12th AIAA/CEAS Aeroacoustics Conference, 8-10 May 2006, Cambridge, MA, USA

Impedance Models in Time Domain including the Extended Helmholtz Resonator Model

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The problem of translating a frequency domain impedance boundary condition to time domain involves the Fourier transform of the impedance function. This requires extending the definition of the impedance not only to all real frequencies but to the whole complex frequency plane. Not any extension, however, is physically possible. The problem should remain causal, the variables real, and the wall passive. This leads to necessary conditions for an impedance function.

Various methods of extending the impedance that are available in the literature are discussed. A most promising one is the so-called z-transform by Ozyoruk & Long, which is nothing but an impedance that is functionally dependent on a suitable complex exponent $\mathrm{e}^{-\mathrm{i}\omega\kappa}$. By choosing κ a multiple of the time step of the numerical algorithm, this approach fits very well with the underlying numerics, because the impedance becomes in time domain a delta-comb function and gives thus an exact relation on the grid points.

An impedance function is proposed which is based on the Helmholtz resonator model, called *Extended Helmholtz Resonator Model*. **This has the advantage that relatively easily the mentioned necessary conditions can be satisfied in advance. At a given frequency, the impedance is made exactly equal to a given design value. Rules of thumb are derived to produce an impedance which varies only moderately in frequency near design conditions.**

An explicit solution is given of a pulse reflecting in time domain at a Helmholtz resonator impedance wall that provides some insight into the reflection problem in time domain and at the same time may act as an analytical test case for numerical implementations, like is presented at this conference by the companion paper AIAA-2006-2569 by N. Chevaugeon, J.-F. Remacle and X. Gallez.

The problem of the instability, inherent with the Ingard-Myers limit with mean flow, is discussed. It is argued that this instability is not consistent with the assumptions of the Ingard-Myers limit and may well be suppressed.

1. Introduction

ACOUSTIC liners are usually described in frequency domain by their impedance. If we use a numerical method in time domain, this is to be translated into an equivalent relation. This is not straightforward. For example, it is principally not possible for an non-real impedance to retain this value for all frequencies. We have to extend the impedance as a function in the complex frequency. This requires certain specific conditions. If these are not obeyed, the impedance would be physically impossible. This would manifest itself by, for example, non-physical instabilities.

2. The problem

A wall, applied with locally reacting lining of impedance type, is without mean flow described by

$$\hat{p}(0,\omega) = Z(\omega)\hat{v}(0,\omega), \tag{1}$$

where $\hat{v} = (\hat{\boldsymbol{v}}\cdot\boldsymbol{n})$, $\hat{\boldsymbol{v}}$ is the acoustic velocity vector and $\boldsymbol{n}$ denotes the normal vector of the wall that points *into* the wall. Pressure is scaled on $\rho_0 c_0^2$, velocity on c_0 and impedance on $\rho_0 c_0$, while we use throughout the $\mathrm{e}^{+\mathrm{i}\omega t}$

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convention. In time domain relation (1) becomes

$$p(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} z(t-\tau)v(0,\tau)\,\mathrm{d}\tau \tag{2}$$

where $p(x,t)$ and $v(x,t)$ denote the inverse Fourier transforms of $\hat{p}(x,\omega)$ and $\hat{v}(x,\omega)$, respectively,

$$p(x,t) = \int_{-\infty}^{\infty} \hat{p}(0,\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega, \quad v(x,t) = \int_{-\infty}^{\infty} \hat{v}(0,\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{3}$$

$z(t)$ denotes the inverse Fourier transform of $Z(\omega)$

$$z(t) = \int_{-\infty}^{\infty} Z(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{4}$$

and the integral over $z(t-\tau)v(0,\tau)$ is called a convolution product.

Equivalent other forms are also possible. If we express v in p by using the admittance $Y(\omega) = Z(\omega)^{-1}$, we have

$$v(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(t-\tau)p(0,\tau)\,\mathrm{d}\tau \tag{5}$$

where $y(t)$ denotes the inverse Fourier transform of $Y(\omega)$. If we write $Z(\omega) = A(\omega)/B(\omega)$ and bring B to the left-hand side of (1), we obtain a convolution on both sides

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} b(t-\tau)p(0,\tau)\,\mathrm{d}\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(t-\tau)v(0,\tau)\,\mathrm{d}\tau \tag{6}$$

where $a(t)$ and $b(t)$ denote the inverse Fourier transforms of $A(\omega)$ and $B(\omega)$.

With mean flow, we have (according to Ingard[1] and Myers[2] for a certain limit of mean flow boundary layer and acoustic boundary layer thickness being small compared to the acoustic wavelength)

$$(\hat{\boldsymbol{v}}\cdot\boldsymbol{n}) = \big(\mathrm{i}\omega + \boldsymbol{v}_0\cdot\nabla - \boldsymbol{n}\cdot(\boldsymbol{n}\cdot\nabla\boldsymbol{v}_0)\big)\frac{\hat{p}}{\mathrm{i}\omega Z}. \tag{7a}$$

As the mean flow usually satisfies $(\boldsymbol{v}_0\cdot\boldsymbol{n}) = 0$, this may be recast into

$$(\hat{\boldsymbol{v}}\cdot\boldsymbol{n}) = \big(\mathrm{i}\omega + \boldsymbol{v}_0\cdot\nabla + \boldsymbol{v}_0\cdot(\boldsymbol{n}\cdot\nabla\boldsymbol{n})\big)\frac{\hat{p}}{\mathrm{i}\omega Z}, \tag{7b}$$

where, incidentally, $(\boldsymbol{n}\cdot\nabla\boldsymbol{n})$ is tangential to the wall surface and vanishes for a flat surface. Other forms, that utilize $\nabla\cdot(\rho_0\boldsymbol{v}_0) = 0$, are also possible.[3] If Z is independent of position we get

$$\mathrm{i}\omega Z\hat{v} = \mathrm{i}\omega\hat{p} + \boldsymbol{v}_0\cdot\nabla\hat{p} + \boldsymbol{v}_0\cdot(\boldsymbol{n}\cdot\nabla\boldsymbol{n})\hat{p}, \tag{7c}$$

In time domain this becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} z(t-\tau)\tfrac{\partial}{\partial\tau}v(0,\tau)\,\mathrm{d}\tau = \tfrac{\partial}{\partial t}p + \boldsymbol{v}_0\cdot\nabla p + \boldsymbol{v}_0\cdot(\boldsymbol{n}\cdot\nabla\boldsymbol{n})p, \tag{8}$$

or any other equivalent form. This boundary condition is, however, not free of fundamental problems.[4,5] Although the derivation by Ingard and Myers is perfectly allright under the conditions assumed (acoustic perturbation $\ll$ mean flow boundary layer $\ll$ typical acoustic wavelength, and ignoring a possible hydrodynamic mode), there are indications that the perturbed boundary layer is unstable, spoiling these very conditions. As a result the model shows instabilities that are apparently not seen in reality. These instabilities are therefore to be suppressed by other means, like in [6]. We will come back to this problem in section 8.

When we are primarily interested in the frequency domain solution for a particular, given, frequency, detailed knowledge of $Z(\omega)$ in frequency domain is usually not available and from a designer's point of view not relevant. However, if we aim at achieving this solution by means of a time dependent model, we need to construct for the design impedance Z_0 at $\omega = \omega_0$ a physically possible model $Z(\omega)$ with the property $Z(\omega_0) = Z_0$ that has an explicitly known and convenient inverse Fourier transform that can be used in our solution method.

In particular, if we assume that we have p and v available along the time grid $t = n\Delta$ for $n = 0, 1, \ldots,$ the boundary condition in time domain should be easily expressible in terms of this discrete representation.

The purpose of the present document is to derive a class of impedance realisations that reduce in time domain to a relation between a limited number of the most recent terms of such time series.

3. Fundamental conditions

Since $p(0,t)$ cannot depend on $v(0,t)$ of the future, we must have $z(t-\tau)=0$ for $\tau > t$ and $Z(\omega)$ has to satisfy the causality condition (see Appendix 9)

$$Z(\omega) \text{ is analytic in } \operatorname{Im}(\omega) < 0. \tag{9}$$

Since $v(0,t)$ cannot depend on $p(0,t)$ of the future, the admittance $Z(\omega)^{-1}$ must be causal and thus

$$Z(\omega) \text{ is non-zero in } \operatorname{Im}(\omega) < 0. \tag{10}$$

In a similar way any representation of the form $Z(\omega) = A(\omega)/B(\omega)$ must be such that

$$A(\omega) \text{ and } B(\omega) \text{ are analytic in } \operatorname{Im}(\omega) < 0. \tag{11}$$

Since $p(0,t)$ and $v(0,t)$ are real, $z(t)$ has to be real too, and so $Z(\omega)$ has to satisfy the reality condition

$$Z^*(\omega) = Z(-\omega), \tag{12}$$

where * denotes the complex conjugate. In other words, the real part $\operatorname{Re}(Z)$ (resistance) is even and the imaginary part $\operatorname{Im}(Z)$ (reactance) is odd in ω.

Usually, the impedance wall is a passive wall that absorbs energy at any frequency, i.e. the acoustic intensity into the wall $\mathcal{I} = \frac{1}{2}\operatorname{Re}(\hat{p}\hat{v}^*) = \frac{1}{2}\operatorname{Re}(Z)|\hat{v}|^2$ is positive. As a result, the resistance $\operatorname{Re}(Z)$ has to be positive. In other words, the impedance satisfies the condition of passivity

$$\operatorname{Re}(Z(\omega)) \geq 0 \quad \text{for all } \omega \in \mathbb{R}. \tag{13}$$

It should be noted that violation of the reality condition produces physically impossible results; violation of causality produces instabilities due "decay" in reversed time; violating the passivity condition implies a wall that produces instead of absorbs energy which will also lead to an instability. Next to these unphysical instabilities which are artefacts due to bad modelling, there is apparently with mean flow the possibility of a physical instability, comparable with a Kelvin-Helmholtz instability.[4,5,7] See section 8.

