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Impedance Restrictions on Independent Finger Grippers

M. E. Brokowski and Michael Peshkin

Abstract—The impedance matrices of independent point fingers of a multifingered gripper map to the impedance matrix of a grasped workpart. We find that in a planar geometry, three fingers are enough to allow an unrestricted range of workpart impedances, if finger impedances are selectable. In a spatial geometry however, five fingers are necessary for the broadest range of workpart impedances, and even so there is one impedance matrix that a workpart cannot attain regardless of the number of fingers that grasp it. We find this "unattainable" impedance matrix. We also characterize the impedance restrictions on workparts grasped with fewer than five spatial or three planar fingers.

Index Terms—Compliance, grippers, impedance, multifingered, restrictions, robot.

I. INTRODUCTION

Force-guided assembly allows a robot to use the forces generated during an assembly operation to guide the operations successful completion. One implementation of force-guided assembly utilizes impedance control, wherein a workpart's impedance is specified such that forces resulting from errors in positioning are mapped to motions that reduce the errors [14].

Consider a multifingered gripper where each finger can have a specifiable impedance characteristic. A grasped workpart will have an effective impedance characteristic that is a function of the

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The authors are with the Department of Mechanical Engineering, Northwestern University, Evanston, IL 60208 USA.

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impedances of the fingers and of the grasp geometry (i.e. the points where the fingers contact the workpart). Often, we wish to confer a particular impedance characteristic on a workpart by specifying particular impedance characteristics for the fingers that grasp it. For linear admittance/impedance*, this means that the impedance matrices of the fingers map to an effective impedance matrix of the grasped workpart.

The goal of this paper is to explore what sorts of impedance properties a grasped workpart can have when gripped with a given number of fingers whose impedance we can control.

A. Language

Suppose that we can choose each element of each fingers damping matrix to be whatever we want. If we can confer a particular damping matrix on the workpart by choosing an appropriate set of these finger damping matrices, then we say that the workpart damping matrix is *attainable*.

More formally, let $\{\mathbf{D}_i\}, i = 1 \dots n$, be the set of n damping matrices corresponding to n fingers grasping some workpart and let the damping matrix of the grasped workpart as a function of that set be $\mathbf{D}_{\text{workpart}} = f(\{\mathbf{D}_i\})$. Then $\mathbf{D}_{\text{workpart}}$ is attainable if and only if

$$\exists \{\mathbf{D}_i\} \ni \mathbf{D}_{\text{workpart}} = f(\{\mathbf{D}_i\}).$$

Further, if *all* workpart damping matrices are attainable, each attained by choosing an appropriate $\{\mathbf{D}_i\}$, we say that f is a *full rank mapping*.

B. Scope of This Paper

Using this language, we will determine two things. First, we will determine the number of fingers that must grasp an workpart in order to achieve a full rank mapping from the finger damping matrices to the damping matrix of the workpart.

We will also determine the limitations on the attainable damping/accommodation matrices of the grasped workpart when it is grasped by fewer than the number of fingers needed for a full rank mapping. That is, when we have too few fingers to get any workpart damping/accommodation matrix that we want, which ones can we still get?

We do not address the inverse problem of designing finger accommodations for a particular desired workpart accommodation. In this paper, the damping matrices of the fingers are assumed known, it is the damping characteristics of the grasped workpart that we wish to determine.

We also do not determine optimal grasp geometry. The positions of the fingers on the workpart are unrestricted, but they are assumed to be known.

C. Assumptions

- The robot is rigid outside of its fingers. That is, we are not treating any damping of the robot which cannot be accounted for in the fingers. In addition, the workpart is a rigid, massless body, a reasonable approximation for workparts which are relatively stiff and light compared to the fingers which hold them.
- We speak of damping/accommodation properties, though generalization to impedances, including stiffness and mass, is totally analogous for our rigid massless workparts.

Impedance restrictions on independent finger grippers

Mike Brokowski, Michael A. Peshkin

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- The force/velocity characteristic of each finger may be described entirely by its own damping matrix. That is, the forces felt by one finger do not affect the motion of another.
- Contact friction between workpart and fingers is high enough that slip does not occur.
- We assume contact characteristic of hard, point fingers stably grasping a workpart.

II. BACKGROUND

A. Accommodation and Damping

Accommodation describes a force-to-velocity relationship in mechanical systems. The inverse relationship, the mapping from velocities to forces, is known as *damping*. When the force acting on a damper varies in proportion to its velocity, we have *linear damping*. Thus, a linear damper is characterized by

$$\mathbf{f} = \mathbf{D}\mathbf{v} \quad (1)$$

and, for accommodation

$$\mathbf{v} = \mathbf{D}^{-1}\mathbf{f} = \mathbf{A}\mathbf{f}. \quad (2)$$

For planar systems, if we represent \mathbf{v} as a vector of rotational and translational velocities $[\omega \ v_x \ v_y]^T$ and represent the force vector \mathbf{f} similarly $[\tau \ f_x \ f_y]^T$, then all of the linear force-velocity relationships at a point O can be described with a general *damping matrix* such as the 3×3 matrix $\mathbf{D}_{\text{workpart}}$ shown in (3).

$$\begin{bmatrix} \tau \\ f_x \\ f_y \end{bmatrix} = \begin{bmatrix} d_{\omega\tau} & d_{x\tau} & d_{y\tau} \\ d_{\omega x} & d_{xx} & d_{yx} \\ d_{\omega y} & d_{xy} & d_{yy} \end{bmatrix} \begin{bmatrix} \omega \\ v_x \\ v_y \end{bmatrix}. \quad (3)$$

We note here that a damping matrix will look different to observers in different coordinate systems. When we say the damping matrix at point O we mean the damping matrix as referred to a coordinate system fixed at point O.

