xxxx}$, whereas in two dimensions $\nabla^4 I = I_{xxxx} + 2I_{xxyy} + I_{yyyy}$. The fourth order linear equation

$$I_t = -\nabla^4 I, \quad I|_{t=0} = I_0 \tag{14}$$

is often referred to as a hyper-diffusion flow (also super-diffusion). The fundamental solution of (14) in the frequency domain of (ω) is $e^{-\omega^4 t}$, implying that it is a strongly-low-pass filtering flow that rapidly diminishes high frequency oscillations. (See Fig.5 for plots of the fundamental solution, and Figs. 6, 7 for examples of hyper-diffusion in one and two dimensions.) A nonlinear hyper-diffusion term was added in [32] to the standard Perona-Malik equation [24], to rapidly remove the noise. Note, though, that hyper-diffusion does not obey the minimum-maximum principle (the spatial fundamental solution is not strictly positive and resembles more the ideal lowpass *sinc* function (Fig. 5)). Thus, its implementation for denoising purposes should be executed with care.

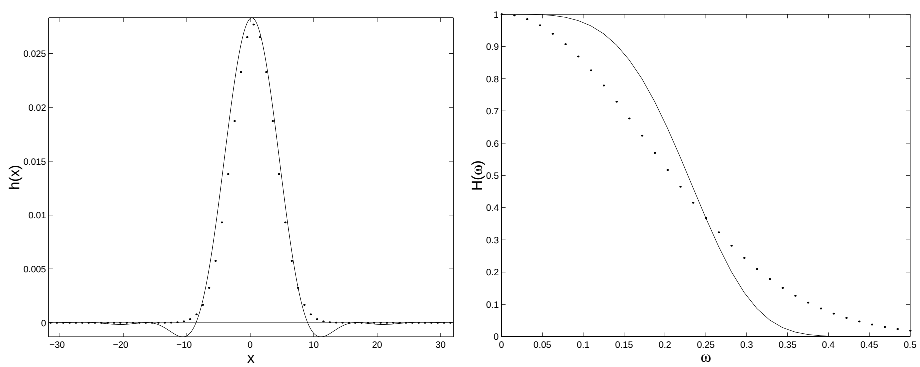

Fig. 5. Fundamental solution of the hyper-diffusion (line) vs. diffusion (dots), plotted in the spatial domain (left) and frequency domains (right). Whereas the diffusion kernel is a Gaussian in both domains, the hyper-diffusion has a sharper frequency cutoff and is not strictly positive in the spatial domain.

The Cahn-Hilliard [3] and Kuramoto-Sivashinsky [16], [30] equations have a hyper-diffusion term, that is stabilizing inverse diffusion processes (along with a first order nonlinearity). These equations were used to model evolution of phase fields in alloy mixtures [3], oscillatory chemical reactions [16] and fronts of premixed flames [30], among other natural phenomena [26], [36].

It was shown in [36] that a nonlinear forward-backward diffusion process with higher order regularization (of hyper-diffusion and a viscous relaxation term) yields a unique solution. Although the equations are different (e.g. the nonlinear diffusion coefficient in [36] is a function of the signal itself ($c = c(I)$) and not of its gradient), we assume that similar results can be

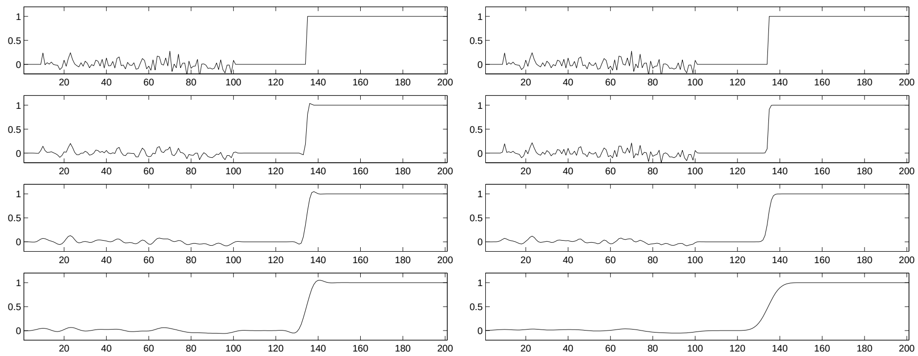

Fig. 6. Comparison of hyper-diffusion (left) and linear diffusion (right) processing of noise and a step edge, given at times 0, 0.1, 1, 10 (from top to bottom, respectively). Hyper-diffusion diminishes high frequency noise more rapidly, while low frequencies decay slower. Also, hyper-diffusion does not obey the minimum-maximum principle (most apparent in the step processing).

obtained in our case.