Altogether we have the following conditions on Z:

Theorem 1
Conditions for a physically representable impedance model $Z(\omega)$ are

1. *$Z(\omega)$ is analytic in $\operatorname{Im}(\omega) < 0$,*
2. *$Z(\omega)$ is non-zero in $\operatorname{Im}(\omega) < 0$,*
3. *$Z(\omega) = Z^*(-\omega)$ for any $\omega \in \mathbb{R}$,*
4. *$\operatorname{Re}(Z(\omega)) \geq 0$ for any $\omega \in \mathbb{R}$.*

If we want to split Z into a numerator/denominator, we have the following condition

Theorem 2
We may Fourier transform the relation $B(\omega)\hat{p}(\omega) = A(\omega)\hat{v}(\omega)$ back into time domain, with $Z(\omega) = A(\omega)/B(\omega)$, only if $A(\omega)$ and $B(\omega)$ are analytic in $\operatorname{Im}(\omega) < 0$.

A. Examples

1. Mass-spring-damper

The mass-spring-damper model, given by

$$Z(\omega) = R + \mathrm{i}\omega m - \frac{\mathrm{i}K}{\omega} \tag{14}$$

must have $R \geq 0$, and has a pole at $\omega = 0$, which should be accounted to the upper half plane, while the zeros are located in the upper half plane as follows:

- if $R = 0$, then $\omega = \pm\sqrt{K/m}$, and $K/m > 0$;
- if $m = 0$, then $\omega = \mathrm{i}K/R$, and $K > 0$;
- if $K = 0$, then $\omega = \mathrm{i}R/m$, and $m > 0$;
- if else, then $\omega = \mathrm{i}(2m)^{-1}R(1 \pm \sqrt{1 - 4mK/R^2})$, and $m > 0$.

In other words, Z satisfies all conditions of theorem 1 if all parameters are interpreted physically and considered only for non-negative values. The corresponding inverse Fourier transform is

$$\frac{z(t)}{2\pi} = R\delta(t) + m\delta'(t) + KH(t) \tag{15}$$

leading to

$$p(t) = Rv(t) + mv'(t) + K\int_{-\infty}^{t} v(\tau)\,\mathrm{d}\tau. \tag{16}$$

If we write

$$Z(\omega) = \frac{\mathrm{i}\omega R + (\mathrm{i}\omega)^2 m + K}{\mathrm{i}\omega} \tag{17}$$

we obtain

$$p'(t) = Rv'(t) + mv''(t) + Kv(t). \tag{18}$$

2. *A rational function in ω*

A useful generalisation of the previous model is an impedance written as a rational function in ω, such that any zero's and poles are located in the upper half plane[a]. To enforce this causality condition is not very difficult, but it is less easy to formulate a general statement how to make sure that the conditions of reality and passivity are satisfied.

For real and positive q, a, b, c, with $2b > c$, we have the following simple form

$$Z(\omega) = \mathrm{i}q\frac{(\omega - a - \mathrm{i}b)(\omega + a - \mathrm{i}b)}{\omega - \mathrm{i}c} \tag{19}$$

with

$$z(t) = 2\pi q(a^2 + (b-c)^2)H(t)\,\mathrm{e}^{-ct}, \tag{20}$$

that satisfies all conditions of causality, reality, and passivity. In time domain it corresponds with the condition

$$p(t) = q(a^2 + (b-c)^2)\int_{-\infty}^{t} \mathrm{e}^{-c\tau}\, v(\tau)\,\mathrm{d}\tau \tag{21}$$

or

$$cp(t) + p'(t) = q(a^2 + b^2)v(t) + 2bqv'(t) + qv''(t). \tag{22}$$

The impedance model has the following properties (see figure 1):

- $\mathrm{Re}(Z(\omega)) \to q(2b - c)$ for $\omega \to \pm\infty$.
- $Z(0) = q(a^2 + b^2)/c$
- $\mathrm{Im}(Z) = 0$ for $\omega = 0$ and, if the square root exists, for $\omega = \pm\sqrt{a^2 + b^2 - 2bc}$.
- If $a^2 + b^2 - 2bc < 0$, $\mathrm{Im}(Z) > 0$ for $\omega > 0$ and $\mathrm{Im}(Z) < 0)$ for $\omega < 0$.
- If $a^2 + b^2 - 2bc > 0$, $\mathrm{Im}(Z)$ attains both positive and negative values for $\omega > 0$, by a loop where the curve crosses itself at $Z = 2qb$
- If $a^2 + b^2 - 2bc = 0$, the curve passes a cusp singularity.

[a]Note that this is often confused with a Padé approximation. For any given (smooth) function, the Padé approximation is a uniquely defined best approximation by a rational function, without any further control of the zero's and poles. So for clarity we will avoid the word "Padé approximation" here.

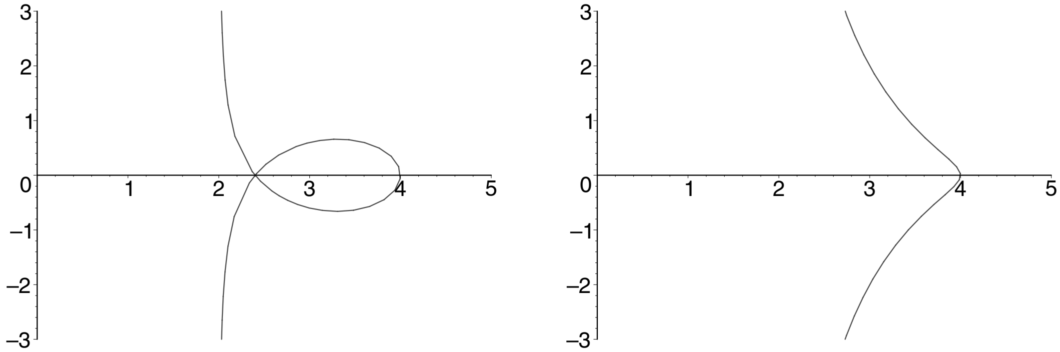


Figure 1. Left: $a = 1, b = 3, c = 1, q = \frac{2}{5}$**, right:** $a = \frac{1}{5}\sqrt{23}, b = 1, c = \frac{6}{5}, q = \frac{5}{2}$**.**

3. Basic Helmholtz Resonator

A very important example is the basic Helmholtz resonator model given by

$$Z(\omega) = R + \mathrm{i}\omega m - \mathrm{i}\cot(\omega L/c_0), \tag{23}$$

which is a good representation of a wall consisting of an array of Helmholtz resonators (see figure 2) with face-sheet resistance R, face-sheet mass reactance ωm, cavity reactance $-\cot(\omega L/c_0)$, cell depth L and sound speed c_0. It is straightforward that $Z(\omega) = Z(-\omega)^*$. With $R, m, L, c_0 \geq 0$, $\mathrm{Re}(Z) = R \geq 0$ is trivially satisfied, while

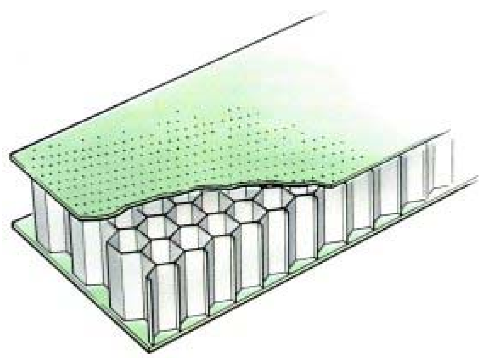


Figure 2. An acoustic liner of Helmholtz resonators.

$Z(\omega)$ is free of zeros in the lower half plane: write $\omega = \xi - \mathrm{i}\eta$, then for $\eta > 0$

$$\mathrm{Re}(Z) = R + m\eta + \frac{\sinh(2\eta L/c_0)}{\cosh(2\eta L/c_0) - \cos(2\xi L/c_0)} > 0 \tag{24}$$

so Z cannot be zero. Z's real poles at $\omega = n\pi c_0/L$ ($n \in \mathbb{Z}$) are to be accounted to the upper half plane. So for $\mathrm{Im}(\omega) < 0$

$$\begin{aligned} Z(\omega) &= R + \mathrm{i}\omega m - \mathrm{i}\cot(\omega L/c_0) \\ &= R + \mathrm{i}\omega m + \frac{\mathrm{e}^{\mathrm{i}\omega L/c_0} + \mathrm{e}^{-\mathrm{i}\omega L/c_0}}{\mathrm{e}^{\mathrm{i}\omega L/c_0} - \mathrm{e}^{-\mathrm{i}\omega L/c_0}} \\ &= R + \mathrm{i}\omega m + 1 + 2\frac{\mathrm{e}^{-2\mathrm{i}\omega L/c_0}}{1 - \mathrm{e}^{-2\mathrm{i}\omega L/c_0}} \\ &= R + \mathrm{i}\omega m + 1 + 2\sum_{n=1}^{\infty} \mathrm{e}^{-2\mathrm{i}n\omega L/c_0}, \end{aligned} \tag{25}$$

and we have for the inverse Fourier transform the δ-comb[8]

$$\frac{z(t)}{2\pi} = R\delta(t) + m\delta'(t) + \delta(t) + 2\sum_{n=1}^{\infty} \delta(t - \lambda_n), \quad \lambda_n = \tfrac{2nL}{c_0}. \tag{26}$$

This yields in time domain the condition

$$p(t)=Rv(t)+mv'(t)+v(t)+2\sum_{n=1}^{\infty}v(t-\lambda_n). \tag{27}$$

Instead of expanding the geometrical series in $\mathrm{e}^{-2\mathrm{i}\omega L/c_0}$ we may also write

$$Z(\omega)=\frac{(R+\mathrm{i}\omega m)(1-\mathrm{e}^{-2\mathrm{i}\omega L/c_0})+1+\mathrm{e}^{-2\mathrm{i}\omega L/c_0}}{1-\mathrm{e}^{-2\mathrm{i}\omega L/c_0}} \tag{28}$$

to arrive at the shorter alternative (without the infinite series)

$$p(t)-p(t-\lambda_1)=R\big(v(t)-v(t-\lambda_1)\big)+m\big(v'(t)-v'(t-\lambda_1)\big)+v(t)+v(t-\lambda_1). \tag{29}$$

4. *Double layer of Helmholtz resonators*

An interesting generalisation that is worthy of further research is the configuration of two Helmholtz resonators on top of each other, separated by a septum.[9] If the top cavity has depth L_1, the bottom cavity has depth L_2, while $L=L_1+L_2$, and the face sheet impedance is given by $Z_1=R_1+\mathrm{i}\omega m_1$ and the septum impedance is $Z_2=R_2+\mathrm{i}\omega m_2$, then we have for the total impedance

$$Z=Z_1+\frac{Z_2\dfrac{\cos(\omega L_1/c_0)\sin(\omega L_2/c_0)}{\sin(\omega L/c_0)}-\mathrm{i}\cot(\omega L/c_0)}{1+\mathrm{i}Z_2\dfrac{\sin(\omega L_1/c_0)\sin(\omega L_2/c_0)}{\sin(\omega L/c_0)}} \tag{30}$$

From its construction we may expect that this impedance function will satisfy all necessary conditions if the parameters are assumed positive. This is, however, difficult to prove. For example, an exhaustive study of all poles and zeros is very difficult for general cell depths L_1 and L_2. It is, however, really feasible for simpler cases like when $m_1=m_2=0$ and $L_1=L_2$.

