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IMMERSED SPHERES IN SYMPLECTIC 4-MANIFOLDS

by Dusa MCDUFF(*)

1. Introduction.

In this paper we discuss conditions under which a symplectic 4-manifold has a compatible Kähler structure. We will say that a symplectic manifold (V,ω) is **Kählerian** if V admits an integrable almost complex structure J such that ω is the Kähler form associated to a Kähler metric on (V,J). It is well known that every symplectic manifold may be given the structure of an almost Kähler manifold, that is, that V may be given an almost complex structure J which is compatible with ω in the sense that

$$\omega(x,Jx)>0, \quad \text{and} \quad \omega(Jx,Jy)=\omega(x,y)$$

for all tangent vectors x,y. (In fact, the set of such J is contractible.) The formula $g_J(x,y) = \omega(x,Jy)$ then defines a metric on V, and the triple g_J, J, ω satisfies exactly the same algebraic relations as in the integrable case. Thus the question boils down to understanding the geometric consequences of the non-integrability of J.

There certainly are non-Kählerian symplectic 4-manifolds. The basic example is due to Kodaira-Thurston, and is a T^2-bundle over T^2, which may be obtained from the space $T^2\times S^1\times[0,1]$ by identifying $(x_1,x_2,x_3,0)$ with

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$(x_1+x_2, x_2, x_3, 1)$. See [TH], [FGG]. On the other hand, Gromov observed in [GR] that many features of the behaviour of the J-holomorphic curves in V do not depend on the integrability (or lack of integrability) of J. (Recall that a map f from a Riemann surface (Σ, j) to the almost complex manifold (V, J) is said to be J-holomorphic if $df \circ j = J \circ df$.) Further, Kähler surfaces which contain holomorphic curves with sufficiently positive normal bundles have very simple structure : they have Kodaira dimension equal to $-\infty$ and so are blow ups of rational or ruled surfaces. Our aim is to see how much of this theory remains true in the symplectic case.

A beginning on this project was made in [RR]. We will summarize the results of this paper here : a further discussion may be found in the survey article [KY]. All manifolds considered here are smooth, compact and without boundary.

The first set of results concern **exceptional spheres** i.e. symplectically embedded 2-spheres of self-intersection -1. (Note that the first Chern class c of (V, J) takes the value 1 on these spheres.) Recall from [BL] that one can "blow down" an exceptional sphere (replacing it not by a point but rather by a symplectic ball) to obtain a well-defined symplectic manifold. Let us say that (V, ω) is **minimal** if it contains no exceptional spheres. Then we showed in [RR] that any symplectic 4-manifold (V, ω) has a minimal reduction $(\overline{V}, \overline{\omega})$ which may be obtained by blowing down a disjoint collection $\Sigma_1, \dots, \Sigma_k$ of exceptional spheres. Further, this reduction $(\overline{V}, \overline{\omega})$ is determined up to symplectomorphism by the set of cohomology classes of the Σ_i. There is still an open question concerning the uniqueness of symplectic blowing up. In particular, it is not clear whether the blow up of a Kählerian manifold is always Kählerian. However, every such blow-up is **pseudo-isotopic** to a Kählerian blow-up, that is, it may be joined to a Kählerian form by a family of non-cohomologous symplectic forms : [BL], [UB].

The second set of results concerns the structure of symplectic manifolds which contain a symplectically embedded 2-sphere S of non-negative self-intersection. (Equivalently, one requires $c(S)$ to be ≥ 2.) We showed that under these conditions the minimal reduction $\overline{V}$ of V is symplectomorphic either to $\mathbf{C}P^2$ with its standard Kähler form (the rational case) or to the total space of a symplectic S^2-bundle over a Riemann surface (the ruled case). In the latter case, one can prove that, for some of the possible cohomology classes, all symplectic forms representing this class are symplectomorphic and Kählerian. In general, all one knows is that the form is pseudo-isotopic to a Kählerian form[(1)].

The above results concern the symplectic analogue of embedded rational curves. In the complex case, there are other classical theorems about holomorphic curves, which may be found in [BPV] III.2.3, III.4.6, VI.6, for example.

THEOREM 1.1. — *If a minimal complex surface Y contains a curve C such that $c(C) = -K_Y \cdot C > 0$ then Y is rational or ruled. (Here K_Y is the canonical divisor.)*

THEOREM 1.2. — *If a complex surface Y has two different minimal reductions then Y is a blow-up of a rational or ruled surface.*

Here one is dealing with possibly singular curves of arbitrary genus. (As we shall see, even though the second theorem looks as though it concerns only embedded 2-spheres, one is quickly led to consider immersed 2-spheres.) The classical proofs of these theorems rely heavily on the Riemann-Roch theorem, which gives numerical criteria for the existence of holomorphic curves. No analogous result is known in the symplectic context. However, Mori gave a more geometric proof in [U] which uses properties of the space of deformations of the curve C. Some but not all of his arguments transfer to the symplectic case. One of the main stumbling blocks is that in the integrable case one uses the fact that the moduli space of curves has a complex structure and that the evaluation map is holomorphic. This is no longer true in the almost complex case : *cf.* the discussion of the tangent space to the moduli space given in [EX], Proposition 4.3. However, when one is dealing with spheres, one can get around this and show that under appropriate circumstances the evaluation

(1) Unfortunately, Theorem 1.3 in [RR] about the structure of symplectic S^2-bundles needs an extra hypothesis. Francois Lalonde pointed out that the argument which proves uniqueness works only for a restricted range of cohomology classes. Further, the condition $a^2(V) > (a(F))^2$ need not hold when M has genus > 0. The trouble is that in [RR] Lemma 4.15, the integral of the form ρ over the section Γ is not zero in general, but depends on the homology class of Γ. For example, in the case of the trivial bundle, one can assume that Γ has class $[M] + k[F]$, so that $\int_M \rho = k \int_F \rho$. The rescaling in Lemma 4.15 then works provided that $\int_M \omega > k \int_F \omega$. In fact, by slightly refining the argument, one can show that all fibered symplectic forms on the product $T^2 \times S^2$ are symplectomorphic to a product form, but in all the other cases of bundles over a Riemann surface of genus > 0 there are some cohomology classes which might perhaps support several different fibered symplectic forms. Note also that the proof of [RR] Lemma 4.16, is not quite right because the complex structure J_1' has too many cusp-curves. However, this is easy to rectify : one just has to replace J_1' by a generic integrable J. All this is discussed further in [RU].

map has positive degree. Again using the fact that one is dealing with spheres, one can get enough information from this evaluation map to prove an analogue of Theorem 1.1 for symplectic spheres and hence establish a symplectic version of Theorem 1.2.

In order to state our results precisely we need the following definition.

DEFINITION 1.3. — *We say that a closed* 2*-manifold* S *is* **positively symplectically immersed** *in* (V, ω) *iff it is symplectically immersed (i.e. the restriction of* ω *to* S *does not vanish) and its only singularities are transverse double points of positive orientation.*

We will see below that these submanifolds S may be parametrized to be J-holomorphic for some ω-tame J. (Recall that J is ω-tame if $\omega(x, Jx) > 0$ for all non-zero x.) Of course, the singularities of a J-holomorphic curve need not all be transverse double points. However, it is proved in [LB] that any J-holomorphic curve can be perturbed so that it is positively symplectically immersed in the above sense. Hence this is a good definition to work with.