E. Energy Minimization Flow

We use the following dissipating energy process:

$$\begin{aligned} &I_t = \mathrm{div}(c_W(|\nabla I|^2)\nabla I) + \lambda(I_0 - I) - \varepsilon \nabla^4 I, \\ &I|_{t=0} = I_0,\ \partial_n I|_{x\in\partial\Omega} = 0,\ \partial_n^2 I|_{x\in\partial\Omega} = 0, \end{aligned} \tag{15}$$

where n is a unit vector, outward normal to the boundary $\partial\Omega$. The second boundary condition is stated in this case for the fourth order PDE to be well defined (in addition to the standard first order Neumann BC).

IV. EXAMPLES

A one dimensional signal resembling a blurred line (two close step edges of opposite signs), with additive noise, was processed (Fig. 8). This example demonstrates a noise removing process sharpens edges. Whereas the two edges are sharpened, the noise was smoothed out. This process can handle multiple types of blurs, both isotropic and anisotropic, simultaneously (Fig. 9). This is in contrast to deconvolution techniques that assume either an *a priori* known or an unknown (blind deconvolution) stationary (generally linear) blurring kernel. In Fig. 10 a blurred flower image is processed. Here an extended version of the processes is implemented, where the parameters controlling the shape of the well are spatially varying, that is, we use

Fig. 7. Hyper-diffusion processing of the cameraman image, given at normalized times 0 (top-left), 0.1 (top-right), 1 (bottom-left), 10.

$k_f(x,y), k_b(x,y)$. This is done in order to have wider sharpening range, where enhancement is accomplished by inducing different thresholds in different locations. We employ an automatic heuristic mechanism to determine these parameters without having any prior information. We define $T(x,y) = g_{\sigma_s} * |\nabla I_0(x,y)|$, which measures the average gradient magnitude in a neighborhood. The potential parameters are in turn adjusted according to $T(x,y)$. The following values were assigned: $k_f(x,y) = 0.5T(x,y)$, $k_b(x,y) = 5T(x,y)$, $\sigma_s = 5$.

Though edges are sharper, there are still some staircasing effects and the edges are not so smooth. A straightforward improvement could be the implementation of tensor diffusivity, instead of a scalar one (as in Weickert's coherence enhancing diffusion [34]), where the sharpening triple-well potential is used across the edge, and some smoothing potential is used along the edge.

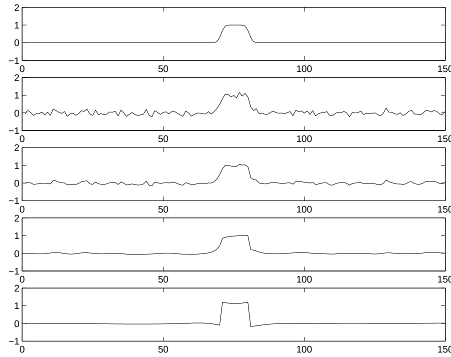

Fig. 8. Line edge with additive white Gaussian noise of standard deviation $\sigma_n = 0.1$ (SNR=7dB).

The numerical implementation consists of two iterative stages: at each time step, the nonlinear FAB diffusion, with a fidelity term, is calculated by a standard 3x3 template. The second stage implements the linear hyper-diffusion, by convolution with a 5x5 kernel (the minimal support required in the case of a fourth order equation). For the triple-well potential we used, in all examples, $\alpha = 2.2k_f/k_b$.

V. Discussion and Conclusions

This study has been concerned with the task of enhancement of important (steep) edges, by increasing their gradients, in order to reverse blurring effects. Such an ill-posed task has to be accomplished without noise amplification to avoid signal 'explosion'. This led to the formulation of a novel approach of signal and image sharpening processes according to a framework of calculus of variations. This led us to propose a gradient-dependent energy functional based on a triple-well potential. The present study extends our FAB diffusion-type process for sharpening of edges while denoising fluctuations and noise [12], in that it formulates it as a variational problem. The variational approach permits incorporation of additional terms into the functional, to account for the importance of additional image attributes. It also facilitates the process of regularization.