4. Physically inspired forms

One might be tempted to believe that impedance models which are directly taken from representations of physical realisations would automatically satisfy all the necessary conditions. In some cases this may be true, but in general the expressions are either too complicated to be useful, or simplified or approximated to such an extent that the fundamental conditions are violated.

For example, we find in [9–11] impedance models of the form

$$Z=-\frac{\mathrm{i}\omega\tau}{\sigma c_0}\left[\frac{2J_1(K)}{K\,J_0(K)}-1\right]+\frac{1}{\sigma}\left\{\left[1-\frac{2}{\gamma}J_1(\gamma)\right]+\mathrm{i}M_1(\gamma)\right\}-\mathrm{i}\cot\left(\frac{\omega L}{c_0}\right) \tag{31}$$

where J_n denotes an n-th order Besselfunction,[12] d the hole diameter, τ the plate thickness, ν the air kinematic viscosity and σ is the porosity (fraction of open area). The parameters K, γ and M_1 are defined by

$$K^2=-\frac{\mathrm{i}\omega d^2}{4\nu},\quad \gamma=\frac{\omega d}{c_0},\quad M_1(\gamma)=\frac{4}{\pi}\int_0^{\pi/2}\sin(\gamma\cos\alpha)\sin(\alpha)^2\,\mathrm{d}\alpha.$$

Although the expression appears to be analytic in a half plane, little can be said about the location of the zeros, while the construction of the inverse Fourier transform is not easy. Since in most aero-engine applications $|K|$ is large and γ is small, the following approximation is used

$$Z=(1+\mathrm{i})\frac{\sqrt{8\omega\nu}}{\sigma c_0}\left(1+\frac{\tau}{d}\right)+\frac{\omega^2d^2}{8\sigma c_0^2}+\frac{\mathrm{i}\omega\tau}{\sigma c_0}-\mathrm{i}\cot\left(\frac{\omega L}{c_0}\right), \tag{32}$$

supplemented by a nonlinear term of the form

$$\cdots+\frac{1-\sigma^2}{\sigma c_0}(|\boldsymbol{v}|+0.3|\boldsymbol{v}_0|). \tag{33}$$

These forms are totally useless for the present purposes, because $\sqrt{\omega}$ and $|\boldsymbol{v}|$ cannot be extended to functions that obey both causality and reality conditions.

If any, only the basic Helmholtz resonator liner (example 3) was found to be useful (see below).

5. Direct numerical approaches

In the literature some direct numerical approaches are presented. We will discuss here three.

A. 3-parameter model

In [13] Tam & Aurialt explored the possibilities of a simple 3-parameter model, which is essentially the mass-spring-damper model discussed in subsection 1. It was applied in (for example) [14, 15]. For a positive reactance they made the choice of $K = 0$ (in our notation), and expressed p in v and v'. For a negative reactance they took $m = 0$, used the split form (18), and expressed p' in v and v'. The causality condition of the impedance Z was correctly associated with the location of the pole at $\omega = 0$, but the causality condition of the admittance Z^{-1} was called "stability" condition, which is confusing.

B. Rational functions in ω

An, at first sight general and universal, approach is a model in the form of a rational function in ω (a further generalisation of the model discussed in subsection 2)

$$Z(\omega) = Z_\infty \frac{\prod_{n=1}^{N}(\omega - \alpha_n)}{\prod_{n=1}^{M}(\omega - \beta_n)} = Z_\infty \frac{\sum_{n=1}^{N} a_n \omega^n}{\sum_{n=1}^{M} b_n \omega^n} \tag{34}$$

In [16, 17] and [18] this form was used for $N = 4$ and $M = 3$, in [19] with $N = M = 2$, and in [20] with $N = M = 2$ and $N = M = 4$. Care is required that the zeros and poles are located in the correct part of the complex plane (i.e. the upper half for the present $\mathrm{i}\omega t$ notation), while at the same time the reality and passivity conditions are fulfilled. In practice this means that Z_∞ is real and α_n (and similarly β_n) are either real or come in pairs such that $\alpha_n^* = -\alpha_{n'}$ (see figure 3).

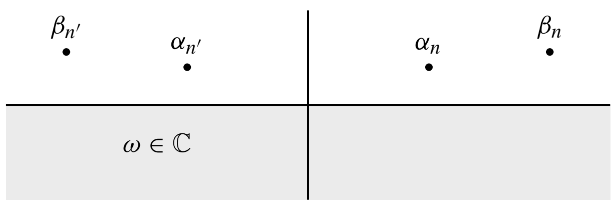


Figure 3. Sketch of location of zeros α_n and poles β_n of $Z(\omega)$.

These extra conditions make it practically difficult to use for Z a Padé approximation, i.e. the best rational-function fit through a given set of data points. This Padé approximation is unique, but there is no guarantee that the result satisfies the fundamental conditions of causality etc. Therefore, it is not appropriate to call this form a Padé expression, like in [19].

The resulting time domain form is either: p expressed in $v, v', \ldots, v^{(M-N)}$, and an integral over v, or: a relation between $p, p', \ldots, p^{(N)}$ and $v, v', \ldots, v^{(M)}$. The occurrence of the derivatives makes this method less attractive in numerical applications.

C. The z-transform, or a rational function in $\mathrm{e}^{-\mathrm{i}\omega\kappa}$

In order to avoid the need of evaluating derivatives, Ozyoruk & Long proposed a model where the frequency dependence is always through the group $\mathrm{e}^{-\mathrm{i}\omega\kappa}$. In the jargon of signal processing, this is called: the z-transform. By taking κ equal to a multiple of the numerical timestep no other data than already available is needed for an exact representation. In our formulation, the idea is as follows.

Consider the following function $F(z)$ in the complex variable z, analytic on the unit disk $|z| < 1$. Although not absolutely necessary[b], it is convenient to assume that F is a rational function. At least in the unit disk, F has a Taylor series expansion.

$$F(z) = F_\infty \frac{\prod_{n=1}^{N}(z - r_n)}{\prod_{n=1}^{M}(z - s_n)} = \sum_{n=0}^{\infty} a_n z^n. \tag{35}$$

[b] To construct such a function in general is related to the Riemann-Hilbert problem[21]

Select a real number κ, and define the impedance function by

$$Z(\omega) = F(\mathrm{e}^{-\mathrm{i}\omega\kappa}). \tag{36}$$

If $|r_n|, |s_n| \geq 1$, $F(z)$ is analytic and non-zero on the unit disc $|z| < 1$, with the result that Z is analytic and non-zero in $\mathrm{Im}(\omega) < 0$. If F_∞ is real, and F's poles and zeros are either real or come in pairs, such that $r_n^* = r_{n'}$ and $s_n^* = s_{n'}$, Z satisfies the reality condition (see figure 4). Automatically all a_n will be real. If we are able to make sure that $\mathrm{Re}(F) \geq 0$ along the unit circle $|z| = 1$ (this is more difficult), the impedance is passive.

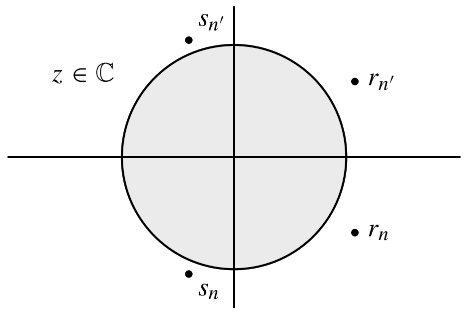


Figure 4. Sketch of location of zeros r_n and poles s_n of $F(z)$.

The inverse Fourier transform of Z is now the δ-comb

$$z(t) = 2\pi \sum_{n=0}^{\infty} a_n \delta(t - n\kappa) \tag{37}$$

which follows from the relation

$$\delta(t - n\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega n\kappa}\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega. \tag{38}$$

The key idea is now to take $\kappa = \Delta t$, or a multiple, to synchronize the δ-comb with the numerical grid of stepsize Δt. We find then in time domain (without mean flow)

$$p(t) = \sum_{n=0}^{\infty} a_n v(t - n\Delta t). \tag{39}$$

Note that the infinite series converges exponentially if the poles and zeros are taken outside (not on) the unit disk. The larger $|r_n| - 1$ and $|s_n| - 1$ are taken, the faster the series will converge.

If the infinite series is inconvenient, we can expand the products and write

$$F(z) = \frac{\sum_{n=1}^{N} c_n z^n}{\sum_{n=1}^{M} d_n z^n} \tag{40}$$

to get (without mean flow)

$$\sum_{n=1}^{M} d_n p(t - n\Delta t) = \sum_{n=1}^{N} c_n v(t - n\Delta t). \tag{41}$$

Note that the basic Helmholtz resonator model is almost of the present form. If $m = 0$ and we identify $2L/c_0 = \Delta t$ we obtain from (28)

$$Z(\omega) = \frac{R(1 - \mathrm{e}^{-\mathrm{i}\omega\Delta t}) + 1 + \mathrm{e}^{-\mathrm{i}\omega\Delta t}}{1 - \mathrm{e}^{-\mathrm{i}\omega\Delta t}} = \frac{(R+1) - (R-1)\,\mathrm{e}^{-\mathrm{i}\omega\Delta t}}{1 - \mathrm{e}^{-\mathrm{i}\omega\Delta t}} \tag{42}$$

The zero in the z-plane is found at $\frac{R+1}{R-1}$, which is indeed less than -1 for $0 < R < 1$ and greater than 1 for $R > 1$. The pole is at $z = 1$.

