Impact of time illiquidity in a mixed market without full observation*

Salvatore Federico [a)]   Paul Gassiat [b)]   Fausto Gozzi [c)]

November 6, 2012

Abstract

We study a problem of optimal investment/consumption over an infinite horizon in a market consisting of two possibly correlated assets: one liquid and one illiquid. The liquid asset is observed and can be traded continuously, while the illiquid one can be traded only at discrete random times corresponding to the jumps of a Poisson process with intensity λ, is observed at the trading dates, and is partially observed between two different trading dates. The problem is a nonstandard mixed discrete/continuous optimal control problem which we face by the dynamic programming approach. When the utility has a general form we prove that the value function is the unique viscosity solution of the HJB equation and, assuming sufficient regularity of the value function, we give a verification theorem that describes the optimal investment strategies for the illiquid asset. In the case of power utility, we prove the regularity of the value function needed to apply the verification theorem, providing the complete theoretical solution of the problem. This allows us to perform numerical simulation, so to analyze the impact of time illiquidity in this mixed market and how this impact is affected by the degree of observation.

Keywords: Investment-consumption problem, liquidity risk, optimal stochastic control, Hamilton-Jacobi-Bellman equation, viscosity solutions, regularity of viscosity solutions.

MSC 2010 Classification: 93E20, 91G80, 35D40, 35B65.
JEL Classification: C61, G11.

Contents

*This research has been partially supported by the PRIN research project “Metodi deterministici e stocastici nello studio di problemi di evoluzione” of the Italian Minister of University and Research.

[a)] Dipartimento di Economia, Management e Metodi Quantitativi, Università di Milano, Italy. E-mail: salvatore.federico@unimi.it. (Part of this research was done when this author was post-doc at the LPMA - Université Paris 7 financed by Alma Research.)

[b)] Institut für Mathematik, TU Berlin. E-mail: gassiat@math.tu-berlin.de

[c)] Dipartimento di Economia e Finanza, Libera Università degli Studi Sociali “Guido Carli”, Roma, Italy. E-mail: fgozzi@luiss.it

1 Introduction

Following the seminal works of Merton on portfolio management, a classical assumption in mathematical finance is to suppose that assets may be continuously traded by the agents operating in the market. However, this assumption is unrealistic in practice, especially in the case of less liquid markets, where investors cannot buy and sell assets immediately, and have to wait some time before being able to unwind a position.

In the recent years, several works have studied the impact of this type of illiquidity on the investors. Rogers and Zane [23], Matsumoto [20], Pham and Tankov [21] (see also [5, 22]) consider an investment model where the discrete trading times are given by the jump times of a Poisson process with constant intensity $\lambda > 0$. Bayraktar and Ludkovski [3] study a portfolio liquidation problem in a similar context.

The aforementioned works focus on an agent investing exclusively in an illiquid asset. However, in practice it is common to have several correlated tradable assets with different liquidity. For instance an index fund over some given financial market will be usually much more liquid than the individual tracked assets, while sharing a positive correlation with those assets. An investor in this market will then have the possibility of hedging his exposure in the less liquid assets by investing in the index and rebalancing his position frequently.

To our knowledge only few papers consider the case of a market composed by two (possibly correlated) assets, one liquid and one illiquid. This is the case of Longstaff [19], who analyzes a two agents portfolio problems in a market composed by a liquid asset and another asset that becomes non tradable only for a given time period. Moreover, Schwartz and Tebaldi [24] consider a market composed by a liquid asset that can be traded continuously, and an illiquid asset that cannot be traded and is liquidated at a terminal date. Finally, we have to mention the recent paper by Ang, Papanikolaou and Westerfield [1] where the authors, in the standard framework of maximizing an infinite horizon discounted power utility from consumption, take a less restrictive point of view on the tradability of the illiquid asset assuming, as in [5, 12, 20, 21, 23], that the illiquid asset may be traded at discrete random times.

Following the line of these papers, here we also consider a market composed by a liquid asset and an illiquid one. In particular, as in [1] the illiquid asset can be traded at some discrete random dates. The main novelties of our paper in the context of mixed liquid/illiquid market case are the following: on one hand, *we consider the case of incomplete observation on the illiquid asset price between trading dates*; on the other hand, *we provide a complete theoretical framework that allows to cover also more general form of utility functions, not restricted to CRRA type.*

To be more precise, we study a problem of optimal investment/consumption over an infinite horizon in a market consisting of a liquid and an illiquid asset. The liquid asset is continuously observed and can be continuously traded. The illiquid asset is correlated with the liquid one with correlation parameter $\rho \in (-1, 1)$ and can be traded only at discrete random times corresponding to the jumps of a Poisson process with intensity $\lambda > 0$. About the observation, we assume that the illiquid asset can be observed at the trading dates (as in [5, 12, 20, 21]), but we introduce a new feature in the model - with respect to the aforementioned literature - allowing to consider the possibility of a partial information between trading dates. We introduce a parameter $\gamma \in [0, 1]$ measuring the observation of the illiquid asset between two trading dates. The limit cases for this parameter, i.e. $\gamma = 0$ and $\gamma = 1$, correspond respectively to the no observation case (as it is done in [5, 21, 22], but only for the single asset case) and to the full observation case (as in [1, 23, 24]). In this sense our model is more general and flexible than the other ones proposed by the literature: first, it allows to put together the presence of two correlated liquid/illiquid assets and the incomplete observation of the illiquid one; second, to merge two extreme (full vs. none observation) cases and consider intermediate situations where the agent has a partial information about the state of the illiquid asset between two trading dates (see Remark 2.2).

The mathematical problem is, as expected, a nonstandard mixed discrete/continuous optimal control problem, which is in our case more difficult to tackle than in the aforementioned papers due to the presence of the features described above. By means of a suitable use of Dynamic Programming (DP) (extending what is done in [21]), we show that the stochastic control problem between trading times can be written as an infinite horizon stochastic time-inhomogeneous control problem. Then we apply the usual machinery of

DP for such kind of problems and, using some results of [7][1], we characterize the value function $\widehat{V}$ of this auxiliary problem as the unique viscosity solution of a Hamilton-Jacobi-Bellman equation. At this stage, the viscosity characterization only allows to provide the optimal feedback allocation in the illiquid asset. In order to go further in the solution and characterize the optimal feedback allocation in the liquid asset as well as the optimal feedback consumption strategy, we need to prove a regularity result for $\widehat{V}$. Such (nonstandard) regularity result is provided in the special case of power utility[2] and allows to give a full theoretical solution to the problem. This solution is made implementable by a numerical scheme and numerical results are then provided and discussed for different values of the relevant parameters γ, λ, ρ.

The plan of the paper is as follows. Section 2 describes the market model and formulates the investment/consumption problem. In Section 3 we show how, by a suitable dynamic programming principle, the problem can be reduced to a standard continuous time stochastic control problem; then we present some useful properties satisfied by the value functions - the original one and the auxiliary one - and prove an analytical characterization of them by means of viscosity solutions; finally, we provide the characterization of the optimal investment in the illiquid asset. In Section 4 we solve completely the problem in the case of power utility and provide an iterative scheme to solve the problem numerically. Finally, Section 5 is devoted to the discussion of the numerical results we obtained.

2 Model and optimization problem

In this section we present the model and the optimization problem we deal with. Let us consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, on which are defined:

- A Poisson process $(N_t)_{t\geq 0}$, with intensity $\lambda > 0$. We denote by $(\tau_k)_{k\geq 1}$ its jump times; moreover we set $\tau_0 = 0$.
- Two independent standard Brownian motions $(B_t)_{t\geq 0}$, $(W_t)_{t\geq 0}$, independent also of the Poisson process $(N_t)_{t\geq 0}$.

2.1 Market model

The market model we consider on the probability space above consists of two risky assets with correlation $\rho \in (-1, 1)$:

- A liquid risky asset that can be traded continuously; it is described by a stochastic process denoted by L_t whose dynamics is

$$dL_t = L_t\,(b_L dt + \sigma_L dW_t), \tag{1}$$

[1] Actually, in [7] these results are proved for the case $\gamma = 0$. However, their extension to the general case $\gamma \in [0, 1]$ is straightforward, see Subsection 3.1.

[2] Our assumption on the utility function covers only the case of positive power, differently from [1]. However the methods used here can be employed by suitable modifications to cover the case of negative power as well (see also Remark 2.4).

where $b_L, \sigma_L > 0$.

- An illiquid risky asset that can only be traded at the trading times τ_k; it is described by a stochastic process denoted by I_t, whose dynamics is

$$dI_t \;=\; I_t \left(b_I dt + \sigma_I \left(\rho dW_t + \sqrt{1-\rho^2} dB_t\right)\right), \tag{2}$$

  where $b_I, \sigma_I > 0$.

Without loss of generality we assume $L_0 = I_0 = 1$. We also suppose that in the market is present a riskless asset with deterministic dynamics. Just for simplicity we assume that the interest rate of this asset is constant and equal to 0.

Remark 2.1 *If the riskless asset interest rate is not* 0*, one needs just to add an extra term in all the equations. Moreover, in the special case of power utility treated in Section 4, the assumption that the riskless rate return is* 0 *can be done without loss of generality, as in this case the interest rate can be discarded in the discount factor of the objective functional (the constant* β *in* (8) *below) by a suitable change of variables (see Remark 2, p. 189, in [14]).*

2.2 Information

The information setting we consider is the following. We assume that:

- the liquid asset L is continuously observed;
- the illiquid asset I is observed at the trading random times $(\tau_k)_{k\geq 0}$;
- the illiquid asset I is only partially observed in the time interval (τ_k, τ_{k+1}).

In order to deal with and make formal the last issue, we suppose that the Brownian motion B_t can be split as

$$B_t \;=\; \gamma B_t^{(1)} + \sqrt{1-\gamma^2} dB_t^{(2)}, \quad \gamma \in [0,1],$$

where $B^{(1)}, B^{(2)}$ are mutually independent Brownian motions also independent of W, N, and we assume that $B^{(1)}$ is observed and $B^{(2)}$ is unobserved.

Let $(\mathcal{N}_t)_{t\geq 0}$, $(\mathcal{W}_t)_{t\geq 0}$, $(\mathcal{B}_t^{(1)})_{t\geq 0}$ be the filtrations generated respectively by N, W, $B^{(1)}$. Define the σ-algebra

$$\mathcal{I}_t = \sigma\left(I_{\tau_k} \mathbf{1}_{\{\tau_k \leq t\}},\ k \geq 0\right), \quad t \geq 0.$$

Moreover define the filtration

$$\mathbb{G}^0 \;:=\; (\mathcal{G}_t^0)_{t\geq 0}; \qquad \mathcal{G}_t^0 \;=\; \mathcal{N}_t \vee \mathcal{I}_t \vee \mathcal{W}_t \vee \mathcal{B}_t^{(1)} \;=\; \sigma(\tau_k, I_{\tau_k}; \tau_k \leq t) \vee \mathcal{W}_t \vee \mathcal{B}_t^{(1)}.$$

The observation filtration we consider is

$$\mathbb{G} \;=\; (\mathcal{G}_t)_{t\geq 0}; \qquad \mathcal{G}_t \;=\; \mathcal{G}_t^0 \vee \sigma(\mathbb{P}\text{-null sets}).$$

This means that at time t we have:

- a full information on the past of the liquid asset up to time t;
- a full information on the trading dates of the illiquid assets occurred before t and on the values of the illiquid asset at such trading dates;
- a partial information (in the sense described above) on the value of the illiquid asset at time t.

The parameter γ measures how much information on I is available in the random intervals (τ_k, τ_{k+1}). The limit cases are:

- $\gamma = 0$, which corresponds to have no information on the illiquid asset in the interval (τ_k, τ_{k+1}) after the last information on its value at time τ_k; this case fits the model described in [22];
- $\gamma = 1$, which corresponds to the full information case, recovering the setting of [1].

Remark 2.2 *Roughly speaking, the main idea behind our model is that the price of the illiquid asset is observed exactly at the trading times $(\tau_k)_{k\geq 0}$ (as in [5, 21, 22]), while at different times $t \in (\tau_k, \tau_{k+1})$ the agent can observe a process $I_t^{(1)}$ satisfying the equation*

$$dI_t^{(1)} \;=\; I_t^{(1)}\big(b_I dt + \sigma_I(\rho\, dW_t + \sqrt{1-\rho^2}\,\gamma\, dB_t^{(1)})\big), \qquad I_{\tau_k}^{(1)} \;=\; I_{\tau_k},$$

and

$$I_t \;=\; I_t^{(1)} \cdot I_t^{(2)},$$

where

$$dI_t^{(2)} \;=\; I_t^{(2)} \sigma_I \sqrt{1-\rho^2}\sqrt{1-\gamma^2}\, dB_t^{(2)}, \qquad I_{\tau_k}^{(2)} \;=\; 1,$$

is an unobserved noise on our knowledge of I. The idea behind this is that, while the price of I is clearly observed when it is bought/sold, between two trading dates we have only a partial knowledge of it. This partial knowledge is represented by the process $I^{(1)}$ and differs by the "real price" I by a factor $I^{(2)}$.

2.3 Trading/consumption strategies and wealth dynamics

In the setting above, we define the set of admissible trading/consumption strategies in the following way. Consider all the triplets of processes (c, π, α) such that

(h1) $c = (c_t)_{t\geq 0}$ is a continuous-time nonnegative process $(\mathcal{G}_t)_{t\geq 0}$-predictable and with locally integrable trajectories; c_t represents the consumption rate at time t;

(h2) $\pi = (\pi_t)_{t\geq 0}$ is a continuous-time process $(\mathcal{G}_t)_{t\geq 0}$-predictable and with locally square integrable trajectories; π_t represents the amount of money invested in the liquid asset at time t;

(h3) $\alpha = (\alpha_k)_{k\in\mathbb{N}}$, is a discrete process where α_k is $\mathcal{G}_{\tau_k}$-measurable; α_k represents the amount of money invested in the illiquid asset in the interval $(\tau_k, \tau_{k+1}]$.

Given an initial wealth $r \geq 0$ and a triplet (c_t, π_t, α_k) satisfying the requirements (h1)–(h3) above, we can consider the process R_t representing the wealth associated to such strategy. Its dynamics can be defined by recursion on $k \geq 0$ by

$$R_0 = r, \tag{3}$$

$$R_t = R_{\tau_k} + \int_{\tau_k}^{t} \big(\pi_s(b_L ds + \sigma_L dW_s) - c_s ds\big) + \alpha_k \left(\frac{I_t}{I_{\tau_k}} - 1\right), \quad t \in (\tau_k, \tau_{k+1}]. \tag{4}$$

We observe that the process R is in general not $\mathbb{G}$-adapted (unless $\gamma = 1$), as I is not.

We can define also a liquid part X_t (observable, therefore $\mathbb{G}$-adapted)[3] and an illiquid one A_t (partially observable) of the wealth R_t. They are defined in the intervals $[\tau_k, \tau_{k+1})$

$$X_t = R_{\tau_k} - \alpha_k + \int_{\tau_k}^{t} \big(\pi_s(b_L ds + \sigma_L dW_s) - c_s ds\big), \tag{5}$$

$$A_t = \alpha_k \frac{I_t}{I_{\tau_k}}. \tag{6}$$

Obviously we have

$$R_t = X_t + A_t, \qquad \forall t \geq 0.$$

Observe that the process R is continuous, while the processes X, A are not, due to the rebalancing. We also observe that at time τ_k, for any k, the process R does not depend on the value of α_k, while the process X, A do.

As a class of admissible controls we consider the triplets of processes (c, π, α) satisfying the measurability and integrability conditions above and such that the corresponding wealth process R_t is nonnegative (no-bankruptcy constraint). One can see without big difficulty that this requirement is equivalent to require that both the liquid and the illiquid wealth have to be nonnegative at each time, i.e. that $X_t \geq 0$, $A_t \geq 0$ for every $t \geq 0$. So, the admissibility of a strategy (c, π, α) is equivalent to

$$0 \leq \alpha_k \leq R_{\tau_k}, \qquad \forall k \geq 0, \tag{7}$$

$$\int_{\tau_k}^{t} \big(\pi_s(b_L ds + \sigma_L dW_s) - c_s ds\big) \leq R_{\tau_k} - \alpha_k, \qquad \forall t \in [\tau_k, \tau_{k+1}),\ \forall k \geq 0.$$

The class of admissible controls depends on the initial wealth $R_0 = r$. We denote this class by $\mathcal{A}(r)$ noticing that it is not empty for every $r \geq 0$, as the null strategy $(c, \pi, \alpha) = (0, 0, 0)$ belongs to it for every $r \geq 0$.

2.4 Optimization problem

Let $R_0 = r$. The optimization problem consists in maximizing over the set of admissible strategies $\mathcal{A}(r)$ the expected discounted utility from consumption over an infinite horizon.

[3]Notice that X_t is the sum of the money invested in the liquid asset and of the money held in the bank account (which has interest rate 0 by our assumption).

In other terms, chosen a utility function U and a discount factor $\beta > 0$, the optimization problem we consider is the mixed discrete/continuous stochastic control problem

$$\text{Maximize} \quad \mathbb{E}\left[\int_0^\infty e^{-\beta s} U(c_s) ds\right], \quad \text{over } (c, \pi, \alpha) \in \mathcal{A}(r). \tag{8}$$

Assumption 2.3 *The preference of the agent are described by a utility function*

$$U : \mathbb{R}_+ \longrightarrow \mathbb{R}$$

which is continuous, nondecreasing, concave, and such that $U(0) = 0$. Moreover we assume the following growth condition on U: there exist constants $K_U > 0$ and $p \in (0, 1)$ such that

$$U(c) \quad \leq \quad K_U \frac{c^p}{p}. \tag{9}$$

Remark 2.4 *In the applications one is usually interested to work with power utility functions of the form*

$$U(c) \quad = \quad \frac{c^p}{p}, \qquad p \in (-\infty, 1),$$

with the usual agreement that $U(c) = \log c$ when $p = 0$. However, Assumption 2.3 includes only the case $p \in (0, 1)$. On one hand the case of negative exponent is interesting, as it seems to be even more realistic from the point of view of the agents' behavior (see [2]); on the other hand, to be extensively treated it would need much more space. For this reason we will work with Assumption 2.3. Nevertheless we stress that the case $p \leq 0$ can be treated by the same techniques by straightforward modifications. This case is treated in [1] under full observation.

We observe that, due to Assumption 2.3 on U, the Legendre tranform of U

$$\widetilde{U}(w) \quad = \quad \sup_{c \geq 0} \{U(c) - cw\}, \quad w > 0,$$

is finite, nonincreasing and convex. Moreover, the growth condition (9) yields the following growth condition for $\widetilde{U}$: there exists $K_{\widetilde{U}} > 0$ such that

$$\widetilde{U}(w) \quad \leq \quad K_{\widetilde{U}}\, w^{-\frac{p}{1-p}}. \tag{10}$$

According to Remark 2.6 below, we assume the following:

Assumption 2.5 *We assume that*

$$\beta \; > \; k_p, \tag{11}$$

where

$$k_p \quad := \quad \sup_{u_L \in \mathbb{R}, u_I \in [0,1]} \left\{ p(u_L b_L + u_I b_I) - \frac{p(1-p)}{2}(u_L^2 \sigma_L^2 + u_I^2 \sigma_I^2 + 2\rho u_L u_I \sigma_L \sigma_I) \right\}. \tag{12}$$

Remark 2.6 *The assumption on β is related to the investment/consumption problem with the same assets but in a liquid market. Let $p \in (0,1)$ and consider an agent with initial wealth r, consuming at rate c_t and investing in L_t and I_t continuously with respective proportions u_t^L and u_t^I and under the constraint that $u_t^I \in [0,1]$. Suppose, moreover, that the preferences of the agent are represented by the utility function $U^{(p)}(c) = c^p/p$, with $p \in (0,1)$. Let us denote by $\mathcal{A}_{Mert}(r)$ the set of strategies keeping the wealth nonnegative and define the value function*

$$V_{Mert}^{(p)}(r) = \sup_{(u^L,u^I,c)\in\mathcal{A}_{Mert}(r)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U^{(p)}(c_t)dt\right], \tag{13}$$

This is a constrained Merton problem which dominates our problem, in the sense that $V_{Mert}^{(p)}(r)$ is higher than the optimal value of our problem, up to the multiplicative constant K_U of (9)*. One can see (for instance by solving the HJB equation) that $V_{Mert}^{(p)}$ is finite if and only if* (11) *is satisfied and that in this case*

$$V_{Mert}^{(p)}(r) = \left(\frac{1-p}{\beta - k_p}\right)^{1-p} r^p. \tag{14}$$

Therefore, condition (11) *guarantees together with* (9) *finiteness for our problem too.*

Further note that the constrained liquid investment/consumption problem described above can always be reduced to the case where the two assets are independent, because

$$dX_t = X_t\Big(u_t^L \frac{dL_t}{L_t} + u_t^I \frac{dI_t}{I_t}\Big) = X_t\left(\Big(u_t^L + \frac{\rho b_L \sigma_I}{\sigma_L} u_t^I\Big)\frac{dL_t}{L_t} + \pi_t^I \frac{dJ_t}{J_t}\right),$$

where J is the process defined below in (21) *(taking $\gamma = 0$), and the problem is equivalent to an agent investing in L and J, with the same constraint for the proportion invested in I. However, this reduction does not work for the illiquid problem that we consider: neither the observation constraint (the integrand in L being $\mathbb{G}$-adapted) nor the trading constraint (the amount held in the illiquid asset being constant between τ_k and τ_{k+1}) are preserved by this transformation.*

3 Dynamic Programming

We denote the value function of the optimal stochastic control problem (8) by V:

$$V(r) = \sup_{(c,\pi,\alpha)\in\mathcal{A}(r)} \mathbb{E}\left[\int_0^\infty e^{-\beta s} U(c_s)ds\right], \quad r \geq 0. \tag{15}$$

Proposition 3.1 *V is everywhere finite, concave, p-Hölder continuous and nondecreasing. Moreover*

$$V(r) \leq K_V r^p, \quad \textit{for some } K_V > 0. \tag{16}$$

Proof. As we have already observed in Remark 2.6, finiteness and (16) follow from (9) and (11), by comparing with a constrained Merton problem.

