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IMBEDDING COMPACT 3-MANIFOLDS IN E^3

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ABSTRACT. We show that in a large finite disjoint collection of compacta in a closed orientable 3-manifold there is a compactum that imbeds in E^3. However, given a closed 3-manifold M^3, there is a pair of compact 3-manifolds (L, N) such that L contains infinitely many disjoint copies of N but N does not imbed in M^3.

It is sometimes useful to know that a subset of a 3-manifold can be imbedded in E^3. For example, it is sometimes desirable to use a linking argument in a neighborhood of an element of a decomposition. We will prove that in a large enough disjoint collection of compacta in a closed orientable 3-manifold there is a compactum with a neighborhood that imbeds in E^3. We will then show that no closed 3-manifold is an imbedding manifold for nonorientable 3-manifolds in the sense that E^3 is for orientable 3-manifolds.

Definitions. A *surface* is a closed connected 2-manifold. A surface S is *incompressible* in the 3-manifold M^3 means (1) S is not a 2-sphere and if $D \subseteq M^3$ is a disk with $D \cap S = \operatorname{Bd} D$, then $\operatorname{Bd} D$ bounds a disk in S; or (2) S is a 2-sphere that bounds no 3-cell in M^3. The surfaces S_1 and S_2 are said to bound a *parallelity component* U of M^3 and the disjoint collection of surfaces $\{S_1, S_2, \cdots, S_n\}$ if U is a component of $M^3 - \bigcup_{i=1}^n S_i$, $\overline{U}$ is homeomorphic to $S_1 \times I$, and $\operatorname{Bd} \overline{U} = S_1 \cup S_2$.

We assign some constants to a given compact 3-manifold M^3. Let $\beta_1(H_1(M^3, G))$ be the rank of $H_1(M^3, G)$. When $G = \mathbf{Z}$ we sometimes write just β_1. Also let α be the maximum number of disjoint nonparallel incompressible 2-sided surfaces in M^3. We know α exists by [1].

Theorem 1. *Suppose M^3 is a closed orientable 3-manifold and $\{C_1, \cdots, C_{\alpha+\beta_1+2}\}$ is a disjoint collection of compacta in M^3. Then there is a C_i with a neighborhood imbeddable in E^3.*

Proof. Let $\{U_1, \cdots, U_{\alpha+\beta_1+2}\}$ be a disjoint collection of compact poly-

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hedral 3-manifolds such that U_i is a neighborhood of C_i for each i. We will show that some U_i imbeds in E^3.

We first alter the U_i's. If for some i there is a component of $\mathrm{Bd}\, U_i$ that is a compressible surface not a 2-sphere, then we can find a disk D so that $D \cap \mathrm{Bd}\, U_j = \mathrm{Bd}\, D$ for some j, and either $D \subseteq U_j$ or $D \cap (\bigcup_{i=1}^{\alpha+\beta_1+2} U_i) = \mathrm{Bd}\, D$.

In the first case we alter the collection by replacing U_j by $\overline{U_j - N(D)}$, where $N(D)$ is a regular neighborhood of D in U_j, and in the second case we replace U_j by $U_j \cup N(D)$, where $N(D)$ is a regular neighborhood of D in $\overline{M^3 - \bigcup_{i=1}^{\alpha+\beta_1+2} U_i}$. Since each such step reduces the sum $\Sigma_{S \in C}(\chi(S) - 2)^2$, with C the collection of surfaces in $\bigcup_{i=1}^{\alpha+\beta_1+2} \mathrm{Bd}\, U_i$ (where χ stands for Euler characteristic), in a finite number of steps we get a disjoint collection $\{V_1, \cdots, V_{\alpha+\beta_1+2}\}$ of compact 3-manifolds so that each component of each $\mathrm{Bd}\, V_i$ is incompressible in M^3 or a 2-sphere bounding a 3-cell.

