

IMBEDDINGS OF HOMOGENEOUS SPACES WITH MINIMUM TOTAL CURVATURE

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1. Introduction. Let M be an n -dimensional compact differentiable manifold immersed in the Euclidean space R^{n+k} . Let B be the set of unit normal vectors of M . Then B is a bundle of $(k-1)$ -sphere over M and is a manifold of dimension $n+k-1$. Let S be the unit $(n+k-1)$ -sphere about the origin of R^{n+k} . Let $d\sigma$ be the volume element of S and $c_{n+k-1} = \int_S d\sigma$ the volume of S . If we denote by ν the canonical map $B \rightarrow S$, then the total curvature of the immersed manifold M is defined by (cf. Chern and Lashof [3])

$$\frac{1}{c_{n+k-1}} \int_B \nu^*(d\sigma).$$

Since the total curvature defined above depends not only on M but on the immersion $\varphi: M \rightarrow R^{n+k}$, we shall denote it by $\tau(M, \varphi, R^{n+k})$ or simply by $\tau(\varphi)$.

Let F be the set of functions f on M whose critical points are all non-degenerate. The number of the critical points of index i of $f \in F$ will be denoted by $\beta_i(M, f)$. We set

$$\left\{ \begin{array}{l} \beta_i(M) = \min_{f \in F} \beta_i(M, f), \\ \beta(M, f) = \sum_{i=0}^n \beta_i(M, f), \\ \beta(M) = \min_{f \in F} \beta(M, f). \end{array} \right.$$

Since $\beta_i(M, f) = \beta_{n-i}(M, -f)$, we have

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$$\beta_i(M) = \beta_{n-i}(M).$$

Evidently we have also

$$\beta(M) \geq \sum_{i=0}^n \beta_i(M).$$

For an arbitrarily fixed coefficient field, let $b_i(M)$ be the i -th Betti number of M . Then the Morse inequalities state

$$\beta_i(M) \geq b_i(M).$$

Chern-Lashof [4] proved the inequality

$$\tau(M, \varphi, R^{n+k}) \geq \sum_{i=0}^n b_i(M).$$

Kuiper [8] obtained a stronger inequality:

$$\tau(M, \varphi, R^{n+k}) \geq \beta(M).$$

Thus

$$\tau(M, \varphi, R^{n+k}) \geq \beta(M) \geq \sum_{i=0}^n \beta_i(M) \geq \sum_{i=0}^n b_i(M).$$

He also proved that if M is fixed, then for variable immersion $\varphi : M \rightarrow R^{n+k}$ and variable k :

$$\inf \tau(M, \varphi, R^{n+k}) = \beta(M).$$

An immersion $\varphi : M \rightarrow R^{n+k}$ is said to be minimal if $\tau(\varphi) = \beta(M)$. Given a compact manifold M , in general there does not exist a minimal immersion of M . Kuiper pointed out that an exotic sphere cannot be minimally immersed. In fact, if M is an exotic sphere, it admits a function with two isolated singularities (one maximum and one minimum) and hence $\beta(M)=2$. On the other hand, by a theorem of Chern-Lashof [3] an immersed compact manifold M with $\tau(M, \varphi, R^{n+k}) = 2$ is a convex hypersurface in some $R^{n+1} \subset R^{n+k}$, which implies that M is diffeomorphic with a usual sphere. It is an interesting but difficult problem to decide which manifolds can be minimally immersed, since it involves not only topological but differentiable structures of manifolds. Kuiper [9] proved that every orientable closed surface and also

every non-orientable closed surface with Euler number ≤ -2 can be minimally immersed in R^3 and that the real projective plane and the Klein bottle can not be minimally immersed in R^3 . In a earlier paper (Kuiper [8]) he exhibited a minimal immersion of the real projective plane in R^4 .¹⁾

The purpose of this paper is to prove

THEOREM 1. *Every compact homogeneous Kaehler manifold can be minimally imbedded into a Euclidean space.*

We can also estimate the dimension of the receiving Euclidean space. Every compact homogeneous Kaehler manifold M can be written as a coset space G/H of a compact Lie group G , (see references given in § 2). Let S be the semi-simple part of G and C the center of G . Then M can be minimally imbedded into a Euclidean space of dimension $\dim S + \frac{3}{2} \dim C$.