THEOREM 1.4. — (i) *If a compact symplectic* 4*-manifold* (V, ω) *contains a positively symplectically immersed* 2*-sphere* S *with* $c(S) \geq 2$*, then* (V, ω) *is the blow up of a rational or ruled manifold.*

(ii) *If* S *is not embedded, then* V *is rational* (2).

As an almost immediate corollary we obtain :

THEOREM 1.5. — *If a compact symplectic* 4*-manifold* (V, ω) *has two non-symplectomorphic minimal reductions then* (V, ω) *is the blow-up of a rational or ruled symplectic* 4*-manifold.*

Note 1.6. — Just as in the case of embedded spheres, the class of symplectic 4-manifolds considered in the above theorem is closed under blowing up and down and under deformations of ω through non-cohomologous forms(3). Further, under blowing down, the Chern class and

(2) To be consistent with the integrable case, we will call a symplectic manifold rational if it may be obtained from $\mathbb{C}P^2$ by blowing up and down.

(3) Lalonde pointed out that the proof I gave of this in [RR] (5.4), is inadequate because when I was considering blowing down I did not allow for the possibility that the intersection number $C \cdot \Sigma$ might be ≥ 2, so that the blow down of C is no longer embedded. Of course, this case can be dealt with by the methods of the present paper. However, it can also be treated in the framework of [RR], since all we have to do is to reduce it to the corresponding result in the integrable case. The first step is to

number of double points of an immersed sphere do not decrease. Therefore, if one starts off with a sphere whose Chern class is too small, one can try to increase it by blowing down.

Note 1.7. — The hypotheses of Theorem 1.4 exclude the case when $c(S) = 1$. These spheres are rigid, i.e. they have no J-holomorphic deformations, and so our methods give no information. Mori avoids them by considering only "extremal curves", i.e. curves whose homology class lies on the edge of the convex cone generated by the classes of the J-holomorphic curves. An argument due to Grothendieck implies that rigid immersed (but non-embedded) spheres represent classes in the interior of this cone. However, Grothendieck's argument is based on the Riemann-Roch theorem and so is unavailable to us.

We sketch the proofs of Theorems 1.4 and 1.5 in §2. Technical results showing that the evaluation map has positive degree are proved in §3, and results on the structure of the compactified moduli space are given in §4. I wish to thank Simon Donaldson for prompting me to think about these questions, and Claude LeBrun and Gang Tian for discussing Mori's arguments with me. Special thanks are due to Francois Lalonde for reading my paper [RR] so carefully and pointing out the various mistakes and inaccuracies mentioned here. He also made useful comments on a first version of this paper. Finally I wish to thank Michèle Audin and I.R.M.A. for their hospitality and support during part of the work on this paper.

2. Outline of Proofs.

Proof of Theorem 1.5. — Theorem 1.5 is proved by reducing it to Theorem 1.4 as follows. As mentioned above, one may form a minimal reduction $(\overline{V}, \overline{\omega})$ of (V, ω) by blowing down a maximal collection $\{\Sigma_1, \ldots, \Sigma_k\}$ of disjoint exceptional spheres. Now $(\overline{V}, \overline{\omega})$ is determined up to symplectomorphism by the homology classes $E_1, \ldots, E_k$ of these spheres. Therefore, if V has two minimal reductions, it has two different maximal sets of spheres corresponding to different sets of homology classes, say $E_1, \ldots, E_k$

deform ω until it is Kählerian. Then it suffices to show that the homology class of Σ may be represented by a J-holomorphic curve (rather than cusp-curve) for some integrable J. For then, if we blow down this curve, we obtain a Kählerian manifold with $q(X) = P_2(X) = 0$, and this contains a suitable embedded sphere by [BPV] (2.3). But V must be rational (since ruled surfaces do not contain non-embedded holomorphic spheres), and it is easy to check that any integrable J which is good and generic in the sense of [FM] Chap. III will do.

and $E_1', \dots, E_m'$. Recall from [RR] that we associate to a symplectic manifold (V, ω) the contractible family $\mathcal{J}$ of all ω-tame almost complex structures, where $\mathcal{J} = \{J : \omega(x, Jx) > 0$ for all non-zero $x \in TV\}$. This set $\mathcal{J}$ is open in the space of all almost complex structures, and we will assume, often without explicit mention, that our almost complex structures J are in $\mathcal{J}$. Every exceptional sphere may be parametrized so that it is J-holomorphic, for suitable $J \in \mathcal{J}$. Indeed, it is proved in [RR] Lemma 3.1, that there is a dense open subset $\mathcal{U} \subset \mathcal{J}$ such that any homology class E which may be represented by an exceptional sphere has a unique embedded J-holomorphic representative for each $J \in \mathcal{U}$. It follows that we may assume that all the classes $E_1, \dots, E_k$ and $E_1', \dots, E_m'$ mentioned above have embedded J-holomorphic representatives $\{\Sigma_1, \dots, \Sigma_k\}$ and $\{\Sigma_1', \dots, \Sigma_m'\}$. Recall further from [RR] (2.5), that every intersection point of two J-holomorphic curves C and C' always contributes positively to the algebraic intersection number $C \cdot C'$. Thus $p_{ij} = \Sigma_i' \cdot \Sigma_j \geq 0$ for all i, j and $= 0$ iff the spheres are disjoint. Therefore, by maximality, $p_{1j} > 0$ for some j, and, by renumbering, we may suppose that $p = p_{11} > 0$.

If $p = 1$ the argument is quickly finished. By [RR] (2.5), the spheres Σ_1' and Σ_1 intersect exactly once transversally. Therefore, when we blow down Σ_1, the sphere Σ_1' descends to a symplectically embedded sphere with zero self-intersection. But the only symplectic 4-manifolds which contain such a sphere are blow-ups of rational and ruled Kähler manifolds. Since the class of these manifolds is closed under blowings-up and down by [RR] Theorem 1.2, we are done.

If $p > 1$, matters are not quite so easy. We show in Lemma 3.2 below that may assume that the exceptional spheres Σ_1' and Σ_1 are in general position and J-holomorphic for some J which is integrable near Σ_1. Then we may blow down Σ_1 by first constructing the almost complex manifold (V', J') which is the obvious holomorphic blow-down of $\Sigma_1 \subset (V, J)$ (obtained by collapsing Σ_1 to a single point), and then putting a suitable symplectic form on V' : see [RR] Lemma 3.2[(4)]. Since the blowing down map is holomorphic, Σ_1' descends to a J'-holomorphic sphere C which is immersed with one p-fold multiple point x_0. Finally, perturb C to a positively symplectically immersed sphere S in V'. Since $c(S) = c(C) = p + c(\Sigma_1') = p + 1 \geq 3$, the result will now follow from Theorem 1.4, as claimed. Note that if this case actually occurs, V is rational by Theorem 1.4 (ii).

(4) Note that the symplectic form described there needs to be smoothed in the radial directions.

Proof of Theorem 1.4 (ii). — Suppose that the sphere S is not embedded. By Lemma 3.3 we may suppose that S has a J-holomorphic parametrization for some regular J, and then, as in Lemma 3.2, put it in general position with respect to a maximal family of disjoint exceptional spheres. It follows as above that the image $\overline{S}$ of S in the minimal reduction $\overline{V}$ of V may be perturbed to be positively symplectically immersed, but not embedded. If $\overline{V}$ is not rational, we saw in §4 of [RR] that every compatible J on $\overline{V}$ defines a J-holomorphic map $\overline{V} \to B$, where B is a Riemann surface of genus > 0. Thus $\overline{S}$ projects to a point in B, and so must be one of the fiber spheres. But this is impossible since these are embedded.