Lemma 2. *If M^3 is a closed 3-manifold and $\{V_1, \cdots, V_{\alpha+2}\}$ is a disjoint collection of compact connected 3-manifolds in M^3 such that each component of each $\mathrm{Bd}\, V_i$ is incompressible in M^3, then there is an i_0 such that V_{i_0} is the product of a surface and an interval.*

Proof. Let

$$\{S_1, \cdots, S_{\alpha+2+k}\} \tag{1}$$

be the collection of boundary components of the V_i's. We know $M^3 - \bigcup_{i=1}^{\alpha+2+k} S_i$ has at most $\alpha + 3 + k$ components, so $M^3 - \bigcup_{i=1}^{\alpha+2} V_i$ has at most $k + 1$ components $\{W_1, \cdots, W_l\}$ with $l \leq k + 1$. We now change the collection (1) by removing, for each W_j, one surface $S_i \subseteq \mathrm{Bd}\, W_j$. Our collection still contains at least $\alpha + 1$ surfaces and so there are two surfaces, call them S_1 and S_2, such that S_1 and S_2 bound V, a parallelity component of the changed collection [1]. Notice that $S_1 \cup S_2$ is not the entire boundary of any W_j. Then there is an i_0 such that $V_{i_0} \subseteq \overline{V}$. Now by [1, p. 91], $\mathrm{Bd}\, V_{i_0}$ has exactly two components and V_{i_0} is topologically the product of a surface and an interval.

Let $\{X_1, X_2, \cdots, X_{\alpha+2}\}$ be a collection of components of $\bigcup_{i=1}^{\alpha+\beta_1+2} V_i$. We claim that some X_i imbeds in E^3. We first consider all components of $\bigcup_{i=1}^{\alpha+2} \mathrm{Bd}\, X_i$. If some component is a 2-sphere bounding a 3-cell, then either that 3-cell contains an X_i or else we can add the 3-cell to the appropriate X_j to get a new collection of $\alpha + 2$ disjoint compact connected 3-manifolds. Eventually we have either some X_i in a 3-cell or all the boundary components incompressible and Lemma 2 applies to establish the claim.

It follows from the claim that at most $\alpha + 1$ of the V_i's have components that do not imbed in E^3, and therefore at least $\beta_1 + 1$ of the V_i's obtained from the U_i's can be imbedded in E^3. So we may assume that each of

$$\{V_1, \cdots, V_{\beta_1+1}\} \tag{2}$$

can be imbedded in E^3.

Notice that we can go step by step in M^3 from the collection (2) to the collection

$$\{U_1, \cdots, U_{\beta_1+1}\} \tag{3}$$

by performing the operations preceding Lemma 2 backwards, so that each step consists of adding a 1-handle to a compact 3-manifold or removing a regular neighborhood of a properly imbedded arc from (digging a tunnel in) a compact 3-manifold.

Lemma 3. *If we reconstruct* (3) *from* (2) *as above, then there is an* i_0 *so that in the reconstruction of* U_{i_0}, *whenever we attach a* 1*-handle to a connected compact* 3*-manifold, we attach both ends to the same boundary component.*

Proof. Suppose that in the reconstruction of U_i for each i, we attach a 1-handle H_i to the compact connected 3-manifold M_i with one end on boundary component S_i and one end on another boundary component. Then we can draw a simple closed curve $J_i \subseteq M_i \cup H_i$ made up of an arc in H_i and an arc in M_i so that the intersection number of J_i and S_i is one. In subsequent steps we dig all tunnels to miss J_i.

Since we have more than β_1 curves, there is a nontrivial relation in $H_1(M^3, \mathbf{Z})$,

$$n_1[J_1] + \cdots + n_{\beta_1+1}[J_{\beta_1+1}] = 0,$$

where $[J]$ is the element of $H_1(M^3, \mathbf{Z})$ represented by J. We may assume $n_1 \neq 0$ and that J_1 is the last such curve formed in the sequence going from (2) to (3). The relation tells us that the intersection number of S_1 and the 1-manifold $n_1 J_1 \cup \cdots \cup n_{\beta_1+1} J_{\beta_1+1}$ is zero, where $n_i J_i$ is the 1-manifold consisting of n_i copies of J_i near J_i. But $J_i \cap S_1 = \emptyset$ for $i \neq 1$ and the intersection number of J_1 and S_1 is one. We have arrived at a contradiction and the lemma is proved.