For further comments on the imbedding we construct, see the last section.

2. Reduction to the simply connected case. Let M and M' be compact manifolds and $\varphi: M \rightarrow R^N$ and $\varphi': M' \rightarrow R^{N'}$ be immersions with total curvature $\tau(M, \varphi, R^N)$ and $\tau(M', \varphi', R^{N'})$. Then the total curvature $\tau(M \times M', \varphi \times \varphi', R^{N+N'})$ is given by

$$\tau(M \times M', \varphi \times \varphi', R^{N+N'}) = \tau(M, \varphi, R^N) \cdot \tau(M', \varphi', R^{N'});$$

see Kuiper [8] for the proof.

From the definition of $\beta(M)$ and $\beta(M')$, it is clear that

$$\begin{aligned} \beta(M)\beta(M') &\geq \beta(M \times M') \geq \sum_k b_k(M \times M') \\ &= \left(\sum_i b_i(M) \right) \left(\sum_j b_j(M') \right). \end{aligned}$$

Hence if $\beta(M) = \sum_i b_i(M)$ and $\beta(M') = \sum_j b_j(M')$, then

$$\beta(M)\beta(M') = \beta(M \times M').$$

We may now conclude

1) For the total curvature of immersed manifolds, see also an excellent exposition by D. Ferus, Die absolute Totalkrümmung Riemannscher Immersionen, Diplomarbeiten, Bonn 1966.

LEMMA 1. *Let M and M' be compact manifolds minimally immersed in R^N and $R^{N'}$ respectively. If $\beta(M) = \sum_i b_i(M)$ and $\beta(M') = \sum_j b_j(M')$, then $M \times M'$ is minimally immersed in $R^{N+N'}$ in a natural manner.*

Since a torus T is a product of circles, we have

LEMMA 2. *For a torus T , we have the equality $\beta(T) = \sum_i b_i(T)$.*

The following is due to Frankel [5].

LEMMA 3. *Let M be a compact Kaehler manifold with $b_1(M) = 0$. If it admits a Killing vector field X with isolated zeros, then $\beta(M) = \sum_i b_i(M)$.*

Since we need the proof in the next section, we shall describe it briefly. Let ξ be the 1-form corresponding to X under the duality defined by the metric. Let J be the complex structure of M . Then $J\xi$ is a closed 1-form and hence $J\xi = df$ for some function f . Clearly the critical points of f coincide with the zeros of X . Frankel shows that the isolated critical points of f are all non-degenerate and of even index, i.e.,

$$\beta_i(M, f) = 0 \quad \text{for all odd } i.$$

From the Morse relations it follows that

$$b_i(M) = \beta_i(M, f) \quad \text{for all } i,$$

which implies Lemma 3.

LEMMA 4. *A torus T of dimension r can be minimally imbedded into R^N where $N = \frac{3}{2}r$ if r is even and $N = \frac{3}{2}(r-1) + 2$ if r is odd.*

If r is even, we write T as a product of 2-dimensional tori. If r is odd, we write T as a product of 2-dimensional tori and a circle. A 2-dimensional torus can be minimally imbedded into R^3 (in an ordinary doughnut shaped form) and a circle can be minimally imbedded into R^2 in a usual manner. Lemma 4 follows from Lemmas 1 and 2.

The following main lemma will be proved in the next section.

LEMMA 5. *Let $M = G/H$ be a simply connected homogeneous Kaehler*

manifold with G compact. Then $\beta(M) = \sum_i b_i(M)$ and M can be minimally imbedded into R^N where $N = \dim G$.

The following lemma is due to Borel-Remmert [2].

LEMMA 6. *A compact homogeneous Kaehler manifold is a direct product of a complex torus and a compact simply connected homogeneous Kaehler manifold and can be written as a coset space G/H of a compact Lie group G .*

The following lemma is due to Matsushima [12].