Concavity of V comes from concavity of U and linearity of the state equation by standards arguments. Also monotonicity is consequence of standard arguments due to monotonicity of U. Finally, p-Hölder continuity follows from concavity and monotonicity of V and from (16). $\square$

Following [21], we state a suitable Dynamic Programming Principle (DPP) to reduce our mixed discrete/continuous problem to a standard one between two trading times.

Proposition 3.2 (DPP) *We have the following equality:*

$$V(r) \quad = \quad \sup_{(c,\pi,\alpha)\in\mathcal{A}(r)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds + e^{-\beta\tau_1}V\left(R_{\tau_1}\right)\right]. \tag{17}$$

Proof. The proof is long and technical, but similar to the one in [22] and we omit it for brevity. However, we note that, unlike in [22], there is some additional random information between 0 and τ_1 brought by W, so that the "shifting" procedure is slightly more technical to achieve. We refer for instance to the proof of the DPP provided in Appendix B of [12], where this shifting procedure is performed in a similar framework. $\square$

Now we will use this DPP to relate our original problem to a standard continuous-time control problem. For each $x \geq 0$, let $\mathcal{A}_0(x)$ be the set of couples of stochastic processes $(c_s, \pi_s)_{s\geq 0}$ such that

- $(c_s)_{s\geq 0}$ is $(\mathcal{W}_s \vee \mathcal{B}_s^{(1)})_{s\geq 0}$-predictable, nonnegative and has locally integrable trajectories;
- $(\pi_s)_{s\geq 0}$ is $(\mathcal{W}_s \vee \mathcal{B}_s^{(1)})_{s\geq 0}$-predictable and has locally square-integrable trajectories;
- $x + \int_0^T (-c_s ds + \pi_s(b_L ds + \sigma_L dW_s)) \geq 0$, for every $T \geq 0$.

By Lemma A.1, we obtain that (17) may actually be rewritten as

$$V(r) \quad = \quad \sup_{0\leq a\leq r} \sup_{(c,\pi)\in\mathcal{A}_0(r-a)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds + e^{-\beta\tau_1}V\left(R_{\tau_1}\right)\right]. \tag{18}$$

We want to rewrite in a suitable way the inner optimization problem in (18), i.e.

$$\sup_{(c,\pi)\in\mathcal{A}_0(r-a)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds + e^{-\beta\tau_1}V\left(R_{\tau_1}\right)\right]. \tag{19}$$

First define (see Remark 3.3(i) for explanations on this choice)

$$b_Y \;=\; \gamma^2 b_I + (1-\gamma^2)\frac{\rho b_L \sigma_I}{\sigma_L}, \qquad b_J \;=\; (1-\gamma^2)\big(b_I - \frac{\rho b_L \sigma_I}{\sigma_L}\big), \tag{20}$$

and given $x, y \geq 0$, $(c,\pi) \in \mathcal{A}_0(x)$, define the processes $J, \tilde{X}^{x,c,\pi}, \tilde{Y}^y$ as solutions to the SDEs

$$dJ_t \;=\; J_t\left(b_J dt + \sigma_I\sqrt{1-\rho^2}\sqrt{1-\gamma^2}dB_t^{(2)}\right), \quad J_0 = 1, \tag{21}$$

$$d\tilde{X}_s \;=\; -c_s ds + \pi_s(b_L ds + \sigma_L dW_s), \quad \tilde{X}_0 = x, \tag{22}$$

$$d\tilde{Y}_s \;=\; \tilde{Y}_s\left(b_Y dt + \sigma_I\big(\rho dW_s + \sqrt{1-\rho^2}\;\gamma dB_t^{(1)}\big)\right), \quad \tilde{Y}_0 = y. \tag{23}$$

Then we have, for every $t \in [0, \tau_1)$

$$X_t \;=\; \tilde{X}_t^{r-\alpha_0,c,\pi}, \qquad A_t \;=\; \tilde{Y}_t^{\alpha_0} \cdot J_t.$$

Since τ_1 is independent of $(\mathcal{W}_t)_{t\geq 0}$, $(\mathcal{B}_t^{(1)})_{t\geq 0}$, $(\mathcal{B}_t^{(2)})_{t\geq 0}$ and has distribution $\mathcal{E}(\lambda)$, while c, J, $\tilde{X}^{x,c,\pi}, \tilde{Y}^y$ are $(\mathcal{W}_\infty \vee \mathcal{B}_\infty^{(1)} \vee \mathcal{B}_\infty^{(2)})$-measurable, we have

$$\begin{aligned}
& \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s} U(c_s) ds + e^{-\beta\tau_1} V(R_{\tau_1}) \;\Big|\; \mathcal{W}_\infty \vee \mathcal{B}_\infty^{(1)} \vee \mathcal{B}_\infty^{(2)}\right] \\
=\;& \int_0^\infty \lambda e^{-\lambda t}\left(\int_0^t e^{-\beta s} U(c_s) ds + e^{-\beta t} V(\tilde{X}_t^{r-a,c,\pi} + J_t \cdot \tilde{Y}_t^a)\right) dt \\
=\;& \int_0^\infty e^{-\beta s} U(c_s) \int_s^\infty \lambda e^{-\lambda t} dt\, ds + \int_0^\infty \lambda e^{-(\lambda+\beta)t} V(\tilde{X}_t^{r-a,c,\pi} + J_t \cdot \tilde{Y}_t^a) dt \\
=\;& \int_0^\infty e^{-(\beta+\lambda)t}\left(U(c_t) + \lambda V(\tilde{X}_t^{r-a,c,\pi} + J_t \cdot \tilde{Y}_t^a)\right) dt,
\end{aligned}$$

where in the second equality we have used Fubini's theorem.

On the other hand, since J is independent of $(\mathcal{W}_t)_{t\geq 0}$ and $(\mathcal{B}_t^{(1)})_{t\geq 0}$, while c, $\tilde{X}^{x,c,\pi}, \tilde{Y}^y$ are $(\mathcal{W}_\infty \vee \mathcal{B}_\infty^{(1)})$-predictable, conditioning the equality above with respect to $\mathcal{W}_\infty \vee \mathcal{B}_\infty^{(1)}$ we get

$$\begin{aligned}
& \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s} U(c_s) ds + e^{-\beta\tau_1} V(R_{\tau_1}) \;\Big|\; \mathcal{W}_\infty \vee \mathcal{B}_\infty^{(1)}\right] \\
=\;& \mathbb{E}\left[\int_0^\infty e^{-(\beta+\lambda)t}\left(U(c_t) + \lambda V(\tilde{X}_t^{r-a,c,\pi} + J_t \cdot \tilde{Y}_t^a)\right) dt \;\Big|\; \mathcal{W}_\infty \vee \mathcal{B}_\infty^{(1)}\right], \\
=\;& \int_0^\infty e^{-(\beta+\lambda)t}\left(U(c_t) + \lambda G[V](t, \tilde{X}_t^{r-a,c,\pi}, \tilde{Y}_t^a)\right) dt,
\end{aligned}$$

where

$$G[V](t,x,y) \;:=\; \mathbb{E}\left[V(x + yJ_t)\right]. \tag{24}$$

In conclusion, we may rewrite (19) as

$$\sup_{(c,\pi)\in\mathcal{A}_0(r-a)} \mathbb{E}\left[\int_0^\infty e^{-(\beta+\lambda)t}\left(U(c_t) + \lambda G[V](t, \tilde{X}_t^{r-a,c,\pi}, \tilde{Y}_t^a)\right) dt\right]. \tag{25}$$

The new form of the problem involves (24). It is useful to define G as a linear operator from the space $\mathcal{M}_1(\mathbb{R}_+;\mathbb{R})$ of measurable functions with at most linear growth to the space of measurable functions $\mathcal{M}(\mathbb{R}_+^3;\mathbb{R})$ and stress its dependence on γ (it depends on γ through J). So we define

$$\begin{aligned}
G^\gamma :\; \mathcal{M}_1(\mathbb{R}_+;\mathbb{R}) \;&\longrightarrow\; \mathcal{M}(\mathbb{R}_+^3;\mathbb{R}) && (26)\\
\psi \;&\longmapsto\; G^\gamma[\psi](t,x,y) := \mathbb{E}\left[\psi(x + yJ_t)\right]. && (27)
\end{aligned}$$

Useful properties of G^γ are listed in Proposition A.2 in Appendix.

The problem (25) is a continuous non autonomous stochastic control problem over an infinite horizon that we call auxiliary problem. One can apply a standard dynamic programming approach to this problem. To do that, as usual we define the same problem for generic initial data t, x, y. For each $t, x \geq 0$, let $\mathcal{A}_t(x)$ be the set of couples of stochastic processes $(c_s, \pi_s)_{s\geq t}$ such that

- $(c_s)_{s\geq t}$ is $(\mathcal{W}_s \vee \mathcal{B}_s^{(1)})_{s\geq t}$-predictable, nonnegative and has locally integrable trajectories;

- $(\pi_s)_{s\geq t}$ is $(\mathcal{W}_s \vee \mathcal{B}_s^{(1)})_{s\geq t}$-predictable and has locally square-integrable trajectories;

- $x + \int_t^T (-c_s ds + \pi_s(b_L ds + \sigma_L dW_s)) \geq 0, \quad \forall T \geq t.$

Let $x, y \geq 0$. Given $(c,\pi) \in \mathcal{A}_t(x)$, let $(\tilde{X}_s^{t,x,c,\pi})_{s\geq 0}$, $(\tilde{Y}_s^{t,y})_{s\geq 0}$ be the solutions to the SDE's[4]

$$
\begin{aligned}
d\tilde{X}_s &= -c_s ds + \pi_s(b_L ds + \sigma_L dW_s), \quad \tilde{X}_t = x, &(28)\\
d\tilde{Y}_s &= \tilde{Y}_s\left(b_Y dt + \sigma_I\big(\rho dW_s + \sqrt{1-\rho^2}\,\gamma dB_t^{(1)}\big)\right), \quad \tilde{Y}_t = y. &(29)
\end{aligned}
$$

Define the value function

$$
\widehat{V}(t,x,y) = \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)}\left(U(c_s) + \lambda G^\gamma[V](s, \tilde{X}_s^{t,x,c,\pi}, \tilde{Y}_s^{t,y}\right) ds\right]. \tag{30}
$$

Associating to every locally bounded function $\hat{v}$ on $\mathbb{R}_+^3$ the function $\mathcal{H}\hat{v}$ defined on $\mathbb{R}_+$ by

$$
[\mathcal{H}\hat{v}](r) = \sup_{0\leq a\leq r} \hat{v}(0, r-a, a),
$$

by the arguments above we may rewrite the original value function (15) as (see also (18) and (30))

$$
V(r) = [\mathcal{H}\widehat{V}](r). \tag{31}
$$

The problems (30)-(31) are coupled since $\widehat{V}$ is defined in terms of V in (30), and, viceversa, V is expressed in terms of $\widehat{V}$ by (31).

Remark 3.3 *(i) We explain our particular choice for the drifts b_Y and b_J in* (20)*. First of all we clarify that what we need for our argument is a couple of processes $(\tilde{Y}, J)$ such that: (i) $\tilde{Y}\cdot J = A$ on $[0,\tau_1)$, where A is defined in* (6)*; (ii) $\tilde{Y}$ is $\mathbb{G}$-adapted; (iii) J is independent of $\mathbb{G}$.*

Therefore, it is natural to consider the processes driven by SDE's (21) *and* (23)*, with b_Y, b_J that can be chosen freely under the constraint $b_J + b_Y = b_I$. Define the constants*

$$
\begin{aligned}
k_{L,Y,p} := \sup_{u_L\in\mathbb{R}, u_Y\in[0,1]} \Big\{ & p(u_L b_L + u_Y b_Y) \\
& - \frac{p(1-p)}{2}(u_L^2\sigma_L^2 + u_Y^2\sigma_I^2(\rho^2 + \gamma^2(1-\rho^2)) + 2\rho u_L u_Y \sigma_L\sigma_I)\Big\},
\end{aligned} \tag{32}
$$

$$
k_{J,p} := \sup_{u_J\in[0,1]}\left\{p b_J u_J - \frac{p(1-p)}{2}\sigma_I^2(1-\rho^2)(1-\gamma^2)u_J^2\right\}. \tag{33}
$$

[4]Note that while we take the state variable $\tilde{X}, \tilde{Y}$ starting at t, the process J remains the same as in (21) and enters in G^γ making the auxiliary problem nonautonomous.

These constants naturally appear respectively in Lemma A.3 and Proposition A.2-(v). Combining these two results with (16)*, one gets an estimate on the growth of $\widehat{V}$ (precisely estimate* (37) *below) under the condition that $\beta > k_{L,Y,p} + k_{J,p}$. In Lemma A.4 in the Appendix it is proved that, for our choice of b_Y, b_J, we have*

$$k_{L,Y,p} + k_{J,p} \quad = \quad k_p, \tag{34}$$

which is the minimum possible value of $k_{L,Y,p} + k_{J,p}$. So, our choice of drifts allows then to treat the auxiliary problem without further restriction on β other than Assumption 2.5.

(ii) *We notice that the auxiliary problem* (30) *is not autonomous due to the dependence of $G^\gamma[V]$ on time. However, in the case of full observation ($\gamma = 1$), one has $J \equiv 1$, and $G^1[V](t,x,y) = V(x+y)$. In this case, consistently with [1], we get an autonomous problem*[5]

$$\sup_{(c,\pi)\in\mathcal{A}_0(x)} \mathbb{E}\left[\int_0^\infty e^{-(\beta+\lambda)s}\left(U(c_s) + \lambda V\left(\tilde{X}_s^{0,x,\pi,c} + \tilde{Y}_s^{0,y}\right)\right) ds\right].$$

Therefore, time inhomogeneity of our auxiliary problem is due to the lack of full information*.* *It is also worth to outline here that in* (30) *we take the discount $e^{-(\beta+\lambda)(s-t)}$ in place of the more natural $e^{-(\beta+\lambda)s}$, as it allows to get rid of the exponential terms in the HJB equation.*

(iii) *The value function $\widehat{V}$ is the analogue of the value function of [1, 24]. Indeed, in [1, 24], the initial time is not supposed to be a trading time (of course this can be done also by us without loss of generality), so the initial endowments x, y in the liquid and illiquid asset cannot be rebalanced optimally at $t = 0$ - in other terms, the analogue of the optimization* (31) *is not performed at $t = 0$ in [1, 24].*

3.1 HJB equation and viscosity characterization of $\widehat{V}$

In this section we characterize $\widehat{V}$ as unique viscosity solution of an associated Hamilton-Jacobi-Bellman (HJB) equation. We start stating some qualitative properties of $\widehat{V}$. The proof can be found in [7] in the case $\gamma = 0$. In the general case the proof proceeds exactly in the same way and we omit it for brevity.

Proposition 3.4 *$\widehat{V}(t,\cdot)$ is concave with repect to (x,y) and nondecreasing with respect to x and y for every $t \geq 0$. Moreover it satisfies the boundary condition*

$$\widehat{V}(t,0,y) \quad = \quad \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)}\lambda G^\gamma[V](s,0,\tilde{Y}_s^{t,y})ds\right], \quad \forall t \geq 0,\ \forall y \geq 0. \tag{35}$$

In particular, since by Assumption 2.3 it is $U(0) = 0$, due to Proposition A.2(v) and (16)*, we have*

$$\widehat{V}(t,0,0) \quad = \quad 0, \quad \forall t \geq 0. \tag{36}$$

[5] With respect to [1], our dynamic programming approach to the problem is different. Indeed, apart the fact that we deal with general utility functions (which is not relevant for that), our approach seems to be the only possible to deal with the issue of possible partial information ($\gamma < 1$). Hence, the differential problem we get is different from the one derived in [1], also when our control problem coincides with the control problem of [1] ($\gamma = 1$ and power utility). Nevertheless, as suggested by the intuition, our differential problem must be autonomous as well as the one of [1], when the two control problems coincide.

Finally, $\widehat{V}$ is continuous on $\mathbb{R}^3_+$ and satisfies, for some $K_{\widehat{V}} > 0$, the growth condition

$$0 \quad \leq \quad \widehat{V}(t,x,y) \quad \leq \quad K_{\widehat{V}} e^{k_{J,p}t}(x+y)^p, \qquad \forall (t,x,y) \in \mathbb{R}^3_+. \tag{37}$$

By standard arguments of stochastic control (see e.g. [25, Ch. 4]), we can associate to $\widehat{V}$ an HJB equation, which in this case reads as[6]

$$-\hat{v}_t + (\beta+\lambda)\hat{v} - \lambda G[\mathcal{H}\hat{v}] - \sup_{c\geq 0, \pi\in\mathbb{R}} H_{cv}(y, D_{(x,y)}\hat{v}, D^2_{(x,y)}\hat{v};\ c, \pi) \quad = \quad 0, \tag{38}$$

where for $(y,q,Q) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathcal{S}_2$ (where $\mathcal{S}_2$ denotes the space of symmetric 2×2 matrices), $c \geq 0, \pi \in \mathbb{R}$, the function H_{cv} is defined by

$$\begin{aligned} & H_{cv}(y,q,Q;c,\pi) \\ = \ & U(c) + (\pi b_L - c)q_1 + b_Y y q_2 + \frac{\sigma_L^2 \pi^2}{2} Q_{11} + \pi\rho\sigma_I\sigma_L y Q_{12} + (\rho^2 + \gamma^2(1-\rho^2))\frac{\sigma_I^2}{2} y^2 Q_{22}. \end{aligned}$$

Note that $\sup_{c\geq 0,\pi\in\mathbb{R}} H_{cv}(y,q,Q;c,\pi)$ is finite if $q_1 > 0$, $Q_{11} < 0$, in which case we have

$$\begin{aligned} & \sup_{c\geq 0, \pi\in\mathbb{R}} H_{cv}(y,q,Q;c,\pi) \\ & \qquad = \ \widetilde{U}(q_1) - \frac{(b_L q_1 + \rho\sigma_L\sigma_I y Q_{12})^2}{2\sigma_L^2 Q_{11}} + b_Y y q_2 + (\rho^2 + \gamma^2(1-\rho^2))\frac{\sigma_I^2}{2} y^2 Q_{22}. \end{aligned}$$

We are going to characterize $\widehat{V}$ as unique constrained viscosity solution to (38) according to the following definition.

Definition 3.5 **(1)** An upper-semicontinuous (resp. lower-semicontinuous) function v is a viscosity subsolution (resp. supersolution) to (38) at $(t_0,x_0,y_0) \in \mathbb{R}^3_+$ if

$$\begin{aligned} & -\varphi_t(t_0,x_0,y_0) + (\beta+\lambda)\varphi(t_0,x_0,y_0) - \lambda G[\mathcal{H}v](t_0,x_0,y_0) \\ & - \sup_{c\geq 0,\pi\in\mathbb{R}} H_{cv}(y_0, D_{(x,y)}\varphi(t_0,x_0,y_0), D^2_{(x,y)}\varphi(t_0,x_0,y_0);\ c,\pi) \quad \leq \quad 0 \quad (\text{resp. } \geq 0) \end{aligned}$$

for any function $\varphi \in C^{1,2}(\mathbb{R}^3_+;\mathbb{R})$ such that $\varphi(t_0,x_0,y_0) = v(t_0,x_0,y_0)$ and $\varphi \geq v$ (resp. $\leq$) on $[t_0, t_0+\varepsilon) \times \mathcal{O}$, for some neighboorhood $\mathcal{O} \subset \mathbb{R}^2_+$ of (x_0,y_0) and $\varepsilon > 0$.