The absence of derivatives makes the present method very desirable, especially if we are able to define a parameterised family of impedance models satisfy the fundamental conditions automatically. In the next section we will propose such a family, based on the Helmholtz model of above.

6. Extended Helmholtz resonator model

Inspired by the Helmholtz resonator model and the z-transform method we will now propose an impedance model that is rich enough to represent any impedance at any given design frequency, needs no additional time derivatives in its time-domain representation and is guaranteed physical, i.e. satisfies the fundamental conditions. We will call this the *Extended Helmholtz Resonator Model*.

A. Frequency domain

In comparison with the basic Helmholtz resonator of subsection 3 we have applied the following changes.

1. Damping $\varepsilon > 0$ in the cavity's fluid;
2. A varying cavity reactance by parameter $\beta > 0$;
3. Cavity depth $\sim$ multiple of time step: $2L/c_0 = \nu\Delta t$;

resulting into

$$Z(\omega) = R + \mathrm{i}\omega m - \mathrm{i}\beta\cot\left(\tfrac{1}{2}\omega\nu\Delta t - \mathrm{i}\tfrac{1}{2}\varepsilon\right). \tag{43}$$

See figure 5 for an example of the model as a function of ω. The trajectory slowly spirals upwards due to the

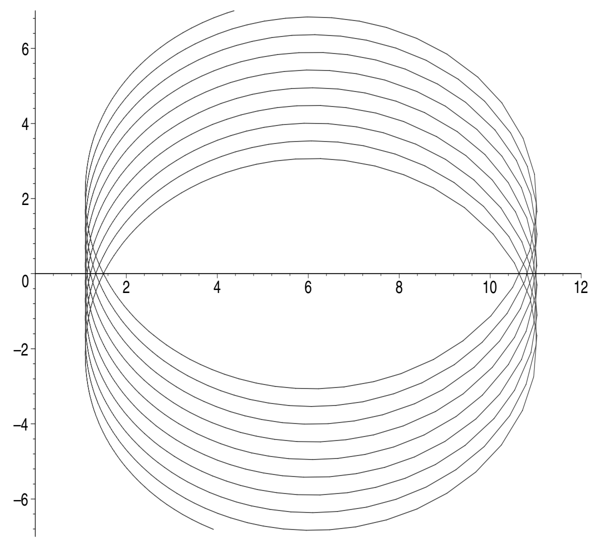


Figure 5. Extended Helmholtz resonator impedance as a function of frequency. $R = 1, m = 0.15, \beta = 1, \varepsilon = 0.1, \Delta t = 1, \nu = 1, \omega = -4.9\pi \ldots 4.9\pi$.

small but non-zero m. The relatively small parameter ε is responsible for the large radius of one turn. It is straightforward to check that the model satisfies the reality condition. The model is passive:

$$0 < \mathrm{Re}(Z) = R + \beta\frac{\sinh(\varepsilon)}{\cosh(\varepsilon) - \cos(\omega\nu\Delta t)} \in [R + \beta\tanh(\tfrac{1}{2}\varepsilon),\ R + \beta\coth(\tfrac{1}{2}\varepsilon)]. \tag{44}$$

The model is causal, since the poles are located at

$$\omega = \frac{1}{\nu\Delta t}(2n\pi + \mathrm{i}\varepsilon), \quad n \in \mathbb{Z}, \tag{45}$$

i.e. in the upper complex half plane, while for $\omega = \xi - \mathrm{i}\eta$ we have

$$\mathrm{Re}(Z) = R + m\eta + \beta\frac{\sinh(\eta\nu\Delta t + \varepsilon)}{\cosh(\eta\nu\Delta t + \varepsilon) - \cos(\xi\nu\Delta t)} > 0 \tag{46}$$

for any $\eta > 0$. So for $\mathrm{Im}(\omega) < \varepsilon/\nu\Delta t$ we have

$$Z(\omega) = R + \mathrm{i}\omega m + \beta + 2\beta \sum_{n=1}^{\infty} \mathrm{e}^{-\mathrm{i}\omega n\nu\Delta t - \varepsilon n}, \tag{47}$$

leading to

$$\frac{z(t)}{2\pi} = R\delta(t) + m\delta'(t) + \beta\delta(t) + 2\beta \sum_{n=1}^{\infty} \mathrm{e}^{-\varepsilon n}\, \delta(t - n\nu\Delta t). \tag{48}$$

B. Constructing an impedance that satisfies a design condition

Finally we have to fix the parameters for a given design impedance $Z = Z_0 = R_0 + \mathrm{i}X_0$ at a frequency $\omega = \omega_0 > 0$. So we have to find parameter values that satisfy

$$R_0 = R + \beta \frac{\sinh(\varepsilon)}{\cosh(\varepsilon) - \cos(\omega_0\nu\Delta t)} \tag{49a}$$

$$X_0 = \omega_0 m - \beta \frac{\sin(\omega_0\nu\Delta t)}{\cosh(\varepsilon) - \cos(\omega_0\nu\Delta t)} \tag{49b}$$

Suppose that a wave period $T = 2\pi/\omega_0$ is divided in N parts (typically $N = 20$), i.e. $T = N\Delta t$. So

$$\omega_0\nu\Delta t = \frac{2\pi\nu}{N}. \tag{50}$$

If $X_0 = 0$, we are ready by taking $m = \beta = 0$, so in the following we will assume that $X_0 \neq 0$. From a numerical-algorithmic point of view, a most convenient choice is

$$m = 0 \tag{51}$$

because it relieves us of the task to calculate $v'(t)$ without flow, and no second derivative with mean flow. In that case we have no other way to comply with the sign of X_0 than by selecting $\sin(\omega_0\nu\Delta t)$ opposite in sign of X_0:

$$\text{if} \quad X_0 < 0, \quad \text{we need to take} \quad \nu \in \{1, \ldots, \lceil \tfrac{1}{2}N \rceil - 1\}, \tag{52a}$$

$$\text{if} \quad X_0 > 0, \quad \text{we need to take} \quad \nu \in \{\lfloor \tfrac{1}{2}N \rfloor + 1, \ldots, N - 1\}. \tag{52b}$$

If we now choose ε between the following limits

$$0 \leq \varepsilon \leq \operatorname{arsinh}\left(-\frac{R_0}{X_0} \sin(\omega_0\nu\Delta t) \right), \tag{53}$$

then β is determined from X_0 and R from R_0 as follows.

$$\beta = -X_0 \frac{\cosh(\varepsilon) - \cos(\omega_0\nu\Delta t)}{\sin(\omega_0\nu\Delta t)} \tag{54a}$$

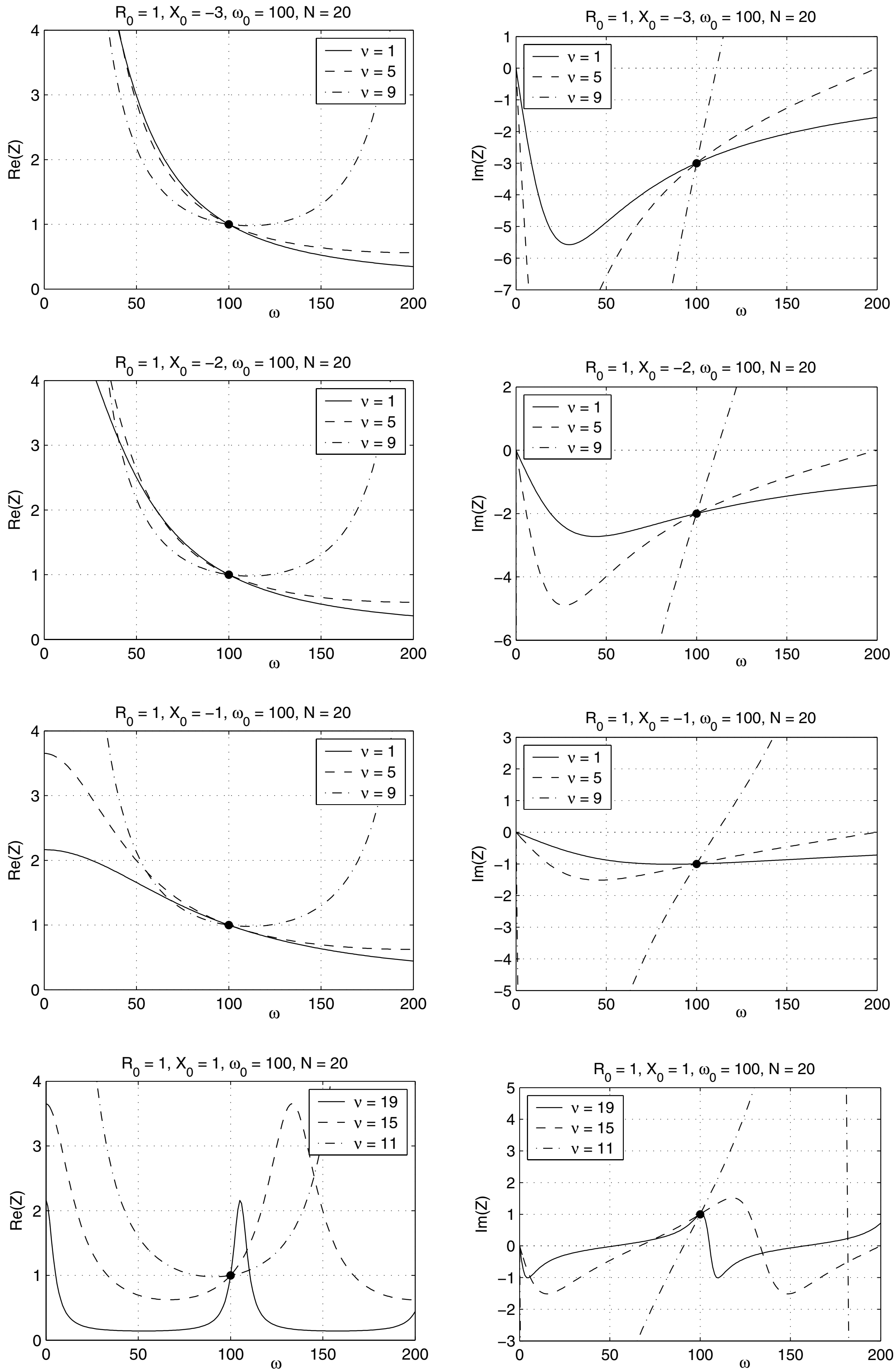
$$R = R_0 + X_0 \frac{\sinh(\varepsilon)}{\sin(\omega_0\nu\Delta t)}. \tag{54b}$$

The restriction on ε is to make sure that $R \geq 0$.