A generalized damping matrix for spatial cases is shown below.

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} d_{\omega_x\tau_x} & d_{\omega_y\tau_x} & d_{\omega_z\tau_x} & d_{v_x\tau_x} & d_{v_y\tau_x} & d_{v_z\tau_x} \\ d_{\omega_x\tau_y} & d_{\omega_y\tau_y} & d_{\omega_z\tau_y} & d_{v_x\tau_y} & d_{v_y\tau_y} & d_{v_z\tau_y} \\ d_{\omega_x\tau_z} & d_{\omega_y\tau_z} & d_{\omega_z\tau_z} & d_{v_x\tau_z} & d_{v_y\tau_z} & d_{v_z\tau_z} \\ d_{\omega_x f_x} & d_{\omega_y f_x} & d_{\omega_z f_x} & d_{v_x f_x} & d_{v_y f_x} & d_{v_z f_x} \\ d_{\omega_x f_y} & d_{\omega_y f_y} & d_{\omega_z f_y} & d_{v_x f_y} & d_{v_y f_y} & d_{v_z f_y} \\ d_{\omega_x f_z} & d_{\omega_y f_z} & d_{\omega_z f_z} & d_{v_x f_z} & d_{v_y f_z} & d_{v_z f_z} \end{bmatrix} \times \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ v_x \\ v_y \\ v_z \end{bmatrix}. \quad (4)$$

B. Motivation

Two important concerns have driven interest in impedance control in robotics. Most robots are position-controlled but have limited positional accuracy. By implementing accommodation at the workpart, the force generated by collisions can be made to map into a velocity in the direction opposite that of the force, allowing the error to be overcome without damage. In such a case, accommodation has been used to ensure *consistent* behavior, so that no geometric constraints are violated (e.g. [14]).

However, impedance control is not limited to ensuring that errors in robot position do not cause damage; it is also used to map the forces produced by these errors into *error-corrective* motions [14], [17] which ensure proper completion of placement tasks.

A typical accommodation mapping using linear damping is given by $\mathbf{v} = \mathbf{v}_0 + \mathbf{A}\mathbf{f}$ where \mathbf{A} is the accommodation matrix, \mathbf{f} is the force vector due to some contact error, \mathbf{v}_0 is the nominal robot velocity, and \mathbf{v} is the resulting total velocity of the workpart.

There are two distinct paradigms used in analyzing accommodating manipulators. The first is to design the nominal velocity (the path, \mathbf{v}_0) for successful task completion, assuming that we know the impedance (\mathbf{A}) of the manipulator. This approach is commonly referred to as *fine motion planning* [8], [9]. The second paradigm, known as *force-guided assembly*, designs the impedance (\mathbf{A}) for success, knowing the nominal path (\mathbf{v}_0) [14].

It is this second utilization of accommodation control which motivates this paper as we examine what sorts of workpart accommodation matrices are possible to design, given that a specified number of fingers are grasping the workpart.

III. RELATED WORK

A. Force-Guided Assembly

The implementation of this second paradigm of force-guided error-corrective assembly has been examined by several authors. Whitney [20] describes the use of a *remote center of compliance* (RCC) wrist for a special task, the chamfered peg-in-hole insertion. Others examine more general assembly tasks. Peshkin [14] studies assembly using linear accommodation using nondiagonal accommodation matrices. Peshkin and Schimmels [17] determine a systematic approach to identify the bounds of force-guided assembly and a systematic approach to design a manipulator's accommodation matrix. Lui and Asada [10] utilize neural networks to implement nonlinear accommodation mappings as do Gullapalli *et al.* [3]. Vougioukas and Gottschlich [19] examine an approach for automatic generation of compliance mappings using a computer to generate and test error configurations. Ahn *et al.* [1] implement a learning-based system on a SCARA robot.

B. Multifingered Gripping

Mason and Salisbury [16] study grasping of a workpart by multifingered grippers and discuss point finger and soft finger contact as well as forces necessary for force and form closure. Payandeh and Goldenberg [13] examine the inverse problem of determining finger impedances for a desired workpart impedance. Several authors examine conditions necessary to achieve force- and form-closure and related grasps [15], [18].

Much work has gone into studying the internal forces fingers must exert on workparts in order to hold a workpart without slip at the workpart-finger contact points and without violating joint-torque limits of the fingers. Metrics for quantifying these conditions as well as several other measures of grasp effectiveness have aided the evaluation and optimization of multifingered grasps [2]–[7], [12].

Melchiorri [11] examines the mapping from joint velocities to part velocities in manipulation devices with different contact constraints, categorizing the vector spaces involved in the mapping in several ways. Melchiorri's work examining vector mappings is similar to the work presented here except that 1) he looks at force-to-force and velocity-to-velocity mappings rather than impedance mappings and 2) he does not examine spatial point fingers.

