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IMMERSING HOMOGENEOUS SPACES IN EUCLIDEAN SPACE

Howard Hiller *

In [4], Lam proves, among other things, an immersion result for the real flag manifold

$$G_{\mathbb{R}}(n_1,\ldots,n_s) = O(n_1+\ldots+n_s)/O(n_1)\times\ldots\times O(n_s)$$

where $O(n)$ is the real orthogonal group. His result is a special case of the following more general observation.

Proposition 1. Let G be a compact, connected semisimple Lie group and H a closed subgroup. Then either G/H is a π-manifold (and so immerses in codimension one) or G/H immerses in $\mathbb{R}^{\dim(\mathfrak{g})}$, where $\mathfrak{g}$ is the Lie algebra of G.

Remark 2. The dimension of the ambient Euclidean space is independent of the subgroup H, so one expects the strongest results for small H. But if H is a maximal torus (all $n_i = 1$ in above example) then G/H is a π-manifold [1]. See Remark 4 below.

Proof. A real vector bundle over G/H is determined by an action of H on a real vector space. The tangent bundle $T(G/H)$ comes from the adjoint action of H on $\mathfrak{g}/\mathfrak{h}$, $\mathfrak{h}$ = Lie H. Let η denote the bundle over G/H coming from the adjoint action of H on $\mathfrak{g}$. Crearly η is trivial since the action extends to all of G. There is a bundle epimorphism $\eta \to T(G/H)$

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which necessarily splits (see for example [5,p.46]). Since $\dim(\eta) = \dim(\theta)$, the theorem of Hirsch [3] yields the desired immersion.

Corollary 3. (Lam) The manifold $G_{\mathbb{R}}(n_1,\ldots,n_s)$ inmerses in $R^{\binom{n}{2}}$; where $n = n_1+\ldots+n_s$.

Proof. Observe $\dim\ O(n) = \binom{n}{2}$.

Remark 4. That these immersions are really interesting follows from [2] where it was shown that if $s = 2$, $n_1 = n_2$, one obtains a best possible immersion for the real Grassmannian.

Similarly, one obtains in the complex and quaternionic cases.

Corollary 5. (a) $G_{\mathbb{C}}(n_1,\ldots,n_s)$ immerses in $\mathbb{R}^{n^2}$

(b) $G_{\mathbb{H}}(n_1,\ldots,n_s)$ immerses in $\mathbb{R}^{2n^2+n}$

These results are not as strong as those of Lam [4].

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Mathematisches Institut
der Georg-August-Universität
Bunsenstr. 3-5

3400 Göttingen