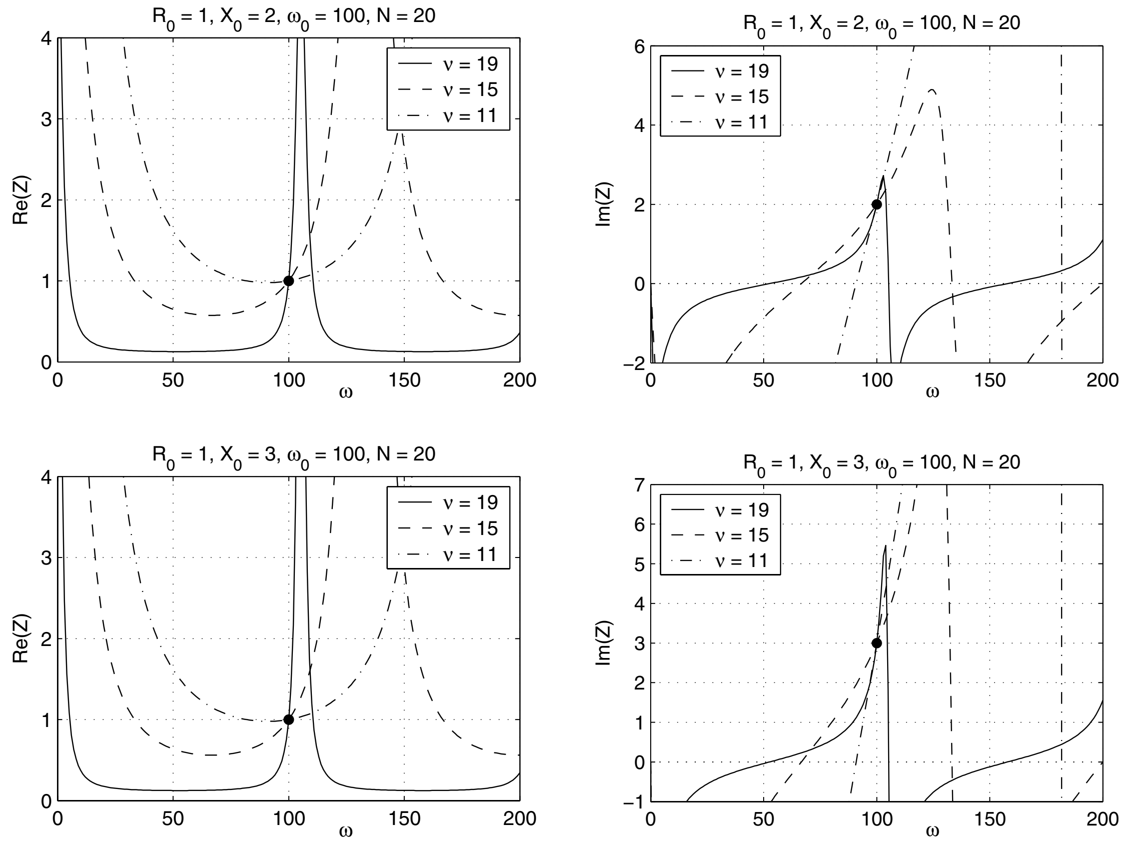
C. Parameter study

A smooth behaviour of Z in $\omega \in \mathbb{R}$ is obtained by pushing the poles (45) away from the real axis, by choosing ε as large as is reasonable, for example $\varepsilon = 0.9 \operatorname{arsinh}(-(R_0/X_0)\sin(\omega_0\nu\Delta t))$.

In order to investigate the effect of the choice of parameter ν on the smoothness and variability of Z, we made a series of plots of $\mathrm{Re}(Z)$ and $\mathrm{Im}(Z)$, as a function of ω, for various ν and Z_0. In all cases we selected $\varepsilon = 0.9 \operatorname{arsinh}(-(R_0/X_0)\sin(\omega_0\nu\Delta t))$. We took $R_0 = 1$, $\omega_0 = 100$ and $N = 20$ fixed, and varied X_0 and ν as: $X_0 = -3, -2, -1, 1, 2, 3$ and $\nu = 1, 5, 9$ for $X_0 < 0$, and $\nu = 19, 15, 11$ for $X_0 > 0$. The results for $0 \leq \omega \leq 2\omega_0$ are shown below.



R0 = 1, X0 = −3, ω0 = 100, N = 20
ν = 1
ν = 5
ν = 9
Re(Z)
ω
R0 = 1, X0 = −3, ω0 = 100, N = 20
ν = 1
ν = 5
ν = 9
Im(Z)
ω
R0 = 1, X0 = −2, ω0 = 100, N = 20
ν = 1
ν = 5
ν = 9
Re(Z)
ω
R0 = 1, X0 = −2, ω0 = 100, N = 20
ν = 1
ν = 5
ν = 9
Im(Z)
ω
R0 = 1, X0 = −1, ω0 = 100, N = 20
ν = 1
ν = 5
ν = 9
Re(Z)
ω
R0 = 1, X0 = −1, ω0 = 100, N = 20
ν = 1
ν = 5
ν = 9
Im(Z)
ω
R0 = 1, X0 = 1, ω0 = 100, N = 20
ν = 19
ν = 15
ν = 11
Re(Z)
ω
R0 = 1, X0 = 1, ω0 = 100, N = 20
ν = 19
ν = 15
ν = 11
Im(Z)
ω



The conclusion that may be drawn (finer details depend on the other parameters), is that for $X_0 < 0$ the best choice, i.e. where Z varies not too much near Z_0, is for $\nu = 1$. For $X_0 > 0$ on the other hand, it is better to be halfway between $\nu = \frac{1}{2}N$ and N.

D. Time domain

In time domain we have thus

$$p(t) = (R+\beta)v(t) + 2\beta \sum_{n=1}^{\infty} \mathrm{e}^{-\varepsilon n}\, v(t - n\nu\Delta t). \tag{55}$$

or

$$p(t) - \mathrm{e}^{-\varepsilon}\, p(t-\nu\Delta t) = (R+\beta)v(t) - (R-\beta)\,\mathrm{e}^{-\varepsilon}\, v(t-\nu\Delta t). \tag{56}$$

It is not immediately clear which form is preferred in view of the manipulations of p that are needed with mean flow.

Without flow we could have had $m \neq 0$. With flow, however, we have already a derivative, so here it is better avoided. We have

$$p'(t) + \boldsymbol{v}_0\cdot\nabla p(t) + \boldsymbol{v}_0\cdot(\boldsymbol{n}\cdot\nabla\boldsymbol{n})p(t) = (R+\beta)v'(t) + 2\beta\sum_{n=1}^{\infty}\mathrm{e}^{-\varepsilon n}\, v'(t-n\nu\Delta t). \tag{57}$$

or

$$\begin{aligned}\big[p' + \boldsymbol{v}_0\cdot\nabla p + \boldsymbol{v}_0\cdot(\boldsymbol{n}\cdot\nabla\boldsymbol{n})p\big]_t - \mathrm{e}^{-\varepsilon}\big[p' + \boldsymbol{v}_0\cdot\nabla p + \boldsymbol{v}_0\cdot(\boldsymbol{n}\cdot\nabla\boldsymbol{n})p\big]_{t-\nu\Delta t}& \\ = (R+\beta)v'(t) - (R-\beta)\,\mathrm{e}^{-\varepsilon}\, v'(t-\nu\Delta t).& \end{aligned} \tag{58}$$

E. A double layer extended Helmholtz model

Based on the conjecture (this is difficult to prove in general) that the double layer Helmholtz resonator model (30), section 4, satisfies always the fundamental conditions, a generalisation to a Double Layered Extended Helmholtz Resonator model may be of interest, for example if we need two design impedances at two frequencies within the same model. If we identify $2\omega L_1/c_0 = \nu_1 \Delta t$, $2\omega L_2/c_0 = \nu_2 \Delta t$, add damping ε_1 and ε_2, write $\nu = \nu_1 + \nu_2$ and $\varepsilon = \varepsilon_1 + \varepsilon_2$, take $m_1 = m_2 = 0$ and introduce a parameter β, we may arrive at something like

$$Z(\omega) = R_1 + \frac{R_2 \dfrac{\cos(\frac{1}{2}\omega\nu_1\Delta t - \mathrm{i}\frac{1}{2}\varepsilon_1)\sin(\frac{1}{2}\omega\nu_2\Delta t - \mathrm{i}\frac{1}{2}\varepsilon_2)}{\sin(\frac{1}{2}\omega\nu\Delta t - \mathrm{i}\frac{1}{2}\varepsilon)} - \mathrm{i}\beta\cot(\frac{1}{2}\omega\nu\Delta t - \mathrm{i}\frac{1}{2}\varepsilon)}{1 + \mathrm{i}R_2 \dfrac{\sin(\frac{1}{2}\omega\nu_1\Delta t - \mathrm{i}\frac{1}{2}\varepsilon_1)\sin(\frac{1}{2}\omega\nu_2\Delta t - \mathrm{i}\frac{1}{2}\varepsilon_2)}{\sin(\frac{1}{2}\omega\nu\Delta t - \mathrm{i}\frac{1}{2}\varepsilon)}} =$$
$$\frac{(2R_1 - R_1R_2 + R_2 - 2\beta)\,\mathrm{e}^{-\mathrm{i}\omega\nu-\varepsilon} + (R_1 - 1)R_2\,\mathrm{e}^{-\mathrm{i}\omega\nu_1-\varepsilon_1} + (R_1 + 1)R_2\,\mathrm{e}^{-\mathrm{i}\omega\nu_2-\varepsilon_2} - (2R_1 + R_1R_2 + R_2 + 2\beta)}{(2 - R_2)\,\mathrm{e}^{-\mathrm{i}\omega\nu-\varepsilon} + R_2\,\mathrm{e}^{-\mathrm{i}\omega\nu_1-\varepsilon_1} + R_2\,\mathrm{e}^{-\mathrm{i}\omega\nu_2-\varepsilon_2} - (2 + R_2)} \tag{59}$$

The numerator and denominator can be Fourier transformed as before, leading to time domain conditions similar to (56) and (58).