Previous work in multifingered gripping focuses on several areas, including the relationship between the motions of fingers and of a grasped workpart; force- and form-closure for a particular grasp; internal forces exerted by fingers onto a grasped workpart; and the inverse problem of finding optimal grasp geometry or optimal force distribution. In the present paper, we examine the impedance

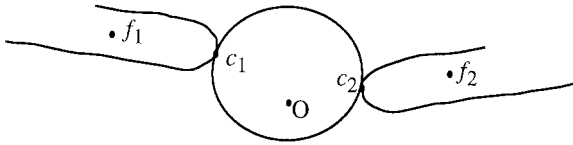


Fig. 1. Two fingers grasp a disk-shaped workpart at contact points c_1 and c_2 . The damping matrices of the fingers are known at points f_1 and f_2 . How do we determine the effective damping matrix of the workpart at point O ?

restrictions that occur in the *forward* mapping from finger damping matrices to a workpart damping matrix. We determine whether a full rank mapping exists with a given number of fingers and what sorts of workpart damping matrices are still attainable when a workpart is grasped with too few fingers to assure a full rank mapping.

IV. DETERMINING EFFECTIVE DAMPING MATRICES FOR A GRASPED WORKPART

To examine whether a given number of fingers is sufficient to achieve a full rank mapping, we must first describe how the damping matrices of the individual fingers map to the damping matrix of the grasped workpart. In this section, we present this mapping (Section IV-A) and then “vectorize” the result of this mapping (Section IV-A) to achieve the form that we examine in Section V.

A. Moving and Adding Damping Matrices

Suppose that n fingers grasp a workpart. The fingers touch a workpart at contact points $\{c_1, c_2, \dots, c_n\}$. Let the damping matrices of the fingers at those points be $\{D_1, D_2, \dots, D_n\}$.¹ We are interested in the effective damping matrix D_{workpart} of the workpart at a point O . (Fig. 1 depicts an example with two fingers.)

It can be shown that the effective workpart damping matrix at point O is given by

$$D_{\text{workpart}} = \sum_{i=1}^n (\mathbf{H}_{c_i}^O \mathbf{J}^{-1})^T D_i (\mathbf{H}_{c_i}^O \mathbf{J}^{-1}). \quad (5)$$

Where \mathbf{H} is a matrix representing the point contact constraints on moments and rotations for hard, point contact [16]. These constraint matrices “reduce” the finger damping matrices, leaving only the purely translational elements nonzero. We note that

$$\mathbf{H}_{\text{planar}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad (6a)$$

$$\mathbf{H}_{\text{spatial}} = \begin{bmatrix} 0_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix}. \quad (6b)$$

$\mathbf{H}_{c_i}^O \mathbf{J}$ is the Cartesian transformation matrix (following the notation in [2]) which relates the instantaneous velocity from O to c_i (so that $\mathbf{v}_O = \mathbf{H}_{c_i}^O \mathbf{J} \mathbf{v}_{c_i}$). For instance, the transformation matrix from A to B in spatial cases is given by

$$\begin{aligned} \mathbf{B} \mathbf{J} &= \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{R}_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix} \quad \text{where} \\ \mathbf{R}_{3 \times 3} &= \begin{bmatrix} 0 & -r_{B/Az} & r_{B/Ay} \\ r_{B/Az} & 0 & -r_{B/Ax} \\ -r_{B/Ay} & r_{B/Ax} & 0 \end{bmatrix} \end{aligned} \quad (7)$$

for velocity vectors as defined in Section II-A and where $r_{B/Ax}$, $r_{B/Ay}$, and $r_{B/Az}$ are the components of vector $r_{B/A}$, from A to B .

¹Recall that we mean by this that the damping matrices of the fingers referred to coordinate systems fixed at points $\{f_1, f_2, \dots, f_n\}$ are $\{D_1, D_2, \dots, D_n\}$.

B. “Vectorizing” the Matrices; the Geometry Transform Matrix B

Ordinarily, matrices map vectors to vectors. Here, we are interested in how matrices map to other matrices, particularly how finger damping matrices map to workpart damping matrices. In order to use familiar tools in our study of matrix to matrix mappings, we may recast our subject matrices as vectors. We then study the larger matrices which relate them.

For notational convenience, we define a “vectorize” function vec . Suppose we have a $p \times q$ matrix \mathbf{A} . Then we say that

$$\text{vec}(\mathbf{A}) = [a_{11} \ a_{12} \ \dots \ a_{pq}]^T. \quad (8)$$

We will apply this to D_{workpart} to create, for example, a 36-vector out of a 6×6 matrix.

We are also interested in vectorizing the damping matrices of the fingers. To vectorize a set of matrices, we expand our definition of vec somewhat. Let $\{D_i\}, i = 1 \dots n$, be a set of n matrices whose purely translational submatrices are $\{T_i\}, i = 1 \dots n$. For example, if

$$D_1 = \begin{bmatrix} d_{\omega\tau} & d_{x\tau} & d_{y\tau} \\ d_{\omega x} & d_{xx} & d_{yx} \\ d_{\omega y} & d_{xy} & d_{yy} \end{bmatrix} \quad \text{then} \quad T_1 = \begin{bmatrix} d_{xx} & d_{yx} \\ d_{xy} & d_{yy} \end{bmatrix}. \quad (9)$$

For such a set of matrices, we define the vectorize function to be

$$\text{vec}\{D_i\} = [\text{vec}(T_1) \ \text{vec}(T_2) \ \dots \ \text{vec}(T_n)]^T. \quad (10)$$

So that $\text{vec}\{D_i\}$ is a vector of all of the purely translational elements in the set of matrices $\{D_i\}$.

