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IMPEDANCE CONCEPT IN WAVE GUIDES*

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1. Introduction. The impedance concept is the foundation of engineering transmission theory. If wave guides are to be fully utilized as transmission systems or parts thereof, their properties must be expressed in terms of appropriately chosen impedances or else a new transmission theory must be developed. The gradual extension of the concept has necessitated a broader point of view without which an exploitation of its full potentialities would be impossible.

In the course of various private discussions, I have found that there exists some uneasiness with regard to the applicability of the concept at very high frequencies. In part this may be attributed to relative unfamiliarity with the wave guide phenomena and in part to the evolution of the concept itself. Some particular aspects of the concept have to be sacrificed in the process of generalization and although these aspects may be logically unimportant, they frequently become psychological obstacles to understanding in the early stages of the development. For this reason I am going to devote several sections of this paper to a general discussion of the impedance concept before passing to more specific applications; then by way of illustration I shall prove that an infinitely thin perfectly conducting iris between two *different* wave guides behaves as if between the admittances of its faces there existed an ideal transformer. This theorem is a generalization of another theorem which I proved several years ago to the effect that when the two wave guides are alike, the iris behaves as a shunt reactor. Actual calculation of the admittances and the transformer ratio depends on the solution of an appropriate boundary value problem.

More generally, wave guide discontinuities are representable by T-networks. In some special cases these networks lack series branches and in other cases, the shunt branch.

2. Evolution of concepts. Concepts evolve. It is a long way from the primitive to the modern number concept. The primitive number was an integer, a *concrete integer* at that. In some primitive languages there is no word corresponding to "two." There are words meaning "two men," "two horses," etc.; but the concept of "two" applying either to men, or to horses, is lacking. To a primitive mind the difference between a class comprised of two men and a class comprised of two horses overshadowed the similarity. Seeing similarities requires a degree of abstraction. A resistance to abstract ideas seems to be a characteristic of human minds even in modern times; only the modern mind is quicker to overcome it. An example, pertinent to

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our present discussion, is the following excerpt from a paper *Derivation and discussion of the general solution of the current flowing in a circuit containing resistance, self-inductance and capacity with any impressed electromotive force*, by Frederick Bedell and Albert C. Crehore, published in the Journal A.I.E.E., **9**, 340 (1892):

> "From the analogy of this equation to Ohm's law, we see that the expression $[R^2+(1/C\omega-L\omega)^2]^{1/2}$ is of the nature of a resistance, and is the apparent resistance of a circuit containing resistance, self-inductance and capacity. This expression would quite properly be called 'impedance' but the term impedance has for several years been used as a name for the expression $[R^2+L^2\omega^2]^{1/2}$, which is the apparent resistance of a circuit containing resistance and self-inductance only. We suggest, therefore, that the word 'impediment' be adopted as a name for the expression $[R^2+(1/C\omega-L\omega)^2]^{1/2}$ which is the apparent resistance of a circuit containing resistance, self-inductance and capacity, and the term impedance be retained in the more limited meaning it has come to have, that is, $[R^2+L^2\omega^2]^{1/2}$, the apparent resistance of a circuit containing resistance and self-inductance only."

The name "impediment" was not adopted. Apparently, it was soon understood that if one really wished to emphasize the difference between the impedances of various circuits, one could simply describe the circuits and, therefore, for most purposes, it was best to emphasize the similarity rather than the difference. And only ten years ago there were some who objected to the use of the word impedance for the ratio E/H in an electromagnetic wave and who wanted a new word for it.

The word "number" now includes fractions, negative numbers, irrational numbers and complex numbers; the impedance is now a complex number, and not its absolute value as originally intended. There are mechanical impedances, acoustic impedances, electromechanical impedances, and finally impedances associated with any wave no matter what its physical nature happens to be. The impedance is now the force/response ratio when the force and response are harmonic functions of time and are represented by complex exponentials. Around this concept has grown the transmission theory of force and response in linear systems. The principal tool of this theory is the theory of functions of a complex variable. This theory is used for engineering purposes as in the design of filters, equalizers, and other transmission systems with prescribed desired properties; and with equal advantage it may be used for general transmission studies. In this paper I am particularly concerned with fundamental ideas applied to wave guides and wave guide elements.

3. General discussion of impedance and admittance. Superficially, it may seem that the impedance concept does not apply to wave guides or if it does it is quite different from the concept as applied to ordinary transmission lines. Actually there is no significant difference; whatever difference there exists is largely psychological rather than logical. In wave guides a characteristic impedance has to be associated with each transmission mode. At first the existence of various transmission modes may strike one as a feature which distinguishes high frequency wave guides from low frequency "ordinary" transmission lines; but soon one will realize that even in ordinary transmission lines it is usual to distinguish between different modes of transmission. Consider, for instance, parallel wires at the same height above ground; there are two obvious transmission modes recognized by communication engineers; in one the currents in the wires are equal and flow in opposite directions and in the other they are equal, flow in the same direction and return through ground. It is the existence of these two transmission modes that accounts for the important engineering difference between balanced and unbalanced transmission lines. Similarly, there are two obvious transmission modes in a shielded parallel pair. The field patterns are different for different transmission modes and the characteristic impedances are usually different. The existence of transmission modes is not peculiar to hollow tubes and other structures which have become prominent in high frequency transmission; high frequency transmission studies make us merely aware of the fact that any physical wave guide, whether a coaxial pair or a shielded pair or a hollow tube, admits of an infinite number of transmission modes with their characteristic field patterns, characteristic impedances, and propagation constants.