7. An exact solution

In order to understand the behaviour of an impedance in time-domain, we will investigate a concrete reflection problem with the extended Helmholtz resonator impedance type, introduced in section 6. The following analysis generalises [8].

Consider the 1D acoustic wave equations (in dimensional form) for velocity perturbations v, perturbations pressure p, and density perturbations ρ

$$\frac{\partial\rho}{\partial t} + \rho_0\frac{\partial v}{\partial x} = 0, \qquad \rho_0\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = 0, \qquad p = c_0^2\rho. \tag{60}$$

Consider an arbitrary plane wave $p_i = \rho_0 c_0^2 f(t - x/c_0)$ incident from $x < 0$, and reflecting into $p_r = \rho_0 c_0^2 g(t + x/c_0)$ by an impedance wall at $x = 0$, with impedance $Z(\omega)$ given by

$$Z(\omega) = \rho_0 c_0\Big[R + \mathrm{i}\omega m - \mathrm{i}\cot\big(\omega L/c_0 - \mathrm{i}\tfrac{1}{2}\varepsilon\big)\Big]. \tag{61}$$

The total acoustic field is given for $x < 0$ by:

$$p(x,t) = \rho_0 c_0^2\big(f(t - x/c_0) + g(t + x/c_0)\big), \tag{62a}$$
$$v(x,t) = c_0\big(f(t - x/c_0) - g(t + x/c_0)\big). \tag{62b}$$

The reflected wave g is determined via the impedance condition.

$$\rho_0 c_0\big(f(t) + g(t)\big) = \frac{1}{2\pi}\int_{-\infty}^{\infty} z(t-\tau)\big(f(\tau) - g(\tau)\big)\,\mathrm{d}\tau. \tag{63}$$

where

$$\frac{z(t)}{2\pi} = \rho_0 c_0\Big[R\delta(t) + m\delta'(t) + \delta(t) + 2\sum_{n=1}^{\infty}\mathrm{e}^{-\varepsilon n}\,\delta\big(t - 2nL/c_0\big)\Big]. \tag{64}$$

For any incident wave starting at some finite time ($t = 0$) we have $f(t) = 0$ for $t < 0$, so that all in all the infinite integral reduces to an integration over the interval $[0, t]$. For any time t, $g(t)$ is built up from $f(t)$ and the history of f and g along $[0, t]$.

A. Solution

Assuming f and g to be sufficiently smooth and using the sifting property of the delta-functions, the above relation reduces to

$$(R+2)g(t) + mg'(t) = Rf(t) + mf'(t) + 2\sum_{n=1}^{\infty}\mathrm{e}^{-\varepsilon n}\big(f(t_n) - g(t_n)\big), \tag{65}$$

where $t_n = t - 2nL/c_0$. If $m \neq 0$, this differential equation in g has solution

$$g(t) = f(t) - \frac{2}{m}\int_{-\infty}^{t} \exp\Big(-\frac{R+2}{m}(t-\tau)\Big)\Big[f(\tau) - \sum_{n=1}^{\infty} \mathrm{e}^{-\varepsilon n}\big(f(t_n) - g(t_n)\big)\Big]\,\mathrm{d}\tau. \tag{66}$$

If $m = 0$ we have

$$g(t) = \frac{R}{R+2} f(t) + \frac{2}{R+2}\sum_{n=1}^{\infty} \mathrm{e}^{-\varepsilon n}\big(f(t_n) - g(t_n)\big). \tag{67}$$

Note that the integral really starts from whenever $f \neq 0$, say, from $\tau = 0$, which means that the summation is finite and ends at

$$N(t) = \lfloor tc_0/2L \rfloor. \tag{68}$$

If we let f tend to a pulse, we can evaluate the integral in the limit $f(t) = \delta(t)$ explicitly to get ($m \neq 0$)

$$g(t) = \delta(t) - \frac{2}{m}\sum_{n=0}^{N(t)} \exp\Big(-\frac{R+2}{m}t_n - \varepsilon n\Big) L_n^{(-1)}(2t_n/m) \tag{69}$$

and ($m = 0$)

$$g(t) = \frac{R}{R+2}\delta(t) + \frac{4}{R(R+2)}\sum_{n=1}^{N(t)} \Big(\frac{R}{R+2}\Big)^n \mathrm{e}^{-\varepsilon n}\,\delta(t_n). \tag{70}$$

where $L_n^{(-1)}$ is a generalised Laguerre polynomial:[12,22]

$$L_n^{(-1)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n-1}{k-1} x^k \tag{71}$$

with $L_0^{(-1)}(x) = 1$, $L_1^{(-1)}(x) = -x$, etc. It may be noted that $L_n^{(-1)}(x)$ has the generating function

$$\begin{aligned}\mathcal{G}(x,z) = \exp\Big(-\frac{xz}{1-z}\Big) &= 1 - xz + (-x + \tfrac{1}{2}x^2)z^2 + (-x + x^2 - \tfrac{1}{6}x^3)z^3 \\ &\quad + (-x + \tfrac{3}{2}x^2 - \tfrac{1}{2}x^3 + \tfrac{1}{24}x^4)z^4 + (-x + 2x^2 - x^3 + \tfrac{1}{6}x^4 - \tfrac{1}{120}x^5)z^5 \\ &\quad + (-x + \tfrac{5}{2}x^2 - \tfrac{5}{3}x^3 + \tfrac{5}{12}x^4 - \tfrac{1}{24}x^5 + \tfrac{1}{720}x^6)z^6 + \mathcal{O}(z^7)\end{aligned} \tag{72}$$

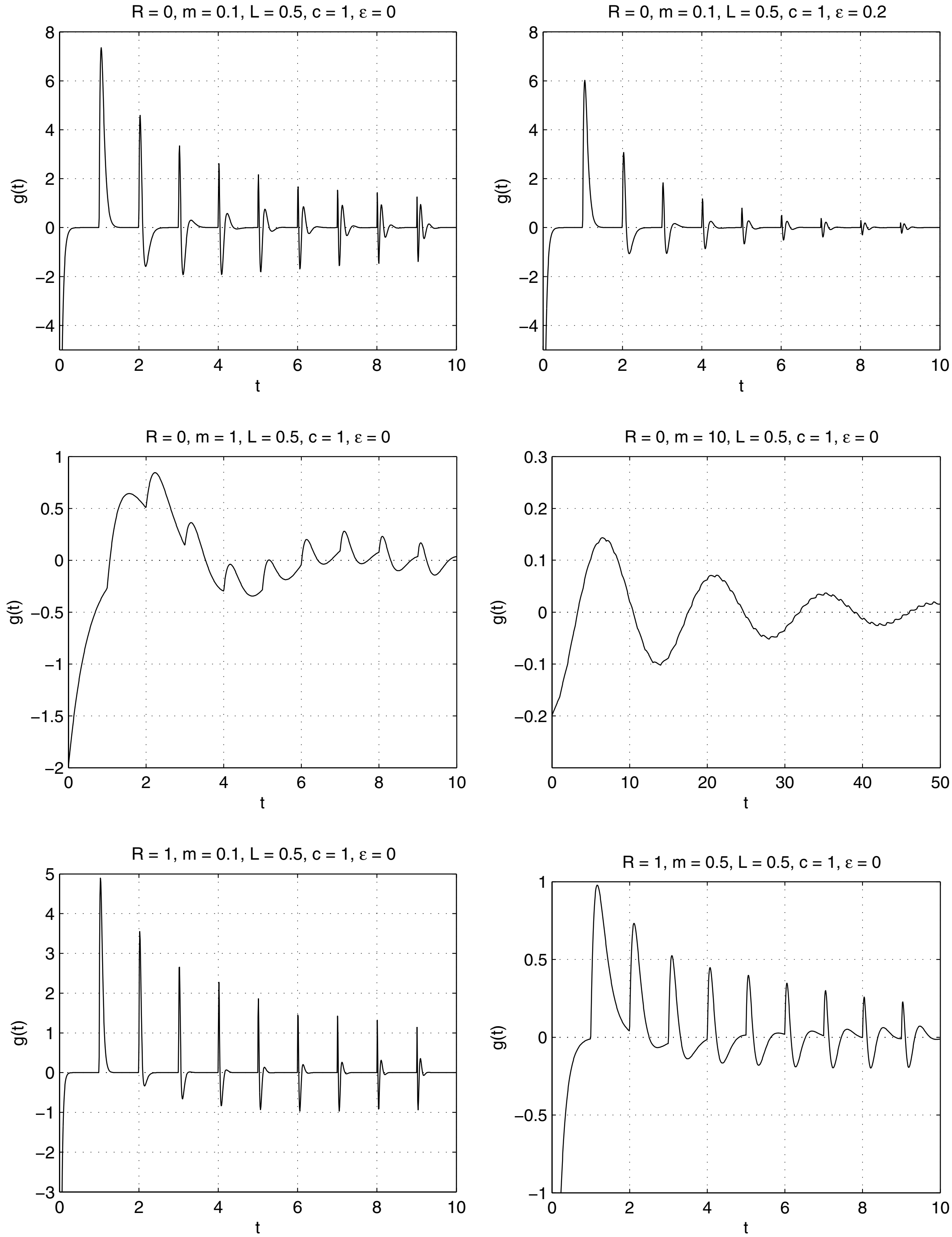
From this solution (69, 70) we can solve for any initial profile by writing

$$f(t) = f_0(t) = \int_{-\infty}^{\infty} f_0(t-\tau)\delta(\tau)\,\mathrm{d}\tau \tag{73}$$