Examining (5), we can see that the damping matrix of the workpart D_{workpart} is linearly related to the damping matrices of each of the fingers. We can vectorize the workpart damping matrix D_{workpart} and the set of n finger damping matrices $\{D_i\}, i = 1 \dots n$. Doing this, we rewrite (5) as

$$\text{vec}(D_{\text{workpart}}) = \mathbf{B} \text{vec}\{D_i\}. \quad (11)$$

We note that, for n fingers grasping a workpart, a planar $\text{vec}(D_{\text{workpart}})$ is the 9×1 column vector representing the nine elements of the 3×3 matrix D_{workpart} and a spatial $\text{vec}(D_{\text{workpart}})$ is a 36×1 column vector. Similarly, a planar $\text{vec}\{D_i\}$ is a $4n \times 1$ column vector of the $4n$ translational elements of $\{D_i\}$ and a spatial $\text{vec}\{D_i\}$ is a $9n \times 1$ column vector of the $9n$ translational elements of $\{D_i\}$. Consequently, a planar \mathbf{B} will be a $9 \times 4n$ geometry transform matrix accounting for the geometry introduced by the grasp in (5) while a spatial \mathbf{B} will be a $36 \times 9n$ matrix.

Now, (11) is our vectorized version of (5) describing the relationship between the damping matrices of the fingers and the resulting effective damping matrix of the workpart. We can now examine this relationship with the well-known tools used to analyze vector mappings.

V. ANALYZING THE GEOMETRY TRANSFORM MATRIX B

We would like to be able to specify the damping characteristics of a grasped workpart by varying the damping characteristics of the fingers which grasp the workpart. By examining properties of \mathbf{B} such as its rank and nullspace we can learn about the limitations of the mapping from $\{D_i\}$ to D_{workpart} .

A. The Rank of B

If the rank of \mathbf{B} is less than the number of elements in D_{workpart} then there are some restrictions on what damping characteristics are attainable by a workpart even if we have full control over the damping characteristics of the fingers.

TABLE I

| # fingers n | $\{D_i\}$ | Rank of B | Rank needed for full rank mapping | Rank Deficiency |
|------------------|-----------|----------------|--------------------------------------|--------------------|
| 2 | general | 7 | 9 | 2 |
| 3 | general | 9 | 9 | none |
| 2 | symmetric | 5 | 6 | 1 |
| 3 | symmetric | 6 | 6 | none |

TABLE II

| # fingers n | $\{D_i\}$ | Rank of B | Rank needed for full rank mapping | Rank Deficiency |
|------------------|-----------|----------------|--------------------------------------|--------------------|
| 3 | general | 24 | 36 | 12 |
| 4 | general | 30 | 36 | 6 |
| 5 | general | 35 | 36 | 1 |
| more than 5 | general | 35 | 36 | 1 |
| 3 | symmetric | 15 | 21 | 6 |
| 4 | symmetric | 18 | 21 | 3 |
| 5 | symmetric | 20 | 21 | 1 |
| more than 5 | symmetric | 20 | 21 | 1 |

More formally, the number of independently specifyable elements in the workpart damping matrix D_{workpart} is equal to or less than the rank of the geometry transform matrix B .

By a simple counting argument, we expect there to be a minimum number of fingers needed to achieve a full rank mapping. For example, consider a planar workpart which has nine elements in its damping matrix. We would expect that two fingers would be insufficient to allow independent specification of all nine, because the purely translational elements of the two fingers' damping matrices have only eight independent elements themselves. However, we will see that, even with what might seem to be a sufficient number of fingers according to this counting argument, it is possible that the fingers' damping matrices are coupled in some way and do not act independently to allow for a full rank mapping to the workpart's damping matrix.

B. Ranks of Various B Matrices

Here we show the ranks of B for several cases of workparts grasped by different numbers of fingers. We also show the rank needed to achieve a full rank mapping and the difference between the two. The deficiency is the number of restrictions on the part's damping resulting from lack of a full rank mapping. For example, a deficiency of 2 means there are two distinct mathematical restrictions on the elements of D_{workpart} , where each restriction may involve several elements, but not necessarily imply that two of the elements of D_{workpart} are zero.

1) *Planar Cases:* Table I displays ranks of B matrices for planar geometry. We show the ranks for both symmetric as well as the more general asymmetric finger damping matrices because symmetric impedances are common in applications. Note, however, that the rank shown as needed for a full rank mapping to a symmetric workpart assumes that we are using symmetric fingers. It is possible for there to be symmetric workpart matrices to which we cannot map with two general fingers.

The general cases are not terribly surprising since they say that three fingers are needed to fully specify the part's damping matrix. We would conclude the same thing from our counting argument.

In the symmetric cases, counting shows that there are six independently choosable elements in the symmetric fingers' matrices and six elements in the workpart's resulting symmetric matrix. Therefore, it might seem possible to specify every element in the workpart's damping matrix when it is grasped with just 2 fingers. However, the rank of B for this case is five, so there is still one restriction on the elements of the workpart's damping matrix. Three fingers must be used to achieve a full rank mapping, just as in the general case.

2) *Spatial Cases:* Table II displays ranks of B matrices for spatial geometry. As for the planar cases, we show the ranks for both symmetric and general finger damping matrices.

There is a particularly interesting phenomenon in the spatial cases which does not occur in the planar cases. In the planar cases, there

were rank deficiencies in B when the workpart was held with too few fingers, but, by adding more fingers (until we had at least three total), we could always achieve a full rank mapping.

Not so in the spatial cases. We see in the above table that, though the five-fingered case shows a deficiency of one in both the asymmetric and the symmetric cases, adding additional fingers does not eliminate the deficiency. That is, five fingers is the best we can do and additional fingers will not result in a full rank mapping. There is always at least one restriction on the spatial damping matrices that we can attain, regardless of how many fingers we use. We now turn our attention to determining what that restriction is.