Another cause of worry to some is a degree of indeterminacy connected with an impedance and its associated quantities. The characteristic impedance of a wave guide may be defined in a number of ways giving different values. For each oscillation mode a cavity resonator behaves as an ordinary circuit comprised of inductance and capacitance; but different values are obtained, depending on how L and C are defined. This indeterminacy is really inherent in these conceptions but in elementary theory it is not stressed for the simple reason that no occasion arises for such stressing. In the final analysis, this indeterminacy is of the same kind as that involved in the essential arbitrariness of units and is related to the fact that properties of analytic functions are not affected by a constant factor. Putting it in the language of transmission theory, the essential properties of impedance functions are not affected by ideal transformers. If we have a closed box containing an electric network with two accessible terminals and if we measure a resistance R across these terminals, we cannot be certain that the box contains a resistance R; it may contain a resistance $R/10$ which is then boosted to R by an ideal transformer. It does not really matter which is the case. Similarly, if the measurement seems to indicate that in the box we have a tuned circuit with an inductance L in series with a capacitance C, we may actually have a tuned circuit with an inductance $\frac{1}{2}L$ and the capacitance $2C$ in the secondary of an ideal transformer which then doubles the impedance. More generally, the impedance function is defined by its zeros, infinities, and other singularities *except for a constant.*

If V is the voltage across an impedance Z, I the current through Z, and W the complex power, then

$$V = ZI, \qquad W = \tfrac{1}{2}VI^* = \tfrac{1}{2}ZII^*, \qquad W = \frac{VV^*}{2Z^*}, \tag{3-1}$$

where the asterisk is used to designate conjugate complex numbers. Now suppose that our voltmeters and ammeters contain concealed ideal transformers; then "Z" will have different values in the above equations and we shall have

$$V = Z_{V,I}I, \qquad W = \tfrac{1}{2}Z_{W,I}II^*, \qquad W = \frac{VV^*}{2Z^*_{W,V}} \cdot \tag{3-2}$$

These new equations are in effect various definitions of impedance and admittance

$$Z_{V,I} = \frac{V}{I}, \qquad Z_{W,I} = \frac{2W}{II^*}, \qquad Z_{W,V} = \frac{VV^*}{2W^*},$$
$$Y_{V,I} = \frac{1}{Z_{V,I}}, \qquad Y_{W,I} = \frac{1}{Z_{W,I}}, \qquad Y_{W,V} = \frac{1}{Z_{W,V}}. \tag{3-3}$$

Ordinarily, we make sure that there are no concealed ideal transformers in our measuring instruments. Furthermore, at low frequencies we can measure the voltage across the total capacitance[1] and the total current through the inductance. There seems to be no question about the meaning of "V" and "I" and it so happens that in this case we are led to equations (3-1). However, in a section of a transmission line or in a cavity resonator the capacitance and inductance are not localized and we are forced to recognize the existence of a certain amount of indeterminacy. There is no harm in this indeterminacy; it does not really matter in which of the following two forms we decide to write the expression for power

$$W = \tfrac{1}{2}ZII^*, \quad \text{or} \quad W = \tfrac{1}{2}(n^2Z)\frac{I}{n}\frac{I^*}{n}, \tag{3-4}$$

so long as we know how to compute it.

Just as ideal transformers in our "ammeters" and "voltmeters" transform equations (3-1) into equations (3-2) in the case of "ordinary" networks, they may be used to transform equations (3-2) into (3-1) in the case of wave guides and networks with distributed constants.

4. General impedance relations. Eliminating V, I, and W from (3-2), we have the following equation connecting various impedances

$$Z_{W,I}Z^*_{W,V} = Z_{V,I}Z^*_{V,I}. \tag{4-1}$$

If the impedances are real, then

$$Z_{W,I}Z_{W,V} = Z^2_{V,I}. \tag{4-2}$$

In equations (3-2) V and I may be arbitrarily chosen values of the voltage and current associated with a given impedor. If we choose a given definition for I, we can define a voltage

$$V_{W,I} = Z_{W,I}I \tag{4-3}$$

for which equations (3-1) will hold and the impedance $Z_{W,I}$ will become the only impedance associated with the impedor. We can also define

$$I_{W,V} = \frac{V}{Z_{W,V}}, \tag{4-4}$$

so that again we shall have equations (3-1) with $Z_{W,V}$ as the sole impedance.

Since the power is an invariant we have

$$V_{W,I}I^* = VI^*_{W,V} \quad \text{or} \quad \frac{V_{W,I}}{V} = \frac{I^*_{W,V}}{I^*}. \tag{4-5}$$

[1] Or almost across the total capacitance.