(2) We say that a continuous function v is a *constrained viscosity solution* to (38) if it is a subsolution on $\mathbb{R}^3_+$, a supersolution on $\mathbb{R}_+ \times (0,+\infty) \times \mathbb{R}_+$ and satisfies the boundary condition

$$\hat{v}(t,0,y) \quad = \quad \mathbb{E}\Big[\int_t^\infty e^{-(\beta+\lambda)(s-t)} \lambda G^\gamma[\mathcal{H}\hat{v}](s,0,\tilde{Y}_s^{t,y})ds\Big], \quad \forall t \geq 0,\ \forall y \geq 0. \tag{39}$$

[6]Note that the HJB equation associated to the auxiliary problem would have as third term $-\lambda G[V]$ and then should be coupled with (31). This is the approached followed by [21, 22]. Here we follow a different approach by inserting directly (31), which yields a nonlocal term in the equation.

Remark 3.6 ***(i)*** *To simplify some proofs later on (in particular for regularity properties of the solution), we have chosen a definition of viscosity solution that slightly differs from the usual one. Indeed, in the standard definition of viscosity solution for parabolic PDEs (e.g. [4]), the test functions φ should be above v in a neighborhood of (t_0, x_0, y_0), while in our definition we are only concerned with their position for $t \geq t_0$. We notice that our definition is in principle more restrictive than the usual one, as we are enlarging the set of test functions. However, as soon as comparison is satisfied for the standard definition the two definitions are actually equivalent, see [13] for results in this direction.*

(ii) *The concept of constrained viscosity solution we use naturally comes from the stochastic control problem. The boundaries $\{x = 0,\ y \geq 0\}$ and $\{x \geq 0,\ y = 0\}$ are both absorbing for the control problem (in the sense that starting from these boundaries, the trajectories of the control problem remain therein), but they have different features. Indeed starting from the boundary $\{x \geq 0,\ y = 0\}$ the control problem degenerates in a one dimensional control problem; the associated HJB equation is nothing else but our HJB equation restricted to this boundary and this is why we require viscosity sub- and supersolution properties to the value function at this boundary. Instead starting from the boundary $\{x = 0,\ y \geq 0\}$ there is no control problem (since $\mathcal{A}_t(0) = \{(0,0)\}$) and the natural condition to impose is a Dirichlet boundary condition.*

Theorem 3.7 *$\widehat{V}$ is the unique constrained viscosity solution to* (38) *satisfying the growth condition* (37)*.*

Proof. The proof that $\widehat{V}$ is a viscosity subsolution on $\mathbb{R}^3_+$ and a viscosity supersolution on $\mathbb{R}_+ \times (0, +\infty)^2$ follows exactly the same arguments of, e.g., [25, Ch. 4], once we note that such arguments look at test functions only for $t \geq t_0$. The Dirichlet boundary condition (39) is verified due to (35) and (31). The growth condition (37) has been already proved. It remains to show that $\widehat{V}$ is a supersolution when $y = 0$. In this case, as noticed in Remark 3.6(ii), the control problem degenerates in a one-dimensional problem and again standard arguments apply to this control problem, giving the viscosity supersolution property.

Uniqueness is consequence of the comparison principle Proposition 3.8 below, whose proof can be found in the case of standard definition of viscosity solution in [7] when $\gamma = 0$. In the general case there is no special change to perform in the proof, so we omit it. $\square$

Proposition 3.8 *Let v_1 (resp. v_2) be a viscosity subsolution (resp. supersolution) to* (38) *on $\mathbb{R}_+ \times (0, +\infty) \times \mathbb{R}_+$. Assume that v_1, v_2 satisfy the growth condition* (37)*, and the boundary condition*

$$v_1(t, 0, y) \quad \leq \quad \mathbb{E}\Big[\int_t^\infty e^{-(\beta+\lambda)(s-t)} \lambda G[\mathcal{H} v_1](s, 0, Y_s^{t,y}) ds\Big] \tag{40}$$

(resp. $\geq$ for v_2). Then $v_1 \leq v_2$ on $\mathbb{R}^3_+$.

3.2 Optimal policies

In this section we show how to use the results obtained in the previous sections to construct optimal policies for the original problem. Unfortunately, since we do not know a priori if the value function $\widehat{V}$ is smooth, we cannot write the optimal feedback for the controls (c,π). However $\widehat{V}$ can be computed numerically as viscosity solution of (38) (see Section 4.6) and then an optimal allocation $(\alpha_k^*)_{k\geq 0}$ for the illiquid asset can be derived. Due to (31), at time $t=0$ an optimal allocation α_0^* in the illiquid asset must be such that

$$\alpha_0^* \in \operatorname{argmax}_{0\leq a\leq r} \widehat{V}(0, r-a, a). \tag{41}$$

We notice that we cannot exclude that $\alpha_0^*=0$ at this stage.

The expression of the optimal allocation in the illiquid asset (41) can be generalized to the random trading dates τ_k, for $k\geq 1$. At such trading dates, an optimal allocation must be such that

$$\alpha_k^* \in \operatorname{argmax}_{0\leq a\leq R_{\tau_k}} \widehat{V}(0, R_{\tau_k}-a, a). \tag{42}$$

This fact comes from the Markov property of our controlled system, whose proof we omit for brevity.

Assuming sufficient regularity[7] for $\widehat{V}$ we can prove a verification type result providing the optimal feedback for the controls (c,π).

Hereafter, by $C^{1,k}$ we will denote the class of functions which are once differentiable with respect to the time variable and k-times differentiable with respect to the space variables, with continuous derivatives.

Theorem 3.9 *Suppose that*

$$\begin{cases} (i) \quad \widehat{V}\in C^{1,2}(\mathbb{R}_+\times(0,+\infty)^2;\mathbb{R}), \\ (ii) \quad \widehat{V}(\cdot,\cdot,0)\in C^{1,2}(\mathbb{R}_+\times(0,+\infty);\mathbb{R}). \end{cases} \tag{43}$$

Suppose that there exist measurable feedback maps C^*,Π^* such that

$$C^*(t,x,y) \begin{cases} \in \mathit{argmax}_{c\geq 0}\left\{U(c)-c\widehat{V}_x(t,x,y)\right\}, & \mathit{if}\ x>0, \\ =0, \quad \mathit{if}\ x=0, \end{cases} \tag{44}$$

$$\Pi^*(t,x,y)$$
$$\begin{cases} \in \mathit{argmax}_{\pi\in\mathbb{R}}\left\{\pi b_L\widehat{V}_x(t,x,y)+\frac{\sigma_L^2\pi^2}{2}\widehat{V}_{xx}(t,x,y)+\pi\rho\sigma_I\sigma_L y\widehat{V}_{xy}(t,x,y)\right\}, & \mathit{if}\ x>0, \\ =0, \quad \mathit{if}\ x=0, \end{cases} \tag{45}$$

and that the closed loop equation

$$\begin{cases} d\tilde{X}_s = -C^*(s,\tilde{X}_s,\tilde{Y}_s^{t,y})ds+\Pi^*(s,\tilde{X}_s,\tilde{Y}_s^{t,y})(b_Lds+\sigma_LdW_s), \\ \tilde{X}_t = x, \end{cases} \tag{46}$$

admits a unique nonnegative solution $\tilde{X}^{*,t,x}$. Define the feedback strategies

$$c_s^* = C^*(s,\tilde{X}_s^{*,t,x},\tilde{Y}_s^{t,y}), \qquad \pi_s^* = \Pi^*(s,\tilde{X}_s^{*,t,x},\tilde{Y}_s^{t,y}). \tag{47}$$

Then $(c^*,\pi^*)\in\mathcal{A}_t(x)$ and is an optimal control for the auxiliary problem (30).

[7] Such regularity will be proved in the next section in the case of power utility.

Proof. First of all we prove that $(c^*, \pi^*) \in \mathcal{A}_t(x)$. Note that the fact that the SDE (46) has a well-defined solution implies that c^* and π^* satisfy the required integrability conditions, and moreover, by uniqueness of solutions of (46)

$$\tilde{X}^{t,x,c^*,\pi^*} \;=\; \tilde{X}^{*,t,x} \geq 0, \tag{48}$$

concluding the proof of admissibility.

To prove optimality we distinguish two cases: $y > 0$ and $y = 0$.

Case $y > 0$. In this case we have $Y_s^{t,y} > 0$ a.s. for every $s \geq t$. Therefore, due to the assumption of regularity (43)-(i), we may apply Dynkin's formula to

$$s \longmapsto e^{-(\beta+\lambda)(s-t)} \widehat{V}(s, \tilde{X}_s^{*,t,x}, \tilde{Y}_s^{t,y})$$

in the interval $[t, \tau \wedge T)$ for all $T > t$, where $\tau = \inf\{s \geq t \mid \tilde{X}_s^{*,t,x} = 0\}$. Since $\widehat{V}$ solves (38) in $\mathbb{R}_+ \times (0, +\infty)^2$ in classical sense, by definition of C^*, Π^* we get, arguing as in standard verification theorems,

$$\begin{aligned} &\widehat{V}(t,x,y) - \mathbb{E}\left[e^{-(\beta+\lambda)(\tau\wedge T)} \widehat{V}(\tau \wedge T, \tilde{X}^{*,t,x}_{\tau\wedge T}, \tilde{Y}^{t,y}_{\tau\wedge T})\right] \\ &\qquad = \mathbb{E}\left[\int_t^{\tau\wedge T} e^{-(\beta+\lambda)(s-t)} \left(U(c_s^*) + \lambda G^\gamma[V](s, \tilde{X}_s^{*,t,x}, \tilde{Y}_s^{t,y})\right) ds\right]. \end{aligned}$$

Splitting on the sets $A_T = \{\tau < T\}$ and A_T^c, we write

$$\begin{aligned} &\widehat{V}(t,x,y) - \mathbb{E}\left[\mathbf{1}_{A_T}\, e^{-(\beta+\lambda)\tau} \widehat{V}(\tau, 0, \tilde{Y}_\tau^{t,y}) + \mathbf{1}_{A_T^c}\, e^{-(\beta+\lambda)T} \widehat{V}(T, \tilde{X}_T^{*,t,x}, \tilde{Y}_T^{t,y})\right] \\ &\qquad = \mathbb{E}\Bigg[\mathbf{1}_{A_T} \int_t^\tau e^{-(\beta+\lambda)(s-t)} \left(U(c_s^*) + \lambda G^\gamma[V](s, \tilde{X}_s^{*,t,x}, \tilde{Y}_s^{t,y})\right) ds \\ &\qquad\qquad + \mathbf{1}_{A_T^c} \int_t^T e^{-(\beta+\lambda)(s-t)} \left(U(c_s^*) + \lambda G^\gamma[V](s, \tilde{X}_s^{*,t,x}, \tilde{Y}_s^{t,y})\right) ds\Bigg]. \end{aligned} \tag{49}$$

By (35) and using the Markov property of $\tilde{Y}$ and the fact that A_T is $\mathcal{G}_\tau$ measurable,

$$\begin{aligned} \mathbf{1}_{A_T} \widehat{V}(\tau, 0, \tilde{Y}_\tau^{t,y}) &= \mathbf{1}_{A_T}\, \mathbb{E}\left[\int_\tau^\infty e^{-(\beta+\lambda)(s-\tau)} \lambda G^\gamma[V](s, 0, \tilde{Y}_s^{\tau, \tilde{Y}_\tau^{t,y}}) ds \;\Big|\; \mathcal{G}_\tau\right] \\ &= \mathbb{E}\left[\mathbf{1}_{A_T} \int_\tau^\infty e^{-(\beta+\lambda)(s-\tau)} \lambda G^\gamma[V](s, 0, \tilde{Y}_s^{t,y}) ds \;\Big|\; \mathcal{G}_\tau\right]. \end{aligned} \tag{50}$$

Noting that $\tilde{X}^{*,t,x} \equiv 0$ from τ on and moving the term corresponding to (50) to the right hand side in (49), we get

$$\begin{aligned} &\widehat{V}(t,x,y) - \mathbb{E}\left[\mathbf{1}_{A_T^c}\, e^{-(\beta+\lambda)T} \widehat{V}(T, \tilde{X}_T^{*,t,x}, \tilde{Y}_T^{t,y})\right] \\ &\qquad = \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)} \left(U(c_s^*) + \lambda G^\gamma[V](s, \tilde{X}_s^{*,t,x}, \tilde{Y}_s^{t,y})\right) ds\right] \\ &\qquad\quad - \mathbb{E}\left[\mathbf{1}_{A_T^c} \int_T^{+\infty} e^{-(\beta+\lambda)(s-t)} \left(U(c_s^*) + \lambda G^\gamma[V](s, \tilde{X}_s^{*,t,x}, \tilde{Y}_s^{t,y})\right) ds\right]. \end{aligned} \tag{51}$$

Now we take $T \to \infty$ in (51). The second term of the left hand side converges to 0 by dominated convergence due to (11), (34), (37) and (98). On the other hand, since $\mathcal{J}(t,x;c^*,\pi^*) < \infty$, we have almost surely

$$\int_t^{+\infty} e^{-(\beta+\lambda)(s-t)} \left(U(c_s^*) + \lambda G^\gamma[V](s, \tilde{X}_s^{*,t,x}, \tilde{Y}_s^{t,y}) \right) ds \quad < \quad \infty.$$

Hence,

$$s \ \longmapsto\ \mathbf{1}_{[T,+\infty)}(s)\, e^{-(\beta+\lambda)(s-t)} \left(U(c_s^*) + \lambda G^\gamma[V](s, \tilde{X}_s^{*,t,x}, \tilde{Y}_s^{t,y}) \right)$$

is almost surely dominated by a function which is integrable on $[t,+\infty)$ and goes to 0 when $T \to \infty$. This shows, by dominated convergence that almost surely

$$\int_T^{+\infty} e^{-(\beta+\lambda)(s-t)} \left(U(c_s^*) + \lambda G^\gamma[V](s, \tilde{X}_s^{*,t,x}, \tilde{Y}_s^{t,y}) \right) ds \ \longrightarrow\ 0.$$

Then by dominated convergence on Ω we get that the last term of the right hand side of (51) goes to 0.

Case $y = 0$. In this case $\tilde{Y}_s^{t,y} \equiv 0$ a.s. for every $s \geq 0$. This means that the boundary $\{x \geq 0,\ y = 0\}$ is a trap for the control problem, which indeed lives in a one-dimensional space. The value function of this control problem is $\widehat{V}(\cdot,\cdot,0)$ and the HJB equation is still (38) restricted on the half-line $\{x \geq 0,\ y = 0\}$ (all the terms containing the derivatives with respect to y indeed disappear). Due to the regularity assumption (43)-(ii), a verification argument similar (indeed easier) to the one above yields the claim. □

In the next section we will be able to show that the assumptions of the theorem above are fulfilled in the case of power utility. We can put together (42) and Theorem 3.9 obtaining an optimal control (c^*,π^*,α^*) for the original problem. The result again follows from arguments exploiting the Markovian property of our controlled system, we only state it without giving a proof for sake of brevity.

Theorem 3.10 *Assume that the hypotheses of Theorem 3.9 hold. Then an optimal control* (α^*, c^*, π^*) *is given in closed loop form as*

$$\begin{cases} \alpha_k^* \ \in\ \mathit{argmax}_{0 \leq a \leq R_{\tau_k}} \widehat{V}(0, R_{\tau_k} - a, a), & k \geq 0, \\ c_s^* \ =\ C^*(s - \tau_k, \tilde{X}_s^{*,\tau_k,R_{\tau_k}-\alpha_k^*}, \tilde{Y}_s^{\tau_k,\alpha_k^*}), & s \in [\tau_k, \tau_{k+1}), \quad k \geq 0, \\ \pi_s^* \ =\ \Pi^*(s - \tau_k, \tilde{X}_s^{*,\tau_k,R_{\tau_k}-\alpha_k^*}, \tilde{Y}_s^{\tau_k,\alpha_k^*}), & s \in [\tau_k, \tau_{k+1}), \quad k \geq 0. \end{cases}$$

When U is strictly concave we have uniqueness of optimal controls.

Proposition 3.11 *Assume that the assumptions of Theorem 3.9 hold. If* U *is strictly concave, then* V *and* $\widehat{V}(t,\cdot,\cdot)$ *are strictly increasing (*$\widehat{V}$ *with respect to both* x, y*) and strictly concave, and we have uniqueness of optimal controls.*

Proof. The fact that U is strictly concave implies also that it is strictly increasing. This yields also strict monotonicity of the value function V (using a standard argument that can be found, e.g., in [26, Prop. 2.1] or [6, Prop. 4.7]).

Now, let $0 \leq r_1 < r_2$ and take two optimal controls (c^1, π^1, α^1), (c^2, π^2, α^2) for r_1, r_2 respectively, their existence being provided by Theorem 3.10. Since V is strictly increasing, it must be $c^1 \neq c^2$. Let $\lambda \in (0,1)$ and set $r_\lambda = \lambda r_1 + (1-\lambda) r_2$ and

$$(c^\lambda, \pi^\lambda, \alpha^\lambda) \;=\; \lambda(c^1, \pi^1, \alpha^1) + (1-\lambda)(c^2, \pi^2, \alpha^2).$$

Then by linearity of the state equation (3)-(4) we have $(c^\lambda, \pi^\lambda, \alpha^\lambda) \in \mathcal{A}(r_\lambda)$. On the other hand, since U is strictly concave and $c^1 \neq c^2$, we get $V(r_\lambda) > \lambda V(r_1) + (1-\lambda)V(r_2)$, which proves that V is strictly concave.

Strict concavity of V implies also strict concavity of $G^\gamma[V](t, \cdot, \cdot)$ for every $t \geq 0$. In the same way strict monotonicity of V implies strict monotonicity of $G^\gamma[V]$ with respect to x, y, which implies strict monotonicity of $\widehat{V}$ with respect to x, y. Then, arguing as before, one gets strict concavity of $\widehat{V}(t, \cdot, \cdot)$ too.

Let now $t, x, y \geq 0$ and suppose that (c^1, π^1), (c^2, π^2) are optimal controls for the auxiliary problem starting from (t, x, y). As a consequence of strict concavity of U, we get $c_1^* = c_2^* =: c^*$. As a consequence of strict concavity of $G^\gamma[V]$, we get $\tilde{X}^{t,x,c^*,\pi_1^*} = \tilde{X}^{t,x,c^*,\pi_2^*}$. From that, one easily derives $\pi_1^* = \pi_2^*$.

Now consider the original problem with initial value r and take two optimal controls (c^1, π^1, α^1), (c^2, π^2, α^2). Strict concavity of $\widehat{V}$ implies that $a \mapsto \widehat{V}(0, r-a, a)$ has a unique maximizer in $[0, r]$, so $\alpha_0^1 = \alpha_0^2 =: \alpha_0$. On the other hand the controls $(c^1, \pi^1)|_{[0,\tau_1)}$ and $(c^2, \pi^2)|_{[0,\tau_1)}$ are (locally) optimal for the auxiliary problem in $[0, \tau_1)$ with initial value $(x, y) = (r - \alpha_0, \alpha_0)$. By uniqueness of optimal controls for the auxiliary problem, it must be $(c^1, \pi^1) = (c^2, \pi^2)$ in $[0, \tau_1)$. By the Markov property, the same argument applies to generic intervals $[\tau_k, \tau_{k+1})$, so the proof is complete. □

4 Power utility

In this section we consider the problem when the utility function is

$$U(c) \;=\; \frac{c^p}{p}, \qquad p \in (0,1).$$

In this case the Legendre transform of U is the function

$$\tilde{U}(w) \;=\; \frac{1-p}{p}\, w^{-\frac{p}{1-p}}, \qquad w > 0.$$

We notice that in the case $p \leq 0$ the problem is investigated in [1], but supposing full observation of the illiquid asset, corresponding to the case $\gamma = 1$ in our setting.