C. The Left Nullspace of B

Since we know from examining the rank of B that there are matrices that we cannot attain using a given number of fingers, the next obvious question is *what are those matrices?* This is what the left nullspace of B helps us determine. We can determine orthogonal bases for the left nullspace of B which reveal the restrictions on the attainable workpart damping matrices.

The property of the left nullspace concerning us is that, for any attainable workpart matrix D_{workpart} , the dot product of the vectorized D_{workpart} with any vector in the left nullspace of B is zero. That is,

Let $\mathcal{D}_{\text{attainable}}$ be the space of all vectorized attainable workpart damping matrices (the *column space* or *range* of B) and let $\mathcal{D}_{\text{null}}$ be the set of vectors in the left nullspace of B , then

$$\begin{aligned} \forall D_{\text{workpart}} \ni \text{vec}(D_{\text{workpart}}) \in \mathcal{D}_{\text{attainable}} \quad \text{and} \\ \forall D_n \ni \text{vec}(D_n) \in \mathcal{D}_{\text{null}} \\ \text{vec}(D_{\text{workpart}}) \cdot \text{vec}(D_n) = 0. \end{aligned} \quad (12)$$

A consequence of this property is that there is no way for B to map any finger matrices into the left nullspace of B , implying that, for any workpart damping matrix in the left nullspace of B , there is no possibility of choosing a set of finger damping matrices $\{D_i\}$ to achieve that workpart matrix.

1) *Grasp-Dependent Left Nullspaces:* The left nullspace of B is spanned by basis vectors which are vectorized damping matrices. These nullspace basis vectors are of two types. The first type is grasp-dependent. Such vectors depend on the particular positions of the contact points $\{c_1, c_2, \dots, c_n\}$ at which the fingers grasp the workparts.

Consider a simple example of a grasp-dependent nullspace vector for the case where a planar workpart is grasped by two horizontally

opposed symmetric fingers, each located equally distant from the part's center O (so that $r_{2x} = -r_{1x}$ and $r_{1y} = r_{2y} = 0$).

We can determine the grasp-dependent left nullspace vector, $\text{vec}(\mathbf{D}_{\text{null}})$, satisfying (12). Recall that vectors here are vectorized matrices; when returned to matrix form, our nullspace vector is

$$\mathbf{D}_{\text{null}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -r_{1x}^2 \end{bmatrix}. \quad (13)$$

We can take the dot product of this nullspace vector with the vectorized form of the general 3×3 planar damping matrix presented earlier in (1). Doing this, we find

$$d_{\omega\tau} = r_{1x}^2 d_{yy}. \quad (14)$$

Which implies that, regardless of the damping matrices of the two symmetric fingers, the torsional resistance to rotation ($d_{\omega\tau}$) will be coupled to the resistance to motion along the y -axis (d_{yy}) by the length of the effective moment arm, which is not surprising.

Unfortunately, the grasp-dependent nullspace vectors are, in general, algebraically complicated and are usually quite challenging to interpret, particularly for the spatial cases.

2) *Grasp-Independent Nullspaces*: The second type of left nullspace vector does not depend upon the particular geometry of the contact points. Nullspace vectors of this type are restrictions on what workpart damping matrices are attainable regardless of where the fingers grasp the workpart.

For example, in the planar case where we specify three symmetric fingers grasping a workpart, the three orthogonal left nullspace vectors (in matrix form) that result for the workpart are given by

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (15a)$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (15b)$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \quad (15c)$$

Dotting these vectors with a general 3×3 matrix, we note that the attainable workpart damping matrices must be such that

$$d_{21} = d_{12} \quad (16a)$$

$$d_{31} = d_{13} \quad \text{and} \quad (16b)$$

$$d_{23} = d_{32}. \quad (16c)$$

Therefore, such nullspace vectors imply that, regardless of where we place the symmetric fingers, the workpart will have a symmetric damping matrix. Again, a sensible result.

From our examination of ranks, we note that the rank of \mathbf{B} for spatial cases never rises above 35. This implies that, regardless of how many fingers grasp a spatial workpart, we can never achieve a full rank mapping, as we could in the planar case by using three or more fingers to grasp the workpart. An interesting nullspace vector is the one for spatial workparts held by five or more fingers. The matrix form of this vector turns out to be

$$\mathbf{D}_{\text{null}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \quad (17)$$

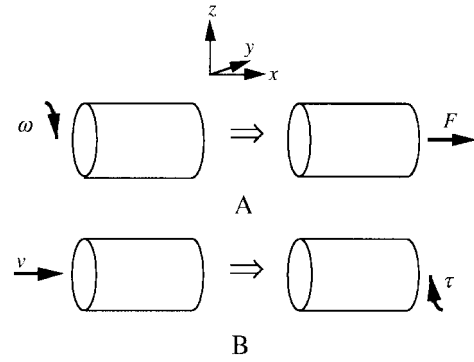


Fig. 2. According to the circulant restriction for a system with a damping only along and about x -axis, if a positive rotation results in a positive force, then a positive velocity will result in a negative torque.