From (4-3), (4-4), and (4-5) we have

$$\left|\frac{V_{W,I}}{V}\right| = \left|\frac{I_{W,V}}{I}\right| = \sqrt{\frac{Z_{W,I}}{Z_{W,V}}} = \sqrt{\frac{Y_{W,V}}{Y_{W,I}}}. \tag{4-6}$$

It is now evident that we can base our calculations on any particular voltage-current pair and then, whenever desirable, we may pass to any other pair simply by inserting in our transmission diagrams an ideal transformer with a proper impedance transformation ratio.

5. Characteristic impedances and admittances of wave guides. The basic impedance associated with the nth transmission mode in a wave guide is defined as the ratio of the transverse electric to the transverse magnetic intensity

$$K_n = \frac{E_{t,n}}{H_{t,n}}. \tag{5-1}$$

It is called the *wave impedance* or the *specific impedance* and enters in the expression for the average power flow per unit area in the direction of the guide

$$W_S = \tfrac{1}{2}E_{t,n}H_{t,n}^* = \tfrac{1}{2}K_nH_{t,n}H_{t,n}^*. \tag{5-2}$$

The reciprocal of this impedance is the *wave admittance*

$$M_n = \frac{1}{K_n}; \tag{5-3}$$

the power flow is then

$$W_S = \tfrac{1}{2}M_n^*E_{t,n}E_{t,n}^*. \tag{5-4}$$

In wave guides with perfectly conducting walls the various transmission modes carry power independently of each other. The field patterns are "orthogonal" to each other and may be "normalized"; that is, the transverse intensities for a typical mode may be expressed as follows

$$E_{t,n} = V_nF_n(u, v), \qquad H_{t,n} = I_nF_n(u, v), \qquad I_n = M_nV_n, \tag{5-5}$$

where

$$\iint [F_n(u, v)]^2dS = 1, \qquad \iint F_m(u, v)F_n(u, v)dS = 0, \quad \text{if} \quad m \neq n, \tag{5-6}$$

u and v are suitable coordinates in the transverse plane of the wave guide and the integration is extended over the entire cross-section. The coefficients V_n and I_n may be called respectively the *normalized voltage* and *normalized magnetomotive force* or *normalized current* associated with the nth mode.

Calculating the total power carried in the nth mode, we obtain

$$W = \tfrac{1}{2}K_nI_nI_n^* = \tfrac{1}{2}M_n^*V_nV_n^*. \tag{5-7}$$

Thus, if we express our transmission formulas in terms of normalized voltages and currents, the same impedance coefficient appears in the alternative expressions (3-1) for power and this impedance is also the ratio of the normalized voltage to the normalized current.

Before going on let us see just what the above formulas mean in one or two special cases. Consider a wave guide consisting of two parallel metal strips of width a, separated by distance b, Fig. 1. In the dominant mode the electric intensity is perpendicular to the metal plates and is distributed almost uniformly except near the edges and in the external region where the field is weak and little power is carried by the wave. Neglecting the edge effect, we shall assume that the electric intensity is constant

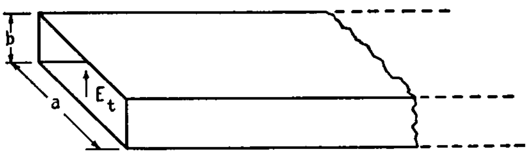

FIG. 1.

$$E_t = E_0. \tag{5-8}$$

The normalized distribution pattern is given by

$$F_0(x, y) = \frac{1}{\sqrt{ab}}, \tag{5-9}$$

and, therefore,

$$E_t = V_0 F_0(x, y), \qquad V_0 = E_0\sqrt{ab}. \tag{5-10}$$

The wave impedance for transverse electromagnetic waves is $K_0 = \sqrt{\mu/\epsilon}$ and therefore

$$H_t = I_0 F_0(x, y), \qquad I_0 = H_0\sqrt{ab} = \frac{V_0}{K_0}\cdot \tag{5-11}$$

In air K_0=approximately 377 ohms. The transverse voltage V between the plates and the longitudinal current I are

$$V = bE_0 = V_0\sqrt{b/a}, \qquad I = aH_0 = I_0\sqrt{a/b}; \tag{5-12}$$

consequently the characteristic impedance on the voltage-current basis is

$$K_{V,I} = \frac{V}{I} = \frac{bV_0}{aI_0} = \frac{b}{a}K_0. \tag{5-13}$$

For the total power flow we have

$$W = \tfrac{1}{2}K_0 I_0 I_0^* = \tfrac{1}{2}M_0 V_0 V_0^* = \tfrac{1}{2}K_{V,I} II^* = \frac{VV^*}{2K_{V,I}}, \tag{5-14}$$

so that in the present case

$$K_{W,I} = K_{W,V} = K_{V,I} = \frac{b}{a} K_0. \quad (5\text{-}15)$$

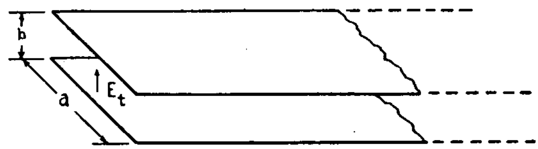

FIG. 2.