Proof of Theorem 1.4 (i) when $c(A) \geq 3$. — When proving Theorem 1.4 it turns out to be easiest to deal with families of spheres in a homology class A with $c(A) = 3$. If we start with a sphere with $c(S) > 3$, we can always reduce $c(S)$ by blowing up points on S. The case $c(S) = 2$ will be dealt with later. We will assume that a maximal family of exceptional spheres in V which are disjoint from S have been blown down, and that V contains no symplectically embedded spheres with Chern class equal to 2 or 3, and will then derive a contradiction.

By Lemma 3.3 any positively symplectically immersed 2-sphere with $c(S) > 0$ in (V, ω) has a J-holomorphic parametrization for some generic ω-tame J. Thus we will suppose that our sphere S in class A is J-holomorphic for some generic J. Since J is generic, the set $M(J, A)$ of all parametrized J-holomorphic spheres in class A is a manifold of dimension $2(c(A)+2) = 10$. Fix $z_0 \in S^2$ and let $G_0 = \{\gamma \in G : \gamma(z_0) = z_0\}$, where G is the Möbius group. Consider the evaluation map

$$e_0 : M(J, A) \times_{G_0} S^2 \to V : \quad (f, z) \mapsto f(z_0).$$

By [RR] Lemma 5.2, the inverse image $W^\circ = e_0^{-1}(\{x_0\})$ is a manifold for generic choice of the point $x_0 \in V$. Notice that W° is an S^2-bundle over a 2-manifold $B^\circ \subset M(J, A)/G_0$, and that there is an evaluation map $e : W^\circ \to V$ given by $e(f, z) = f(z)$. Also, W° has a section $\Sigma^\circ = \{(f, z) \in W^\circ : z = z_0\}$ which is mapped by e to the point x_0.

By the Compactness Theorem, each end of B° corresponds to a degeneration of the A-sphere into a cusp-curve, which is a connected union of at least two J-holomorphic spheres. (More details are given in §4.) Because the Fredholm index of (unparametized) spheres with $c < 1$ is negative, such spheres are not present for generic J. Therefore, an A-sphere with $c(A) = 3$ can decompose either into two components $S_1 \cup S_2$ in classes A_i where $c(A_i) = i$ or into three components each with $c = 1$. As already mentioned, J-holomorphic spheres with $c = 1$ are rigid, i.e. for generic J, they are isolated and there are only finitely many in each homology class. Therefore, if we choose x_0 so that it does not lie on any such sphere, the second type of degeneration will not occur. For the same reason, even if A is a multiple class, i.e. $A = 3A_1$ for some integral class A_1, the A-spheres through x_0 will not degenerate to the union of a A_1-sphere together with a double cover of the same (or a different) sphere. Nor will there be a triply covered sphere in W°. (This is important, because Fredholm theory does not work at multiply-covered points : see [EX] §4.) Thus the only possible cusp-curves have the form $S_1 \cup S_2$, where $c(S_1) = 1, c(S_2) = 2$ and $x_0 \in S_2$. Moreover, S_2 is not a double cover of S_1 and so meets S_1 in a finite number of points. By Lemma 4.1, we may suppose that S_1 and S_2 are in general position. Then, it is possible to understand the degeneration at $S_1 \cup S_2$ by considering it to be the image of a corresponding degeneration of embedded curves. Thus we find :

PROPOSITION 2.1. — (i) *The bundle $\rho : W^\circ \to B^\circ$ is trivial near the ends of B° and so may be completed to a compact S^2-bundle $\rho : W \to B$ by adding one copy of S^2 over each end of B°.*

(ii) *The evaluation map e extends continuously over the manifold $W^\#$ obtained by blowing up one point in each of the added fibers. Thus, if B° has k ends, the extended evaluation map takes the form*

$$e : W^\# = W\#k(\overline{\mathbb{C}P}^2) \to V,$$

where e takes the blown-up points in $W^\#$ to the A_1- components of the cusp-curves in V. Moreover, because none of the blown up points lie on the section Σ°, we may extend Σ° to a section Σ of $W^\#$ which is mapped by e to a single point.

(iii) *The map $e : W^\# \to V$ has positive degree N.*

COROLLARY 2.2 [5]. — *The first Betti number $\beta_1 = rk\ H_1(V, \mathbf{Z})$ of V is zero.*

Proof. — The inclusion $\Sigma \to W^\#$ clearly induces an isomorphism on π_1. Hence, by (ii), e induces the zero map on H_1. But, by Poincaré duality, e, being a map of positive degree, induces an injection on $H^*(V; \mathbf{Q})$. Hence $H^1(V; \mathbf{Q}) = 0$ as claimed.

[5] This was suggested by Lalonde.

The rest of the proof of Theorem 1.4 (i) is a cohomological argument, which shows that the map e cannot exist unless V contains a symplectically embedded sphere with $c = 2$. But then V has the required form by [RR].

Cohomological argument.

Suppose that S has P double points. Then $A \cdot A = 2P + 1$, and, if S_1 has p double points and S_2 has q double points, we have :

$$S_1 \cdot S_1 = 2p - 1, \quad S_2 \cdot S_2 = 2q, \quad S_1 \cdot S_2 = \gamma$$

where

$$P = p + q + \gamma - 1.$$

(There is one extra point of intersection of S_1 with S_2 coming from the degeneration.) Note that $\gamma \geq 1$ because the cusp-curve is connected. We may suppose that P and q are strictly positive, since otherwise there is nothing to prove, but p could in principle be 0.

Case (i) There are degenerations, which all correspond to the same cohomological decomposition $A = A_1 + A_2$ where $A_2 \neq 2A_1$.

Let a, a_1 and a_2 be the Poincaré duals of the homology classes A, A_1 and A_2, and let c be the first Chern class of V. We write $\hat{a}$ etc for the pull-backs of these classes by e. Because e has degree $N > 0$ the following identities hold :

$$\hat{a}_i \cup \hat{c}(W^{\#}) = iN, \qquad \hat{a}_1 \cup \hat{a}_1(W^{\#}) = (2p-1)N,$$
$$\hat{a}_2 \cup \hat{a}_2(W^{\#}) = 2qN, \qquad \hat{a}_1 \cup \hat{a}_2(W^{\#}) = \gamma N.$$

Now choose a basis f, s, e_i for $H^2(W^{\#}, \mathbf{Z})$ which is dual to the basis F, Σ, E_i, where F is the fiber, and the E_i are the blown-up points. Put $\lambda = f^2(W^{\#})$. (Note that the value of λ depends on the class of the section Σ.) Then we have

$$e_*(F) = A, \quad e_*(\Sigma) = 0, \quad e_*(E_i) = A_1, \text{ for all } i.$$

Hence

$$\hat{a}_1 = (2p - 1 + \gamma)f + \sum_i (2p-1)e_i,$$
$$\hat{a}_2 = (2q + \gamma)f + \sum_i \gamma e_i,$$
$$\hat{c} = 3f + \sum_i e_i.$$