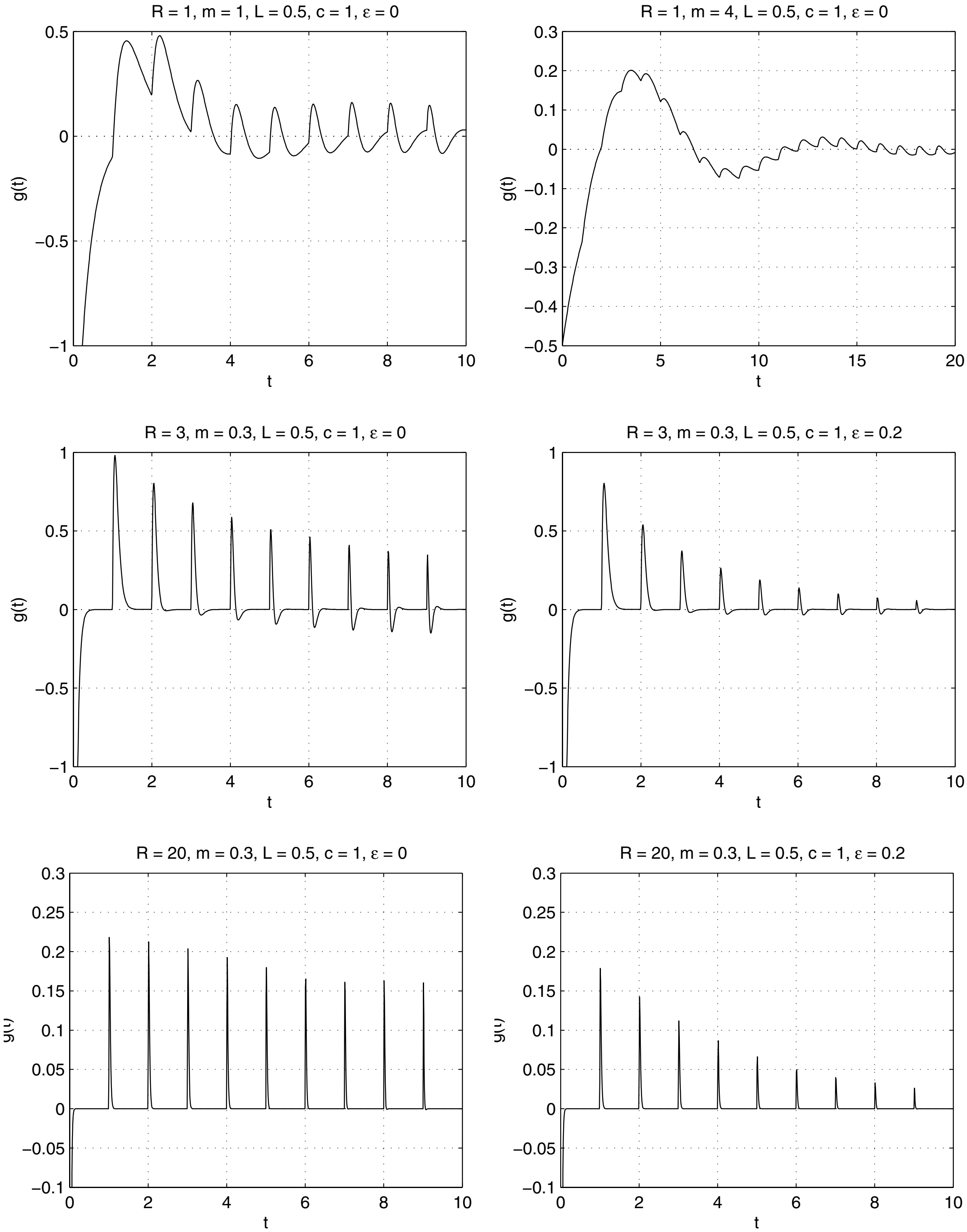
Assuming that $f_0(t) \equiv 0$ for $t < 0$, we have for $m = 0$

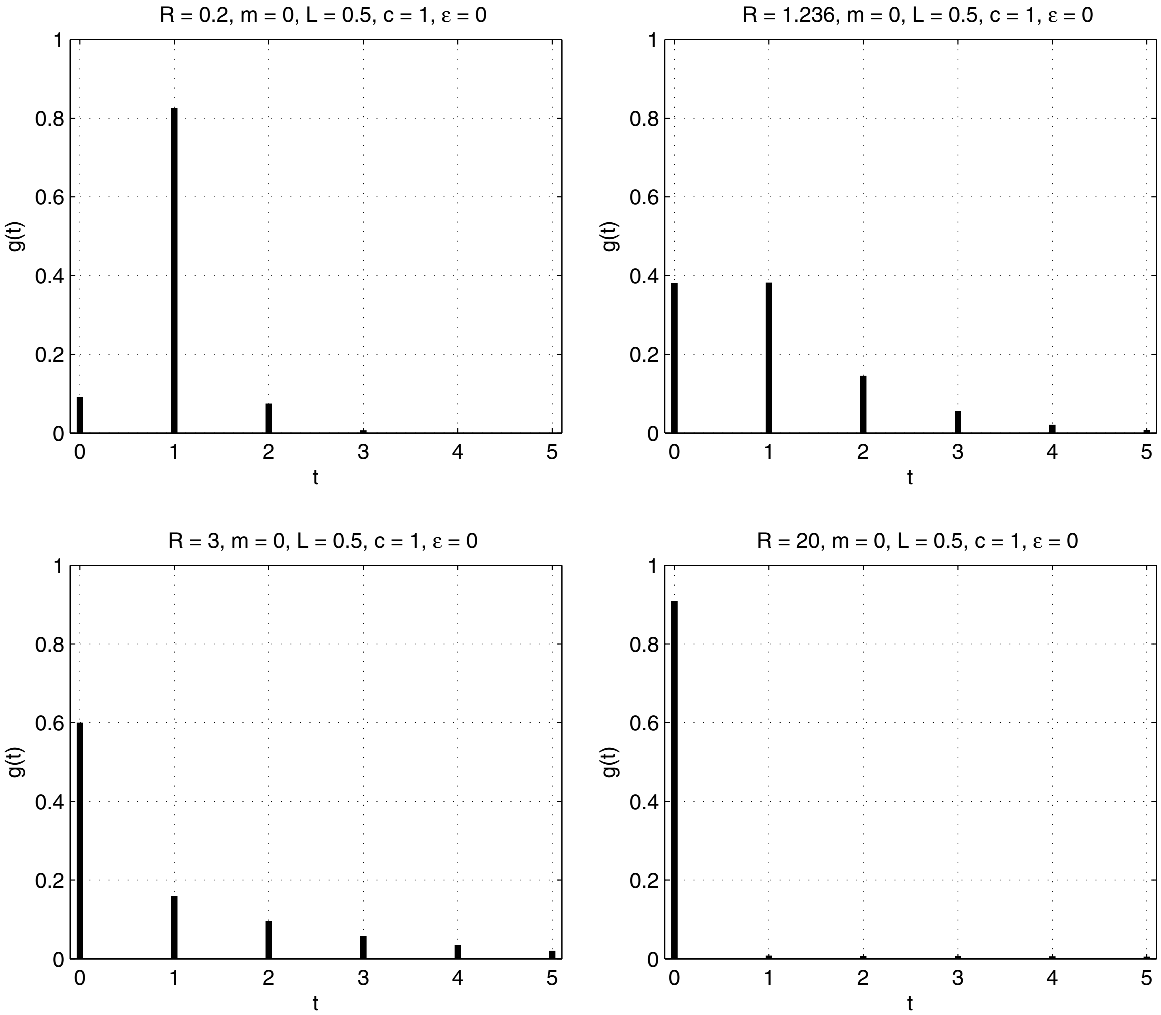
$$\begin{aligned}g(t) &= \int_{-\infty}^{\infty} f_0(t-\tau)\left[\frac{R}{R+2}\delta(\tau) + \frac{4}{R(R+2)}\sum_{n=1}^{N(\tau)}\Big(\frac{R}{R+2}\Big)^n \mathrm{e}^{-2\varepsilon n}\,\delta(\tau_n)\right]\mathrm{d}\tau \\ &= \frac{R}{R+2} f_0(t) + \frac{4}{R(R+2)}\sum_{n=1}^{N(t)}\Big(\frac{R}{R+2}\Big)^n \mathrm{e}^{-2\varepsilon n}\, f_0(t_n).\end{aligned} \tag{74}$$

In the following we present some graphical examples of this solution $g(t)$, i.e. (69) and (70). When $m \neq 0$ the first δ-function is not drawn. When $m = 0$, the δ-functions are drawn like a solid line of a length, proportional to its amplitude.



R = 0, m = 0.1, L = 0.5, c = 1, ε = 0
R = 0, m = 0.1, L = 0.5, c = 1, ε = 0.2
R = 0, m = 1, L = 0.5, c = 1, ε = 0
R = 0, m = 10, L = 0.5, c = 1, ε = 0
R = 1, m = 0.1, L = 0.5, c = 1, ε = 0
R = 1, m = 0.5, L = 0.5, c = 1, ε = 0
g(t)
t


R = 1, m = 1, L = 0.5, c = 1, ε = 0
R = 1, m = 4, L = 0.5, c = 1, ε = 0
R = 3, m = 0.3, L = 0.5, c = 1, ε = 0
R = 3, m = 0.3, L = 0.5, c = 1, ε = 0.2
R = 20, m = 0.3, L = 0.5, c = 1, ε = 0
R = 20, m = 0.3, L = 0.5, c = 1, ε = 0.2
g(t)
t



8. Fundamental problems related to the Ingard-Myers correction for mean flow

From analogy with the Helmholtz instability along an interface between two media of different velocities, there are serious indications[4,5,7,23] that the boundary condition, that results from the Ingard-Myers limit[1,2] of a vanishing mean flow boundary layer along a lined wall, allows instabilities for any frequency. These instabilities are probably artefacts of the *linear* model of acoustic perturbations that are small compared to the boundary layer, which on its turn is small compared to a typical wave length. It is clear that a model, only valid for infinitesimally small perturbations close to the wall, that allows an instability, bites its own tail once the instability has grown to a size such that the vicinity of the wall cannot be portrayed linearly.