Consider now the $TE_{1,0}$-wave in a rectangular wave guide, Fig. 2; for this wave the field is given by

$$E_t = E_1 \sin\frac{\pi x}{a}, \qquad H_t = H_1 \sin\frac{\pi x}{a},$$
$$E_1 = K_1 H_1, \qquad H_1 = M_1 E_1, \qquad K_1 = \sqrt{\mu/\epsilon}\left(1 - \frac{\lambda^2}{4a^2}\right)^{-1/2}. \quad (5\text{-}16)$$

The normalized field distribution function is

$$F_1(x, y) = \sqrt{\frac{2}{ab}} \sin\frac{\pi x}{a}, \quad (5\text{-}17)$$

so that

$$E_t = V_1 F_1(x, y), \qquad H_t = I_1 F_1(x, y),$$
$$V_1 = E_1\sqrt{ab/2}, \qquad I_1 = H_1\sqrt{a/2b}. \quad (5\text{-}18)$$

In this case the maximum transverse voltage V across the guide and the total longitudinal current I are given by

$$V = E_1 b = V_1\sqrt{2b/a},$$
$$I = \int_0^a H_t dx = \frac{4}{\pi} I_1\sqrt{ab/2}. \quad (5\text{-}19)$$

From these equations we have

$$K_{V,I} = \frac{V}{I} = \frac{\pi b}{2a} K_1. \quad (5\text{-}20)$$

The power transfer is

$$W = \tfrac{1}{2}K_1 I_1 I_1^* = \tfrac{1}{2}M_1^* V_1 V_1^*; \quad (5\text{-}21)$$

consequently

$$K_{W,I} = \frac{\pi^2 b}{8a} K_1, \qquad K_{W,V} = \frac{2b}{a} K_1. \quad (5\text{-}22)$$

Now let us see what happens when we join two wave guides, each consisting of two parallel metal strips. Suppose that the frequency is so low that we do not have to worry about higher transmission modes. At the junction the transverse voltage and the longitudinal current must be continuous. This requirement is responsible for reflection unless the characteristic impedances of the two guides are equal. The coefficients of reflection and transmission depend on the impedance ratio $K'_{V,I}/K''_{V,I}$ of the two wave guides. As we shall find later the effect of the geometric discontinuity can be calculated equally well by concentrating attention on normalized voltages and currents. With respect to these variables the characteristic impedances of the above wave guides are equal; but at the wave guide junction there will exist an effective ideal transformer with the impedance transformation ratio equal to $K''_{V,I}/K'_{V,I}$. In the case of ordinary low frequency transmission lines we prefer to think in terms of total voltages and currents; to think in terms of normalized voltages and currents would be to make simple matters complicated; but it will presently become evident that, in general, it is advantageous to introduce the normalized variables at least in certain stages of the analysis.

Take an iris in a rectangular wave guide. We know that for frequencies between the lowest cut-off frequency and the next higher, the iris can be represented as a shunt susceptance. The value of this susceptance will depend on its definition; but the ratio to the corresponding characteristic admittance of the guide is an invariant. It is this ratio that appears in transmission formulas involving lumped elements inserted in a uniform transmission line. If, however, the iris is between circular and rectangular wave guides, the ratio of the characteristic impedances of the two guides will also be involved and this ratio depends on whether both impedances are defined on the power-voltage basis or the power-current basis. It is evident, therefore, that in this case the iris cannot behave as a simple shunt susceptance. The theory which we are now evolving permits us to prove that in the more general case the equivalent transducer for the iris consists of two shunt susceptances, corresponding to the two faces of the iris, and an ideal transformer between them. The transformer ratio depends on the particular voltage-current set we happen to choose for our analytical work but our final transmission formulas will be independent of this choice. The degree of arbitrariness involved in the choice of "V" and "I" is of the same kind as that involved in the choice of coordinate systems or of units. In elementary analysis, a particular choice was so natural that a mistaken notion spread abroad that this choice was a necessary one.

6. An iris between two wave guides. Let us now obtain an exact equivalent circuit for an infinitely thin perfectly conducting iris between two wave guides of arbitrary

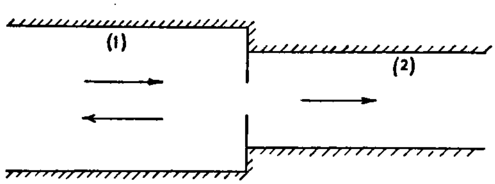

FIG. 3.

cross-section (Fig. 3). The constants of this circuit depend on the particular transmission mode under consideration; that is, there is one equivalent circuit for transition from each transmission mode in wave guide 1 to each mode in wave guide 2. The most important case is that of transition from the dominant mode in one wave guide to the dominant mode in the other, and in the following analysis we shall keep this case specifically in mind; but the analysis applies to any other case. We shall use Cartesian coordinates in our equations; but this does not mean that our analysis is restricted to rectangular guides.

