IMMERSED SURFACES AND PENCILS OF PLANES IN 3-SPACE

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1. Introduction. Let M be a compact connected boundaryless surface and $f:M\rightarrow\mathbb{R}^3$ a smooth immersion transverse to a straight line L. Thus there is an even number p of points $x\in M$ such that $f(x)\in L$. Under further transversality assumptions on f (see §3) there is a finite number q of points x of M such that the plane containing $f(x)$ and L touches $f(M)$ at $f(x)$. These assumptions are mild in the sense that they hold for any f in an open dense subset of the space of smooth immersions under consideration. Suppose that the Gaussian curvature of $f(M)$ is positive at q^+ of these points and negative at q^-, with $q=q^++q^-$. Then

$$p+q^+-q^-=e(M), \qquad (*)$$

where $e(M)$ denotes the Euler number of M.

The proof is an application of the Poincaré-Hopf theorem (see [**1**], [**2**]). Our theorem may be interpreted as a development of the theory of horizon immersions for surfaces. In particular the main result of [**3**] is a consequence of our result.

We illustrate the relation $(*)$ with a few examples. Take the standard embedding of the torus T into $\mathbb{R}^3$, and let L be parallel to the axis of T. In Figure 1, we show three positions of L viewed from above.

I am grateful to Stewart Robertson for suggesting this problem and I thank him and David Chillingworth for many helpful discussions.

2. Notations. In what follows we shall be dealing with compact connected smooth $(=C^\infty)$ boundaryless manifolds. All the maps are smooth unless otherwise stated.

Given manifolds M and N and a map $g:M\rightarrow N$ the derivative of g is denoted by $g_*:TM\rightarrow TN$, and $g_{*x}:T_xM\rightarrow T_{g(x)}N$ denotes the restriction of g_* to the tangent space to M at x. The critical set of g is denoted by $C(g)$.

If $f:M\rightarrow\mathbb{R}^{n+1}$ (where $\dim M=n$) is an immersion, then we denote by T_x the affine tangent n-plane to $f(M)$ at $f(x)$. Such an immersion induces a map $F:M\rightarrow\mathbb{R}^{n+1}_n$, where $\mathbb{R}^{n+1}_n$ denotes the Grassmannian of affine n-planes in $\mathbb{R}^{n+1}$. By the Gaussian curvature of $f(M)$ at $f(m)$ we mean the Gaussian curvature of $f(U)$ at $f(m)$, where U is an oriented open neighbourhood of m and $f|U$ is an embedding.

We may assume that the line L in $\mathbb{R}^3$ is defined by $x_1=x_2=0$. The pencil $\tilde{L}$ of 2-planes containing L, as a submanifold of $\mathbb{R}^3_2$, is diffeomorphic to the 1-dimensional real projective space P^1. For P^1 we shall consider the two standard charts $\psi:V\rightarrow\mathbb{R}$, $\psi':V'\rightarrow\mathbb{R}$, where

$$V(\text{resp. } V')=\{\pi\varepsilon P^1 \mid \pi \text{ is defined by } (1,x)(\text{resp.}(x,1))\}$$

and $\psi(\pi)=x$, $\psi'(\pi)=x$.

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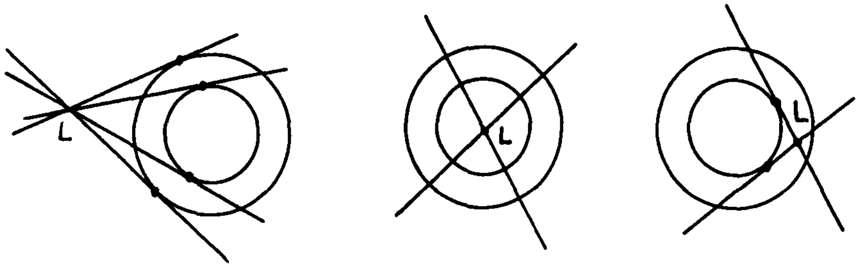

$p=0, q^+=q^-=2$ $p=q^+=q^-=0$ $p=q^-=2, q^+=0$

Figure 1

For any immersion $f:M\to\mathbb{R}^3$, with $\dim M=2$, we shall use three types of chart α, β, γ for M where

$$f\circ\alpha^{-1}(x,y)=(x,y,g(x,y)),$$
$$f\circ\beta^{-1}(x,y)=(x,h(x,y),y),$$
$$f\circ\gamma^{-1}(x,y)=(k(x,y),x,y).$$

Finally $I_2(F,\tilde{L})$ will denote the intersection number mod(2) of F with $\tilde{L}$.