Thus we have :

$$\begin{array}{rclcl} (\hat{a}_1)^2(W^{\#}) & = & (2p-1)N & = & (2p-1+\gamma)^2\lambda - k(2p-1)^2, \\ \hat{a}_1 \cup \hat{c}(W^{\#}) & = & N & = & 3(2p-1+\gamma)\lambda - k(2p-1), \quad \text{and} \\ (\hat{a}_2)^2(W^{\#}) & = & 2qN & = & (2q+\gamma)^2\lambda - k\gamma^2. \end{array}$$

Since $q > 0$ the last equation implies that $\lambda > 0$. Note also that $2p-1+\gamma > 0$. (Otherwise, $p = 0$ and $\gamma = 1$, so that $A \cdot A_1 = 0$. This implies that the A_1-sphere is an embedded exceptional sphere which does not meet A. But all such spheres were blown down.) Then, combining the first two equations, we find that

$$(2p-1)N = (2p-1+\gamma)^2\lambda - k(2p-1)^2 = (2p-1)(3(2p-1+\gamma)\lambda - k(2p-1)),$$

which gives

$$2p-1+\gamma = 3(2p-1).$$

Thus $\gamma = 2(2p-1) = 2x$, say. Also,

$$N = 9x\lambda - kx.$$

Using the identity,

$$2N = \hat{a}_2 \cup \hat{c}(W^{\#}) = 3(2q+\gamma)\lambda - k\gamma = 9\gamma\lambda - k\gamma,$$

we find $q = 2x = \gamma$. In particular, this means that the non-zero class $a_2 - 2a_1$ vanishes on A and on A_1. Thus $\hat{a}_2 - 2\hat{a}_1 = e^*(a_2 - 2a_1) = 0$ which contradicts the injectivity of e^*.

Case (ii) The only J-holomorphic decomposition of A is $A = A_1 + 2A_1$.

We first claim that the second Betti number β_2 of V is 1. For, if not, the intersection form is indefinite (by the injectivity of e^*), and so there is a non-zero element $B \in H_2(V; \mathbb{Q})$ such that $B \cdot A_1 = 0$. But then, if b is its Poincaré dual, $e^*(b) = 0$ which contradicts the injectivity of e^*. Since $c(A_1) = 1$, A_1 must generate the free group $H_2(V; \mathbb{Z})$ and c must generate the dual group $H^2(V; \mathbb{Z})$. Thus $c^2(V) = 1$.

Now, recall that on any almost complex manifold, the class c is related to the Euler characteristic χ and the signature σ by the identity $c^2(V) = 2\chi + 3\sigma$. But this is impossible, because $\chi = 3$ by Corollary 2.2 and $\sigma = 1$.

Case (iii) A has no degenerations.

In this case, $k = 0$ and $\beta_2 = 1$. If we assume that the original sphere S is not embedded, so that $P > 0$ and $A \cdot A = 2P + 1 > 1$, A cannot generate $H_2(V;\mathbb{Z})$. It follows as in case (ii) that c generates $H^2(V;\mathbb{Z})$, so that $c^2(V) = 1$. The argument may now be completed as before.

Case (iv), A has degenerations of different cohomological types.

We will say that a class A_i is J-representable if it can be represented by a J-holomorphic sphere. Assume that J is generic. It follows easily from the Compactness Theorem (see §4) that every J has a neighbourhood $N(J)$ such that the set R of classes B, which have $\omega(B) \leq K = \omega(A)$ and are J'-representable for some $J' \in N(J)$, is finite. Therefore, we may assume that J is regular for all these classes. Also, we will denote a class with $c = i$, for $i = 1, 2$, by A_i or B_i. Pick a class $A_2 \in R$ so that $2q = A_2 \cdot A_2 \leq B_2 \cdot B_2$, where B_2 ranges over all classes in R which have non-zero intersection with some $B_1 \in R$. Then pick A_1 so that $A_2 \cdot A_1 \leq A_2 \cdot B_1$ where B_1 ranges over all classes in R which have non-zero intersection with A_2. By hypothesis, there is some pair A_2, A_1 satisfying these conditions. We may assume that $q > 0$, since otherwise the A_2-sphere is embedded. The following lemma is proved in §4.

LEMMA 2.3. — *If classes A_1 and A_2 with $c(A_1 + A_2) = 3$ are both realised by J-holomorphic spheres, and if $A_1 \cdot A_2 > 0$ there is a J-holomorphic sphere in class $A_1 + A_2$.*

It follows that there is a J-holomorphic sphere in class $B = A_1 + A_2$. Since $c(B) = 3$ there is an evaluation map

$$e : W^{\#} = W\#k(\overline{\mathbb{C}P}^2) \to V$$

which, as before, has positive degree. Because e^* is injective, the intersection form is either indefinite or negative definite on any rank 2 subgroup of $H_2(V)$. If the degeneration $B = A_1 + A_2$ is the only one, we are in case (i) or (ii) above. So suppose that B also decomposes as $B_1 + B_2$. By construction, $B_2 \cdot B_2 = 2q' \geq 2q$ and $\gamma = B_2 \cdot A_2 = (A_2 + A_1 - B_1) \cdot A_2 \leq A_2 \cdot A_2 \leq 2q$. Then the intersection matrix of A_2 and B_2 has the form :

$$\begin{pmatrix} 2q & \gamma \\ \gamma & 2q' \end{pmatrix}$$

which is positive semi-definite. Since A_2 and B_2 span a rank 2 subgroup by construction, this is impossible.

Proof of Theorem 1.4 (i) when $c(A) = 2$.

Case (i) A is a simple class, i.e. it has no degenerations.

In this case the moduli space $M(J,A)/G$ is compact, and the evaluation map

$$e : W = M(J,A) \times_G S^2 \to V$$

has positive degree. (To see this, one argues as in Proposition 3.4, using $S^2 \times S^2$ as the model instead of $\mathbf{C}P^2$.) Hence e^* is injective, and $\beta_2 = 1$ or 2. As before, we suppose that our initial sphere S is not embedded, and so has P double points where $P > 0$. We then derive a contradiction by a homological argument.

Let a be the Poincaré dual of A as before. Then, because $\hat{a}(F) - P\hat{c}(F) = 0$, the class $(\hat{a} - P\hat{c})^2 = 0$, and so $(a - Pc)^2 = 0$ also. It follows that $Pc^2(V) = 2$. Hence $c^2(V) = 1$ or 2.

If $\beta_2 = 2$, we must have $\sigma = 0$ and then the identity $c^2(V) = 2\chi + 3\sigma$ implies that $\chi = 1$. But this contradicts the fact that $\chi = 4 - 2\beta_1$.

So suppose that $\beta_2 = 1$. In this case, $\sigma = 1$ and $\chi = 3 - 2\beta_1$, and it is easy to check that $\beta_1 = 2$. Because $\hat{c}$ is an element of $H^2(W)$ with positive square, cupping with $\hat{c}$ induces an isomorphism $H^1(W, \mathbf{Q}) \to H^3(W, \mathbf{Q})$. Hence the same holds on V, i.e. if u and v generate $H^1(V, \mathbf{Q})$, cu and cv generate $H^3(V, \mathbf{Q})$. Poincaré duality then implies that $uv \neq 0$. But then uv is a non-zero multiple of c, since c generates $H^2(V, \mathbf{Q})$. Thus $(uv)^2 \neq 0$, which is absurd.

Case (ii) There are degenerations.