Suppose that the transverse field of the incident wave at the surface of the iris is

$$\begin{aligned} E_t^i(x, y) &= V_1^i F_1(x, y), \\ H_t^i(x, y) &= M_1 V_1^i F_1(x, y), \end{aligned} \tag{6-1}$$

where V_1^i is the normalized incident voltage. In response to this impressed field, we shall have some field over the aperture of the iris. Let $f(x, y)$ be the tangential electric intensity over the aperture; then in wave guide 2 the "transmitted" tangential electric intensity is

$$\begin{aligned} \hat{E}_t^i(x, y) &= f(x, y) \quad \text{over the aperture,} \\ &= 0 \quad\quad\quad \text{over the screen.} \end{aligned} \tag{6-2}$$

In wave guide 1 the total tangential electric intensity, that is, the sum of the incident and the reflected intensity, must be

$$\begin{aligned} V_1^i F_1(x, y) + E_t^r(x, y) &= f(x, y) \quad \text{over the aperture,} \\ &= 0 \quad\quad\quad \text{over the screen.} \end{aligned} \tag{6-3}$$

The function defined by (6-2) may be expanded into a series of normalized orthogonal functions appropriate to wave guide 2; thus

$$\hat{E}_t^i(x, y) = \sum_{n=1}^{\infty} \hat{V}_n \hat{F}_n(x, y). \tag{6-4}$$

The tangential magnetic intensity is then

$$\hat{H}_t^i(x, y) = \sum_{n=1}^{\infty} \hat{M}_n \hat{V}_n \hat{F}_n(x, y). \tag{6-5}$$

The function defined by (6-3) can be expanded into a series of normalized orthogonal functions appropriate to the wave guide 1; thus

$$V_1^i F_1(x, y) + E_t^r(x, y) = \sum_{n=1}^{\infty} V_n F_n(x, y). \tag{6-6}$$

The reflected tangential intensity is therefore

$$E_t^r(x, y) = (V_1 - V_1^i) F_1(x, y) + \sum_{n=2}^{\infty} V_n F_n(x, y). \tag{6-7}$$

The corresponding tangential magnetic intensity is then

$$H_t^r(x, y) = - M_1 (V_1 - V_1^i) F_1(x, y) - \sum_{n=2}^{\infty} M_n V_n F_n(x, y). \tag{6-8}$$

The transfer of complex power through the aperture must be continuous; therefore

$$2 M_1 V_1^i V_1^* - \sum_{n=1}^{\infty} M_n V_n V_n^* = \sum_{n=1}^{\infty} \hat{M}_n \hat{V}_n \hat{V}_n^*. \tag{6-9}$$

The voltage reflection coefficient q_V is defined as the ratio of the reflected voltage $V_1 - V_1^i$ to the incident voltage V_1^i; it may be obtained from (6-9) if we divide the equation by $2 M_1 V_1 V_1^*$; thus

$$\frac{1}{1+q_V} = \frac{V_1^i}{V_1} = \frac{1}{2}\left[1 + \sum_{n=2}^{\infty} \frac{M_n V_n V_n^*}{M_1 V_1 V_1^*}\right] + \frac{\hat{M}_1 \hat{V}_1 \hat{V}_1^*}{2 M_1 V_1 V_1^*}\left[1 + \sum_{n=2}^{\infty} \frac{\hat{M}_n \hat{V}_n \hat{V}_n^*}{\hat{M}_1 \hat{V}_1 \hat{V}_1^*}\right]. \tag{6-10}$$

Consider now the complex power flow into the second wave guide

$$W^* = \frac{1}{2} \hat{M}_1 \hat{V}_1 \hat{V}_1^* + \frac{1}{2} \sum_{n=2}^{\infty} \hat{M}_n \hat{V}_n \hat{V}_n^* . \tag{6-11}$$

The form of this expression is such that from the input end various transmission modes appear to be in parallel. It is not exactly that the characteristic admittances $\hat{M}_1$, $\hat{M}_2$, $\hat{M}_3$, $\cdots$ are directly in parallel; if we select the first admittance for reference, the others are transformed in the ratio $\hat{V}_n \hat{V}_n^* / \hat{V}_1 \hat{V}_1^*$ before being connected in parallel. In any case the net effect on the input admittance is the same as would be obtained if we had an admittance $\hat{Y}$ in shunt with a transmission line maintaining only the dominant mode. Thus we can write

$$W = \tfrac{1}{2}\hat{M}_1 \hat{V}_1 \hat{V}_1^* + \tfrac{1}{2}\hat{Y} \hat{V}_1 \hat{V}_1^*, \tag{6-12}$$

where

$$\hat{Y} = \sum_{n=2}^{\infty} \hat{M}_n \frac{\hat{V}_n \hat{V}_n^*}{V_1 V_1^*} . \tag{6-13}$$

The ratio of the shunt admittance to the characteristic admittance

$$\frac{Y}{M} = \frac{\hat{Y}}{\hat{M}_1} = \sum_{n=2}^{\infty} \frac{\hat{M}_n \hat{V}_n \hat{V}_n^*}{\hat{M}_1 \hat{V}_1 \hat{V}_1^*} \tag{6-14}$$

is an invariant. It has the same value regardless of a particular basis for definition of admittances and it depends only on the *form* of distribution of the tangential electric intensity over the aperture.