This type of matrix is known as a *circulant* and we will call it \mathbf{C} . It is a nullspace in all of the spatial cases, regardless of how many fingers grasp a workpart or whether or not they are symmetric. From (12), its presence as a left nullspace vector implies that

$$\text{vec}(\mathbf{D}_{\text{workpart}}) \cdot \text{vec}(\mathbf{C}) = 0 \quad (18)$$

or, carrying out this dot product with the general spatial damping matrix given in (4), we find the restriction that applies to *all* spatial $\mathbf{D}_{\text{workpart}}$

$$d_{v_x\tau_x} + d_{v_y\tau_y} + d_{v_z\tau_z} + d_{\omega_x f_x} + d_{\omega_y f_y} + d_{\omega_z f_z} = 0 \quad (19)$$

and

$$d_{v_x\tau_x} + d_{v_y\tau_y} + d_{v_z\tau_z} = 0 \quad (20)$$

for symmetric matrices. We must note that, although the circulant is its own inverse, the restriction on damping does not imply that the same restriction exists for accommodation.

One mechanical interpretation of this circulant restriction is that there is no way to force a finger-held workpart to behave as a simple threaded fastener. For example, suppose we want a workpart to move unimpeded except along and about the x -axis. This implies that we want zero workpart damping elements except for $d_{v_x\tau_x}$ and $d_{\omega_x f_x}$. According to (19), to achieve this our workpart must then have

$$d_{v_x\tau_x} + d_{\omega_x f_x} = 0 \quad \text{or} \quad d_{v_x\tau_x} = -d_{\omega_x f_x}. \quad (21)$$

This implies that the translational resistance to torques must be the *opposite* of the torsional resistance to translations. For example, suppose we choose $d_{v_x\tau_x}$ positive, so that a positive x -axis rotation maps to a positive x -axis force, as shown in Fig. 2(a). Then, according to (21), a corresponding positive x -axis velocity will necessarily map to a negative x -axis torque [Fig. 2(b)], which is the opposite of a simple threaded fastener.

VI. CONCLUSION

We have examined how the damping characteristics of point fingers map to damping characteristics of a grasped workpart. By “vectorizing” the workpart and finger damping matrices, we have formulated the mapping in such a way that we can define a geometry transform matrix \mathbf{B} relating the damping matrices of the fingers to the effective damping matrix of the workpart. The rank of \mathbf{B} allows us to determine how many fingers need grasp a workpart to achieve a full rank mapping. (A full rank mapping would mean that we could confer any desired damping matrix upon the workpart by choosing appropriate finger damping matrices.)

We have determined that planar workparts must be held with a minimum of three fingers in order to achieve full rank mappings. We

have compared this result to a naive counting argument and found that the counting argument erroneously predicts that only two fingers are needed for planar symmetric full rank mappings.

We have found that, for spatial workparts, a five-fingered grasp allows for the maximum range of attainable matrices. However, there remains one restriction on the attainable workpart damping matrices that persists regardless of the number of fingers that grasp the workpart or where they grasp it.

By examining the left nullspace of B , we are able to characterize the restrictions on attainable workpart damping matrices when a workpart is grasped with an insufficient number of fingers to achieve a full rank mapping. Examination of B reveals that the circulant matrix describes the fundamental restriction on all damping matrices attainable by spatial workparts.

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Fault Tolerant Operation of Kinematically Redundant Manipulators for Locked Joint Failures

Christopher L. Lewis and Anthony A. Maciejewski

Abstract—This paper studies the degree to which the kinematic redundancy of a manipulator may be utilized for failure tolerance. A redundant manipulator is considered to be fault tolerant with respect to a given task if it is guaranteed to be capable of performing the task after any one of its joints has failed and is locked in place. A method is developed for determining the necessary constraints which insure the failure tolerance of a kinematically redundant manipulator with respect to a given critical task. This method is based on estimating the bounding boxes enclosing the self-motion manifolds for a given set of critical task points. The intersection of these bounding boxes provides a set of artificial joint limits that may guarantee the reachability of the task points after a joint failure. An algorithm for dealing with the special case of 2-D self-motion surfaces is presented. These techniques are illustrated on a PUMA 560 that is used for a 3-D Cartesian positioning task.

Index Terms—Fault tolerance, Jacobian matrices, kinematically redundant, manipulator kinematics, manipulators, redundant systems.

I. INTRODUCTION

Kinematically redundant manipulators have been proposed for use in the cleanup and remediation of nuclear and hazardous materials, as well as for remote applications such as space or sea exploration [1], [2]. In these applications repairing broken actuators and sensors is impossible and the probability of their failure is increased due to the harsh operating environment [3]–[5]. The redundant degrees of freedom may or may not also be equipped with redundant actuators [6]. The extra degrees of freedom (DOF) of a redundant manipulator may be used to compensate for a failed joint if the manipulator has been properly designed and controlled. The most basic task of a manipulator, i.e., the positioning and/or orienting of the end-effector in the workspace, is described by the forward kinematic equation

$$x = f(\theta) \quad (1)$$

where $x \in R^m$ is the generalized vector of the position and/or orientation of the end-effector and $\theta \in R^n$ is the vector of joint

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C. L. Lewis is with Sandia National Laboratories, Albuquerque, NM 87185 USA.

A. A. Maciejewski is with the School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907 USA.

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Impedance Restrictions on Independent Finger Grippers

M. E. Brokowski Michael Peshkin

Department of Mechanical Engineering
Northwestern University
Evanston, Illinois 60208

Abstract

The impedance matrices of independent point fingers of a multifingered gripper map to the impedance matrix of a grasped workpart. We find that in a planar geometry, three fingers are enough to allow an unrestricted range of workpart impedances, if finger impedances are selectable. In a spatial geometry however, five fingers are necessary for the broadest range of workpart impedances, and even so there is one impedance matrix that a workpart cannot attain regardless of the number of fingers that grasp it. We find this 'unattainable' impedance matrix. We also characterize the impedance restrictions on workparts grasped with fewer than five spatial or three planar fingers.

