Imaging of coherent fields through lenslike systems

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I derive the imaging condition for the complex amplitude of a monochromatic field by a sequence of lens-like elements.

In this Letter I consider the problem of imaging a *coherent* electromagnetic field by a general axisymmetric lenslike system, as illustrated in Fig. 1. The individual elements that compose the system are each describable by an $ABCD$ ray matrix, and the propagation between the input plane (x_0, y_0) and the output plane (x_1, y_1) is thus governed by the overall system matrix:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} A_n & B_n \\ C_n & D_n \end{vmatrix} \cdots \begin{vmatrix} A_1 & B_1 \\ C_1 & D_1 \end{vmatrix}. \tag{1}$$

In an important extension of the eikonal formalism, Baues[1] and Collins[2] showed that the output and input complex fields in our problem are related by

$$f_1(x_1, y_1) = \frac{ik}{2\pi B}\exp(-ikL)\int\int_{\Sigma_0} f_0(x_0, y_0)\exp\left\{\left(-\frac{ik}{2B}\right)\left[D(x_1^2 + y_1^2) - 2x_1x_0 - 2y_1y_0 + A(x_0^2 + y_0^2)\right]\right\} dx_0 dy_0. \tag{2}$$

By rearranging and manipulating the exponent we can rewrite Eq. (2) as

$$\begin{aligned} f_1(x_1, y_1) = {} & \frac{ik}{2\pi B}\exp\left[\left(-\frac{ik}{2B}\right)\left(D - \frac{1}{A}\right)(x_1^2 + y_1^2)\right] \\ & \times \int\int_{\Sigma_0} f_0(x_0, y_0)\exp\left\{\left(-\frac{ik}{2B}\right)\right. \\ & \times \left.\left[A\left(x_0 - \frac{x_1}{A}\right)^2 + A\left(y_0 - \frac{y_1}{A}\right)^2\right]\right\} dx_0 dy_0 . \end{aligned} \tag{3}$$

Here I consider the function $\exp[-(ikA/2B)(x_0 - x_1/A)^2]$.

I will show formally, at the end of this Letter, that

$$Y(x) \equiv \lim_{B\to 0}\sqrt{\frac{i}{2\pi B}}\exp\left(-i\frac{x^2}{2B}\right) = \delta(x), \tag{4}$$

where $\delta(x)$ is the Dirac delta function, so that in the limit $B \to 0$

$$\begin{aligned} f_1(x_1, y_1)_{B\to 0} = {} & \frac{\exp(-ikL)}{A} f_0\left(\frac{x_1}{A}, \frac{y_1}{A}\right) \\ & \times \exp\left[-i\frac{k(DA - 1)}{2AB}(x_1^2 + y_1^2)\right]. \end{aligned} \tag{5}$$

In the case of a lossless system, $AD - BC = 1$, so that Eq. (5) can be written as

$$\begin{aligned} f_1(x_1 y_1)_{B\to 0} = {} & \frac{\exp(-ikL)}{A} f_0\left(\frac{x_1}{A}, \frac{y_1}{A}\right) \\ & \times \exp\left[-i\frac{kC}{2A}(x_1^2 + y_1^2)\right]. \end{aligned} \tag{6}$$

Equation (6) is the central result of this Letter. It shows that when $B = 0$ the output is an exact, scaled replica of the input field except for a quadratic phase factor.[3,4] The image magnification is A. The generalized imaging condition is thus $B = 0$.[5,6]

As an example, I apply the formalism to the simple case of imaging by a single thin lens, as illustrated in Fig. 2. In this case,

$$\begin{aligned} \begin{vmatrix} A & B \\ C & D \end{vmatrix} & = \begin{vmatrix} 1 & d_i \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{vmatrix} \begin{vmatrix} 1 & d_0 \\ 0 & 1 \end{vmatrix} \\ & = \begin{vmatrix} 1 - \frac{d_i}{f} & d_0 d_i\left(\frac{1}{d_0} + \frac{1}{d_i} - \frac{1}{f}\right) \\ -\frac{1}{f} & 1 - \frac{d_0}{f} \end{vmatrix}. \end{aligned} \tag{7}$$

The imaging condition $B = 0$ in this case assumes the familiar form of geometrical optics,

$$\frac{1}{f} = \frac{1}{d_0} + \frac{1}{d_i}, \tag{8}$$

from which it follows directly that $A = 1 - d_i/f = -d_i/d_0 = -M$, where $M \equiv d_i/d_0$ is the magnification factor. In a similar fashion I show that $C/A = M/f$ and $D = -M^{-1}$, so that

$$\begin{aligned} f_1(x, y) = {} & -\frac{1}{M}\exp\left\{-i\left[kL + \frac{k(x^2 + y^2)}{2Mf}\right]\right\} \\ & \times f_0\left(-\frac{x}{M}, -\frac{y}{M}\right). \end{aligned} \tag{9}$$

This special case reduces to a result given by Goodman.[7] Another special case of imaging by a quadratic index fiber is considered in Ref. 5. I have thus derived the imaging relation for a coherent electromagnetic field by a system of lenslike elements.

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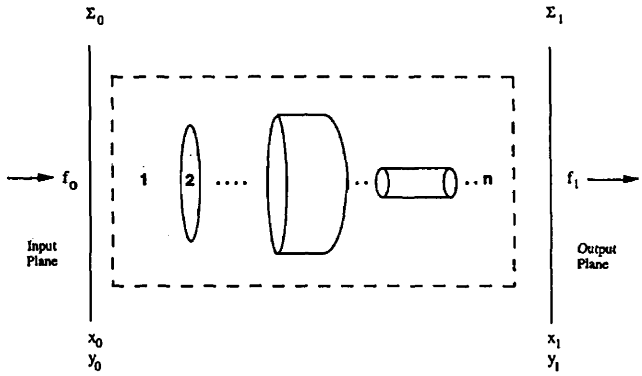

Fig. 1. Generalized lenslike system.