Similarly for the admittance ratio looking from the iris into wave guide 1 we have

$$\frac{Y}{M_1} = \sum_{n=2}^{\infty} \frac{M_n V_n V_n^*}{M_1 V_1 V_1^*} . \tag{6-15}$$

If the frequency is in the interval between the lowest cutoff frequency and the next higher, then M_2, M_3, $\cdots$ are reactive and the shunt admittances are pure susceptances

$$Y = iB, \qquad \hat{Y} = i\hat{B}. \tag{6-16}$$

For frequencies higher than the second cutoff frequency an iris entails some power loss to the dominant wave. The lost power is contributed to one or more higher transmission modes. This is analogous to what happens when a doublet antenna is inserted in shunt with a parallel pair or at the end of it. The plane wave guided by the parallel pair loses power; this power is then carried away by a spherical wave which originates at the junction. One mode of energy transmission is partly transformed into another. Usually there is also an energy exchange between a local field and the plane wave; this results in reactance. At lower frequencies an ordinary coil (or a capacitor) inserted in a transmission line acts just like an iris; electrically it does not matter just what physical means we happen to provide for a local storage of energy.

We now can rewrite (6-10) as follows

$$\frac{1}{1+q_V} = \frac{1}{2}\left(1+\frac{Y}{M_1}\right) + \frac{1}{2}\, nn^* \frac{\hat{M}_1}{M_1}\left(1+\frac{\hat{Y}}{\hat{M}_1}\right), \tag{6-17}$$

where

$$nn^* = \frac{\hat{V}_1\hat{V}_1^*}{V_1V_1^*}. \tag{6-18}$$

The reciprocal of the voltage transmission coefficient is

$$\frac{1}{p_V} = \frac{V_1^i}{\hat{V}_1} = \frac{1}{1+q_V}\cdot\frac{V_1}{\hat{V}_1} = \frac{1}{n}\left[\frac{1}{2}\left(1+\frac{Y}{M_1}\right) + \frac{1}{2}\, mn^* \frac{\hat{M}_1}{M_1}\left(1+\frac{\hat{Y}}{\hat{M}_1}\right)\right]. \tag{6-19}$$

It is a simple matter to prove[2] that for transmission lines coupled as indicated in Fig. 4, p_V and q_V are given precisely by equations (6-17) and (6-19). In Fig. 4 the

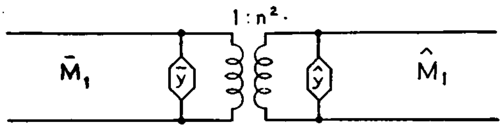

FIG. 4.

transformer ratio $1:n^2$ is indicated for the impedances rather than for the admittances in order to conform to the established practice. If $n=1$, which is always the case when the wave guides on both sides of the iris are the same, the admittances Y and $\hat{Y}$ of the two faces of the iris are just in parallel, and the transformer can be omitted.

The exact numerical values of n, Y/M_1, $\hat{Y}/\hat{M}_1$ are found by solving the appropriate boundary value problems.[3] The approximate values can be obtained quite easily if we assume a reasonable *form* of distribution of the tangential electric intensity over the aperture,[4] and of course, we can always calculate these quantities from measurements of the transmission and reflection coefficients for waves moving from one wave guide into the other. Thus,

$$n^* = \frac{1+q_{\bar{V}}}{1+q_V^+}\,\frac{M_1}{\hat{M}_1} = \frac{1+q_{\bar{V}}}{1+q_V^+}\,\frac{\hat{K}_1}{K_1}, \tag{6-20}$$

where q_V^+ is the voltage reflection coefficient for a wave moving from left to right and $q_{\bar{V}}$ is that for a wave moving in the opposite direction.

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[2] See for instance S. A. Schelkunoff, *Electromagnetic waves*, D. Van Nostrand Company, Inc., New York, 1943, p. 212.

[3] For example, see S. A. Schelkunoff, *The impedance of a transverse wire in a rectangular wave guide*, Quarterly of Applied Mathematics, **1**, 78–85 (1943).

[4] S. A. Schelkunoff, *Electromagnetic waves*, p. 491.

If the iris is not indefinitely thin, there is a section of a wave guide between the two faces of the iris.

While the iris acts effectively as a lumped impedance, the field associated with it is actually distributed. Even if the frequency is such that the iris is reactive, the field extends to some distance on either side of it. Near the cutoff for the second transmission mode this distance may be quite large; but ordinarily the field extends roughly to a distance comparable to the transverse dimensions of the guide. There will exist, therefore, a mutual impedance between those faces of two nearby irises which face each other. For frequencies above the second cutoff, the mutual impedance may, and usually will, exist even between two distant irises. All these considerations do not affect our essential picture of electrical properties of wave guide discontinuities; they affect merely the numerical values of various impedance and admittance functions.