We now complete the proof of Theorem 1. For each V_i in the collection (2) we attempt to mimic in E^3 the step by step reconstruction of U_i from V_i occurring in M^3.

There is no problem if the ends of a 1-handle are to be attached to different components of a compact 3-manifold; we imbed one component in E^3, and then we imbed the other component in the proper complementary domain of the first, being careful to imbed so that the boundary components to which the 1-handle is to be attached are adjacent.

We face a problem only if the ends of a 1-handle are attached to different boundary components of a compact connected 3-manifold. By Lemma 3 we may assume this problem never occurs in the construction of U_1 from V_1, and so U_1 can be imbedded in E^3.

Notice that U_1 is constructed from the product of orientable surfaces with an interval by removing 3-cells from the interior, adding 1-handles, and drilling tunnels.

The referee has pointed out the following corollary to Theorem 1.

Corollary. *Let M^3 be a closed 3-manifold that is not sufficiently large (that is, M^3 contains no incompressible surfaces). If X_1 and X_2 are disjoint compacta in M^3, then one of X_1 and X_2 imbeds in E^3.*

Proof. It follows from [2, p. 774] that M^3 is orientable and $\beta_1(M^3) = 0$. Since M^3 is not sufficiently large, $\alpha = 0$ and the Corollary follows from Theorem 1.

In contrast, if we drop the orientability hypothesis from Theorem 1, then given a closed 3-manifold W^3, we can construct a pair of compact 3-manifolds (L, N) with the following properties:

(1) L contains infinitely many disjoint copies of N, and

(2) N cannot be imbedded in W^3.

We first construct a sequence of auxiliary pairs $\{(L_i, N_i) | i = 1, 2, \cdots\}$. We let $N_1 = L_1 = P \times I$, the product of a projective plane and an interval. If $i \geq 2$, the construction of L_i begins with i copies of $P \times I$. We label them $\{P_j \times I | 1 \leq j \leq i\}$. We then connect $P_j \times 0$ with $P_1 \times 0$ by a 1-handle H_j for each j with $2 \leq j \leq i$. This is L_i. We construct N_i from a copy of L_i by removing regular neighborhoods of $i - 1$ disjoint arcs $\{A_j | 2 \leq j \leq i\}$, where A_j has one end in $P_j \times 1$ and one end in $P_1 \times 1$. The arc A_j runs from $P_j \times 1$ through H_j, around an orientation reversing curve in $P_1 \times I$, and back through H_j, around an orientation reversing curve in $P_j \times I$, and back through H_j to $P_1 \times I$.

Now given W^3 let

$$K = \beta_1(W^3, \mathbf{Z}_3).$$

Then we choose N to be the disjoint union

$$N = N_{2K+1} \cup N_{4K+2} \cup \cdots \cup N_{2^{K+1}K+2^K},$$

and we let L be the disjoint union

$$L = L_{2K+1} \cup L_{4K+2} \cup \cdots \cup L_{2^{K+1}K+2^K}.$$

Property (1). To prove that (L, N) has Property 1 it is clearly enough to imbed infinitely many disjoint copies of N_i in L_i for a given i. As above we write L_i as

$$L_i = \left(\bigcup_{j=1}^{i} P_j \times I\right) \cup \left(\bigcup_{j=2}^{i} H_j\right).$$

We construct $N_{i,1}$, our first copy of N_i, starting with $\bigcup_{j=1}^{i} P_j \times [\frac{1}{4}, \frac{1}{2}]$. Then $P_1 \times \frac{1}{4}$ is joined to $P_j \times \frac{1}{4}$ by a 1-handle running through H_j for $2 \leq j \leq i$. The tunnels are drilled by removing appropriate arcs. The $N_{i,2}$ starts with $\bigcup_{j=1}^{i} P_j \times [3/4, 7/8]$. Then $P_1 \times \frac{3}{4}$ is joined to $P_j \times \frac{3}{4}$ by a 1-handle running through the jth tunnel in $N_{i,1}$ for each j with $2 \leq j \leq i$. The tunnels are then drilled to complete $N_{i,2}$. The rest of the construction is clear.