As above, the only possible degenerations for generic J have the form $S_1 \cup S_2$ where, this time, $c(S_i) = 1$ for both i. Let $A_i = [S_i]$ for $i = 1, 2$. Then, because $A \cdot A = 2P > 0$, and because the classes A, A_i are simultaneously realised by J-holomorphic curves, $A_i \cdot A > 0$ for some i. Hence, by Lemma 2.3 there is a J-holomorphic curve in class $A + A_i$, and we are reduced to the previously considered case when $c(A) = 3$.

3. Immersed spheres.

In this section we prove the technical results about immersed J-holomorphic spheres which were used earlier. We begin with some simple linear algebra.

LEMMA 3.1. — *Let π_1 and π_2 be two transverse planes through $\{0\}$ in $\mathbf{R}^4$ which intersect with positive orientation and are symplectic with respect to the standard linear symplectic form ω_0. Then there is a linear ω_0-tame J which preserves these planes.*

Proof. — Given a subspace π of $\mathbf{R}^4$, let $\pi^\perp$ be its symplectic orthogonal, i.e.

$$\pi^\perp = \{v : \omega(v,w) = 0 \text{ for all } w \in \pi\}.$$

If $\pi_1^\perp = \pi_2$ the result is obvious. So suppose not, and choose a basis e_1, e_2 for π_1 such that $\omega(e_1, e_2) = 1$ and so that the subspaces $\pi_2 \cap \{e_1\}^\perp$ and $\pi_2 \cap \{e_2\}^\perp$ are 1-dimensional. Choose $v_3 \in \pi_2 \cap \{e_1\}^\perp$ and $v_4 \in \pi_2 \cap \{e_2\}^\perp$, normalised so that

$$\omega(v_3, v_4) > 0, \qquad |\,\omega(e_2, v_3)\,| = |\,\omega(e_1, v_4)\,| = a.$$

Then, with appropriate choice of signs,

$$e_3 = v_3 \pm ae_1, \quad e_4 = v_4 \pm ae_2$$

span $\pi_1^\perp$, and, because the intersection of π_1 with π_2 is positive, we may normalise further so that $\omega(e_3, e_4) = 1$. Thus $e_1, \ldots, e_4$ is a standard symplectic basis and, for $b = \pm a$,

$$e_3 = v_3 + be_1, \quad e_4 = v_4 \pm be_2.$$

Now choose J so that $Je_1 = e_2$ and $Jv_3 = v_4$. If $e_4 = v_4 + be_2$, $Je_3 = e_4$ and so J is obviously ω_0-tame. An easy calculation shows that J is ω_0-tame in the other case too. □

Observe that we cannot deal with more than two planes here : this is why a positively symplectically immersed sphere is only allowed to have double points. Also, we cannot necessarily find a ω-compatible J.

Next, we prove the general position result needed for Theorem 1.5. We will say that two J-holomorphic spheres are in **general position** if they are immersed and if their only intersection points are transverse double points. Similarly, two symplectically immersed spheres are in general position if all intersection points are positively oriented transverse double points. In particular, each sphere is positively symplectically immersed. (Note that these definitions are compatible.)

LEMMA 3.2. — *Given two exceptional spheres Σ and Σ' in V, we may assume that they are in general position and are J-holomorphic for some ω-tame J which is integrable near Σ.*

Proof. — By definition, exceptional spheres are embedded. If they are in general position, it is easy to find an ω-tame J such that they are both J-holomorphic : the only possible difficulty occurs at points where two branches cross, and this is dealt with by Lemma 3.1. Then, this J is integrable on the spheres, and it is not hard to see that one can change it off the spheres to make it integrable near Σ.

In order to put the spheres in general position, we first make them J-holomorphic so that they intersect positively. By choosing J in the dense open subset $\mathcal{U}$ of $\mathcal{J}$ which was mentioned in §2, we may also suppose that they are embedded. Therefore, we only have to worry about points x_i where the spheres touch. At these points, one should first use the results of [LB] to perturb small pieces of Σ' near the x_i so that they remain J-holomorphic but intersect Σ transversally, and then patch these J-holomorphic pieces to the rest of Σ' to get a symplectically embedded 2-sphere which intersects Σ positively and in transverse double points, as required. Note that the resulting sphere is J'-holomorphic for some J' arbitrarily close to J. Further, this technique works more generally, because the methods of [LB] allow one to replace singular pieces of a J-holomorphic curve by immersed J-holomorphic pieces : see Lemma 3.5 below. □

We now discuss the relevant Fredholm theory, using the notations of [EL,RR]. Recall that, if F and $\mathcal{J}'$ are suitable spaces of maps and ω-tame almost complex structures, respectively, one can form a Hilbert manifold $\mathcal{M}_A$ consisting of all pairs $(f, J) \in F \times \mathcal{J}'$ such that $f : S^2 \to V$is J-holomorphic and represents A. (Here, as in [EL], in order to remain within the space of C^∞ almost complex structures, we take $\mathcal{J}'$ to be some subset of the space $\mathcal{J}$ of C^∞ ω-tame structures which is closed under an appropriate Hilbert norm.) Further, the projection map

$$P_A : \mathcal{M}_A \to \mathcal{J}'$$

is Fredholm with index $2c(A) + 4$. Thus the inverse image $M(J, A) = P_A^{-1}(J)$ is a manifold for generic $J \in \mathcal{J}'$ and hence for a dense set of $J \in \mathcal{J}$. The words "curve" and "J-holomorphic sphere" will denote either a parametrized curve f or its unparametrized image Imf. A curve is said to be regular iff the corresponding point $(f, J) \in M(J, A)$ is a regular point of P_A. Similarly, we say that an ω-tame J is regular if it is a regular value of P_A for some choice of space $\mathcal{J}'$ and for all the classes A under consideration. (The compactness theorem implies that only finitely many classes are relevant in any situation : see Lemma 4.1 below.)

Since the Mobius group $G = PSL(2, \mathbb{C})$ of reparametrizations is 6-dimensional, an A-sphere can be regular only if $c(A) \geq 1$. It is easy to see that, when J is integrable, this condition is also sufficient: see Lemma 2.8 of [RR](6). One of the main differences between spheres and curves of higher

(6) The statement of this Lemma is not quite right. The necessary and sufficient condition

genus is that in the case of spheres this result remains essentially true even when J is not integrable.

LEMMA 3.3. — *Any positively symplectically immersed 2-sphere with $c(S) > 0$ has a J-holomorphic parametrization for some regular ω-tame J.*

Proof. — By assumption, there is an immersion $\iota_0 : S^2 \to S$ whose only singularities are transverse double points. Write $c(S) = 2 + k$, and let $E \to S^2$ be the complex line bundle over S^2 with Chern number k. Then, ι_0 extends to an immersion $\iota : N \to V$ of some neighbourhood N of the zero section S_0 in E. Because any almost complex structure on a 2-manifold is integrable, we may take N so small that there is an $\iota^*(\omega)$-tame integrable complex structure J_0 on N such that S_0 is J_0-holomorphic. Then, S_0 has a J_0-holomorphic parametrization f_0. By the above remarks, it is J_0-regular, because $c(S) > 0$.