In the above equations we have treated E_t and H_t as if they were scalars; in general, they are vectors. However, the analysis is similar to the above and the final formulae are the same.

In the case of coaxial pairs or wave guides formed by parallel metal strips the dominant wave is transverse electromagnetic. If the edges of the iris are normal to the lines of force for the dominant wave, the voltage between the edges is equal to the transverse voltage across either guide; the total voltages associated with higher transmission modes are equal to zero; and the transformer ratio is unity provided we base our transmission diagram not on the normalized characteristic impedance but on the conventional impedance K which in this case equals $K_{V,I}$, $K_{W,I}$ and $K_{W,V}$.

7. Reactances in series with wave guides. An example of a reactance effectively in series with the wave guide is shown in Fig. 5 which represents a circular wave guide

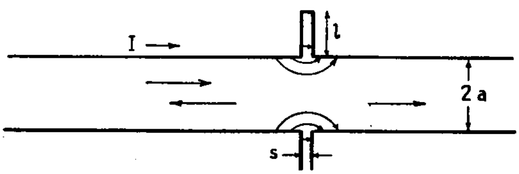

Fig. 5.

and a narrow radial transmission line.[5] Let us suppose that we are concerned with transmission of a $TM_{0,1}$-wave. For this wave the field is circularly symmetric. Magnetic lines are circles coaxial with the tube, and electric lines are in radial planes. It is practically self-evident that the radial line is in series with the guide, and that in parallel with the radial line there is an impedance associated with the gap. If the frequency is between the lowest cutoff frequency and the next higher, this "gap impedance" or fringing impedance is capacitive and is of little importance except when the impedance of the radial transmission line is high. For frequencies above the second cutoff, the gap impedance is in part resistive on account of power transfer from the dominant wave to the higher order waves. As seen from the gap, the impedances of various waves in the guide and the impedance of the radial wave are in parallel; the two halves of the guide are in series; and the impedance diagram looks like that shown

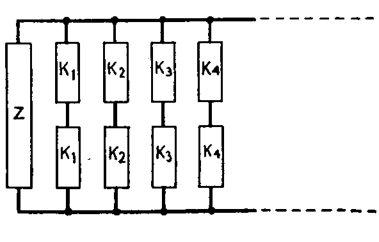

Fig. 6.

[5] S. A. Schelkunoff, U. S. Patent 2,155,508, April 25, 1939.

in Fig. 6. The same diagram is shown in Fig. 7 where the characteristic impedances K_1 and K_1 have been "expanded" into semi-infinite transmission lines; the impedance consisting of $2K_2$, $2K_3$, $\cdots$ in parallel is represented simply as the gap impedance Z_g.

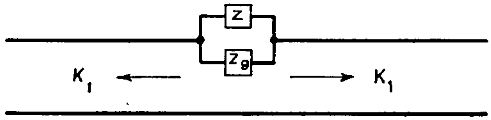

FIG. 7.

Starting with equation (10.17-1) of "Electromagnetic Waves," we can obtain the approximate gap impedance by the method explained there. In this case, however, the following elementary derivation is preferable. To begin with, let us remove the radial line and assume that the electric charge is being transferred across the gap by an impressed voltage V^i. The total conduction current I in the tube is the sum of currents associated with the various transmission modes. Thus for the input current we have

$$I = I_1 + I_2 + I_3 + \cdots . \tag{7-1}$$

The input power is then

$$\begin{aligned} W = \tfrac{1}{2}V^iI^* &= \tfrac{1}{2}V^iI_1^* + \tfrac{1}{2}V^iI_2^* + \cdots \\ &= \tfrac{1}{2}Z_1I_1I_1^* + \tfrac{1}{2}Z_2I_2I_2^* + \cdots , \end{aligned} \tag{7-2}$$

where Z_1, Z_2, $\cdots$ are the input impedances of individual waves; that is,

$$Z_1 = \frac{V^i}{I_1}, \qquad Z_2 = \frac{V^i}{I_2}, \cdots . \tag{7-3}$$

In the above equations we have tacitly assumed that the gap is very small and the current associated with each mode does not vary in the gap. This restriction will presently be removed. The total power contributed to the wave is divided between different modes; one-half of it is carried to the left and the other half to the right. The power carried in one direction in the nth mode is $\tfrac{1}{2}K_{W,I}^{(n)}I_nI_n^*$; thus we have

$$Z_n = 2K_{W,I}^{(n)}. \tag{7-4}$$

Actually the applied voltage is distributed in the interval $(-s/2, s/2)$ around the midpoint $z=0$. Assuming that the distribution is uniform, we may write the contribution to the total current associated with the nth wave at point z due to an elementary voltage at point $\hat{z}$ as follows

$$I_n e^{-\Gamma_n|z-\hat{z}|}\frac{d\hat{z}}{s}, \tag{7-5}$$

where I_n is the amplitude at the source. The total current at point z is then

$$I_n(z) = \frac{I_n}{s}\int_{-s/2}^{s/2} e^{-\Gamma_n|z-\hat{z}|}\frac{d\hat{z}}{s} . \tag{7-6}$$