3. The main result. The proof of the relation stated in §1 will occupy the whole of this section. We start with two elementary observations.

3.1. Suppose $f^{-1}(L)\neq\varnothing$. Then $f\pitchfork L$ at $m\in f^{-1}(L)$ iff T_m intersects L in a point.

3.2. Suppose $F^{-1}(\tilde{L})\neq\varnothing$. Then $F\pitchfork\tilde{L}$ at $m\in F^{-1}(\tilde{L})$ iff

(a) $f(m)\notin L$,

(b) $f(M)$ has non-zero Gaussian curvature at $f(m)$.

Bearing in mind 3.2, we suppose, from now on, that $F\pitchfork\tilde{L}$. Thus the condition $f\pitchfork L$ follows automatically (see §1).

Let $\Delta=f^{-1}(L)$. Define a map $\phi:M\backslash\Delta\to P^1$ by associating to $m\in M\backslash\Delta$ the 2-plane containing $f(m)$ and L. The map ϕ is smooth and $m\in C(\phi)$ iff $T_m\in\tilde{L}$. Since $F\pitchfork\tilde{L}$, the set $C(\phi)\cup\Delta$ is finite.

Consider the vector fields $\operatorname{grad}(\psi\circ\phi)$ and $\operatorname{grad}(\psi'\circ\phi)$. We use these to obtain a well-defined vector field Z on $M\backslash\{\Delta\cup C(\phi)\}$ as follows:

$$Z(m)=\begin{cases}\operatorname{grad}(\psi\circ\phi)(m)/\|\operatorname{grad}(\psi\circ\phi)(m)\| & \text{if} \quad m\in\phi^{-1}(V),\\ -\operatorname{grad}(\psi'\circ\phi)(m)/\|\operatorname{grad}(\psi'\circ\phi)(m)\| & \text{if} \quad m\in\phi^{-1}(V').\end{cases}$$

Take a smooth function $\gamma':M\to\mathbb{R}$, such that

$$\gamma'(x)=0 \quad \text{if} \quad x\in\Delta\cup C(\phi),$$
$$\gamma'(x)>0 \quad \text{if} \quad x\in M\backslash\{\Delta\cup C(\phi)\}.$$

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Using γ' we define another vector field Y by

$$Y(m)=\begin{cases}\gamma'(m)Z(m) & \text{if} \quad m\in M\backslash\{\Delta\cup C(\phi)\},\\ 0 & \text{if} \quad m\in\Delta\cup C(\phi).\end{cases}$$

It is not hard to show that Y is continuous. If $\phi(m)$ is a regular value of ϕ, then $Y(m)$ is orthogonal to the contour of ϕ passing through m.

As we see, we are now in a position which allows us to use the Poincaré-Hopf theorem. To calculate the indices at the zeros of Y we shall distinguish two cases:

(a) m_0 is a zero of Y and $m_0\in C(\phi)$;

(b) m_0 is a zero of Y and $m_0\in\Delta$.

Case (a). Due to the transversality condition we have imposed, both $\psi\circ\phi$ and $\psi'\circ\phi$ are Morse functions. Then the index of m_0 as a zero of Y can be calculated by looking at the index of m_0 as a critical point of either $\psi\circ\phi$ or $-(\psi'\circ\phi)$. It follows from standard calculations that the index is 1 or -1 according as the Gaussian curvature at $f(m_0)$ is positive or negative.

Case (b). Let $m_0\in\Delta$. As $f(m_0)\in L$ and $f\pitchfork L$ it is possible to find a chart $\alpha: U\to U'$ such that $f\mid U$ is an embedding, $f\circ\alpha^{-1}(x,y)=(x,y,g(x,y))$ and no other point of $\Delta\cup C(\phi)$, other than m_0, is in U. The restriction $f\mid U$ is transversal to any $\pi\in\tilde{L}$. For any $m\in U\backslash\{m_0\}$, we have that $Y(m)$ is orthogonal to the contour of $\phi\mid U\backslash\{m_0\}$ through m, and this contour is $(f\mid U)^{-1}(\pi)\backslash\{m_0\}$, where $\phi(m)=\pi$. Moreover $(f\mid U)^{-1}(\pi)$ is a 1-dimensional submanifold of M.