1 Introduction

Force-guided assembly allows a robot to use the forces generated during an assembly operation to guide the operation's successful completion. One implementation of force-guided assembly utilizes impedance control, wherein a workpart's impedance is specified such that forces resulting from errors in positioning are mapped to motions that reduce the errors. [14]

Consider a multifingered gripper where each finger can have a specifyable impedance characteristic. A grasped workpart will have an effective impedance characteristic that is a function of the impedances of the fingers and of the grasp geometry (i.e. the points where the fingers contact the workpart). Often, we wish to confer a particular impedance characteristic on a workpart by specifying particular impedance characteristics for the fingers that grasp it. For linear admittance/impedance*, this means that the impedance matrices of the fingers *map* to an effective impedance matrix of the grasped workpart.

The goal of this work is to explore what sorts of impedance properties a grasped workpart can have when gripped with a given number of fingers whose impedance we can control.

1.1 Language

Suppose that we can choose each element of each finger's damping matrix to be whatever we want. If we can confer a particular damping matrix on the workpart by choosing an appropriate set of these finger damping matrices, then we say that the workpart damping matrix is *attainable*.

More formally, let $\{\mathbf{D}_i\}$, $i = 1 \dots n$, be the set of n damping matrices corresponding to n fingers grasping some workpart and let the damping matrix of the grasped workpart as a function of that set be $\mathbf{D}_{\text{workpart}} = f(\{\mathbf{D}_i\})$. Then $\mathbf{D}_{\text{workpart}}$ is attainable if and only if

$$\exists \{\mathbf{D}_i\} \ni \mathbf{D}_{\text{workpart}} = f(\{\mathbf{D}_i\})$$

Further, if *all* workpart damping matrices are attainable, each attained by choosing an appropriate $\{\mathbf{D}_i\}$, we say that f is a *full rank mapping*.

* Hereafter, we will refer to damping/accommodation, but similar results apply to general admittances/impedances.

1.2 Scope of this work

Using this language, we will determine two things. First, we will determine the number of fingers that must grasp an workpart in order to achieve a full rank mapping from the finger damping matrices to the damping matrix of the workpart.

We will also determine the limitations on the attainable damping/accommodation matrices of the grasped workpart when it is grasped by fewer than the number of fingers needed for a full rank mapping. That is, when we have too few fingers to get any workpart damping/accommodation matrix that we want, which ones can we still get?

We do not address the inverse problem of designing finger accommodations for a particular desired workpart accommodation. In this work, the damping matrices of the fingers are assumed known, it is the damping characteristics of the grasped workpart that we wish to determine.

We also do not determine optimal grasp geometry. The positions of the fingers on the workpart are unrestricted, but they are assumed to be known.

1.3 Assumptions

- The robot is rigid outside of its fingers. That is, we are not treating any damping of the robot which cannot be accounted for in the fingers. In addition, the workpart is a rigid, massless body, a reasonable approximation for workparts which are relatively stiff and light compared to the fingers which hold them.
- We speak of damping/accommodation properties, though generalization to impedances, including stiffness and mass, is totally analogous for our rigid massless workparts.
- The force/velocity characteristic of each finger may be described entirely by its own damping matrix. That is, the forces felt by one finger do not affect the motion of another.
- Contact friction between workpart and fingers is high enough that slip does not occur.
- We assume contact characteristic of hard, point fingers stably grasping a workpart.

2 Background

2.1 Accommodation and Damping

Accommodation describes a force-to-velocity relationship in mechanical systems. The inverse relationship, the mapping from velocities to forces, is known as *damping*. When the force acting on a damper varies in proportion to its velocity, we have *linear damping*. Thus, a linear damper is characterized by

$$\mathbf{f} = \mathbf{D} \mathbf{v} \quad (1)$$

and, for accommodation,

$$\mathbf{v} = \mathbf{D}^{-1} \mathbf{f} = \mathbf{A} \mathbf{f} \quad (2)$$

For planar systems, if we represent \mathbf{v} as a vector of rotational and translational velocities $[\omega \ v_x \ v_y]^T$ and represent the force vector \mathbf{f} similarly $[\tau \ f_x \ f_y]^T$, then all of the linear force-velocity relationships at a point O can be described with a general *damping matrix* such as the 3x3 matrix $\mathbf{D}_{\text{workpart}}$ shown in equation (3).

$$\begin{bmatrix} \tau \\ f_x \\ f_y \end{bmatrix} = \begin{bmatrix} d_{\omega\tau} & d_{x\tau} & d_{y\tau} \\ d_{\omega x} & d_{xx} & d_{yx} \\ d_{\omega y} & d_{xy} & d_{yy} \end{bmatrix} \begin{bmatrix} \omega \\ v_x \\ v_y \end{bmatrix} \quad (3)$$

We note here that a damping matrix will look different to observers in different coordinate systems. When we say “the damping matrix at point O” we mean the damping matrix as referred to a coordinate system fixed at point O.

A generalized damping matrix for spatial cases is shown below.