The power contributed to the nth wave is then

$$W_n = \tfrac{1}{2}V^i \int_{-s/2}^{s/2} I_n^*(z)\,\frac{dz}{s}\,. \tag{7-7}$$

Thus we shall have

$$W_n = \tfrac{1}{2}\chi_n V^i I_n^*, \tag{7-8}$$

where (assuming that Γ_n is real)

$$\chi_n = \frac{1}{s^2}\int_{-s/2}^{s/2} dz \int_{-s/2}^{s/2} e^{-\Gamma_n|z-\hat{z}|}d\hat{z} = \frac{2}{\Gamma_n s} - \frac{2(1-e^{-\Gamma_n s})}{\Gamma_n^2 s^2}\,. \tag{7-9}$$

On the other hand

$$W_n = K_{W,I}^{(n)} I_n I_n^*; \tag{7-10}$$

consequently

$$\frac{V^i}{I_n} = \frac{2K_{W,I}^{(n)}}{\chi_n}\,. \tag{7-11}$$

Since χ_n decreases with increasing n, the successive components of the gap admittance

$$Y_g = Y_2 + Y_3 + Y_4 + \cdots = \frac{\chi_2}{2K_{W,I}^{(2)}} + \frac{\chi_3}{2K_{W,I}^{(3)}} + \frac{\chi_4}{2K_{W,I}^{(4)}} + \cdots \tag{7-12}$$

decrease.

The typical $K_{W,I}^{(n)}$ is given in problem 8.10 on page 509 of "Electromagnetic Waves"

$$K_{W,I}^{(n)} = \frac{\Gamma_n}{4\pi i\omega\epsilon}, \qquad \Gamma_n = \left[\frac{k_n^2}{a^2} - \frac{4\pi^2}{\lambda^2}\right]^{1/2}, \tag{7-13}$$

where k_n is the nth zero of $J_0(x)$. For sufficiently large n, therefore, we have

$$Z_n = \Gamma_n K_{W,I}^{(n)} s = \frac{\Gamma_n^2 s}{4\pi i\omega\epsilon} = \frac{k_n^2 s}{4\pi i\omega\epsilon a^2}\,. \tag{7-14}$$

The impedance of the radial line is approximately

$$Z = 60i\,\frac{s}{a}\tan\frac{2\pi l}{\lambda}\,. \tag{7-15}$$

A more accurate expression in terms of Bessel functions may be found on page 269 of "Electromagnetic Waves."

8. Conclusion. The ideas developed in this paper are adequate for expressing transmission properties of wave guides with discontinuities in terms of impedances and admittances associated with these discontinuities. These impedances are reactive if the frequency is such that the energy in either guide can be transmitted to any distance in only one mode; otherwise, the discontinuities present some resistance for the mode under consideration and a negative resistance to those other modes which participate in transmission of energy. The finding of exact values of impedances requires solution of corresponding boundary value problems; but frequently good approximations can be found by making reasonable a priori assumptions on physical grounds. In fact, the point of view outlined in this paper makes it easy to make such assumptions.

More complex discontinuities can be analyzed into simpler discontinuities. The discontinuity shown in Fig. 8 is equivalent to an ideal transformer between two wave guides; across the left "winding" of which there is a small shunt capacitance[6] and across the right winding there is the capacitance[7] associated with the annular disc

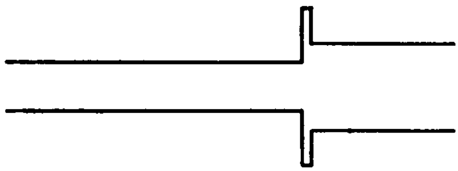

FIG. 8.

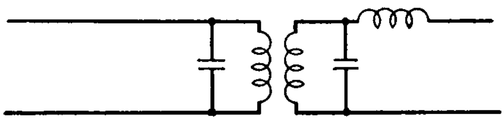

FIG. 9.

looking into the second guide. In parallel with the latter capacitance there is the series combination of the impedance of the radial line and the second guide itself. We may express these ideas by the diagram shown in Fig. 9, where the inductance is used to designate the radial transmission line only because this line, when it is short, is approximately an inductance.

More generally, the discontinuities should be represented by impedances distributed along the guide, as in fact they are. Finally, the section of the guide with the discontinuities may be replaced by an appropriate T-network.

Recently J. R. Whinnery and H. W. Jamieson[8] have obtained explicit expressions for the capacitances of numerous types of "step discontinuities" in transmission lines formed by parallel conducting planes. They show how to apply these results to coaxial conductors. They find theoretical predictions in good agreement with measured values. The equivalent circuits given by Whinnery and Jamieson do not contain ideal transformers; this is because for transmission lines comprised of two conductors, the transformer ratios at discontinuities are equal to unity and the transformers may be omitted.

[6] In the first approximation this capacitance may be neglected.

[7] We assume that we are operating below the second frequency cutoff; otherwise there will also be a conductance.

[8] J. R. Whinnery and H. W. Jamieson, *Equivalent circuits for discontinuities in transmission lines*, I.R.E. Proc., February 1944, pp. 98–114.