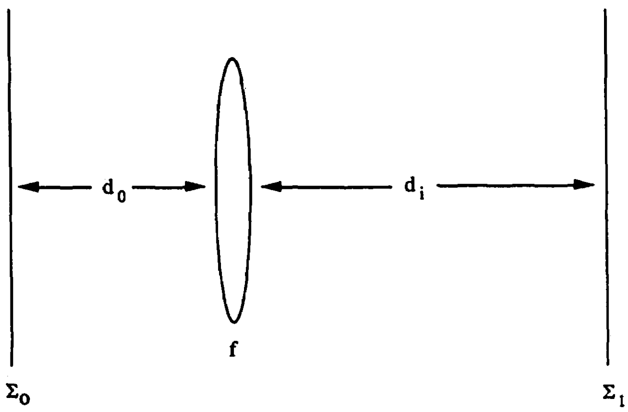

Fig. 2. Imaging by a thin lens.

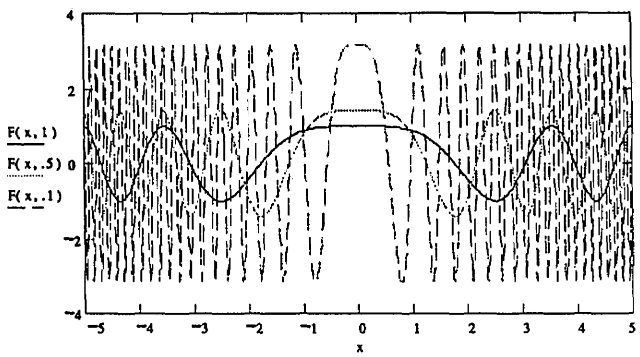

Fig. 3. Function $F(x,B) = (2\pi B)^{-1/2}\cos(x^2/2B)$ for $B = 1, 0.5, 0.1$.

To prove, as claimed above, that

$$Y(t) \equiv \lim_{B\to 0}\left(\frac{i}{2\pi B}\right)^{1/2}\exp\left(-i\frac{2t^2}{2B}\right) = \delta(t), \qquad (10)$$

we need to show that

$$\int_{-\infty}^{\infty} Y(t)\mathrm{d}t = 1 \qquad (11)$$

and that

$$\int_{t_1}^{t_2} Y(t)\mathrm{d}t = 0 \qquad (12)$$

for any interval $t_1 \ldots t_2$ that does not contain the origin. To appreciate qualitatively the nature of the function $(i/2\pi B)^{1/2}\exp(-ix^2/2B)$, consider its projection,

$$F(x,B \equiv \left(\frac{1}{2\pi B}\right)^{1/2}\cos\left(\frac{x^2}{2B}\right). \qquad (13)$$

Figure 3 shows plots of $F(x,B)$ for $B = 1, 0.5, 0.1$. Equation (11) reflects the fact that the main contribution to the integral is from the first few oscillations near the origin (the area under a given number of peaks is independent of B), whereas Eq. (12) follows from the fact that, when $B \to 0$, t_1 and t_2 in the normalized variable $x = t/\sqrt{B}$ of Fig. 3 tend to infinity. More rigorously, I employ the definition of the Fresnel cosine and sine integrals[8] $C(\omega)$ and $S(\omega)$, respectively,

$$C(\omega) = \int_0^{\omega}\cos\left(\frac{\pi}{2}\tau^2\right)\mathrm{d}\tau,$$
$$S(\omega) = \int_0^{\omega}\sin\left(\frac{\pi}{2}\tau^2\right)\mathrm{d}\tau,$$
$$C(\infty) = S(\infty) = 0.5,$$

to express Eq. (11) as

$$\int_{t_1}^{t_2} \mathrm{Y(t)}\,\mathrm{d}t = \frac{1}{2}(1+i)\lim_{B\to 0}\left\{C\left(\frac{t_2}{\sqrt{\pi B}}\right) - C\left(\frac{t_1}{\sqrt{\pi B}}\right) - i\left[S\left(\frac{t_2}{\sqrt{\pi B}}\right) - S\left(\frac{t_1}{\sqrt{\pi B}}\right)\right]\right\}$$
$$= \frac{1}{2}(1+i)\{C(\infty) - C(\infty) - i[S(\infty) - S(\infty)]\} = 0$$

when t_1 and t_2 have the same sign.

In a similar fashion I show that

$$\int_{-\infty}^{\infty} Y(t)\mathrm{d}t = (1+i)[C(\infty) - iS(\infty)] = 1.$$

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References

1. P. Baues, Optoelectronics **1,** 37 (1969).
2. S. B. Collins, J. Opt. Soc. Am. **60,** 1168 (1970).
3. G. N. Lawrence and S. W. Hwang, Appl. Opt. **31,** 5201 (1992).
4. H. T. Yura and S. G. Hanson, J. Opt. Soc. Am. A **4,** 1931 (1987).
5. The identification of the condition $B = 0$ with the imaging condition of a lenslike system is given in A. Yariv, J. Opt. Soc. Am. **66,** 301 (1976) as well as in Ref. 6.
6. A. E. Siegman, *Lasers* (University Science, Mill Valley, Calif., 1986), p. 596.
7. J. W. Goodman, *Introduction to Fourier Optics* (McGraw-Hill, New York, 1968), p. 96.
8. M. Born and Wolf, *Principles of Optics,* 5th ed. (Pergamon, Oxford, 1975), p. 430.