Property (2). Suppose N can be imbedded in W^3. Let $B_{1,2^iK+2^{i-1}}$ be the boundary component of $N_{2^iK+2^{i-1}}$ that intersects $P_1 \times 1$ for each i with $i = 1, 2, \cdots, K+1$, and let $B_{2,2^iK+2^{i-1}}$ be the other boundary component of $N_{2^iK+2^{i-1}}$. By our choice of K there is a smallest r such that $B_{1,2^rK+2^{r-1}}$ separates $W^3 - \bigcup_{i=1}^{r-1} B_{1,2^iK+2^{i-1}}$. We let

$$B_1 = B_{1,2^rK+2^{r-1}}, \qquad B_2 = B_{2,2^rK+2^{r-1}} \quad \text{and} \quad N' = N_{2^rK+2^{r-1}}.$$

Then we let U and V be the components of $M^3 - \bigcup_{i=1}^{r} B_{1,2^iK+2^{i-1}}$ with $N' \subseteq \bar{U}$.

Lemma 4. $\beta_1(H_1(\bar{U}, \mathbf{Z}_3)) \leq \beta_1(H_1(\bar{U} \cap \bar{V}, \mathbf{Z}_3)) + \beta_1(H_1(\bar{U} \cup \bar{V}, \mathbf{Z}_3)) - \beta_1(H_1(\bar{V}, \mathbf{Z}_3))$.

Proof. The Mayer-Vietoris sequence

$$\to H_1(\bar{U} \cap \bar{V}, \mathbf{Z}_3) \xrightarrow{i^*} H_1(\bar{U}, \mathbf{Z}_3) \oplus H_1(\bar{V}, \mathbf{Z}_3) \xrightarrow{j^*} H_1(\bar{U} \cup \bar{V}, \mathbf{Z}_3) \to$$

generates the short exact sequence

$$0 \to H_1(\bar U \cap \bar V, \mathbf{Z}_3)|\mathrm{Ker}\, i^* \to H_1(\bar U, \mathbf{Z}_3) \oplus H_1(\bar V, \mathbf{Z}_3) \to \mathrm{Im}\, j^* \to 0,$$

so we can think of $H_1(\bar U \cap \bar V, \mathbf{Z}_3)|\mathrm{Ker}\, i^*$ as a subgroup of $H_1(\bar U, \mathbf{Z}_3) \oplus H_1(\bar V, \mathbf{Z}_3)$ and

$$(H_1(\bar U, \mathbf{Z}_3) \oplus H_1(\bar V, \mathbf{Z}_3))|(H_1(\bar U \cap \bar V, \mathbf{Z}_3)|\mathrm{Ker}\, i^*) = \mathrm{Im}\, j^*.$$

Then $H_1(\bar U, \mathbf{Z}_3) \oplus H_1(\bar V, \mathbf{Z}_3)$ has as many generators as $H_1(\bar U \cap \bar V, \mathbf{Z}_3)|\mathrm{Ker}\, i^*$ and $\mathrm{Im}\, j^*$ combined, which implies the conclusion of the lemma.

Lemma 5. *If $\bar V$ is a compact 3-manifold with nonempty boundary, then $\beta_1(H_1(\bar V, \mathbf{Z})) \geq 1 - \frac{1}{2} \times (\mathrm{Bd}\, \bar V)$.*

Proof. We know by [3, p. 233] that $2x(\bar V) = x(\mathrm{Bd}\, \bar V)$ (where x stands for Euler characteristic). For compact 3-manifolds, $x(\bar V) = \beta_0 - \beta_1 + \beta_2 - \beta_3$ and $\beta_0 \geq 1$ and $\beta_3 = 0$. So we have $1 - \beta_1 + \beta_2 \leq \frac{1}{2}X(\mathrm{Bd}\, \bar V)$ or $\beta_1 \geq 1 - \frac{1}{2}X(\mathrm{Bd}\, \bar V)$.