As in the previous lemma, J_0 can be adjusted near the inverse images of the double points of ι_0 so that it is the pull-back of some integrable complex structure J defined near S. Extend J so that it is ω-tame on V. Because J has been constructed in a special way, it need not be regular. However, there are regular almost complex structures on V arbitrarily close by, and so, because S_0 is regular, we may choose one, J' say, whose pull-back J_0' to N has the property that S_0 is isotopic to a J_0'-holomorphic embedded sphere S_0'. We may suppose that S_0' is so C^1-close to S_0 that its image S' under ι is isotopic to S by a small isotopy g_t of V. (We do not assume that g_t preserves ω.) Then, $S = g_1 S'$ is $(g_1)_* J'$-holomorphic. Further, if g_t is sufficiently C^1-small, it is easy to see that $J_1 = (g_1)_* J'$ is ω-tame and so regular. (Note that if J is regular, so is $g_*(J)$, provided that it is tame.) □

Let us write $\mathcal{M}_A(x_0)$ for the space $\{(f, J) \in \mathcal{M}_A : f(z_0) = x_0\}$. The proof of Lemma 5.2 in [RR] shows that this is a Hilbert manifold. Therefore $M(J, A, x_0) = M(J, A) \cap \mathcal{M}_A(x_0)$ is also a manifold for generic J. As in §2, we define $W_J^\circ = M(J, A, x_0) \times_{G_0} S^2$, where G_0 is the subgroup of G formed by elements which fix z_0. Thus, when $c(A) = 3$ and J is generic, the spaces $M(J, A, x_0)$ and W_J° are smooth oriented manifolds of dimension 6 and 4 respectively.

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is that $H^1(\Sigma, \nu_\Sigma) = 0$. When $g > 1$ the condition $c_\nu > 2(g-1)$ is sufficient but not necessary : it is equivalent to the curve Imf being regular for all choices of integrable J. When $g - 1 \leq c_\nu \leq 2(g-1)$ it is regular for some integrable J, and not for others.

Recall from Proposition 4.3 of [EX], that every moduli space $M(J, A)$ of curves has a natural orientation, which derives from a stable almost complex structure which is defined over compact subsets of $M(J, A)$. We want to show :

PROPOSITION 3.4. — *The evaluation map $e_J : W_J^\circ \to V$ is orientation preserving at each of its regular points.*

This is easy to see in the integrable case, for then the moduli space W_J° has a complex structure which is compatible with its orientation, and e_J is holomorphic and hence orientation preserving. In the non-integrable case we exploit the fact that our curves are spheres. The first ingredient is the following lemma from [LB].

LEMMA 3.5. — *Pairs (f, J), where f is an immersion whose only singularities are transverse double points, are dense in $\mathcal{M}_A(x_0)$.*

Note. — The procedure given in [LB] for perturbing a singular curve to an immersed curve changes J, but only in small spherical shells around the critical points and not at these points themselves. Therefore, we may choose this perturbation to be so small that both J and its perturbation J' are C^1-close and lie in the same space $\mathcal{J}'$. The perturbed curve is also constructed to be C^1-close to the original one.

LEMMA 3.6. — *Let $c(A) = 3$ and consider a pair $(f, J) \in \mathcal{M}_A(x_0)$ such that f is an immersion whose only singularities are transverse double points which are disjoint from x_0.*

(i) *For any J, regular or not, (f, J) is a manifold point of $M(A, J, x_0)$, and so has a neighbourhood $U(f, J) \subset M(J, A, x_0)$ which is homeomorphic to an open subset of $\mathbf{R}^6$. If J is regular, this homeomorphism is compatible with the smooth structure on $U(J, A)$.*

(ii) *When J is regular, the evaluation map $e_J : W_J^\circ \to V$ is a local orientation-preserving homeomorphism near all points $(f, z), z \neq z_0$.*

Proof of Proposition 3.4. — Let J be a regular value for the projection operator $P_A : \mathcal{M}_A(x_0) \to \mathcal{J}'$ and let $(f, z) \in W_J^\circ$ be a regular point of e_J. Then, by Lemma 3.5, arbitrarily close to J and (f, z) we can find a regular value J' of P_A, and a point $(f', z') \in W_{J'}^\circ$, which is regular for $e_{J'}$ and such that f' satisfies the hypotheses of Lemma 3.6. Then e'_J preserves orientation on $U(f', J')$ by Lemma 3.6 (ii). Since the orientation on $M(J, A)$

is defined by means of the linearization of P_A, it depends continuously on the first derivatives of f and J. Thus, because we can take f' and J' to be arbitrarily close to f and J in the C^1-norm, e_J must also preserve orientation. □

Proof of Lemma 3.6.

(i) As in Lemma 3.3 above, let N be a neighbourhood of the zero-section S_0 of a complex line bundle with Chern number 1, and let $\iota : N \to V$ extend the immersion f. Because $c(A) = 3$ we may identify N with a neighbourhood of a complex projective line in $\mathbb{C}P^2$ in such a way that $\iota^*(\omega)$ extends to the standard Kähler form τ on $\mathbb{C}P^2$. Further, we may extend $(\iota^{-1})_*(J)$ to a τ-tame almost complex structure J_0 on $\mathbb{C}P^2$. Then every J-holomorphic A-curve in V which is sufficiently close to f may be identified with a J_0-holomorphic L-curve on $\mathbb{C}P^2$ where L denotes $[\mathbb{C}P^1]$, and so it suffices to consider the case when f is an embedding onto a complex line S_0 in $\mathbb{C}P^2$, and J is any almost complex structure on $\mathbb{C}P^2$ which makes f J-holomorphic.

Consider the evaluation map

$$ev_J : M(J,L) \times_G (S^2 \times S^2) \to \mathbb{C}P^2 \times \mathbb{C}P^2 : \quad (f, z_1, z_2) \mapsto (f(z_1), f(z_2)).$$

Because this has degree 1 when J is the standard integrable structure, and because L-curves have no degenerations, it has degree 1 for all regular J. Further, because $L \cdot L = 1$, Positivity of Intersections (see [RR] (2.5)) implies that two distinct J-holomorphic L-curves intersect transversally at a single point. It follows that, if x_0 is any point of $\mathbb{C}P^2$ and J is regular, ev_J induces an bijective smooth map :

$$e_J : \{(f,z) \in M(J,L,x_0) \times_{G_0} S^2 : z \neq z_0\} \to \mathbb{C}P^2 - \{x_0\}.$$

Even if J is not regular, this map is continuous and injective. To see that it is surjective, suppose given $x \in \mathbb{C}P^2$ and an arbitrary J. Choose a sequence J_j of regular almost complex structures converging to J, and choose $f_j \in M(J_j, A, x_0)$ so that $f_j(z_1) = x$ for some $z_1 \neq z_0$. By the Compactness Theorem (see next section), a subsequence of the f_j must converge, and the limit cannot be a cusp-curve since there are no L-cusp-curves in $\mathbb{C}P^2$. Therefore, the limit is an element of $M(J, L, x_0)$ which maps onto x. This proves (i).

Note. — In fact, by using the local blowing-up argument of Lemma 3.5 in [BL], one can show that e_J is a local diffeomorphism. For, if it were not, one could produce two distinct L-curves which are holomorphic for some $\overline{J}$ and which intersect at x_0 and at $f(z)$. It follows that we may give the neighbourhoods $U(f, J)$ a smooth structure for all J. But we shall not need that here.