4.1 Reduction to one space variable

Due to the fact that the utility function U is homogeneous, the function V is a power function and the function $\widehat{V}$ is homogeneous in the variables x, y. So (38) can be reduced to a PDE involving just a one-dimensional state variable. Indeed, since

$$\mathcal{A}(\xi r) \;=\; \xi\mathcal{A}(r), \quad \forall \xi \geq 0,\ \forall r \geq 0,$$

taking into account the fact that U is homogeneous of degree p, it is straightforward to show that for some $\Phi_0 > 0$

$$V(r) \;=\; \Phi_0 r^p, \quad \forall r \geq 0, \tag{52}$$

Moreover, since we have also

$$\mathcal{A}_t(\xi x) \;=\; \xi\mathcal{A}_t(x), \quad \forall \xi \geq 0,\ \forall x \geq 0,$$

taking into account (52) and (93), it is straightforward to show that

$$\widehat{V}(t, \xi x, \xi y) \;=\; \xi^p\ \widehat{V}(t, x, y), \quad \forall \xi > 0,\ \forall x, y \geq 0. \tag{53}$$

Hence, following e.g. [24], we perform the change of variables[8]

$$z \;=\; \frac{x}{y}, \qquad x \geq 0, \quad y > 0. \tag{54}$$

By this change of variable and (53), we can rewrite $\widehat{V}$ in separated form as

$$\widehat{V}(t, x, y) \;=\; y^p\ \Phi\left(t, z\right), \quad t \geq 0,\ z \geq 0,\ y > 0, \tag{55}$$

where

$$\Phi(t, z) \;=\; \widehat{V}(t, z, 1), \quad t \geq 0,\ z \geq 0. \tag{56}$$

Set, for $y > 0$,

$$\tilde{c} \;=\; \frac{c}{y}, \quad \tilde{\pi} \;=\; \frac{\pi}{y}. \tag{57}$$

Denote by $\mathcal{M}_p(\mathbb{R}^2_+, \mathbb{R})$ the space of measurable functions ψ such that $|\psi(t, z)| \leq C(1+|z|)^p$ and consider the nonlinear functional

$$\begin{aligned} \mathcal{H}_0 :\ \ \mathcal{M}_p(\mathbb{R}^2_+, \mathbb{R}) &\longrightarrow \mathbb{R}, \\ \psi \qquad &\longmapsto \mathcal{H}_0[\psi] := \sup_{z \geq 0} \frac{\psi(0, z)}{(1+z)^p}. \end{aligned}$$

Then one can check that $V(r) = \mathcal{H}[\widehat{V}](r) = \mathcal{H}_0[\Phi] r^p$, so $\Phi^0 = \mathcal{H}_0[\Phi]$ and setting

$$f^\gamma(t, z) \;=\; G^\gamma[\xi \mapsto \xi^p](t, z, 1)$$

[8]Here we restrict the analysis to the case, $y > 0$. In the case $y = 0$, the problem is already a one-dimensional Merton type problem for which one can deal with explicit solutions to the HJB equation (see Subsection 4.4). A necessary and sufficient condition ensuring that $y = 0$ is never optimal is given in Proposition 4.8 below.

and taking also into account (93) we get

$$G^\gamma[\mathcal{H}\widehat{V}](t,x,y) \;=\; y^p\, G^\gamma[\mathcal{H}\widehat{V}](t,z,1) \;=\; y^p\, f^\gamma(t,z)\mathcal{H}_0[\Phi]. \tag{58}$$

Plugging (53)--(58) (with the formal argument φ in place of Φ) into (38) and dividing by y^p, we get an equation for Φ in $\mathbb{R}_+ \times (0,+\infty)$:

$$-\varphi_t + (\beta+\lambda)\varphi - \lambda f^\gamma(t,z)\mathcal{H}_0[\varphi] - \sup_{\tilde{c}\geq 0,\, \tilde{\pi}\in\mathbb{R}} H^0_{cv}(z,\varphi,\varphi_z,\varphi_{zz}) \;=\; 0, \tag{59}$$

where

$$\begin{aligned} H^0_{cv}(z,\varphi,\varphi_z,\varphi_{zz};\tilde{c},\tilde{\pi}) &= (\rho^2+\gamma^2(1-\rho^2))\frac{\sigma_I^2}{2}\Big(p(p-1)\varphi+2(1-p)z\varphi_z+z^2\varphi_{zz}\Big)+b_Y(p\varphi-z\varphi_z) \\ &\quad + U(\tilde{c}) - \tilde{c}\varphi_z + \tilde{\pi}\left((b_L+\rho\sigma_I\sigma_L(p-1))\varphi_z - \rho\sigma_I\sigma_L z\varphi_{zz}\right) + \tilde{\pi}^2\frac{\sigma_L^2}{2}\varphi_{zz}, \end{aligned}$$

We notice that $\mathcal{H}_0[\varphi]$ is a nonlocal term in (59).

By monotonicity and concavity of $\widehat{V}(t,\cdot,\cdot)$ we deduce also monotonicity and concavity of $\Phi(t,\cdot)$. Hence, we have $\Phi_z \geq 0$, $\Phi_{zz} \leq 0$ in the sense of sub(super)differentials. Now we introduce a change of variables to simplify the computations, denoting

$$\tilde{\theta} \;=\; \tilde{\pi} - \frac{\rho\sigma_I}{\sigma_L}z,$$

so (59) can be rewritten after suitable simplifications as

$$-\varphi_t + K_\lambda\varphi - K_3 z\varphi_z - \lambda f^\gamma(t,z)\mathcal{H}_0[\varphi] - \frac{K_4^2}{2}z^2\varphi_{zz} - \sup_{\tilde{c}\geq 0,\, \tilde{\theta}\in\mathbb{R}} H^1_{cv}(\varphi_z,\varphi_{zz};\tilde{c},\tilde{\theta}) \;=\; 0, \tag{60}$$

where

$$H^1_{cv}(\varphi_z,\varphi_{zz};\tilde{c},\tilde{\theta}) \;=\; U(\tilde{c}) - \tilde{c}\varphi_z + \tilde{\theta}K_1\varphi_z + \frac{1}{2}\tilde{\theta}^2K_2^2\varphi_{zz}. \tag{61}$$

and

$$\begin{cases} K_\lambda \;=\; \beta+\lambda+\dfrac{\rho^2\sigma_I^2}{2}p(1-p) - p\rho\dfrac{b_L\sigma_I}{\sigma_L} - \gamma^2 p\left(b_I - \dfrac{\rho b_L\sigma_I}{\sigma_L}\right), \\ K_1 \;=\; b_L - \rho\sigma_I\sigma_L(1-p), \\ K_2 \;=\; \sigma_L, \\ K_3 \;=\; \gamma^2\left(-b_I + \dfrac{\rho b_L\sigma_I}{\sigma_L} + (1-\rho^2)(1-p)\sigma_I^2\right), \\ K_4 \;=\; -\sigma_I\gamma\sqrt{1-\rho^2}. \end{cases} \tag{62}$$

Since $\Phi_z \geq 0$ and $\Phi_{zz} \leq 0$ in the sense of sub(super)differentials, we can rewrite (60) as

$$-\varphi_t + K_\lambda\varphi - K_3 z\varphi_z - \lambda f^\gamma(t,z)\mathcal{H}_0[\varphi] - \widetilde{U}(\varphi_z) + \frac{1}{2}\frac{K_1^2}{K_2^2}\frac{\varphi_z^2}{\varphi_{zz}} - \frac{K_4^2}{2}z^2\varphi_{zz} \;=\; 0, \tag{63}$$

with the convention

$$\frac{h}{0} \;=\; \begin{cases} 0, & \text{if } |h| = 0, \\ +\infty, & \text{if } |h| > 0. \end{cases}$$

By (35) we get the boundary condition

$$\Phi(t,0) \;=\; \mathcal{H}_0[\Phi] \int_t^\infty e^{-K_\lambda(s-t)}\lambda f^\gamma(s,0)ds \;=\; \Phi^0 \int_t^\infty e^{-K_\lambda(s-t)}\lambda f^\gamma(s,0)ds, \tag{64}$$

and by (37) we get the growth condition

$$\Phi(t,z) \;\leq\; K_{\widehat{V}} e^{k_{J,p}t}(1+z)^p. \tag{65}$$

Due to the results of Section 3 and to the argument above, we get the following (according to the definition of viscosity solution of [4]):

Proposition 4.1 *The function Φ is continuous on $\mathbb{R}^2_+$, strictly increasing and strictly concave in z, and is the unique viscosity solution over $\mathbb{R}_+ \times (0,+\infty)$ of* (60) *fulfilling the boundary condition* (64) *and the growth condition* (65).

4.2 Smoothness of the value function

In this subsection we show that the value function Φ is smooth. We can freeze the nonlocal term in (59) and, by using Proposition 4.1 and standard comparison results for viscosity solutions of second order parabolic PDE's (see, e.g., [4]), we get the following:

Proposition 4.2 Φ *is the unique viscosity solution over $\mathbb{R}_+ \times (0,+\infty)$ of*

$$-\varphi_t + K_\lambda\varphi - K_3 z\varphi_z - \lambda f^\gamma(t,z)\Phi^0 - \frac{K_4^2}{2}z^2\varphi_{zz} - \sup_{\tilde{c}\geq 0,\,\tilde{\theta}\in\mathbb{R}} H^1_{cv}(z,\varphi,\varphi_z,\varphi_{zz}) \;=\; 0, \tag{66}$$

fulfilling the boundary condition (64) *and the growth condition* (65).

The nature of the HJB equation (66) is sensitive to the value of γ. More precisely one has to distinguish, from the point of view of PDE's theory, the cases $\gamma = 0$ and $\gamma \neq 0$. Indeed, in the case $\gamma \neq 0$ we have $K_4 > 0$ and the PDE is a fully nonlinear *nondegenerate* parabolic equation, while in the case $\gamma = 0$ we have $K_4 = 0$ and the PDE is *degenerate*. While a good regularity theory of solutions is available in the nondegenerate case, in the degenerate case only the theory of viscosity solution applies in general. In the latter case, the possibility of getting good regularity results (like classical solutions) strongly relies in the specific structure of the equation. In both cases we can prove that the solution is smooth enough to apply Theorem 3.9. Due to the considerations above, the cases $\gamma = 0$ and $\gamma \neq 0$ must be treated in a different way: the degenerate case $\gamma = 0$ requires a passage to a dual problem, which is explained in detail in [8]; on the other hand in the nondegenerate case $\gamma \neq 0$ the known regularity theory can be used, but needs some technical nonstandard results to localize the equation and restrict the set of controls to a compact one. These results, on which relies the proof of the following regularity result, are provided in the Appendix.

Theorem 4.3 $\Phi \in C^{1,3}(\mathbb{R}_+ \times (0,+\infty);\mathbb{R})$ *and* $\Phi_z \in C^{1,2}(\mathbb{R}_+ \times (0,+\infty);\mathbb{R})$. *Moreover* $\Phi_z > 0$, $\Phi_{zz} < 0$ *in* $\mathbb{R}_+ \times (0,+\infty)$.

Proof. As we said, we prove the claim in the case $\gamma \neq 0$, referring to [8] for the case $\gamma = 0$. We note that in the latter case we get more regularity: indeed $\Phi \in C^{1,\infty}(\mathbb{R}_+ \times (0,+\infty);\mathbb{R})$ and all the space derivatives lie in the same space.

Given $(\bar{t},\bar{z}) \in \mathbb{R}_+ \times (0,+\infty)$ and $\varepsilon \in (0,\bar{z})$, consider the set $D^\varepsilon(\bar{t},\bar{z})$ defined in (101) and let $\mathcal{P}(D_\varepsilon(\bar{t},\bar{z}))$ be the parabolic boundary of $D_\varepsilon(\bar{t},\bar{z})$ defined as

$$\mathcal{P}(D_\varepsilon(\bar{t},\bar{z})) \ := \ \{\bar{t}+\varepsilon\} \times [\bar{z}-\varepsilon, \bar{z}+\varepsilon] \cup [\bar{t}-\varepsilon, \bar{t}+\varepsilon] \times \{\bar{z}-\varepsilon, \bar{z}+\varepsilon\}.$$

Then, by Proposition A.6 and [10, Ch. V, Cor. 8]), Φ is the unique continuous viscosity solution on $\mathcal{P}(D_\varepsilon(\bar{t},\bar{z}))$ of the HJB equation (112) - which is the same as (66) but with constraints on the set of the variables $\tilde{c}, \tilde{\theta}$ - with Dirichlet continuous boundary condition

$$\varphi \ = \ \Phi, \qquad \text{on} \quad \mathcal{P}(D_\varepsilon(\bar{t},\bar{z})), \tag{67}$$

On the other hand, by [17, Th. 3, Sec. 6.4, p. 301] (see also Example 8, Section 6.1, p. 279, of the same book) there exists a solution $C^{1,2}(D_\varepsilon(\bar{t},\bar{z});\mathbb{R})$ of (112) with boundary condition (67). Since such solution must be also a viscosity solution, we conclude that

$$\Phi \ \in \ C^{1,2}(D_\varepsilon(\bar{t},\bar{z});\mathbb{R}). \tag{68}$$

Moreover, by Lemma A.5 we also have for $(t,z) \in D_\varepsilon(\bar{t},\bar{z})$

$$\Phi_z(t,z) \ \geq \ m_\varepsilon \ > \ 0, \qquad \Phi_{zz}(t,z) \ \leq \ -\delta_\varepsilon \ < \ 0. \tag{69}$$

Due to (68)-(69) and Proposition 4.1, and by arbitrariness of $(\bar{t},\bar{z}) \in \mathbb{R}_+ \times (0,+\infty)$, we get that $\Phi \in C^{1,2}(\mathbb{R}_+ \times (0,+\infty);\mathbb{R})$ is a classical solution of (63) and $\Phi_z > 0$, $\Phi_{zz} < 0$ in $\mathbb{R}_+ \times (0,+\infty)$.

Now we apply Lemma A.7 which allows us to formally differentiate (63) and conclude that Φ_z is a viscosity solution to[9]

$$\begin{aligned} -g_t + \Big(K_\lambda + K_3 + \frac{K_1^2}{K_2^2}\Big)g + (K_3 - K_4^2)zg_z + \tilde{U}'(g)g_z \\ -\Big(\frac{K_4^2}{2}z^2 + \frac{K_1^2}{2K_2^2}\frac{g^2}{g_z^2}\Big)g_{zz} + \lambda\Phi_0 f_z \ &= \ 0, \end{aligned} \tag{70}$$

with Dirichlet continuous boundary condition

$$g \ = \ \Phi_z, \qquad \text{on} \quad \mathcal{P}(D_\varepsilon(\bar{t},\bar{z})). \tag{71}$$

Again by [10, Ch. V, Cor. 8.1], the function Φ_z is the unique viscosity solution to this problem. On the other hand equation (70) is a quasilinear uniformly parabolic equation, so it admits a solution of class $C^{1,2}(D_\varepsilon(\bar{t},\bar{z});\mathbb{R})$ (see, e.g., [18, Th. 12.22] - with the assumptions of Th. 12.16 of the same book). As above, by uniqueness we deduce that $\Phi_z \in C^{1,2}(D_\varepsilon(\bar{t},\bar{z});\mathbb{R})$, hence in particular that $\Phi \in C^{1,3}(D_\varepsilon(\bar{t},\bar{z});\mathbb{R})$. By arbitrariness of $(\bar{t},\bar{z}) \in \mathbb{R}_+ \times (0,+\infty)$, we get the final claim. $\square$

[9] Although $\tilde{U}'(g)g_z$ and g^2/g_z^2 are not well-defined for $g = 0$ or $g_z = 0$, we may use (69) to replace these terms by bounded continuous functions of (g, g_z) coinciding with them whenever $m_\varepsilon \leq g \leq M_\varepsilon$, $g_z \leq -\delta_\varepsilon$.

4.3 An auxiliary closed loop equation

In this subsection we study a closed loop equation associated to the feedback maps provided by the maximization in the HJB equation (60) (for $z > 0$), and by the fact that $\mathcal{A}_t(0) = \{(0,0)\}$ (for $z = 0$):

$$\widetilde{C}^*(s,z) \;=\; \begin{cases} (U')^{-1}\left(\Phi_z(s,z)\right), & \text{if } z > 0, \\ 0, & \text{if } z = 0, \end{cases} \qquad \widetilde{\Theta}^*(s,z) \;=\; \begin{cases} -\frac{K_1 \Phi_z(s,z)}{K_2 \Phi_{zz}(s,z)}, & \text{if } z > 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Of course these maps are measurable in the couple (s,z). Moreover, due to Theorem 4.3, the maps $\widetilde{C}^*(s,\cdot)$, $\widetilde{\Theta}^*(s,\cdot)$ are locally Lipschitz continuous in $(0,+\infty)$, uniformly in $s \in [0,T]$ for all $T > 0$. We associate to these maps the closed loop equation

$$\begin{cases} dZ_s \;=\; -\widetilde{C}^*(s,Z_s)ds + \widetilde{\Theta}^*(s,Z_s)\left(\widehat{K}_1 ds + K_2 dW_s\right) + Z_s(\widehat{K}_3 ds + K_4 dB_s^{(1)}) \\ Z_t \;=\; z, \end{cases} \tag{72}$$

where $\widehat{K}_1 = b_L - \rho\sigma_I\sigma_L$ and $\widehat{K}_3 = \gamma^2\left(-b_I + \frac{\rho b_L \sigma_I}{\sigma_L} + (1-\rho^2)\sigma_I^2\right)$. We note that $\widehat{K}_1$ and $\widehat{K_3}$ are not the constants appearing in the HJB equation (60). We comment on that in Remark 4.5 below.

Proposition 4.4 *Given $(t,z) \in \mathbb{R}^2_+$, there exists a unique solution $Z^{t,z,*} \geq 0$ to* (72).

Proof. *Existence.* If $z = 0$ the claim is clear, just by taking $Z^{t,z,*} \equiv 0$. Let $z > 0$ and $T > 0$. Due to local Lipschitz continuity properties of $\widetilde{C}^*(s,\cdot), \widetilde{\Theta}^*(s,\cdot)$, using standard SDE's theory (see, e.g., [16, Ch. 5, Th. 2.9]), we get for each $\varepsilon \in (0,z)$ the existence of a unique solution $Z^{t,z,\varepsilon} \in [\varepsilon, \varepsilon^{-1}]$ in the stochastic interval $[t, \tau_\varepsilon^T)$, where τ_ε^T is implictely defined in terms of the solution itself as

$$\tau_\varepsilon^T \;=\; \inf\,\{s \in [t,T] \mid Z_s^{t,z,\varepsilon} \leq \varepsilon \text{ or } Z_s^{t,z,\varepsilon} \geq \varepsilon^{-1}\},$$

with the convention $\inf \emptyset = T$. Of course, if $\varepsilon < \varepsilon'$, we have $\tau_\varepsilon^T > \tau_{\varepsilon'}^T$ and

$$Z_s^{t,z,\varepsilon} \;\equiv\; Z_s^{t,z,\varepsilon'} \;\text{ on }\; [t,\tau_{\varepsilon'}^T), \qquad \forall\, 0 < \varepsilon < \varepsilon'. \tag{73}$$

Set

$$\tau^T \;=\; \lim_{\varepsilon \downarrow 0} \tau_\varepsilon^T.$$

Then by (73) there exists a unique solution $Z^{t,z,*} \geq 0$ in the interval $[t,\tau^T)$. We now show that this solution can be extended to the whole interval $[t,T]$, which, due to the arbitrariness of T, will imply the claim. By a Girsanov transformation, there exists a probability $\mathbb{Q}^T$ locally equivalent to $\mathbb{P}$, and $\mathbb{Q}^T$-Brownian motions $\tilde{W}$, $\tilde{B}^{(1)}$ such that (72) may be rewritten as

$$dZ_s \;=\; -\widetilde{C}^*(s,Z_s)ds + K_2\widetilde{\Theta}^*(s,Z_s)d\tilde{W}_s + K_4 Z_s d\tilde{B}_s^{(1)}.$$

By nonnegativity of $\widetilde{C}^*$ and $Z^{t,z,*}$, the process $Z^{t,z,*}$ is a nonnegative $\mathbb{Q}^T$-supermartingale on $[t,\tau^T)$. It can be extended to a $\mathbb{Q}^T$-supermartingale (L^1 bounded) on $[t,T]$ by setting

$Z^{t,z,*} \equiv 0$ in $[\tau^T, T]$. Hence, by Doob's convergence Theorem (the usual proof for deterministic intervals - see e.g. Theorem 6.18 in [15] - can be easily adapted to our stochastic interval $[t, \tau^T)$), there exists a finite random variable $Z^{t,z,*}_{\tau^T}$ such that

$$\lim_{s \nearrow \tau^T} Z^{t,z,*}_s \; = \; Z^{t,z,*}_{\tau^T}, \qquad \mathbb{Q}^T\text{-a.s..} \tag{74}$$

Since $\mathbb{Q}^T \sim \mathbb{P}$, we also have

$$\lim_{s \nearrow \tau^T} Z^{t,z,*}_s \; = \; Z^{t,z,*}_{\tau^T}, \qquad \mathbb{P}\text{-a.s..} \tag{75}$$

Immediately (75) yields the desired extension on $\{\tau^T = T\}$. Let us now consider the set $\{\tau^T < T\}$. On this set we have $Z^{t,z,*}_{\tau^T_\varepsilon} \in \{\varepsilon, \varepsilon^{-1}\}$, so that by (75) necessarily $Z^{t,z,*}_{\tau^T} = 0$ almost surely therein, getting

$$\lim_{s \nearrow \tau} Z^{t,z,*}_s \; = \; 0 \quad \text{ a.s. on } \; \{\tau^T < T\}. \tag{76}$$

Therefore, we may now extend $Z^{t,z,*}$ to a solution defined over $[t, T]$ on $\{\tau^T < T\}$ by setting

$$Z^{t,z,*}_s \; \equiv \; 0, \quad \text{for } s \in [\tau^T, T].$$

Uniqueness. The solution is clearly unique on the stochastic interval $[t, \tau^T)$ defined in the existence part. On the set $\{\tau^T < T\}$, when it reaches 0, it must stay there, since it is a nonnegative $\mathbb{Q}^T$-supermartingale. Therefore we have uniqueness on $[t, T]$ and, by arbitrariness of T, on $[t, +\infty)$. □

Remark 4.5 *The constants $\widehat{K}_1$ and $\widehat{K}_3$ that we take in (72) are different from the constants K_1 and K_3 in the HJB equation (60). These are indeed the right constants that allow to come back to the auxiliary control problem in the couple $(\tilde{X}, \tilde{Y})$ of Section 3, as we will show in the next subsection.*

To explain better this fact, we note the following:

(1) What we did in the previous subsection was to reduce the solution to the HJB in two variables to the solution of the HJB in one variable, using homogeneity of U and simplifying the variable y.