LEMMA 7. *Let $M = G/H$ be a homogeneous Kaehler manifold with G compact. Then $G = S \times C$ where S is semi-simple and C is the center of G . Moreover S contains H so that $G/H = (S/H) \times C$ is the decomposition described in Lemma 6, i.e., S/H is a simply connected homogeneous Kaehler manifold and C is a complex torus.*

It is now clear that the proof of Theorem is reduced to that of the main lemma (lemma 5).

3. Proof of Lemma 5. Let $M = G/H$ be a simply connected homogeneous Kaehler manifold on which a compact Lie group G acts effectively. Let \mathfrak{g} denote the Lie algebra of G whose elements are considered as Killing vector fields on M . Let \mathfrak{g}^* denote the space of 1-forms corresponding to the Killing vector fields $X \in \mathfrak{g}$ under the duality defined by the metric. Let $\Delta = d\delta + \delta d$ denote the Laplacian. We define a space E of functions on M by

$$E = \{ \delta(J\xi) ; \xi \in \mathfrak{g}^* \} .$$

Let E^* be the dual space of E and $\varphi : M \rightarrow E^*$ the evaluation map, i.e.,

$$\langle \varphi(x), f \rangle = f(x) \quad \text{for } x \in M \quad \text{and } f \in E .$$

We shall show that φ gives a minimal imbedding of M into E^* .

A compact simply connected homogeneous Kaehler manifold M carries an Einstein-Kaehler metric (cf. Borel [1] and Koszul [7]). If ξ is a 1-form corresponding to a Killing vector field of M , then (cf. Yano and Bochner [14; p. 33])

$$\Delta(J\xi) = J\Delta(\xi) = 2J\xi .$$

Since $J\xi$ is closed (cf. Matsushima [13; lemme 4]), we have

$$d\delta J\xi = 2J\xi.$$

This shows that the mapping $\xi \in \mathfrak{g}^* \rightarrow \delta J\xi \in E$ is a linear isomorphism and that the critical points of the function $f = \delta J\xi$ coincide with the zeros of ξ . From the proof of Lemma 3 described above, we have

LEMMA 8. *Let $f = \delta J\xi \in E$ be a function having only isolated points. Then the critical points of f are all non-degenerate and of even index. Hence*

$$\mathcal{B}(M, f) = \sum_{\text{even } i} \mathcal{B}_i(M, f) = \sum_{\text{even } i} b_i(M) = \mathcal{B}(M).$$

Note that the last equality follows from the first two and from

$$\mathcal{B}(M, f) \geq \mathcal{B}(M) \geq \sum_i b_i(M).$$

Since G is transitive on M , it follows that the set

$$\{(df)_x; f \in E\} = \{(d\delta J\xi_x; \xi \in \mathfrak{g}^*)\} = \{(J\xi)_x; \xi \in \mathfrak{g}^*\}$$

coincides with the cotangent space of M at x . Hence the evaluation map $\varphi: M \rightarrow E^*$ is an immersion.

The linear isomorphism $\xi \in \mathfrak{g}^* \rightarrow \delta J\xi \in E$ induces the dual linear isomorphism $E^* \rightarrow \mathfrak{g}$. Let ρ be the representation of G on E^* corresponding to the adjoint representation of G on \mathfrak{g} . It is easy to verify that the immersion φ is equivariant with ρ in the sense that

$$\rho(s)\varphi(x) = \varphi(s(x)) \text{ for } s \in G \text{ and } x \in M.$$

The group G acts on $\varphi(M)$ in a natural manner; $s \in G$ sends $\varphi(x)$ into $\rho(s)\varphi(x) = \varphi(s(x))$. Let $o \in M$ be the origin corresponding to the coset H of G/H and let H^* be the isotropy subgroup of G acting on $\varphi(M)$ at $\varphi(o)$, i.e.,

$$H^* = \{s \in G; \rho(s)\varphi(o) = \varphi(o)\} = \{s \in G; \varphi(s(o)) = \varphi(o)\}.$$

Since $\varphi: M \rightarrow E^*$ is an equivariant immersion, it follows that $\varphi: M \rightarrow \varphi(M)$ is a covering projection. In other words, the natural map $G/H \rightarrow G/H^*$ is a covering projection.