(ii) We must show that, when J is regular, the evaluation map e_J : $U(f, J) \times_{G_0} S^2 \to V$ preserves orientation at all points $(f, z), z \neq z_0$. As in Lemma 3.3, we may assume that J is regular both for A-curves through x_0 on V and for L-curves near F and through x_0 on $\mathbb{C}P^2$. Further, because the natural orientation on the moduli space is induced by a stable almost complex structure which is defined intrinsically at each point by the Fredholm operator P_A, the identification of a neighbourhood of Imf in $M(J, A)$ with a space of curves on $\mathbb{C}P^2$ preserves this orientation. Therefore, it suffices to consider the case when f is an embedded curve in $\mathbb{C}P^2$. But then $U(f, J) \times_{G_0} S^2$ sits inside the compact manifold $M(J, L, x_0) \times_{G_0} S^2$ and e_J extends to a smooth map $\hat{e}_J : M(J, L, x_0) \times_{G_0} S^2 \to \mathbb{C}P^2$. Since degree is a cobordism invariant, the degree of $\hat{e}_J$ is independent of J, and so is 1. Because $\hat{e}_J$ is injective away from the points (f, z_0) it is orientation preserving at all points $\{(f, z) : z \neq z_0\}$, as required. □

4. Degenerations.

Suppose that the moduli space $M(J, A)/G$ of unparametrized J-holomorphic A-spheres is not compact. The **Compactness Theorem** (see [PW], [WO], [YE]) states that any sequence f_j of A-curves which goes out to infinity in the moduli space $M(J, A)$ has a subsequence which converges in the C^1-topology on the complement of a finite subset $Y \subset S^2$ to a J-holomorphic map $f_\infty : S^2 - Y \to V$. By removal of singularities, f_∞ extends smoothly over the whole of S^2 and so is a J-holomorphic B-sphere for some class B. Moreover, one or more "bubbles" form at the points of Y. These are J-holomorphic spheres obtained by rescaling f_j conformally. (The points of Y are precisely the points where the derivative df_j blows up.) In general, the collection of these spheres forms a "bubble-tree" (see [PW]), i.e. they can be joined together according to any tree, but in the case at hand the structure of the limiting cusp-curve is very simple.

LEMMA 4.1. — *When $c(A) = 3$ and we are considering the family of spheres through a fixed point x_0, then for regular J a cusp-curve has exactly two components S_1 and S_2 which we may suppose labelled so that $c(S_i) = i$. There are only a finite number of possible cusp-curves, and we may suppose that each is in general position, and that $x_0 \in S_2$.*

Proof. — The first statement was proved in §2. To prove the others, observe first that it is an easy consequence of the Compactness Theorem that, for any $K > 0$ and any $J \in \mathcal{J}$, there is a neighbourhood $N(J)$ of J such that only finitely many cohomology classes B with $\omega(B) \leq K$ may be represented by J-holomorphic spheres for some $J \in N(J)$. We may suppose that J is regular for each of these classes. Then, for each decomposition $A = A_1 + A_2$ where A_i has such a representation, there are only finitely many A_1-spheres. Let $Z \subset V$ be the set of points lying on one of these A_1-spheres. We claim that, if $y \in V - Z$ is generic, there are only finitely many A_2-spheres through y. For if not there is a sequence f_j say, of distinct A_2-curves through y. By the Compactness Theorem, this has a subsequence which converges either to a curve or to a cusp-curve. Since a cusp-curve in class A_2 must be the union of two rigid curves with $c = 1$, and since y by hypothesis does not lie on such a curve, the limit must be a curve. Thus the evaluation map

$$e_{A_2} : M(J, A_2) \times_G S^2 \to V$$

is proper over $V - Z$. In particular, if y is a regular value of e_{A_2}, the inverse image $e_{A_2}^{-1}(y)$ is compact and hence finite. (Note that dim $M(J, A_2) = 2c(A_2) + 4 = 8$.) Thus, since x_0 (or equivalently J) is generic, there are only finitely many A_2-spheres through x_0. Hence, there are only finitely many J-holomorphic A-cusp-curves through x_0. We may put each of them in general position using the methods of Lemma 3.2. The perturbed cusp-curves are J'-holomorphic for some J' near J in $\mathcal{J}'$. Moreover, because J is regular, we may choose J' so close to J that these cusp-curves are the only J'-holomorphic A-cusp-curves through x_0. □

We will suppose from now on that $c(A) = 3$, and will assume that all our spheres go through the point x_0. Let us first look more closely at the process of convergence. When a sequence f_j of A-spheres converges to an A-cusp-curve, the unparametrized curves $\mathrm{Im} f_j$ converge in the Hausdorff topology of sets to the set $S_1 \cup S_2$. Morever, the parametrized curves themselves also converge away from the bubble points. In the case under consideration, this can happen in one of three ways :

(i) f_∞ represents one of the spheres, and there is exactly one bubble, which represents the other;

(ii) f_∞ is a constant map and there are two bubble points, each representing one of the spheres;

(iii) f_∞ is a constant map and there is one bubble point, at which both spheres bubble off.

It is easy to check that, given a convergent sequence $\{f_j\}$ of any one of these types, one can choose elements $\gamma_j \in G$ such that the reparametrized sequence $f_j \circ \gamma_j$ has any other of these types. Thus, by reparametrizing we can assume that the convergence is of type (i) In fact, since all our spheres go through the point x_0, we can further fix the parametrization as follows. Choose a pair of antipodal points z_0, z_1 on S^2, choose a J-holomorphic isomorphism $\theta : T_{z_0}S^2 \to T_{x_0}V$, and choose a little 2-disc D which meets S_1 transversally at some point x_1 and is disjoint from S_2. Then each f_j has a unique reparametrization f'_j such that :

$$(1) \qquad f'_j(z_0) = x_0; \quad df'_j(z_0) = \theta; \quad f'_j(z_1) \in D .$$

With this parametrization, the bubble point must be z_1. (To see this, note that, if the bubble point is z, the f'_j converge on $S^2 - \{z\}$ to a map whose image is either S_1 or S_2. But the image has to be S_2. For, $z \neq z_0$ since $df(z_0)$ is bounded, and $df(z_0) = x_0 \in S^2$. Therefore, since $f'_j(z_1)$ is close to $x_1 \in S_1$, the bubble point is z_1, as claimed.)

Now consider the map f_∞ corresponding to this parametrization, and let $x = f_\infty(z_1)$. Then $x \in S_1 \cap S_2$. Further, f_∞ is uniquely determined by the choice of z_0, θ, and x, i.e. it does not depend on the choice of sequence $\{f_j\}$. On the other hand, the parametrization of the "bubble" S_1 cannot be chosen independently of the sequence $\{f_j\}$: if we reparametrize these to f'_j as above and then rescale to maps whose derivative at z_1 has norm 1, we get a sequence which converges to a parametrization h of S_2. Then because z_1 is antipodal to z_0, $h(z_1) = x_1$ and $h(z_0) = x$. However, there is no way to fix $dh(z_1)$: it has norm 1 but there is an unknown phase factor $\lambda \in S^1$ which depends on the initial sequence, i.e. on the way the cusp-curve is being approached in B°.

For short, we will denote the elements of B° by f, even though they are really equivalence classes of maps.