(2) It is possible to do this reduction directly on the control problem. To do this one has to define the new state process Z as the ratio of the old state variables, find an equation for it and rewrite the utility functional in terms of it. Concerning the equation for Z, let $x \geq 0$, $y > 0$, let $(c, \pi) \in \mathcal{A}_0(x)$ and set $\tilde{X} = \tilde{X}^{0,x,c,\pi}$, $\tilde{Y} = \tilde{Y}^{0,y}$. Denoting $Z_s = \frac{\tilde{X}_s}{\tilde{Y}_s}$, one may check that Z solves the SDE

$$dZ_s \;\; = \;\; -\tilde{c}_s ds + \tilde{\theta}_s(\widehat{K}_1 ds + K_2 dW_s) + Z_s(\widehat{K}_3 ds + K_4 dB^{(1)}_s), \tag{77}$$

where $\tilde{c}_s = \frac{c_s}{\tilde{Y}_s}$ and $\tilde{\theta}_s = \frac{\pi_s}{\tilde{Y}_s} - \frac{\rho\sigma_I}{\sigma_L} Z_s$, so with the same constants as the ones appearing in (72). Then, by homogeneity of U one may rewrite the objective functional of the auxiliary control problem of Section 3 as

$$\mathbb{E}\left[\int_0^\infty e^{-(\beta+\lambda)s} (\tilde{Y}_s)^p \left(U(\tilde{c}_s) + \lambda G^\gamma[V](s, Z_s, 1)\right) ds\right]. \tag{78}$$

Similarly to what we did for the HJB equation, we now need to "simplify" the term Y^p. The natural way to do that in this context is to change probability. Indeed consider the probability $\widehat{\mathbb{P}}$ corresponding to the density process $\frac{(\tilde{Y}_s)^p}{\mathbb{E}[(\tilde{Y}_s)^p]}$. Under this probability the processes

$$\widehat{W}_s \ := \ W_s - p\rho\sigma_I s, \qquad \widehat{B}^{(1)}_s := B^{(1)}_s - \gamma\sqrt{1-\rho^2}s$$

are standard Brownian motions, (77) *writes equivalently as*

$$dZ_s \ = \ -\tilde{c}_s ds + \tilde{\theta}_s(K_1 ds + K_2 d\widehat{W}_s) + Z_s(K_3 ds + K_4 d\widehat{B}^{(1)}_s). \tag{79}$$

and (78) *can be rewritten as*

$$y^p \ \mathbb{E}^{\widehat{\mathbb{P}}}\left[\int_0^\infty e^{-K_\lambda s}\left(U(\tilde{c}_s) + \lambda f(s, Z_s) ds\right)\right]. \tag{80}$$

As expected, the HJB equation associated to the control problem with state equation (79) *and objective functional* (80) *is exactly* (60).

4.4 Back to the original problem

As a consequence of Theorem 4.3 and Proposition 4.4 we can apply Theorem 3.9. Indeed we have the following:

Proposition 4.6 *If $U(c) = c^p/p$, all the assumptions of Theorem 3.9 are satisfied.*

Proof. (43)(i) follows from the smoothness of Φ (Theorem 4.3) and (55). Concerning (43)(ii), we observe that, when $y = 0$, we get from (30), (52) and (24)

$$\widehat{V}(t,x,0) \ = \ \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)}\left(\frac{c_s^p}{p} + \lambda\Phi^0\cdot[\tilde{X}^{t,x,c,\pi}_s]^p\right) ds\right]. \tag{81}$$

Since this is a standard homogeneous Merton type problem we get $\widehat{V}(t,x,0) = K_0 x^p$ for some $K_0 > 0$, which proves in particular (43)-(ii).

Let us show now that also the assumptions regarding the existence of the feedback maps and the solvability of the closed loop equation in Theorem 4.3 are satisfied in this case. Set

$$\widetilde{\Pi}^* = \ \widetilde{\Theta}^*\Big(s, \frac{x}{y}\Big) + \frac{\rho\sigma_I}{\sigma_L} z,$$

and, for any $x \geq 0$, $y > 0$,

$$C^*(s,x,y) \ = \ y\ \widetilde{C}^*\Big(s, \frac{x}{y}\Big), \qquad \Pi^*(s,x,y) \ = \ y\ \widetilde{\Pi}^*\Big(s, \frac{x}{y}\Big).$$

The maps C^*, Π^* satisfy the assumptions (44)-(45) of Theorem 3.9. Moreover, one can check by integration by parts that Z is a solution to the closed-loop equation (72) if and only if $X := Z\,\tilde{Y}^{t,y}$ is a solution to the SDE (46). So, also the assumption of Theorem 3.9 about existence and uniqueness of nonnegative solution to the closed loop equation is satisfied due to Proposition 4.4. The case $y = 0$ can be treated separately, and more easily, working on the degenerate Merton type control problem (81). $\square$

4.5 Two properties of the optimal allocation in the illiquid asset

Here we prove two important properties of the optimal allocation strategy in the illiquid asset. The first one concerns the fact that the ratio $\frac{X^*_{\tau_k}}{Y^*_{\tau_k}}$ is constant.

Proposition 4.7 *If $U(c) = c^p/p$, the optimal rebalancing proportion $\frac{X^*_{\tau_k}}{Y^*_{\tau_k}}$ at the trading times of the illiquid asset does not depend on the current value of the wealth. Indeed*

$$\frac{X^*_{\tau_k}}{Y^*_{\tau_k}} \quad = \quad z^* \quad := \quad argmax_{z \geq 0} \, \frac{\Phi(0,z)}{(1+z)^p},$$

where the value z^* above is well defined under the convention that $z^* = +\infty$ if the supremum of $\frac{\Phi(0,z)}{(1+z)^p}$ is not attained, in which case there is no investment in the illiquid asset.

Proof. First of all, we observe that, by strict convexity of Φ the function $a \mapsto a^p \Phi(0, \frac{r}{a} - 1)$ has a unique maximum point on $[0, r]$, which we call $a^*(r)$. Then, due to the homogeneity of $\widehat{V}$, we have

$$\frac{X^*_{\tau_k}}{Y^*_{\tau_k}} \quad = \quad \frac{R_{\tau_k} - a^*(R_{\tau_k})}{a^*(R_{\tau_k})}. \tag{82}$$

Now we notice that by definition of a^*

$$\frac{R_{\tau_k}}{a^*(R_{\tau_k})} - 1 \; = \; \mathrm{argmax}_{z \geq 0} \, \frac{\Phi(0,z)}{(1+z)^p},$$

which yields the claim. □

The second property is concerning with the possibility of having nonzero optimal investment in the liquid and illiquid asset.

Proposition 4.8

1. *$\alpha^*_0 < r$ (if and only if $r > 0$).*
2. *$\alpha^*_0 > 0$ if and only if $\frac{b_I}{\sigma_I} > \frac{\rho b_L}{\sigma_L}$.*[10]

Proof. 1. Of course if $r = 0$ then one needs to have $\alpha^*_0 = 0$ due to the state constraint. Instead assume that $r > 0$ and assume, by contradiction, that $\alpha^*_0 = r$. Then this would yield $z^* = 0$, hence, due to Proposition 4.7 above, $\alpha^*_k = R_{\tau_k}$ for all $k \in \mathbb{N}$. We should conclude that the liquid asset would be always 0, implying by the state constraint $c^*_t \equiv 0$, which cannot be optimal as $V(r) > 0$.

2. *Necessity.* Consider the Merton problem described in Remark 2.6, where the agent invests continuously in L and I, with the constraint that the proportion invested in I be in $[0, 1]$, and with value function denoted by $V^{M,2}$. The optimal investment proportions in L and I for this problem are given by :

$$(u^*_L, u^*_I) \quad = \quad \underset{u_L \in \mathbb{R},\, u_I \in [0,1]}{\mathrm{argmax}} \left\{ p(u_L b_L + u_I b_I) - \frac{p(1-p)}{2} (u_L^2 \sigma_L^2 + u_I^2 \sigma_I^2 + 2\rho u_L u_I \sigma_L \sigma_I) \right\}$$

[10] This condition is the same as the one in the Merton (liquid) problem with two assets (the same result is obtained in [1] in the case of full observation).

Taking first the supremum on u_L, one can see that $u_I^* = 0$ if and only if $b_I \leq \frac{\rho b_L \sigma_I}{\sigma_L}$. In this case, denoting by $V^{M,1}$ the value function for an agent investing only in L, we have $V^{M,2} = V^{M,1}$. Since obviously $V^{M,1} \leq V \leq V^{M,2}$, we obtain $V = V^{M,1}$, and the optimal strategy for our original problem never invests in the illiquid asset I.

Sufficiency. Assume $\frac{b_I}{\sigma_I} > \frac{\rho b_L}{\sigma_L}$. By the homogeneity property due to the power utility, it is enough to show that $h'(0^+) > 0$, where $h(a) = \widehat{V}(0, 1-a, a)$. Noting that

$$\begin{aligned}
h'(0^+) &= \lim_{\eta \to 0} \frac{\widehat{V}(0, 1-\eta, \eta) - \widehat{V}(0,1,0)}{\eta} \\
&= \lim_{\eta \to 0} \frac{(1-\eta)^p}{\eta} \Big(\widehat{V}(0, 1, \frac{\eta}{1-\eta}) - \widehat{V}(0, 1 + \frac{\eta}{1-\eta}, 0) \Big) \\
&= \Big(\lim_{\eta \to 0} (1-\eta)^{p-1} \Big) \, \Big(\lim_{\delta \to 0} \frac{1}{\delta} \big(\widehat{V}(0,1,\delta) - \widehat{V}(0, 1+\delta, 0) \big) \Big) \\
&= \widehat{V}_y(0, 1, 0^+) - \widehat{V}_x(0, 1^+, 0),
\end{aligned}$$

we will show that the latter is strictly positive.

Consider the auxiliary problem with initial data $(t, x, y) = (0, 1, 0)$. In this case the problem is the Merton type problem (81) with homogeneous value function $\widehat{V}(0, x, 0) = K_0 x^p$, so $\widehat{V}_x(0, 1^+, 0) = pK_0$. By solving the HJB equation for this problem, one can see that K_0 is the unique positive solution to

$$\Big(\beta + \lambda - \frac{p b_L^2}{2(1-p)\sigma_L^2} \Big) K_0 - (1-p) p^{-\frac{1}{1-p}} K_0^{-\frac{p}{1-p}} = \lambda \Phi^0, \tag{83}$$

and the corresponding optimal wealth process $\tilde{X}^*$ is given by

$$d\tilde{X}_t^* = -c_t^* dt + \pi_t^* \frac{dL_t}{L_t}, \tag{84}$$

where

$$c_t^* = p^{-\frac{1}{1-p}} K_0^{-\frac{p}{1-p}} \, \tilde{X}_t^*, \qquad \pi_t^* = \frac{b_L}{\sigma_L^2 (1-p)} \, \tilde{X}_t^*. \tag{85}$$

Considering an agent with initial wealth $(1, \delta)$ following the same investment/consumption strategy we get

$$\widehat{V}(0, 1, \delta) \geq \mathbb{E}\Big[\int_0^\infty e^{-(\beta+\lambda)t} \big(U(c_t^*) + \lambda G[V](t, \tilde{X}_t^*, \tilde{Y}_t^{0,\delta}) \big) dt \Big].$$

Therefore

$$\begin{aligned}
\frac{\widehat{V}(0,1,\delta) - \widehat{V}(0,1,0)}{\delta} &\geq \frac{\lambda}{\delta} \, \mathbb{E} \left[\int_0^\infty e^{-(\beta+\lambda)t} \big(G[V](t, \tilde{X}_t^*, \tilde{Y}_t^{0,\delta}) - G[V](t, \tilde{X}_t^*, 0) \big) dt \right] \\
&= \lambda \Phi^0 \int_0^\infty e^{-(\beta+\lambda)t} \, \mathbb{E} \left[\frac{(\tilde{X}_t^* + \tilde{Y}_t^{0,\delta} J_t)^p - (\tilde{X}_t^*)^p}{\delta} \right] dt.
\end{aligned}$$

Letting $\delta \to 0$, applying Fatou's lemma, and observing that $\tilde{Y}^{0,\delta} J_t = \delta I_t$, from the inequality above we get

$$
\begin{aligned}
\widehat{V}_y(0,x,0^+) &\geq p\Phi^0\lambda \int_0^\infty e^{-(\beta+\lambda)t}\mathbb{E}\left[(\tilde{X}_t^*)^{p-1}I_t\right]dt \\
&= p\Phi^0\lambda \int_0^\infty \exp\Big(-\big(\lambda\frac{\Phi^0}{K_0} - (b_I - \frac{\rho b_L \sigma_I}{\sigma_L})\big)t\Big)dt \\
&> pK_0 \quad = \quad \widehat{V}_x(0,1^+,0),
\end{aligned}
$$

where in the equality in the middle we have used (1), (2), (83), (84) and (85), and in the strict inequality we have used $\frac{b_I}{\sigma_I} > \frac{\rho b_L}{\sigma_L}$. The proof is complete. $\square$

The result above says in particular that, when the two assets are uncorrelated, there is investment in the illiquid asset even if the Sharpe ratio of the liquid asset is higher of the one of the illiquid asset, i.e. there is still diversification in the allocation of the portfolio between the two assets.

4.6 Numerical approximations

In this subsection we present an iterative scheme to approximate Φ^0 and Φ. This procedure is illustrated more extensively in the case $\gamma = 0$ in [7, 11], where the value functions V and $\widehat{V}$ are approximated. Here we describe such procedure for $\gamma \in [0,1]$ with regard to Φ^0 and Φ, i.e. for the power utility case, as it will be used in the next section to produce numerical results and comments in this special - but relevant - case.

Let us look at (59). Because of the nonlocal term $\mathcal{H}_0[\varphi]$, we cannot approximate directly the value function Φ as viscosity solution of a PDE, but we need to define an iterative scheme. Fix $T > 0$. Starting with

$$\Phi^{0,0,T} \quad = \quad 0. \tag{86}$$

we define inductively the sequence $(\Phi^{0,n,T}, \Phi^{n,T})$ as follows:

- Given $n \geq 0$ and $\Phi^{0,n,T}$, we define $\Phi^{n,T}$ on $\mathbb{R}^2_+$ as the unique (constrained viscosity) solution on $[0,T] \times \mathbb{R}_+$ to

$$-\Phi_t^{n,T} + (\beta+\lambda)\Phi^{n,T} - \lambda\Phi^{0,n,T} f^\gamma(t,z) - \sup_{\tilde{c}\geq 0, \tilde{\pi}\in\mathbb{R}} \tilde{H}_{cv}(z, \Phi^{n,T}, \Phi_z^{n,T}, \Phi_{zz}^{n,T}; \tilde{c}, \tilde{\pi}) = 0, \tag{87}$$

  with boundary condition

$$\Phi^{n,T}(t,0) \quad = \quad \Phi^{0,n,T}\int_t^T e^{-K_\lambda(s-t)}\lambda f^\gamma(s,0)ds,, \tag{88}$$

  and terminal condition

$$\Phi^{n,T}(T,z) \quad = \quad 0. \tag{89}$$

- Given $n \geq 0$ and $\Phi^{n,T}$, we define $\Phi^{0,n+1,T}$ by

$$\Phi^{0,n+1,T} \quad = \quad \mathcal{H}_0[\Phi^{n,T}]. \tag{90}$$

Then one can prove (see [7, 11] for details) that

$$\lim_{n\to\infty,\ T\to\infty} \Phi^{0,n+1,T} = \Phi^0, \qquad \lim_{n\to\infty,\ T\to\infty} \Phi^{n+1,T} = \Phi.$$

The rate of convergence above is sensitive to the value of λ (see [7]):

- The bigger is λ, the slower is the convergence in n.
- The smaller is λ, the lower is the convergence in T.

5 Discussion

In this section we provide and discuss some numerical experiments we performed in the case of power utility by means of the iterative approximation procedure that we have described in Subsection 4.6. Many more tests and more discussion could be done looking deeply at the properties of the optimal paths.[11] Since the purpose of this paper is mainly methodological, here we limit ourselves to discuss some key features to show how our methodology can be applied.

We have taken as values of the parameters

$$\beta = 0.2, \quad p = 0.5, \quad b_L = 0.15, \quad \sigma_L = 1, \quad b_I = 0.2, \quad \sigma_I = 1.$$

Let us explain our choice for these parameters. First, the illiquid asset should have a higher Sharpe ratio than the liquid one. This is economically intuitive, and moreover ensures that for any value of the correlation ρ, it will always be optimal to invest something in the illiquid asset. Second, we want that the optimal investment proportion in I be in $[0, 1]$, so as to observe mainly the impact of the constraint on trading dates rather than the impact of the proportion constraint (induced by discretization). As in [21, 12] this is obtained for parameters corresponding to a high risk-return market.

We then vary the other relevant parameters γ, λ, ρ, representing respectively the observation of the illiquid asset between two trading times, the liquidity of the market, and the correlation between the two assets. We have solved the PDE's (87) using an explicit finite-difference scheme for parabolic viscosity solutions (see [10, Ch. IX]). More precisely, this has been done after a change of variable

$$\mathbb{R}_+ \to [0,1), \quad z \mapsto \tilde{z} = \frac{z}{z+1} = \frac{x}{x+y},$$

inducing a corresponding transformation $\Phi \mapsto \tilde{\Phi}$, in order to work on the bounded domain $[0, 1)$. We have taken T between 1 and 5 (depending on λ according to what said at the end of Subsection 4.6) and used a uniform grid on $[0,T]\times[0,1]$ with time step length $5\cdot10^{-4}$ and space step length 0.02. The numbers $f^\gamma(t,\tilde{z})$ were computed beforehand at each point of the grid using an L^2-optimal quantization grid for the gaussian law with $N = 5000$ points. Finally, the derivatives have been approximated by finite difference.

[11] See e.g. the tests performed in [1].

5.1 Value function and cost of illiquidity

In this subsection we study the cost of illiquidity by looking at some quantities related to the value function V.

The first (absolute) way to measure the cost of illiquidity is, of course, by looking at the differences of the value functions corresponding to different values of λ. In Figure 1 we represent the value function $V(1)$[12] as function of the correlation parameter ρ and with fixed $\gamma = 0$ (no observation between trading times). The different lines correspond to different values of the liquidity parameter λ. We also sketched the graph of the constrained and unconstrained Merton problem with the two assets L, I, with I considered as liquid as well. We observe that the illiquidity, measured by λ, has a considerable impact on the value function for negative correlation. This is expected, since to deal with negative correlation the agent is naturally driven to hedge the fluctuations of one asset by assuming an opposite position in the other one. However, this is not always possible when one has constraint on the strategies. This effect can be already seen comparing the constrained and the unconstrained Merton model, and it is amplified by the presence of illiquidity, which induces further constraints on the strategies. To this regard it is worth to stress that we cannot expect to have always convergence to Merton unconstrained problem for $\lambda \to \infty$. Indeed, the presence of illiquidity, even in the case of high λ - high trading frequency - immediately induces a constraint on the investment strategies in I in order to satisfy the state constraint - see (7). This fact may produce a gap between the fully liquid case and the illiquid case (even if the trading frequency is very high). Actually, the limit case for $\lambda \to \infty$ corresponds to the *constrained* fully liquid Merton problem, i.e. the Merton problem with the constraint that the investment $\alpha = (\alpha_t)_{t\geq 0}$ in I does not admit borrowing or short selling: $0 \leq \alpha_t \leq R_t$ for every $t \geq 0$.[13]

On the other hand, the impact of the observation parameter γ on the value function $V(1)$ is observed to be low, both in absolute value and in percentage change. This low impact can be observed without regard to the liquidity parameter λ and to the correlation parameter ρ, as shown by Table 1 for the extreme cases $\gamma = 0, 1$ for $\rho = 0$. We performed the same analysis for other risk aversion (not reported here for brevity; we did it also in the case of higher risk aversion, in particular for negative values of p) as well as for other values of the correlation ρ, and still we have observed a low impact of γ on the results. We observe that the change of γ has naturally negligible impact both when $\lambda \to 0$ and $\lambda \to +\infty$. Indeed, when $\lambda = 0$ the illiquid asset is not traded; when $\lambda = +\infty$ the time distance between trading times goes to 0, so the observation between trading times plays no role. So, as we expect, the impact of γ no the value function is higher for intermediate values of λ.