To accomplish the desired task, two additional terms were added to the general energy functional: a standard fidelity term and the square magnitude of the Laplacian, serving as a high order regularizing term. The energy minimization associated with the resultant functional leads to a hyper-diffusion flow; a fourth order process that exhibits strong low-pass filtering, and

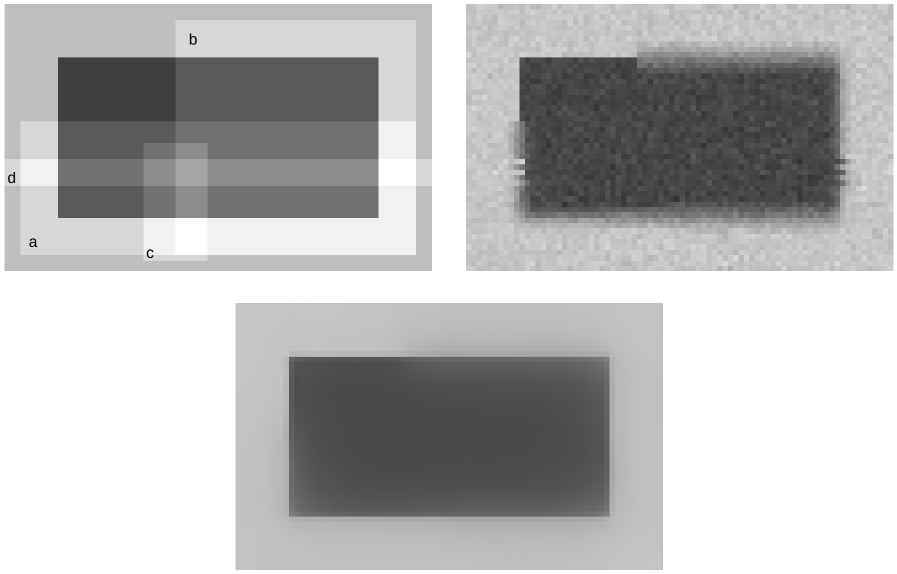

Fig. 9. Processing of a non-stationarily blurred step image, contaminated by additive noise. Top left: Degradation function, highlighting regions of different types of degradations: (a) Isotropic Gaussian blur ($\sigma = 2$), (b) Anisotropic exponential blur, $e^{-|x|+|y|/5}$, (c) 5x5 uniform averaging blur, (d) Jagginess. Regions overlapped by a few filters were processed by all of them. Top right - degraded image, with added Gaussian white noise of std $\sigma_n = 0.03$ and uniform white noise in the band $[-0.05, 0.05]$ (SNR=15dB). Bottom - processed image. Process parameters: $k_f = 0.02$, $k_b = 0.5$, $\lambda = 0.01$, $\epsilon = 0.1$. Image is 50x80 pixels, with original gray-level values of 0.25 (box) and 0.75 (background).

attenuates high frequency oscillations that are characteristic of inverse diffusion. The hyper-diffusion process eliminates the effect of enhancement of isolated points, otherwise sharpened by the triple-well potential. Moreover, edges become more coherent. As the weight of this smoothing term increases, the sharpening affects become less apparent. Additional effects of hyper-diffusion on the general process are yet to be further analyzed and understood. Also, under current investigation are issues related to stability properties and well-posedness of the equations.

The proposed approach of triple-well potentials can be generalized to deal with color images, using the Beltrami framework [31]. It can be further generalized and extended for processing and enhancement of additional image features.

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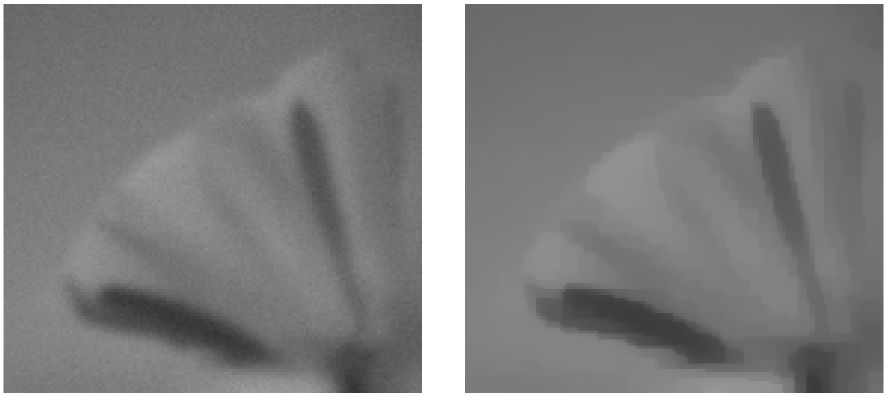

Fig. 10. Processing of a Gaussianly blurred flower image, ($\sigma = 2$), contaminated by white Gaussian noise (SNR=15dB). Left - input image, right - processed image.

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