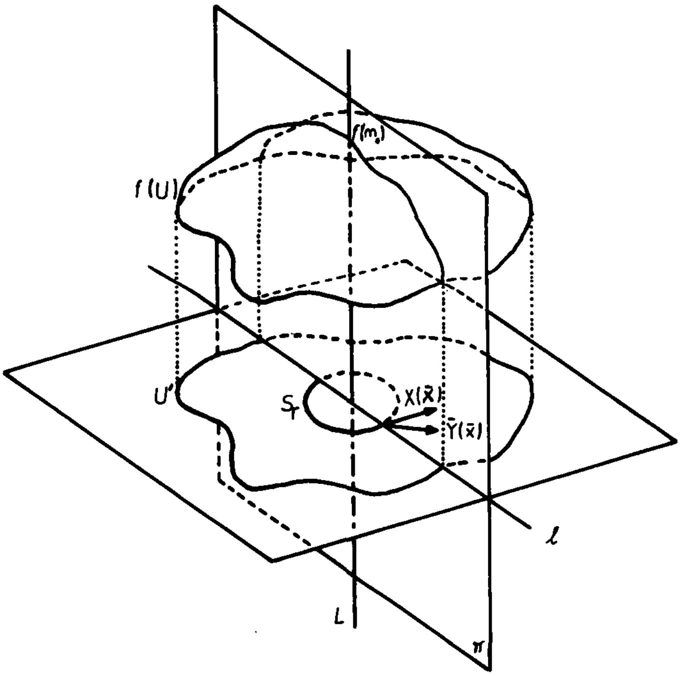

Figure 2

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If π is as shown in Figure 2, $(f \mid U)^{-1}(\pi)$ is diffeomorphic to $U' \cap l$, where $l = (\mathbb{R}^2 \times \{0\}) \cap \pi$. Using the chart α we get the "pullback" $\bar{Y}$ of Y, defined on U'. This vector field is never zero except at the origin. The important remark to make about $\bar{Y}$ is that because $Y(m)$ is orthogonal to $i_{*m}(T_m((f \mid U)^{-1}(\pi)))$ then

$$\bar{Y}(\bar{x}) \notin i_{*\bar{x}}(T_{\bar{x}}(U' \cap l)),$$

for any m such that $\alpha(m) = \bar{x} \neq (0, 0)$. Here we have denoted the obvious inclusions by i, and $\bar{x} = (x, y)$.

Let S_r be a small sphere round the origin such that $S_r \subset U'$. We define a vector field X of unit norm on S_r and tangent to S_r such that at a particular point $\bar{x}$, $X(\bar{x})$ and $\bar{Y}(\bar{x})$ form an acute angle. Looking at the vector fields as maps into $\mathbb{R}^2$,

$$X : S_r \to S^1 \subset \mathbb{R}^2, \qquad \bar{Y} \mid S_r : S_r \to \mathbb{R}^2,$$

we see that they are homotopic, a homotopy $H : S_r \times [0, 1] \to \mathbb{R}^2$ being defined by

$$H(\bar{x}, t) = tX(\bar{x}) + (1 - t)\bar{Y}(\bar{x}).$$

The map H is never zero and therefore induces a homotopy between X and $\bar{Y}_1 : S_r \to S^1$, where $\bar{Y}_1(\bar{x}) = \bar{Y}(\bar{x})/\|\bar{Y}(\bar{x})\|$. Consequently the maps X and $\bar{Y}_1$ have the same degree and the degree of X is 1. Hence the index of Y at m_0 is 1.

Having calculated the indices, we can deduce the relation $(*)$ in §1 immediately.

4. Consequences of the main result. The theorem we have just proved yields the following two corollaries.

COROLLARY 4.1. *Assume* $F \pitchfork \tilde{L}$. *Then* $I_2(F, \tilde{L}) \equiv e(M) (\text{mod } 2)$.

Proof. By intersection theory, $I_2(f, L) = 0$. The result follows now from $(*)$ in §1.

The next corollary is just the theorem on the existence of horizon maps for surfaces [**3**].

COROLLARY 4.2. *If f is a horizon immersion then M is diffeomorphic to S^2, $S^1 \times S^1$ or the Klein bottle. If M is diffeomorphic to S^2 then $\#f^{-1}(L) = 2$; otherwise, $\#f^{-1}(L) = 0$.*

Proof. If f is a horizon immersion then $F \pitchfork \tilde{L}$ and $F^{-1}(\tilde{L}) = \emptyset$. Because $\#f^{-1}(L)$ is even it follows from $(*)$ in §1 that $e(M)$ is even and greater than or equal to zero.

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