For comparison purposes we have also computed the value of $V(1)$ in the model of [21], i.e. with no liquid asset and no observation on the illiquid asset (and with the same values of the parameters b_I and σ_I). We see that in this case adding the illiquid asset, $V(1)$ grows

[12] For another initial wealth r the same measure can be performed by the relation $V(r) = V(1)r^p$.

[13] Of course, when the optimal solution of unconstrained Merton problem satisfies this constraint, constrained and unconstrained Merton problems are equivalent and we have the convergence to the (unconstrained) Merton problem for $\lambda \to \infty$.

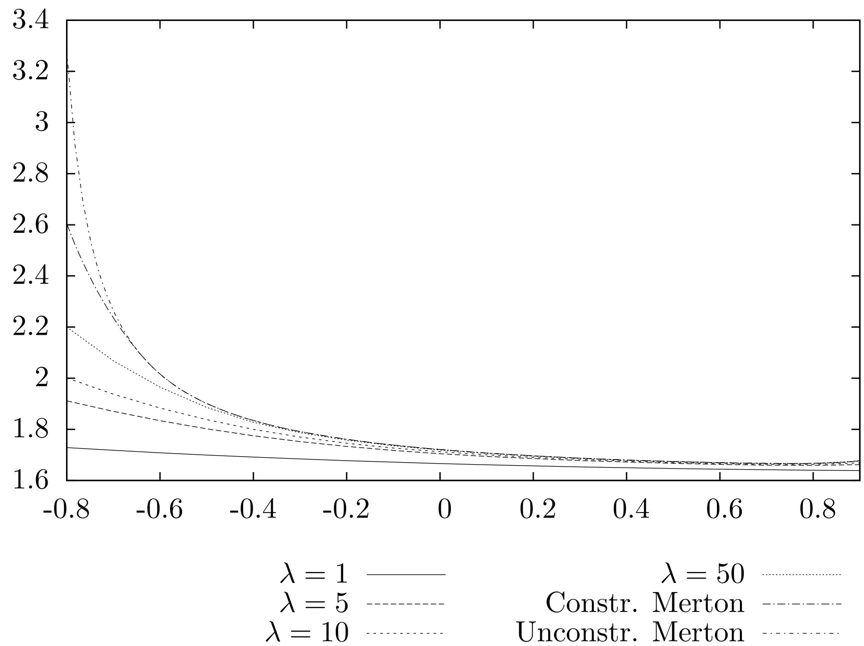

Figure 1: Value function $V(1)$ as a function of ρ with fixed $\gamma = 0$.

λ	1	5	10	50	Merton
$\gamma = 0$	1.66641	1.70493	1.71257	1.71945	1.72133
$\gamma = 1$	1.66995	1.71121	1.71656	1.72036	1.72133

Table 1: $V(1)$ for various γ, λ and fixed $\rho = 0$.

of about 3% without relevant changes with respect to λ.

λ	1	5	10	50	Merton
$V(1)$	1.61973	1.65377	1.65987	1.66526	1.66667

Table 2: $V(1)$ for various λ in the case of no liquid asset (and $\gamma = 0$).

Another way of measuring the cost of illiquidity is to define it - see [21] - as the extra amount of initial wealth $e(r)$ needed to reach the same level of expected utility as an investor without trading restrictions and initial capital x. Hence, it is then computed as the solution to the equation

$$V(r + e(r)) \;=\; V_M(r),$$

where V_M is the value function of the corresponding unconstrained Merton problem.

Tables 3-4-5 reproduce the value of $e(1)$[14] for different values of λ and for $\gamma = 0, 1$, respectively for $\rho = 0$, $\rho = 0.5$, $\rho = -0.5$. As in the case of $V(1)$, the impact of the parameter γ is low in an absolute value, reaching its maximum for intermediate values of λ. Moreover, we also observe that the relative impact $\frac{e^{\gamma=0}(1) - e^{\gamma=1}(1)}{e^{\gamma=1}(1)}$, i.e. the percentage

[14] In our setting of power utility, the cost of liquidity $e(x)$ is proportional to x. We therefore study the cost of liquidity per unit of initial wealth $r = 1$.

change of $e(1)$ passing from $\gamma = 1$ to $\gamma = 0$, is much higher than in the case of $V(1)$.

λ	1	5	10	50
$\gamma = 0$	0.067	0.0193	0.0103	0.00218
$\gamma = 1$	0.062	0.0119	0.0056	0.00112

Table 3: $e(1)$ for various γ, λ and fixed $\rho = 0$.

λ	1	5	10	50
$\gamma = 0$	0.0337	0.00892	0.00462	0.00095
$\gamma = 1$	0.0303	0.00491	0.00237	0.00051

Table 4: $e(1)$ for various γ, λ and fixed $\rho = 0.5$.

λ	1	5	10	50
$\gamma = 0$	0.2511	0.1127	0.0700	0.0161
$\gamma = 1$	0.2493	0.1030	0.0614	0.0120

Table 5: $e(1)$ for various γ, λ and fixed $\rho = -0.5$.

Also in this case we have performed the analysis on the impact of the parameter γ on the results for other for other risk aversion, observing a low impact.

5.2 Optimal policies

Now we look at the optimal policies. Hereafter, without loss of generality (as we are in the power utility case) we assume that the initial wealth is $r = 1$ - hence, investment expressed as amount coincides with investment expressed as portfolio proportion at time $t = 0$.

In Figure 2 below we represent the optimal allocation in the illiquid asset as proportion of the wealth - i.e. the quantity $\hat{z} = 1 - \tilde{z}^*$ - as function of the correlation ρ for $\gamma = 0$ (no observation between trading times). The different lines correspond to different values of λ. When λ is very low, clearly the optimal investment proportion in the illiquid asset is close to 0; on the other hand, when λ is very high, it is close to the one of the constrained Merton problem. In this last case the investment in the "illiquid" asset is higher for higher (in absolute value) correlation, as expected. So, for increasing values of λ, the corresponding graphs lie between these two extreme cases, clearly increasing with λ. in particular, the optimal investment in the illiquid asset will be lower for lower liquidity, which is quite intuitive.

The impact of the observation parameter on the optimal allocation proportion in the illiquid asset $\hat{z}^*$ is represented in Table 6 for different values of λ considering the extreme cases $\gamma = 0$ and $\gamma = 1$ and setting $\rho = 0$. Clearly the agent will invest more in the illiquid asset if he can observe it continuously. Moreover the impact of γ is negligible in the extreme cases $\lambda = 1$, $\lambda = 50$ and of the order of 6% when $\lambda = 3, 5, 10$. So, as for $V(1)$ and $e(1)$, the impact of the parameter γ is higher for intermediate values of λ. We performed the same analysis for other values of ρ and observed similar results.

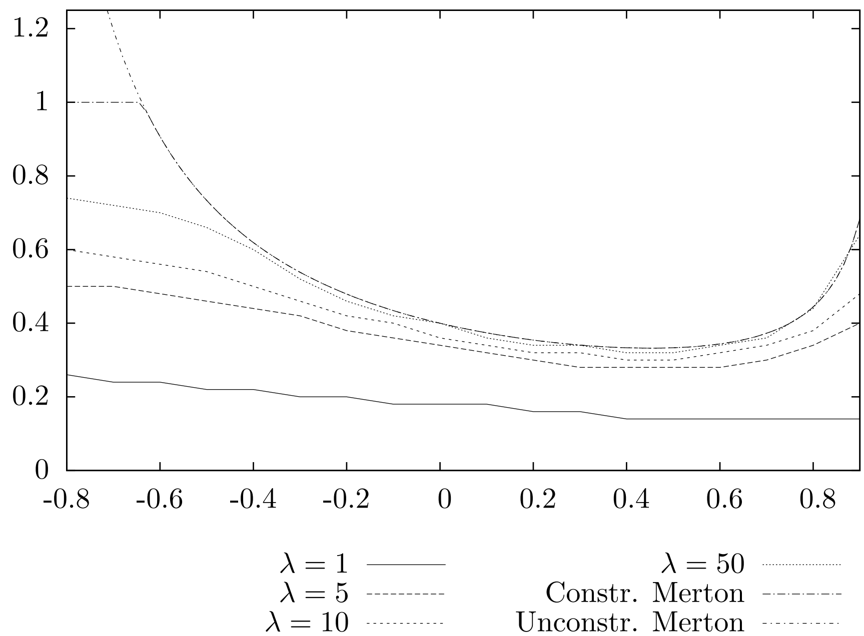

Figure 2: Optimal investment proportion $\hat{z}$ in the illiquid asset as function of ρ for $\gamma = 0$.

	$\lambda = 1$	$\lambda = 3$	$\lambda = 5$	$\lambda = 10$	$\lambda = 50$	Constr./Unconstr. Merton
$\gamma = 0$	0.18	0.3	0.34	0.36	0.4	0.4
$\gamma = 1$	0.18	0.32	0.36	0.38	0.4	0.4

Table 6: Optimal investment (proportion) in the illiquid asset $\hat{z}^*$ with $\rho = 0$.

Let us go now to the analysis of the feedback maps defining the optimal consumption and the optimal investment in the liquid asset. We want to see the influence of the portfolio proportion invested in the illiquid asset on these maps, so it is meaningful to study them with respect to the variable $\hat{z} = \frac{y}{x+y}$. Hence, we look at the functions

$$\widehat{C}^*(t, \hat{z}) := \hat{z}\widetilde{C}^*\left(t, \frac{1}{\hat{z}} - 1\right), \qquad \widehat{\Pi}^*(t, \hat{z}) := \hat{z}\widetilde{\Pi}^*\left(t, \frac{1}{\hat{z}} - 1\right).$$

indeed, by homogeneity, given the total amount $x + y$, we have

$$C^*(t, x, y) := (x + y)\, \widehat{C}^*(t, \hat{z}), \qquad \Pi^*(t, \hat{z}) := (x + y)\, \widehat{\Pi}^*(t, \hat{z}).$$

So if we set the total initial capital equal to 1, the functions $\widehat{C}^*(0, \hat{z}), \widehat{\Pi}^*(0, \hat{z})$ yield respectively the consumption and the optimal amount invested in the liquid asset at time 0. We therefore plot the graphs of $\widehat{C}^*(0, \hat{z}), \widehat{\Pi}^*(0, \hat{z})$.[15] Note that we must pay attention in comparing these maps for different values of the parameters γ, ρ for $t > 0$. Indeed they

[15] In our model, in practice at time $t = 0$ the agent following an optimal strategy will always have fixed proportion $\hat{z}^*$ invested in the illiquid asset, as we are assuming $t = 0$ being a trading date. Anyway, it is still meaningful to look at the graphs of $\widehat{C}^*(0, \cdot)$, $\widehat{\Pi}^*(0, \cdot)$, as this corresponds to suppose that the $t = 0$ is not a trading date - see [1, 24] and Remark 3.3(iii) - and one can be interested in the analysis of the optimal consumption and of the optimal investment in the illiquid asset as function of the initial repartition $\hat{z}$.

are defined on the variable $\hat{z}$, which refers to the stochastic process $\hat{Z} = 1/(Z+1)$. But Z itself depends on the parameters γ, ρ, so the feedbacks maps $\widehat{C}^*, \widehat{\Pi}^*$ do not read the same input for different values of the parameters γ, ρ. Nevertheless, at time $t = 0$ the meaning of this variable is clear, as it is then simply the proportion of the total wealth invested in the illiquid asset. Computing these functions for different values of γ we observe very small changes in the shape and in the values (less than 5 % relative change for all values), so we plot their graphs only in the case $\gamma = 0$.

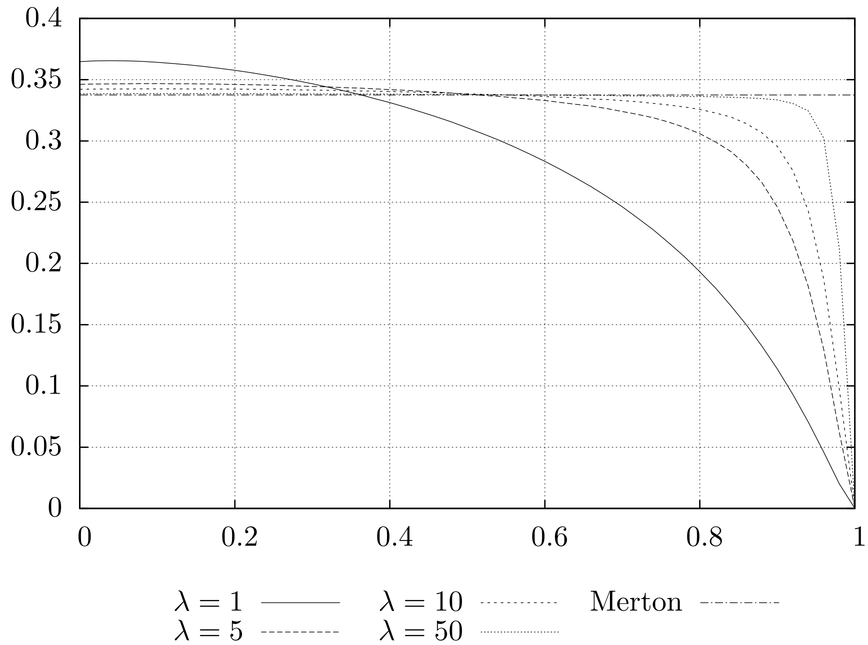

Figure 3: Optimal consumption rate $\widehat{C}^*(0, \cdot)$ as function of $\hat{z}$ with $\rho = 0$, $\gamma = 0$.

In Figure 3 we plot $\widehat{C}^*(0, \cdot)$ only for $\rho = 0$, as different values give similar shapes. As in [12] we observe that the influence of λ on the optimal consumption rate depends on the proportion of illiquid investment $\hat{z}$: when $\hat{z}$ is close to 1, hence most of the portfolio is constituted of illiquid wealth, the investor faces the risk of "having nothing more to consume". As a consequence, the further away the next trading date is, the smaller the consumption rate should be, thus $\widehat{C}^*$ is increasing in λ. When $\hat{z}$ is far from 1, the opposite happens: when λ is smaller, the investor will not be able to invest optimally to maximize future income and should consume more quickly.

In Figures 4 - 5 - 6, we plot the optimal proportional investment in the liquid asset $\widehat{\Pi}(0, \cdot)$, when ρ is respectively 0, -0.5 and 0.5. The Merton line corresponds to the case when I may be traded continuously, while in the case $\lambda = 0$ the proportion invested in I is actually lost. Notice that when λ increases, $\widehat{\Pi}_\lambda(0, \hat{z})$ goes from $\widehat{\Pi}_0(0, \hat{z})$ to $\widehat{\Pi}_M(0, \hat{z})$, increasingly or decreasingly depending on the value of ρ.

Finally, it would be interesting to look at the impact of the parameters on the investor's behavior for $t > 0$. Analyzing that is more complicated: this cannot be performed by a

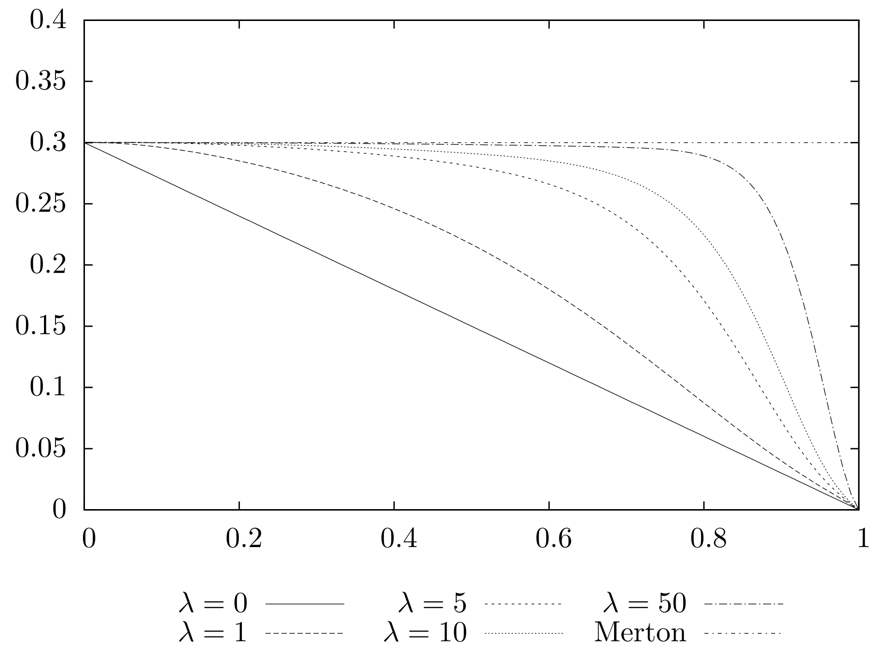

Figure 4: Optimal investment (proportion) in the liquid asset $\widehat{\Pi}^*(0,\cdot)$ as function of $\hat{z}$ with $\rho = 0$, $\gamma = 0$.

simple comparison of the feedback maps, since, as we have observed, these maps do not read the same variable as input for different values of the parameters. To overcome this problem one would need to perform Monte-Carlo simulations to study the distributions of the optimal investor's portfolio and strategies. We observe also that, since our (auxiliary) control problem is not autonomous, we cannot look at the stationary distribution as in [1]. However such approach would be intensive numerically, and for simplicity here we just consider the dependence of the feedback maps on the extra observation[16] $B^{(1)}$ for different values of γ, which still allows to illustrate the effect of partial observation.

Since we are interested mainly in the impact of γ on the strategies, we fix the other parameters, e.g. fix $\lambda = 5$, $\rho = 0$. We consider an agent who at time $t = 1$ has a liquid wealth composed of $\tilde{X}_1 = 0.5$ units, while having invested $\alpha_0 = \tilde{Y}_0 = 0.5$ units in illiquid wealth at time 0 (clearly, we assume that τ_1 has not occurred yet). We plot the optimal consumption rate and the optimal investment in the illiquid asset as function of the additional information, i.e. of $B_1^{(1)}$, which determines, together with γ, the value of $\tilde{Y}_1$.[17] To be more explicit, we have to compute $C^*(1, \tilde{X}_1, \tilde{Y}_1)$, $\Pi^*(1, \tilde{X}_1, \tilde{Y}_1)$ in terms of $B_1^{(1)}$ and γ. Since from (23) we have

$$\tilde{Y}_1 \quad = \quad \tilde{Y}_0 \; e^{b_Y - \frac{(1-\rho^2)\gamma^2}{2} + \sqrt{1-\rho^2}\;\gamma B_1^{(1)}}, \tag{91}$$

[16]We remark that in our model, what one really observes are the the processes $L, I^{(1)}$, so it would make sense to study the feedback maps as functions of them. However, our main interest is to study the dependence of the feedback strategy on γ. Since $I^{(1)}$ is still dependent on γ, to avoid the same problem pointed out above, we study their dependence on $B^{(1)}$.

[17]This is true since $\rho = 0$, so that $\tilde{Y}$ does not depend on W.