By Lemmas 4 and 5 we have

$$\beta_1(H_1(\bar U, \mathbf{Z}_3)) \leq \sum_S (2^i K + 2^{i-1} - 1) + K - \left(1 + \frac{1}{2} \sum_S (2^i K + 2^{i-1} - 2)\right)$$

(where $i \in S$ if $B_{1, 2^i K + 2^{i-1}} \subseteq \mathrm{Bd}\, \bar V$)

$$= \frac{1}{2}\left(\sum_S (2^i K + 2^{i-1} - 2)\right) + K + |S| - 1$$

$$\leq \frac{1}{2}\left(\sum_{i=1}^{r} 2^i K + 2^{i-1} - 2\right) + K + r - 1 = 2^r K + 2^{r-1} - 1\tfrac{1}{2}.$$

We know that $\beta_1(H_1(B_1, \mathbf{Z}_3)) = 2^r K + 2^{r-1} - 1$, so there is a $g \in H_1(B_1, \mathbf{Z}_3)$ with $g \neq 0$ and $j_* g = 0$ in $H_1(\bar U, \mathbf{Z}_3)$, where j_* is the inclusion induced homomorphism.

Now consider the Mayer-Vietoris sequence

$$\to H_1(B_2, \mathbf{Z}_3) \xrightarrow{r_*} H_1(N', \mathbf{Z}_3) \oplus H_1(\overline{U - N'}, \mathbf{Z}_3) \xrightarrow{s_*} H_1(\bar U, \mathbf{Z}_3).$$

Since $s_*(i_* g, 0) = j_* g = 0$, where i_* is here the homomorphism induced by the inclusion of $\mathrm{Bd}\, N'$ in N', there is an $l \in H_1(B_2, \mathbf{Z}_3)$ such that $r_* l = (i_* g, 0)$ or $i_* l = i_* g$ in $H_1(N', \mathbf{Z}_3)$.

We now investigate $H_1(N', \mathbf{Z})$. Let a_j be the orientation reversing curve in $P_j \times 1$ and b_j be the orientation reversing curve in $P_j \times 0$ for each j with $1 \leq j \leq 2^r K + 2^{r-1}$. If the curves are oriented properly and if $[a_j]$ and $[b_j]$ represent the homology classes of a_j and b_j in $H_1(N', \mathbf{Z})$, then $[b_j] = 3[a_j]$ in $H_1(N', \mathbf{Z})$ by our construction of the tunnels in N'. Therefore $[b_j] = 0$ in $H_1(N', \mathbf{Z}_3)$. Since the $[b_j]$'s generate $H_1(B_2, \mathbf{Z}_3)$ we have $i_* H_1(B_2, \mathbf{Z}_3) = 0$ in $H_1(N', \mathbf{Z}_3)$ and $i_* g = i_* l = 0$ in $H_1(N', \mathbf{Z}_3)$ so $\beta_1(H_1(N', \mathbf{Z}_3)) \leq 2^r K + 2^{r-1} - 1$. But since $i_* g = 0$ in $H_1(N', \mathbf{Z}_3)$,

$$\beta_1(H_1(N', \mathbf{Z}_3)) \leq 2^r K + 2^{r-1} - 2.$$

However by Lemma 5

$$\beta_1(H_1(N', \mathbf{Z}_3)) \geq \beta_1(H_1(N', \mathbf{Z})) \geq 1 - \tfrac{1}{2}X(\mathrm{Bd}\, N')$$
$$= 1 - \tfrac{1}{2}(2(2 - 2^r K - 2^{r-1})) = 2^r K + 2^{r-1} - 1.$$

The last two inequalities are incompatible, and Property (2) is established.

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