Following Lichnerowicz [11; p.166] we shall show that $H^* = H$ thereby proving that $\varphi : M \rightarrow \varphi(M)$ is one-to-one. The group H^* is the isotropy subgroup of G at $\varphi(o)$ in the representation ρ . Under the identification $E^* \approx \mathfrak{g}$, H^* is the isotropy subgroup of G at $\varphi(o)$ in the adjoint representation, i.e., the centralizer of the 1-parameter subgroup generated by $\varphi(o) \in \mathfrak{g}$. Since G is compact, the closure of this 1-parameter subgroup in G is a torus and H^* is the centralizer of this torus in G . It follows that H^* is connected and hence $H^* = H$.

Now the fact that the imbedding $\varphi : M \rightarrow E^*$ is minimal follows from Lemma 8 and from the following lemma of Kuiper [10].

LEMMA 9. *In general let φ be an immersion of M into a vector space E^* . Consider each element f of the dual space E as a function $f \circ \varphi$. Then the immersion φ is minimal if and only if*

$$\beta(M, f) = \beta(M)$$

for every function $f \in E$ having only isolated non-degenerate critical points on M .

4. Some comments on the minimal imbedding φ . Throughout this section $M = G/H$ will be a compact simply connected homogeneous Kaehler manifold and G will be a compact semi-simple Lie group.

The minimal imbedding φ of M constructed in § 3 may be considered as an imbedding of M into the Lie algebra \mathfrak{g} of G . The imbedding φ is then equivariant with the adjoint representation of G .

There is no proper affine subspace of E^* which contains $\varphi(M)$. Otherwise there would be a nonzero function $f \in E$ which is constant on M . Since $f = \delta J \xi$ for some $\xi \in \mathfrak{g}^*$ and $0 = df = 2J\xi$, f must vanish identically on M and hence f is the zero element of E . This is absurd.

If $2m$ denotes the real dimension of M , then $\dim E^* = \dim G = \dim M + \dim H \leq \dim M + \dim U(m) = 2m + m^2$, and the equality is attained when M is the complex projective space.

For the complex projective space $P_m(C)$ the minimal imbedding φ may be described as follows. Let (z^0, z^1, \dots, z^m) be a homogeneous coordinate system of $P_m(C)$ with the condition $z^0 \bar{z}^0 + z^1 \bar{z}^1 + \dots + z^m \bar{z}^m = 1$. Let $R^{(m+1)^2}$ be a Euclidean space with coordinate system (X^h, X^{hk}, Y^{hk}) where $h, k = 0, \dots, m$ and $h \neq k$. Consider the imbedding $P_m(C) \rightarrow R^{(m+1)^2}$ defined by

$$X^h = \sqrt{2} z^h \bar{z}^h, X^{hk} = z^h \bar{z}^k + \bar{z}^h z^k, Y^{hk} = i(z^h \bar{z}^k - \bar{z}^h z^k).$$

Then $P_m(C)$ lies in the hyperplane of $R^{(m+1)^2}$ defined by

$$X^0 + \cdots + X^m = \sqrt{2}.$$

It can be shown that the imbedding of $P_m(\mathbb{C})$ into this hyperplane is the same as φ . I found this imbedding in Hodge [6; p. 151].²⁾

Since $P_{2m-1}(\mathbb{C})$ can be written also as $Sp(m)/Sp(m-1) \times U(1)$, $P_{2m-1}(\mathbb{C})$ can be minimally imbedded into R^N where $N = \dim Sp(m) = m(2m+1)$.

ADDED IN PROOF. Generalizing the minimal imbedding of a complex projective space described in §4, S. S. Tai has recently discovered minimal imbeddings of real and quaternionic projective spaces.

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²⁾ Hodge attributes this imbedding to G. Mannoury, Nieuw Archief voor Wiskunde 4(1898), 112.