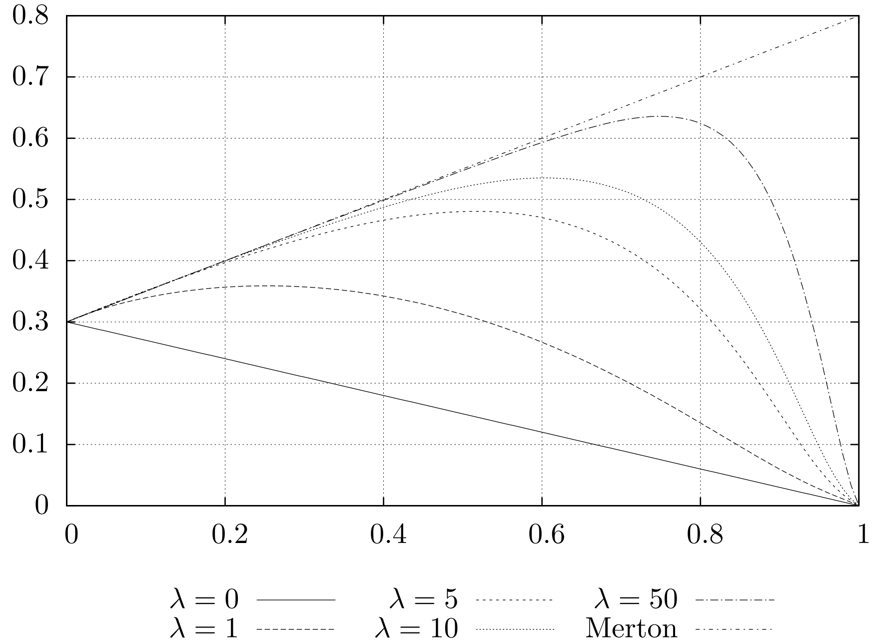

Figure 5: Optimal investment (proportion) in the liquid asset $\widehat{\Pi}(0, \cdot)$ as function of $\hat{z}$ with $\rho = -.5$, $\gamma = 0$.

by substitution we get the function to plot.

In Figure 7 we observe the consumption rate: we see that the optimal consumption is increasing in $B^{(1)}$, which illustrates the unsurprising fact that when the agent knows that his illiquid investment is doing well, he should consume more. This effect is stronger, the more information the agent on the illiquid asset has (i.e., for γ close to 1). Note that the impact of γ here is higher than at time 0, as expected, since when time passes the importance of the additional information should increase.

Figure 8 shows the optimal amount to invest in the risky asset. Again we observe the same behaviour with respect to $B^{(1)}$ and γ.

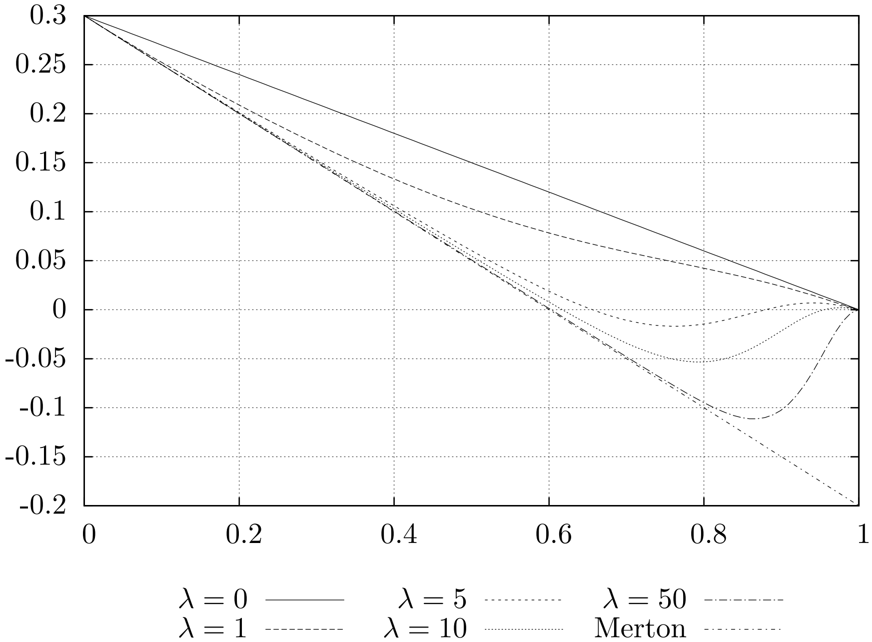

Figure 6: Optimal investment (proportion) in the liquid asset $\widehat{\Pi}(0,\cdot)$ in function of $\hat{z}$ with $\rho = .5$, $\gamma = 0$.

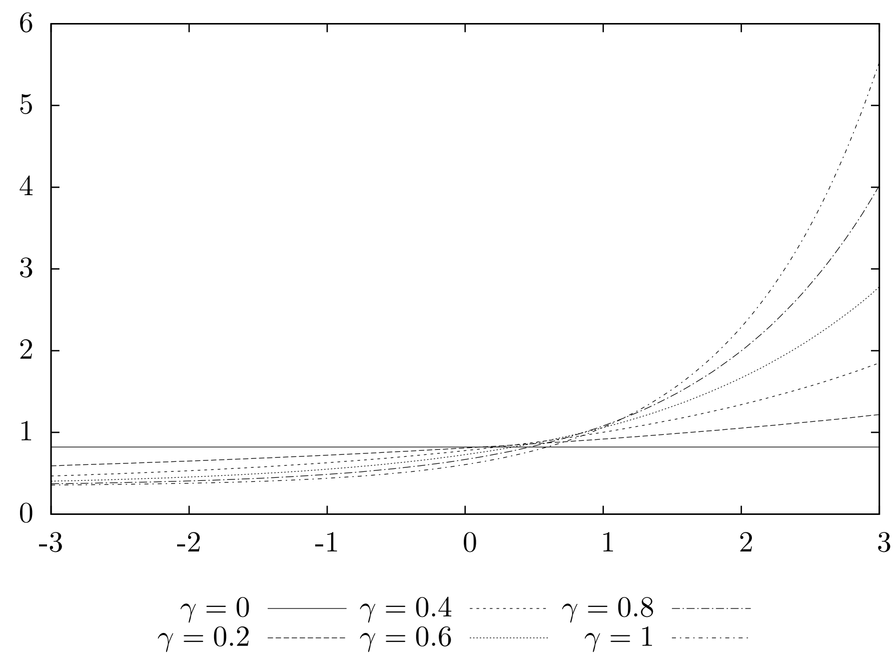

Figure 7: Optimal consumption rate as function of $B_1^{(1)}$ for various γ (setting $\lambda = 5$, $\rho = 0$, $X_1 = .5$, $Y_0 = .5$).

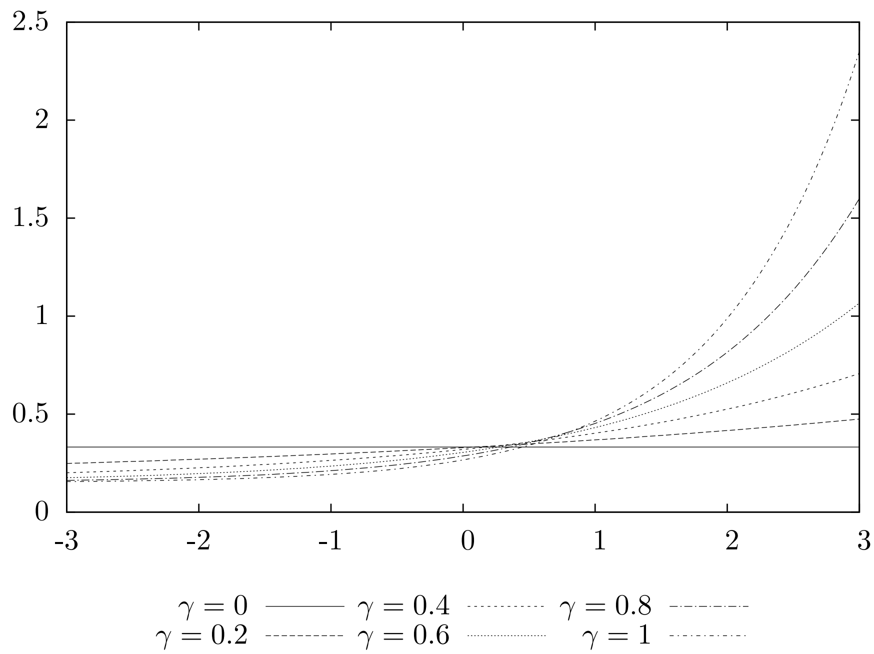

Figure 8: Optimal proportion of liquid wealth to invest in the liquid asset in function of $B_1^{(1)}$ for various γ (setting $\lambda = 5$, $\rho = 0$, $X_1 = .5$, $Y_0 = .5$).

A Appendix

Lemma A.1 *Given $r \geq 0$, for any $(c, \pi, \alpha) \in \mathcal{A}(r)$, there exists $(c^0, \pi^0) \in \mathcal{A}_0(r - \alpha_0)$ such that*

$$(c, \pi)\mathbf{1}_{\{t \leq \tau_1\}} \quad = \quad (c^0, \pi^0)\mathbf{1}_{\{t \leq \tau_1\}}, \quad d\mathbb{P} \otimes ds \ \ a.e. \ . \tag{92}$$

Proof. First, using the definition of $\mathbb{G}$, by a simple monotone class argument, for any (c, π) which is $(\mathcal{G}_t)_{t \geq 0}$-predictable we may find (c^0, π^0) which is $(\mathcal{W}_t \vee \mathcal{B}_t^{(1)})_{t \geq 0}$-predictable satisfying (92). It is straightforward to see that the admissibility constraint $(c, \pi, \alpha) \in \mathcal{A}(r)$ implies $(c^0, \pi^0) \in \mathcal{A}_0(r - \alpha_0)$, and the proof is complete. □

Proposition A.2

(i) G^γ is well defined on the set of measurable functions having at most linear growth.

(ii) G^γ is linear and positive, in the sense that it maps positive functions into positive ones. As a consequence G^γ is increasing in the sense that

$$\phi \ \leq \ \psi \implies G^\gamma[\phi] \ \leq \ G^\gamma[\psi].$$

(iii) G^γ maps increasing functions in functions which are increasing with respect to both x and y.

(iv) G^γ maps concave functions in functions which are concave with respect to (x, y).

(v) If $\psi(r) = r^p$, $p \in (0, 1)$, then ($k_{J,p}$ is defined in (33))

$$G^\gamma[\psi](t, \xi x, \xi y) \ = \ \xi^p G^\gamma[\psi](t, x, y), \quad \forall t \geq 0, \ \forall (x, y) \in \mathbb{R}_+^2, \ \forall \xi \geq 0. \tag{93}$$

$$0 \ \leq \ G^\gamma[\psi](t, x, y) \ \leq \ e^{k_{J,p}t}(x + y)^p, \quad \forall t \geq 0, \ \forall (x, y) \in \mathbb{R}_+^2. \tag{94}$$

(vi) Let $p \in (0, 1]$, and ψ a p-Hölder continuous function on $\mathbb{R}_+$. Then for all $t \geq 0$, $x, x', y, y' > 0$, and $0 \leq h \leq 1$, there exists some constant $C \geq 0$ such that

$$|G^\gamma[\psi](t, x, y) - G^\gamma[\psi](t, x', y)| \ \leq \ C|x - x'|^p, \tag{95}$$

$$|G^\gamma[\psi](t, x, y) - G^\gamma[\psi](t, x, y')| \ \leq \ Ce^{k_{J,p}t}|y - y'|^p, \tag{96}$$

$$|G^\gamma[\psi](t, x, y) - G^\gamma[\psi](t + h, x, y)| \ \leq \ C_1 e^{k_{J,p}t} y^p h^{p/2}, \tag{97}$$

Proof. See [7] for the case $\gamma = 0$. The general case is completely analogous, so we omit it for brevity. □

Lemma A.3 *Let $p \in (0, 1)$ and $k_{L,Y,p}$, $k_{J,p}$ defined as in (32)-(33). For $(t, x, y) \in [0, +\infty)$, $(c, \pi) \in \mathcal{A}_t(x)$,*

$$\mathbb{E}\left[(\tilde{X}_s^{t,x,c,\pi} + \tilde{Y}_s^{t,y})^p\right] \ \leq \ e^{k_{L,Y,p}(s-t)}(x + y)^p, \tag{98}$$

for all $s \geq t$. In particular, combining (98) with Proposition A.2(v) and denoting $\varphi(r) = r^p$,

$$\mathbb{E}\left[G^\gamma[\varphi](s, \tilde{X}_s^{t,x,c,\pi}, \tilde{Y}_s^{t,y})\right] \ \leq \ e^{k_{J,p}t} e^{(k_{L,Y,p} + k_{J,p})(s-t)}(x + y)^p. \tag{99}$$

Proof. See [7] for the case $\gamma = 0$. The general case is completely analogous, so we omit it for brevity. $\square$

Lemma A.4 *Set*

$$f(u_L, u_I) \quad := \quad p(u_L b_L + u_I b_I) - \frac{p(1-p)}{2}(u_L^2\sigma_L^2 + u_I^2\sigma_I^2 + 2\rho u_L u_I \sigma_L \sigma_I).$$

Recalling (12), we have then $k_p = \sup_{u_L\in\mathbb{R}, u_I\in[0,1]} f(u_L, u_I)$. *For any* b'_Y, b'_J *such that* $b'_Y + b'_J = b_I$, *define*

$$f_{b'_Y}(u_L, u_Y) := \; p(u_L b_L + u_Y b'_Y) - \frac{p(1-p)}{2}(u_L^2\sigma_L^2 + u_Y^2\sigma_I^2(\rho^2 + \gamma^2(1-\rho^2)) + 2\rho u_L u_Y \sigma_L\sigma_I),$$

$$f_{b'_J}(u_J) \quad := \quad p b'_J u_J - \frac{p(1-p)}{2}\sigma_I^2(1-\rho^2)(1-\gamma^2)u_J^2,$$

and $k'_{L,Y,p} := \sup_{u_L\in\mathbb{R}, u_Y\in[0,1]} f_{b'_Y}(u_L, u_Y)$, $k'_{J,p} := f_{b'_J}(u_J)$. *Then* $k_p \le k'_{L,Y,p} + k'_{J,p}$, *and this inequality is an equality if we choose*

$$b'_Y \quad = \quad \gamma^2 b_I + (1-\gamma^2)\frac{b_L\rho\sigma_I}{\sigma_L}. \tag{100}$$

Proof. Since $f_{b'_Y}(u_L, u_I) + f_{b'_J}(u_I) = f(u_L, u_I)$, then from the definition of $k_p, k'_{L,Y,p}, k'_{J,p}$ we have

$$k_p \quad = \sup_{u_L\in\mathbb{R}, u_I\in[0,1]} \left(f_{b'_Y}(u_L, u_I) + f_{b'_J}(u_I)\right) \quad \le \quad k'_{L,Y,p} + k'_{J,p}.$$

Since the maximizers of $f, f_{b'_Y}, f_{b'_J}$ always exist, the inequality above becomes equality if and only if there exist a maximizer (u_L^*, u_Y^*) of $f_{b'_Y}$ and a maximizer u_J^* of $f_{b'_J}$ such that $u_Y^* = u_J^*$.

In the case $\gamma \in (0,1)$, by strict convexity of $f'_{b'_Y}$ and $f_{b'_J}$, these maximizers are unique and may be computed explicitly with the first-order condition obtaining

$$\begin{aligned} u_J^* \quad &= \quad \mathrm{Proj}_{[0,1]}\left(\frac{b'_J}{(1-p)\sigma_I^2(1-\rho^2)(1-\gamma^2)}\right), \\ u_Y^* \quad &= \quad \mathrm{Proj}_{[0,1]}\left(\frac{b'_Y - \frac{b_L\rho\sigma_I}{\sigma_L}}{(1-p)\sigma_I^2(1-\rho^2)\gamma^2}\right), \end{aligned}$$

Using that $b'_Y + b'_J = b_I$, (100) may be rewritten as

$$\frac{b'_J}{(1-\gamma^2)} \quad = \quad \frac{b'_Y - \frac{b_L\rho\sigma_I}{\sigma_L}}{\gamma^2},$$

which implies $u_J^* = u_Y^*$.

To conclude, it remains to notice that under (100), for $\gamma = 0$ (resp. $\gamma = 1$), $f_{b'_Y}$ does not depend on u_Y (resp. $f_{b'_J}$ does not depend on u_J), so that clearly in these cases we may choose $u_Y^* = u_J^*$. $\square$

Given $(\bar{t}, \bar{z}) \in \mathbb{R}^+ \times (0, +\infty)$ and $\varepsilon \in (0, \bar{z})$, we denote

$$D_\varepsilon(\bar{t}, \bar{z}) \quad := \quad [\bar{t}, \bar{t}+\varepsilon) \times (\bar{z}-\varepsilon, \bar{z}+\varepsilon) \subset \mathbb{R}^+ \times (0, +\infty). \tag{101}$$

Lemma A.5 *Let $(\bar{t}, \bar{z}) \in \mathbb{R}^+ \times (0, +\infty)$ and $\varepsilon \in (0, \bar{z})$.*

1. *There exist $N_\varepsilon > 0$ such that for each $(t, z) \in D_\varepsilon(\bar{t}, \bar{z})$*

$$\limsup_{h \to 0^+} \left| \frac{\Phi(t+h, z) - \Phi(t, z)}{h} \right| \;\; \leq \;\; N_\varepsilon. \tag{102}$$

2. *$\Phi(t, \cdot) \in C^1((\bar{z} - \varepsilon, \bar{z} + \varepsilon); \mathbb{R})$ for every $t \in [\bar{t}, \bar{t} + \varepsilon)$ and there exist $m_\varepsilon, M_\varepsilon > 0$ such that*

$$m_\varepsilon \;\; \leq \;\; \Phi_z(t, z) \;\; \leq \;\; M_\varepsilon, \quad \forall (t, z) \in D_\varepsilon(\bar{t}, \bar{z}). \tag{103}$$

3. *$\Phi(t, \cdot)$ is twice differentiable a.e. in $(\bar{z} - \varepsilon, \bar{z} + \varepsilon)$ for every $t \in [\bar{t}, \bar{t} + \varepsilon)$. Moreover, denoting by $\mathcal{O}_t^\varepsilon \subset (\bar{z} - \varepsilon, \bar{z} + \varepsilon)$ the set where $\Phi(t, \cdot)$ is twice differentiable, there exists $\delta_\varepsilon > 0$ such that*

$$\Phi_{zz}(t, z) \;\; \leq \;\; -\delta_\varepsilon, \quad \forall t \in [\bar{t}, \bar{t} + \varepsilon), \; z \in \mathcal{O}_t^\varepsilon. \tag{104}$$

Proof. 1. Setting

$$\mathcal{J}(t, x, y; c, \pi) \; = \; \mathbb{E}\left[\int_0^\infty e^{-(\beta+\lambda)s} \left(U(c_s) + \lambda G^\gamma[V](t+s, \tilde{X}_s^{0,x,c,\pi}, \tilde{Y}_s^{0,y}\right) ds\right]. \tag{105}$$

since the equations for $\tilde{X}, \tilde{Y}$ are autonomous, we have

$$\widehat{V}(t, x, y) \; = \; \sup_{(c,\pi) \in \mathcal{A}_0(x)} \mathcal{J}(t, x, y; c, \pi). \tag{106}$$

Since we are in the power case, we have $G^\gamma[V](t, x, y) = \Phi^0 \, \mathbb{E}(x + yJ_t)^p$. Applying Dynkin's formula to $\Phi^0(x + yJ_t)^p$, we see that $G^\gamma[V](\cdot, x, y)$ is differentiable and we get the estimate

$$\left| \frac{\partial}{\partial t} G^\gamma[V](t, x, y) \right| \;\; \leq \;\; C_{J,p} G^\gamma[V](t, x, y), \tag{107}$$

where

$$C_{J,p} \;\; = \;\; |b_J| p + \frac{1}{2} p(1-p)\sigma_J^2.$$

So we can differentiate (105) with respect to t and, using (107) and the nonnegativity of U, we then get

$$\begin{aligned} \left| \frac{\partial}{\partial t} \mathcal{J}(t, x, y; c, \pi) \right| \;\; & \leq \;\; C_{J,p} \, \mathbb{E}\left[\int_0^\infty e^{-(\beta+\lambda)s} \lambda G^\gamma[V](t+s, \tilde{X}_s^{0,x,c,\pi}, \tilde{Y}_s^{0,y}) ds\right] \\ & \leq \;\; C_{J,p} \, \widehat{V}(t, x, y). \end{aligned} \tag{108}$$

Estimate (108) is uniform in $(c, \pi) \in \mathcal{A}_0(x)$, so from (106) and the fact that

$$|\widehat{V}(t+h, x, y) - \widehat{V}(t, x, y)| \;\; \leq \;\; \sup_{(c,\pi) \in \mathcal{A}_0(x)} |\mathcal{J}(t+h, x, y; c, \pi) - \mathcal{J}(t, x, y; c, \pi)|,$$

we get the claim with

$$N_\varepsilon \;\; = \;\; C_{J,p} \cdot \sup_{D_\varepsilon(\bar{t}, \bar{z})} \widehat{V}(t, x, y).$$