PROPOSITION 4.2. — *There is an injective correspondence between the ends of B° and pairs (C, x), where $C = S_1 \cup S_2$ is an A-cusp-curve as above, and $x \in S_1 \cap S_2$.*

Proof. — Let $B_{C,\epsilon}$ be the set of all elements f of B° whose image lies in an ϵ-neighbourhood of C. By the Compactness Theorem, the union of all such sets $B_{C,\epsilon}$ covers a neighbourhood of infinity in B°. Because there are only finitely many choices for C, one can choose ϵ so small that each end E of B° is covered by just one of the sets $B_{C,\epsilon}$. Further, it is clear that each $B_{C,\epsilon}$ breaks into a finite number of disjoint pieces $B_{C,\epsilon,x}$ each corresponding to one of the points $x \in S_1 \cap S_2$. Therefore, each end of B° corresponds to some pair (C, x), and it remains to see that there is only one end of B° which corresponds to a given pair (C, x). This will follow if we prove :

LEMMA 4.3. — *There is $\epsilon > 0$ and a neighbourhood $N(x)$ of x such that, for all $y \in V - N(x)$ which are sufficiently close to C there is a unique element $f_y \in B_{C,\epsilon,x}$ which goes through y.*

Note. — It will be clear from the proof of the lemma that this element f_y varies continuously with y, and hence they all belong to the same end of B°.

Proof of Lemma 4.3. — Suppose first that the spheres S_i are embedded and have one intersection point x. Then, as in the proof of Lemma 3.6, we may identify them with corresponding J_0-holomorphic spheres in the manifold $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ where J_0 is a complex structure obtained from the standard structure on $\mathbb{C}P^2$ by blowing up a point. Let τ be a Kähler form on X in the cohomology class corresponding to ω, and let E, L be the homology classes of the exceptional sphere and of $\mathbb{C}P^1$, respectively. Using the adjunction formula of [RR], one easily checks that the only homology classes which can be represented by J_0-holomorphic spheres are $L + k(L - E), k \in \mathbb{Z}$. Since E is J_0-holomorphic, we must have $k \geq -1$. Thus J_0 belongs to the open dense set $\mathcal{U} \subset \mathcal{J}$ consisting of all tame J which admit no J-holomorphic sphere with $c < 1$. We showed in [BL] that for each $J_0' \in \mathcal{U}$ there is a unique J_0'-holomorphic E-sphere Σ', and X is fibered by J_0'-holomorphic spheres. Further, there is a unique embedded J_0'-holomorphic L-sphere through every pair y_1, y_2 of points in $X - \Sigma'$ which do not lie on the same fiber of this fibration. (The arguments in [BL] can be simplified by using the adjunction formula of [RR].) Thus the family of spheres through x_0 has the required property.

If the spheres S_1, S_2 have double points or have several points of intersection, we embed a lifting N of a neighbourhood of C into X, as in Lemma 3.3, choosing $\iota : N \to nbhd(C)$ so that ι takes the intersection point of the cusp-curve $C_0 = \iota^{-1}C$ in N to x. The result then follows as before. □

Proof of Proposition 2.2.

(i) The fact that the conditions (1) fix a unique parametrization implies that the bundle $\rho : W^\circ \to B^\circ$ is trivial over each set $B_{C,\epsilon}$. But by the above a neighbourhood of each end of B° is contained in some $B_{C,\epsilon}$. Thus

(i) holds.

(ii) Consider the end E associated to the pair (C, x). Let D be a little 2-disc transverse to C as before, and for each $y \in D$ parametrize the unique curve f_y of Lemma 4.3 so that conditions (1) are satisfied. Then we may identify the evaluation map $e_E : \rho^{-1}(E) \to V$ with the map

$$\rho^{-1}E = (D - \{0\}) \times S^2 \to V : (y, z) \mapsto f_y(z).$$

This map clearly factors through the map $\iota : N - C_0 \to V - C$. Since ι extends over C_0 it suffices to prove the claim for curves in N rather than V. Thus, by embedding N in X, we are reduced to the model case when $A = L$ is the class of a complex line in X and we are looking at the behaviour of the family of J-holomorphic lines through the point x_0 as it approaches the cusp-curve C_0.

Observe that the moduli space W° is fibered by 2-spheres, while in $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ the corresponding spheres all go through the fixed point x_0. Therefore, to get a picture of the completion of W°, we blow up the point x_0 in X to obtain a family of disjoint spheres. It is not necessary to be very careful since we are only trying to show that the extension e is continuous.

Here are some more details. Even though J need not be integrable near x_0 one can use the complex structure in the tangent space $T_{x_0}X$ to define a blow up $\tilde{X}$ of X at x_0, in such a way that each embedded J-holomorphic curve through x_0 lifts to $\tilde{X}$ in the usual way. In particular, the cusp-curve C_0 lifts to a cusp-curve $\tilde{C}_0$ made from 2 exceptional curves in $\tilde{X}$. Further, because $A \cdot A = 1$, no two of our A-curves are tangent at x_0 (see [RR] (2.5)), and so they lift to a family $\mathcal{F}$ of disjoint curves in $\tilde{X}$. In fact, if X is $\mathbb{C}P^2$ blown up at y_0, it is useful to think of $\tilde{X}$ as being formed by first blowing up x_0 in $\mathbb{C}P^2$ to get a space fibered by the curves in $\mathcal{F}$, and then blowing up y_0 to get the cusp-curve $\tilde{C}_0$. With this picture the claim (ii) becomes obvious. We may identify the end $\rho^{-1}(E)$ of W° with a deleted neighbourhood of the cusp-curve $\tilde{C}_0$ and then think of the evaluation map $e_E : \rho^{-1}(E) \to X$ as the projection $\phi : \tilde{X} \to X$. Since ϕ extends over $\tilde{C}_0$, we are done. □

Proof of Lemma 2.3. — We must show that, if the classes A_1 and A_2 with $c(A_1 + A_2) > 0$ are both realised by J-holomorphic spheres and if $A_1 \cdot A_2 > 0$, there is a J-holomorphic sphere in class $A = A_1 + A_2$. The first step is to put the spheres S_i which realise the classes A_i into general position, which can be done using the techniques of Lemma 3.2. Note that this step changes J by an arbitrarily small amount to J'.

Next, transfer the problem to $X = \mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ by considering a neighbourhood N of the union of two embedded spheres which covers a neighbourhood of $S_1 \cup S_2$ in V, and then embedding N in X. (Here one has to choose one of the intersection points of S_1 with S_2 to play the role of the cuspidal point x.) The pull-back of J' to N extends to an almost complex structure J_0 on X. Since there is a J_0-holomorphic exceptional sphere by construction, J_0 belongs to the subset $\mathcal{U}$ considered in Lemma 4.3. Therefore, as in that lemma, there are plenty of embedded J_0-holomorphic L-spheres in N, and these map to immersed J'-holomorphic A-curves in V.

Thus for J' arbitrarily close to J, there are immersed J'-holomorphic A spheres. This is all we actually need for the proof of Theorem 1.4. However, the following argument proves the lemma as actually stated. (In fact, it is not hard to show that there is a J-holomorphic A-sphere for every regular J.) Choose a sequence J_j' converging to J such that for each J there is an immersed J_j'-holomorphic A-sphere C_j' as above. Since these spheres are regular by Lemma 3.6, we may perturb the J_j' to make them regular for A-spheres. Then, by Proposition 2.1 the evaluation maps $e(J_j')$ are essentially onto, i.e. there is a J_j'-holomorphic A-sphere through every point in V which does not lie on one of the finite number of J_j'-holomorphic A-cusp-curves. Therefore, we may suppose that the C_j' all go through some point y which does not lie on a J-holomorphic A-cusp-curve. By the Compactness Theorem, a subsequence of these C_j' converges, and its limit must be an A-sphere rather than an A-cusp-curve since it goes through y.

Note. — Obviously this technique works more generally, in situations when $c \neq 3$.

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