Therefore, even if the model is unstable, it is not clear that this instability should be included in the results. Not only experimentally there seems to be no direct evidence yet, but also theoretically an instability conflicts with its own assumptions. Except for impedances near the imaginary axis[7] the predicted unstable mode has invariably a large growth rate, making its presence physically inconsistent.

Our linear model is based on the assumption of perturbations which are small compared to any other reference scale. This is especially the case for the Ingard-Myers boundary condition, where the wall streamline is supposed to undergo small deflections η, much smaller than the thin viscous boundary layer of thickness, say, δ_b. For definiteness we recall the following derivation. A thin layer of zero mean flow is postulated[c] near the wall, separated from the main mean flow by a vortex sheet. The vortex sheet position at, say, $y = \delta_b$, is perturbed by the incident sound field to $y = \delta_b + \eta(x, t)$ (see figure 6).

[c]It may be noted that Eversman and Beckemeyer showed in [24] that (assuming the mean flow velocity nowhere being equal to the modal phase velocity) a continuous mean flow profile of vanishing boundary layer thickness also yields the Ingard-Myers condition.

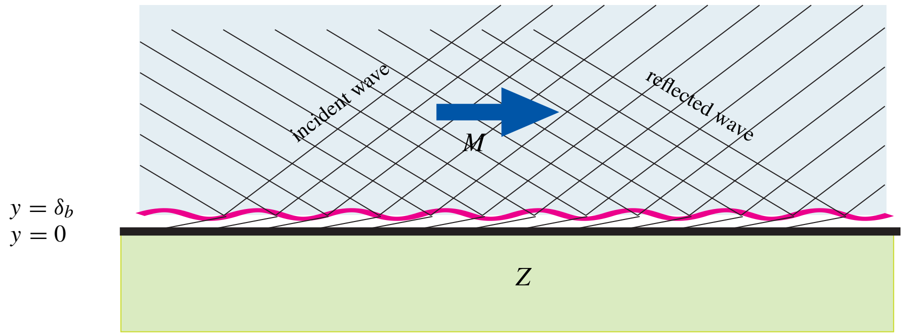


Figure 6. Sketch of the geometrical assumptions for the Ingard-Myers boundary condition

A particle moving along the vortex sheet flow side has, after linearisation, a vertical velocity component v_f given by (in obvious complex short-hand notation)

$$y = \delta_b + \eta(x)\,\mathrm{e}^{\mathrm{i}\omega t} - 0: \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\eta(x(t),t) \simeq (\mathrm{i}\omega + M\tfrac{\partial}{\partial x})\eta(x) = v_f(x,\delta_b),$$

while on the wall side we get for the vertical velocity v_w

$$y = \delta_b + \eta(x)\,\mathrm{e}^{\mathrm{i}\omega t} + 0: \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\eta(x(t),t) \simeq \mathrm{i}\omega\eta(x) \qquad = v_w(x,\delta_b).$$

Across the vortex sheet we have continuity of pressure

$$y = \delta_b + \eta(x)\,\mathrm{e}^{\mathrm{i}\omega t}: \quad p_f(x,\delta_b) = p_w(x,\delta_b).$$

At the wall we have the impedance condition for harmonic perturbations

$$y = 0 \qquad : \quad p_w(x,0) \ = -Zv_w(x,0).$$

All this is valid under the acoustic approximation of η being small. Next we add the approximation of a vanishing boundary layer thickness $\delta_b \to 0$. Then pressure and velocity in the layer $0 < y < \delta_b$ become uniform in y and we find, after eliminating η, the Ingard-Myers condition relating p_f and v_f

$$(\mathrm{i}\omega + M\tfrac{\partial}{\partial x})p_f(x,0) = -\mathrm{i}\omega Z v_f(x,0).$$

So we have a double limit of vanishing η and δ_b, with the essential condition $0 < \eta \ll \delta_b \ll 1$. Clearly, this condition is eventually not met for exponentially growing deflections of the wall streamline and in these cases a nonlinear model is required.

We conclude that as long as we are dealing with the present linear model, including the instability seems worse than excluding it. It seems reasonable to suppress the instability as it occurs in the time-domain problem, like is proposed by Chevaugeon, Remacle and Gallez.[6]

9. Conclusions

The problem of translating an impedance boundary condition to time domain involves the Fourier transform of the impedance function. This requires extending the definition of the impedance not only to all real frequencies but to the whole complex frequency plane. Not any extension, however, is physically possible. The problem should remain causal, the variables real, and the wall passive. This leads to necessary conditions for an impedance function.

Various methods of extending the impedance that are available in the literature are discussed. A most promising one is the so-called z-transform, which is nothing but an impedance that is functionally dependent on a suitable complex exponent $\mathrm{e}^{-\mathrm{i}\omega\kappa}$. By choosing κ a multiple of the time step of the numerical algorithm, this approach fits very well with the underlying numerics, because the impedance becomes in time domain a delta-comb function and gives thus an exact relation on the grid points.

In the present paper a recipe is proposed for an impedance which is based on the Helmholtz resonator model. This has the advantage that relatively easily the mentioned necessary conditions can be satisfied in advance. At given frequency, the impedance is made exactly equal to a given design value. Rules of thumb are derived to produce an impedance which varies only moderately in frequency near design condition.

If more than one design frequencies with corresponding impedances are required within the same impedance model, we could try the Double Layer Extended Helmholtz Resonator model (section E), or resort to the much more general approach of,[16] as sketched in section C (equation 35 and so forth). Its application needs more care, however, because this formulation does not automatically satisfy the fundamental conditions of theorem 1. In both cases, determining the constants can be done practically only numerically, for example by a least squares minimisation procedure.

An explicit solution is given of a pulse reflecting in time domain at a Helmholtz resonator impedance wall that provides some insight into the reflection problem in time domain and at the same time may act as an analytical test case for numerical implementations.

It is discussed that including the instability, that appears to push forward in time domain with the present linear model of liner with vanishing mean flow boundary layer, seems worse than excluding it.

Appendix

Causality

No physical process can exist for all time. A process $f(t)$ that starts by some cause at some finite time $t = t_0$, while it vanishes before t_0, is called *causal*. By analytic continuation in ω we can show that the corresponding Fourier transform

$$\hat{f}(\omega) = \int_{t_0}^{\infty} f(t)\ \mathrm{e}^{-\mathrm{i}\omega t}\ \mathrm{d}t \tag{75}$$

has the property that $\hat{f}(\omega)$ is analytic in the lower complex half-plane $\mathrm{Im}(\omega) < 0$. So this is a necessary condition on $\hat{f}$ for f to be causal. A sufficient condition is the following *causality condition*.[25]

Theorem 3 (Causality Condition) *If $\hat{f}(\omega)$ is analytic in* $\mathrm{Im}(\omega) \le 0$, $|\hat{f}(\omega)|^2$ *is integrable along the real axis, and there is a real t_0 such that $\hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t_0} \to 0$ uniformly with regard to* $\arg(\omega)$ *for $|\omega| \to \infty$ in the lower complex half plane, then $f(t)$ is causal, and vanishes for $t < t_0$.*

Proof. It is no restriction for the proof to assume $t_0 = 0$. Consider, for $t < 0$, the integral $\int \hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega$ along the real contour $[-R, R]$ closed via a semi-circle of radius R in the lower complex half plane. As the integrand is analytic the integral is zero. Let $R \to \infty$. The contribution I_R of the integral along the semi-circle tends to zero, because

$$|I_R| \le \int_0^{\pi} |\hat{f}(\omega)|\,\mathrm{e}^{-|t|R\sin\theta}\,R\,\mathrm{d}\theta \le 2R\max_{\theta}|\hat{f}(\omega)| \int_0^{\frac{1}{2}\pi} \mathrm{e}^{-|t|R2\theta/\pi}\,\mathrm{d}\theta \to 0, \tag{76}$$

where $\omega = R\,\mathrm{e}^{-\mathrm{i}\theta}$. So the contribution from the real axis, being equal to $2\pi f(t)$, is also zero. □

Notes.

- The *lower* complex half-space becomes the *upper* half-space if the opposite Fourier sign convention is taken.
- The theorem is also applicable to non-convergent Fourier integrals that can be interpreted as generalised functions, provided a suitable limit is taken.

Example 1 The Fourier transform $\hat{f}(\omega) = (p + \mathrm{i}\omega)^{-1}$ is causal if $p > 0$, as may be confirmed by the inverse transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega t}}{p + \mathrm{i}\omega}\,\mathrm{d}\omega = H(t)\,\mathrm{e}^{-pt} = \begin{cases} \mathrm{e}^{-pt} & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases} \tag{77}$$

In the limit of no damping ($p \downarrow 0$) the singularity at $\omega = \mathrm{i}p$ moves to $\omega = 0$ on the real axis. The integral is to be interpreted via a suitable indentation of the contour *under* the pole in order to retain causality.

Acknowledgements

This work was carried out under the Messiaen European collaborative project "Messiaen" under the Sixth Framework Programme (Project Co-ordinator Jean-Louis Migeot). The financial support is greatly acknowledged. The present paper is the companion of paper AIAA-2006-2569 [6], to be presented at the present conference, entitled "*Discontinuous Galerkin Implementation of the Extended Helmholtz Resonator Impedance Model in Time Domain*" by N. Chevaugeon, J.-F. Remacle and X. Gallez. The author would like to express his appreciation for the interest and fruitful discussions with Xavier Gallez, Nicolas Chevaugeon, Jean-François Remacle, Yussuf Özyörük, and the partners of the Messiaen project.

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