2. Let $(t,z) \in D_\varepsilon(\bar t, \bar z)$. Since $\Phi(t,\cdot)$ is concave, there exist the left and right derivatives $\Phi_z^-(t,z)$, $\Phi_z^+(t,z)$, and $\Phi_z^-(t,z) \geq \Phi_z^+(t,z)$. To show that $\Phi(t,\cdot)$ is differentiable at z we then must prove that the above inequality is actually an equality. Suppose by contradiction that $\Phi_z^-(t,z) > \Phi_z^+(t,z)$. Consider the function defined for $z_1 \in (\bar z - \varepsilon, \bar z + \varepsilon)$, $t_1 \in [t, \bar t + \varepsilon)$

$$\varphi^\delta(t_1, z_1) \;=\; \Phi(t,z) + \frac{\Phi_z^-(t,z) + \Phi_z^-(t,z)}{2}(z_1 - z) - \frac{1}{2\delta}(z_1 - z)^2 + (N_\varepsilon + \delta)(t_1 - t).$$

Due to item 1, the function $\Phi - \varphi^\delta$ has a local maximum at (t,z) in $(\bar z - \varepsilon, \bar z + \varepsilon) \times [t, \bar t + \varepsilon)$ for each $\delta > 0$. Therefore the subsolution viscosity property at (t,z) implies

$$\begin{aligned} &- N_\varepsilon - \delta + K_\lambda \Phi(t,z) - K_3 z \frac{\Phi_z^-(t,z) + \Phi_z^-(t,z)}{2} - \lambda \Phi^0 f^\gamma(t,z) + \frac{K_4^2}{2} z^2 \frac{1}{\delta} \\ &\qquad\qquad - \sup_{\tilde c \geq 0,\, \tilde\theta \in \mathbb{R}} H^1_{cv}\left(\frac{\Phi_z^-(t,z) + \Phi_z^+(t,z)}{2}, -\frac{1}{\delta}; \tilde c, \tilde\theta \right) \;\leq\; 0, \end{aligned}$$

Letting $\delta \to 0$ we get a contradiction as $\frac{K_4^2}{2} z^2 \frac{1}{\delta} \to +\infty$, so $\Phi(t,\cdot)$ is differentiable at each $z \in (\bar z - \varepsilon, \bar z + \varepsilon)$ and for every $t \in [\bar t, \bar t + \varepsilon)$. The fact that $\Phi(t,\cdot) \in C^1((\bar z - \varepsilon, \bar z + \varepsilon); \mathbb{R})$ for every $t \in [\bar t, \bar t + \varepsilon)$ follows from concavity. Finally let us show (103). Let $\delta = \frac{\bar z - \varepsilon}{2}$. By concavity of $\Phi(t,\cdot)$ we have for every $z \in (\bar z - \varepsilon, \bar z + \varepsilon)$ and $t \in [\bar t, \bar t + \varepsilon)$

$$\frac{\Phi(t, \bar z - \varepsilon) - \Phi(t, \bar z - \varepsilon - \delta)}{\delta} \;\leq\; \Phi_z(t,z) \;\leq\; \frac{\Phi(t, \bar z + \varepsilon + \delta) - \Phi(t, \bar z + \varepsilon)}{\delta} \tag{109}$$

Since $\Phi(t,\cdot)$ is strictly increasing for each $t \in [\bar t, \bar t + \varepsilon)$ (Proposition 4.1), we have

$$\frac{\Phi(t, \bar z - \varepsilon) - \Phi(t, \bar z - \varepsilon - \delta)}{\delta} \;<\; \infty, \qquad \frac{\Phi(t, \bar z + \varepsilon + \delta) - \Phi(t, \bar z + \varepsilon)}{\delta} \;>\; 0.$$

Calling

$$M_\varepsilon \;=\; \sup_{t \in [\bar t, \bar t + \varepsilon)} \frac{\Phi(t, \bar z - \varepsilon) - \Phi(t, \bar z - \varepsilon - \delta)}{\delta}, \qquad m_\varepsilon \;=\; \inf_{t \in [\bar t, \bar t + \varepsilon)} \frac{\Phi(t, \bar z + \varepsilon + \delta) - \Phi(t, \bar z + \varepsilon)}{\delta},$$

by continuity of Φ, we have $0 < m_\varepsilon \leq M_\varepsilon < \infty$, so the claim follows by (109).

3. Let $(t,z) \in D_\varepsilon(\bar t, \bar z)$. The fact that there exists a set $\mathcal{O}_t^\varepsilon$ with full Lebesgue measure such that $\Phi(t,\cdot)$ is differentiable at the points of $\mathcal{O}_t^\varepsilon$ follows from concavity of $\Phi(t,\cdot)$ and Alexandrov's Theorem. Suppose that $z \in \mathcal{O}_t^\varepsilon$. Let $\delta, \delta_1 > 0$ and consider the function defined for $z_1 \in (\bar z - \varepsilon, \bar z + \varepsilon)$, $t_1 \in [t, \bar t + \varepsilon)$

$$\varphi^\delta(t_1, z_1) \;=\; \Phi(t,z) + \Phi_z(t,z)(z_1 - z) + \frac{1}{2}(\Phi_{zz}(t,z) - \delta)(z_1 - z)^2 - (N_\varepsilon + \delta_1)(t_1 - t).$$

Due to item 1, the function $\Phi - \varphi^\delta$ has a local minimum at (t,z) in $(\bar z - \varepsilon, \bar z + \varepsilon) \times [t, \bar t + \varepsilon)$ for each $\delta > 0$. Therefore the supersolution viscosity property at (t,z) and item 2 imply

$$\begin{aligned} &N_\varepsilon + \delta_1 + K_\lambda \Phi(t,z) - K_3 z m_\varepsilon - \lambda \Phi^0 f^\gamma(t,z) - \frac{K_4^2}{2} z^2 (\Phi_{zz}(t,z) - \delta) \\ &\qquad\qquad - \tilde U(M_\varepsilon) + \frac{1}{2}\frac{K_1^2}{K_2^2}\frac{m_\varepsilon^2}{\Phi_{zz}(t,z) - \delta} \;\geq\; 0. \end{aligned} \tag{110}$$

Notice that, given $a_0, b_0 > 0$ and $c_0 \in \mathbb{R}$, there exists $\alpha_0 > 0$ such that

$$a_0\xi - \frac{b_0}{\xi} \;\leq\; c_0, \quad \xi \;\leq\; 0 \implies \xi \;\leq\; -\alpha_0. \tag{111}$$

Since we know that $\Phi_{zz} \leq 0$, from (110) we have that (111) holds for $\xi = \Phi_{zz}(t,z) - \delta$. So, we get the existence of $\delta_\varepsilon > 0$ independent of $(t,z) \in D_\varepsilon(\bar{t},\bar{z})$ and of δ such that

$$\Phi_{zz}(t,z) \;\leq\; \delta - \delta_\varepsilon.$$

By arbitrariness of δ we have the claim. $\square$

Proposition A.6 *Φ is a viscosity solution in $D_\varepsilon(\bar{t},\bar{z})$ of*

$$-\varphi_t + K_\lambda\varphi - K_3 z\varphi_z - \lambda f^\gamma(t,z)\Phi^0 - \frac{K_4^2}{2}z^2\varphi_{zz} - \sup_{\tilde{c}\in[0,\bar{c}],\,\tilde{\theta}\in[-\bar{\theta},\bar{\theta}]} H^1_{cv}(\varphi_z,\varphi_{zz};\tilde{c},\tilde{\theta}) = 0, \tag{112}$$

where

$$\bar{c} \;=\; (U')^{-1}(m_\varepsilon), \quad \bar{\theta} \;=\; \frac{|K_1|M_\varepsilon}{K_2^2\delta_\varepsilon}.$$

Proof. The fact that Φ is a supersolution of (112) in $D_\varepsilon(\bar{t},\bar{z})$ is a straightforward consequence of the fact that it is a supersolution of (66), as the supremum is taken over a small set in (112). Let us show that it is a subsolution in $D_\varepsilon(\bar{t},\bar{z})$. Take $(t,z) \in D_\varepsilon(\bar{t},\bar{z})$ and let $\varphi \in C^{1,2}(D_\varepsilon(\bar{t},\bar{z});\mathbb{R})$ be such that $\varphi(t,z) = \Phi(t,z)$ and in $\varphi \geq \Phi$ in $D_\varepsilon(\bar{t},\bar{z})$. Since Φ is once differentiable with respect to z, it must be $\varphi_z(t,z) = \Phi_z(t,z)$. Now, if $\varphi_{zz} \leq -\delta_\varepsilon$, then

$$\sup_{\tilde{c}\geq 0,\,\tilde{\theta}\in\mathbb{R}} H^1_{cv}(\varphi_z,\varphi_{zz};\tilde{c},\tilde{\theta}) \;=\; \sup_{\tilde{c}\in[0,\bar{c}],\;\tilde{\theta}\in[-\bar{\theta},\bar{\theta}]} H^1_{cv}(\varphi_z,\varphi_{zz};\tilde{c},\tilde{\theta}), \tag{113}$$

so we have the desired subsolution inequality. Otherwise, suppose that $\varphi_{zz}(t,z) > -\delta_\varepsilon$ and consider the function defined for $z_1 \in (\bar{z}-\varepsilon,\bar{z}+\varepsilon),\; t_1 \in [t,\bar{t}+\varepsilon)$

$$\tilde{\varphi}(t_1,z_1) \;=\; \varphi(t_1,z) + \Phi_z(t_1,z)(z_1-z) - \frac{1}{2}\delta_\varepsilon(z_1-z)^2. \tag{114}$$

We have

$$\tilde{\varphi}(t_1,z) \;\geq\; \varphi(t_1,z) \;\geq\; \Phi(t_1,z), \quad \forall\, t_1 \in [t,\bar{t}+\varepsilon). \tag{115}$$

Now fix $t_1 \in [t,\bar{t}+\varepsilon)$. Consider, for $z_1 \in (\bar{z}-\varepsilon,\bar{z}+\varepsilon)$, the Dini derivative of Φ_z at z_1:

$$D_z^+\Phi_z(t_1,z_1) \;:=\; \limsup_{h\to 0} \frac{\Phi_z(t_1,z_1+h) - \Phi_z(t_1,z_1)}{h}.$$

Since $\Phi(t_1,\cdot)$ is concave, we have

$$D_z^+\Phi_z(t_1,z_1) \;\leq\; 0, \quad \forall\, z_1 \in (\bar{z}-\varepsilon,\bar{z}+\varepsilon). \tag{116}$$

Moreover, by Lemma A.5(3), we have

$$D_z^+\Phi_z(t_1,z_1) \;\leq\; -\delta_\varepsilon, \quad \forall\, z_1 \in \mathcal{O}^\varepsilon_{t_1}. \tag{117}$$

As consequence of (116)-(117), of the fact that $\mathcal{O}^{\varepsilon}_{t_1}$ has full measure and of Lemma 3.3 in [9], we get by integrating two times (117)

$$\Phi(t_1, z_1) \ \leq \ \Phi(t_1, z) + \Phi_z(t_1, z)(z_1 - z) - \frac{1}{2}\delta_\varepsilon (z_1 - z)^2. \tag{118}$$

Combining (118) with (114)-(115) we get

$$\tilde{\varphi}(t, z) \ = \ \Phi(t, z), \qquad \tilde{\varphi} \geq \Phi \ \text{ in } (\bar{z} - \varepsilon, \bar{z} + \varepsilon) \times [t, \bar{t} + \varepsilon).$$

Now, since Φ is a viscosity subsolution of (66), we have

$$-\tilde{\varphi}_t + K_\lambda \tilde{\varphi} - K_3 z \tilde{\varphi}_z - \lambda \Phi^0 f^\gamma(t, z) - \frac{K_4^2}{2} z^2 \tilde{\varphi}_{zz} - \sup_{\tilde{c} \geq 0,\, \tilde{\theta} \in \mathbb{R}} H^1_{cv}(\tilde{\varphi}_z, \tilde{\varphi}_{zz}; \tilde{c}, \tilde{\theta}) \ \leq \ 0, \tag{119}$$

On the other hand, we have

$$\sup_{\tilde{c} \geq 0,\, \tilde{\theta} \in \mathbb{R}} H^1_{cv}(\tilde{\varphi}_z, \tilde{\varphi}_{zz}; \tilde{c}, \tilde{\theta}) \ = \ \sup_{\tilde{c} \in [0, \bar{c}],\ \tilde{\theta} \in [-\bar{\theta}, \bar{\theta}]} H^1_{cv}(\tilde{\varphi}_z, \tilde{\varphi}_{zz}; \tilde{c}, \tilde{\theta}),$$

so that also

$$-\tilde{\varphi}_t + K_\lambda \tilde{\varphi} - K_3 z \tilde{\varphi}_z - \lambda \Phi^0 f^\gamma(t, z) - \frac{K_4^2}{2} z^2 \tilde{\varphi}_{zz} - \sup_{\tilde{c} \in [0, \bar{c}],\ \tilde{\theta} \in [-\bar{\theta}, \bar{\theta}]} H^1_{cv}(\tilde{\varphi}_z, \tilde{\varphi}_{zz}; \tilde{c}, \tilde{\theta}) \ \leq \ 0. \tag{120}$$

Noticing that

$$\varphi(t, z) \ = \ \tilde{\varphi}(t, z), \quad \varphi_z(t, z) \ = \ \tilde{\varphi}_z(t, z), \quad \varphi_{zz}(t, z) \ > \ -\delta_\varepsilon \ = \ \tilde{\varphi}_{zz}(t, z), \tag{121}$$

and taking into account that H^1_{cv} is nondecreasing in the second derivative, combining (120) and (121) we get the desired subsolution inequality for φ and the proof is complete. □

Lemma A.7 *Let $a < b$ and $F : [0, T) \times (a, b) \times \mathbb{R}^3 \to \mathbb{R}$, $(t, x, r, p, q) \mapsto F(t, x, r, p, q)$ be continuous, continuously differentiable in (x, r, p, q), and proper degenerate elliptic (i.e. nondecreasing in r and nonincreasing in q). Let $u \in C^{1,2}([0, T) \times (a, b); \mathbb{R})$ be a classical solution in $[0, T) \times (a, b)$ to the parabolic equation*

$$u_t + F(t, x, u, u_x, u_{xx}) \ = \ 0. \tag{122}$$

Then the space derivative $v := u_x$ is a viscosity solution in $[0, T) \times (a, b)$ to the parabolic equation

$$v_t + \nabla F(t, x, u(t, x), v, v_x) \cdot (1, v, v_x, v_{xx}) = 0, \tag{123}$$

where $\nabla F = (F_x, F_r, F_p, F_q)$ is the gradient of $F(t, \cdot)$.

Proof. For $x \in (a, b)$ and sufficiently small $h > 0$, define $u^h(t, x) = u(t, x + h)$ and $v^h = \frac{u^h - u}{h}$. Then, due to continuous differentiability of u, we have that $v^h \to v$ locally uniformly in $[0,T)\times(a,b)$ when h goes to 0. Furthermore, since u is a solution to (122), and using the differentiability of F, we see that

$$\begin{aligned} v_t^h &= \frac{1}{h}\Big(F(t,x,u^h,u_x^h,u_{xx}^h)-F(t,x,u,u_x,u_{xx})\Big) \\ &= \Big(\nabla F(t,x,u,v^h,v_x^h)+\mathcal{E}^h(t,x)\Big)\cdot(1,v^h,v_x^h,v_{xx}^h), \end{aligned}$$

where

$$\begin{aligned} &\mathcal{E}^h(t,x) \\ := &\int_0^1 \nabla F(t,x+hs,(1-s)u^h(t,x)+su(t,x),(1-s)u_x^h(t,x)+su_x(t,x),(1-s)u_{xx}^h(t,x)+su_{xx}(t,x))ds \\ &-F(t,x,u(t,x),v^h(t,x),v_x^h(t,x)). \end{aligned}$$

By continuity of F and of u, u_x, u_{xx}, and by the fact that v^h (resp. v_x^h) goes to u_x (resp. u_{xx}) as h goes to 0, we see that $\mathcal{E}^h$ goes to 0 as $h\to 0$, locally uniformly in $[0,T)\times(a,b)$.

So, it just remains to apply the stability result for viscosity solutions (see, e.g., [25, Prop. 5.9, Ch .4]), to get that v is a viscosity solution to the limiting equation (123). □

Acknowledgements. The authors are sincerely grateful to Claudio Tebaldi for valuable comments.

References

[1] Ang A., Papanikolaou D., and M. Westerfield (2011) : "Portfolio Choice with Illiquid Assets", Preprint of SSRN.

[2] Beaudry P. and E. van Wincoop (1996) : "The Intertemporal Elasticity of Substitution: An Exploration Using a US Panel of State Data," Economica, London School of Economics and Political Science, Vol. 63, pp. 495-512.

[3] Bayraktar E. and B. Ludkovski (2010) : "Optimal Trade Execution in Illiquid Markets", to appear in *Mathematical Finance*.

[4] Crandall M., Ishii H., and P.L. Lions (1992) : "User's Guide to Viscosity Solutions of Second Order Partial Differential Equations", *Bull. Amer. Math. Soc.*, Vol. 27, 1-67.

[5] Cretarola A., Gozzi F., Pham H., and P. Tankov (2011) : "Optimal consumption policies in illiquid markets", *Finance and Stochastics*, Vol. 15, No. 1, pp. 85-115.

[6] Di Giacinto M., Federico S., and F. Gozzi (2011) : "Pension funds with a minimum guarantee: a stochastic control approach", *Finance and Stochastics*, Vol. 15, No. 2, 297-342.

[7] Federico S. and P. Gassiat : "Viscosity characterization of the value function of an investment-consumption problem in presence of illiquid assets". Submitted, preprint available at `http://www.proba.jussieu.fr/mathdoc/preprints/index.html#2012`

[8] Federico S., Gassiat P., and F. Gozzi : "Utility maximization with current utility on the wealth: regularity of solutions to the HJB equation". Submitted, preprint available at `http://www.proba.jussieu.fr/mathdoc/preprints/index.html#2012`.

[9] Federico S., Goldys B., and F. Gozzi (2011) : "HJB equations for the optimal control of differential equations with delays and state constraints: verification and optimal feedbacks", *SIAM - Journal on Control and Optimization*, Vol. 49, No. 6, pp. 2378–2414.

[10] Fleming W. H. and H. M. Soner (2006) : *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag.

[11] Gassiat P. (2011) : "Modélisation du risque de liquidité et méthodes de quantification appliquées au contrôle stochastique séquentiel", Phd thesis of University Paris Diderot, available at `http://tel.archives-ouvertes.fr/tel-00651357/fr/`

[12] Gassiat P., Gozzi F., and H. Pham : "Investment/consumption problem in illiquid markets with regimes switching", Submitted, preprint available at `http://www.proba.jussieu.fr/mathdoc/preprints/index.html#2011`

[13] Juutinen, P. (2001) : "On the Definition of Viscosity Solutions for Parabolic Equations", *Proc. Amer. Math. Soc.*, Vol. 129, No. 10, pp. 2907-2911.

[14] Kabanov Y., and M. Safarian (2009) : "Markets with Transaction Costs: Mathematical Theory", Springer-Verlag.

[15] Kallenberg O. : *Foundations of modern probability*, Second ed., Probability and its Applications, Springer-Verlag, New York, 2002.

[16] Karatzas I. and S.E. Shreve (1988) : *Brownian motion and stochastic calculus*, Springer-Verlag, New York.

[17] Krylov N.V. (1987) : *Nonlinear elliptic and parabolic equations of the second order*, D. Reidel Publishing Company.

[18] Lieberman G. (1996) : *Second Order Partial Differential Equations*, World Scientific Press, Singapore.

[19] Longstaff F. (2005) : "Asset pricing in markets with illiquid assets", *American Economic Review*, Vol. 99, No. 4, pp. 1119–1144.

[20] Matsumoto K. (2006) : "Optimal portfolio of low liquid assets with a log-utility function", *Finance and Stochastics*, Vol. 10, No. 1, pp. 121–145.

[21] Pham H. and P. Tankov (2008) : "A model of optimal consumption under liquidity risk with random trading times", *Mathematical Finance*, Vol. 18, No. 4, pp. 613–627.

[22] Pham H. and P. Tankov (2009) : "A coupled system of integrodifferential equations arising in liquidity risk model", *Applied Mathematics and Optimization*, Vol. 59, No. 2, pp. 147–173.

[23] Rogers C. and O. Zane (2002) : "A simple model of liquidity effects", in *Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann*, Eds. K. Sandmann and P. Schoenbucher, pp. 161–176.

[24] Schwartz E. and C. Tebaldi (2006) : "Illiquid assets and optimal portfolio choice", NBER Working Paper No. w12633.

[25] Yong J., and X.Y. Zhou (1999) : "Stochastic Controls: Hamiltonian Systems and HJB Equations", Springer Verlag, New York.

[26] Zariphopoulou T. (1994) : "Consumption-Investment Models with Constraints", SIAM Journal of Control and Optimization, Vol. 32, No. 1, pp. 59–85.