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} d_{\omega_x\tau_x} & d_{\omega_y\tau_x} & d_{\omega_z\tau_x} & d_{v_x\tau_x} & d_{v_y\tau_x} & d_{v_z\tau_x} \\ d_{\omega_x\tau_y} & d_{\omega_y\tau_y} & d_{\omega_z\tau_y} & d_{v_x\tau_y} & d_{v_y\tau_y} & d_{v_z\tau_y} \\ d_{\omega_x\tau_z} & d_{\omega_y\tau_z} & d_{\omega_z\tau_z} & d_{v_x\tau_z} & d_{v_y\tau_z} & d_{v_z\tau_z} \\ d_{\omega_x f_x} & d_{\omega_y f_x} & d_{\omega_z f_x} & d_{v_x f_x} & d_{v_y f_x} & d_{v_z f_x} \\ d_{\omega_x f_y} & d_{\omega_y f_y} & d_{\omega_z f_y} & d_{v_x f_y} & d_{v_y f_y} & d_{v_z f_y} \\ d_{\omega_x f_z} & d_{\omega_y f_z} & d_{\omega_z f_z} & d_{v_x f_z} & d_{v_y f_z} & d_{v_z f_z} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ v_x \\ v_y \\ v_z \end{bmatrix} \quad (4)$$

2.2 Motivation

Two important concerns have driven interest in impedance control in robotics. Most robots are position-controlled but have limited positional accuracy. By implementing accommodation at the workpart, the force generated by collisions can be made to map into a velocity in the direction opposite that of the force, allowing the error to be overcome without damage. In such a case, accommodation has been used to ensure that *consistent* behavior, so that no geometric constraints are violated (e.g. [14]).

However, impedance control is not limited to ensuring that errors in robot position do not cause damage; it is also used to map the forces produced by these errors into *error-corrective* motions [14] [17] which ensure proper completion of placement tasks.

A typical accommodation mapping using linear damping is given by $\mathbf{v} = \mathbf{v}_0 + \mathbf{A}\mathbf{f}$ where \mathbf{A} is the accommodation matrix, \mathbf{f} is the force vector due to some contact error, \mathbf{v}_0 is the nominal robot velocity, and \mathbf{v} is the resulting total velocity of the workpart.

There are two distinct paradigms used in analyzing accommodating manipulators. The first is to design the nominal velocity (the path, \mathbf{v}_0) for successful task completion, assuming that we know the impedance (\mathbf{A}) of the manipulator. This approach is commonly referred to as *fine motion planning* [8, 9]. The second paradigm, known as *force-guided assembly*, designs the impedance (\mathbf{A}) for success, knowing the nominal path (\mathbf{v}_0) [14].

It is this second utilization of accommodation control which motivates this work as we examine what sorts of workpart accommodation matrices are possible to design, given that a specified number of fingers are grasping the workpart.

3 Related Work

3.1 Force-Guided Assembly

The implementation of this second paradigm of force-guided error-corrective assembly has been examined by several authors. Whitney [20] describes the use of a *remote center of compliance* (RCC) wrist for a special task, the chamfered peg-in-hole insertion. Others examine more general assembly tasks. Peshkin [14] studies assembly using linear accommodation using non-diagonal accommodation matrices. Peshkin and Schimmels [17] determine a systematic approach to identify the bounds of force-guided assembly and a

systematic approach to design a manipulator's accommodation matrix. Lui and Asada [10] utilize neural networks to implement nonlinear accommodation mappings as do Gullapalli et al [3]. Vougioukas and Gottschlich [19] examine an approach for automatic generation of compliance mappings using a computer to generate and test error configurations. Ahn et al [1] implement a learning-based system on a SCARA robot.

3.2 Multifingered Gripping

Mason and Salisbury [16] study grasping of a workpart by multifingered grippers and discuss point finger and soft finger contact as well as forces necessary for force and form closure. Payandeh and Goldenberg [13] examine the inverse problem of determining finger impedances for a desired workpart impedance. Several authors examine conditions necessary to achieve force- and form-closure and related grasps. [15, 18]

Much work has gone into studying the internal forces fingers must exert on workparts in order to hold a workpart without slip at the workpart-finger contact points and without violating joint-torque limits of the fingers. Metrics for quantifying these conditions as well as several other measures of grasp effectiveness have aided the evaluation and optimization of multifingered grasps. [2, 4, 5, 6, 7, 12]

Melchiorri [11] examines the mapping from joint velocities to part velocities in manipulation devices with different contact constraints, categorizing the vector spaces involved in the mapping in several ways. Melchiorri's work examining vector mappings is similar to the work presented here except that 1) he looks at force-to-force and velocity-to-velocity mappings rather than impedance mappings and 2) he does not examine spatial point fingers.

Previous work in multifingered gripping focuses on several areas, including the relationship between the motions of fingers and of a grasped workpart; force- and form-closure for a particular grasp; internal forces exerted by fingers onto a grasped workpart; and the inverse problem of finding optimal grasp geometry or optimal force distribution. In the present work, we examine the impedance restrictions that occur in the *forward* mapping from finger damping matrices to a workpart damping matrix. We determine whether a full rank mapping exists with a given number of fingers and what sorts of workpart damping matrices are still attainable when a workpart is grasped with too few fingers to assure a full rank mapping.

4 Determining Effective Damping Matrices for a Grasped Workpart

To examine whether a given number of fingers is sufficient to achieve a full rank mapping, we must first describe how the damping matrices of the individual fingers map to the damping matrix of the grasped workpart. In this section, we present this mapping (§ 4.1) and then “vectorize” the result of this mapping (§ 4.1) to achieve the form that we examine in section 5.

4.1 Moving and Adding damping matrices

Suppose that n fingers grasp a workpart. The fingers touch a workpart at contact points $\{c_1, c_2, \dots, c_n\}$. Let the damping matrices of the fingers at those points be $\{D_1, D_2, \dots, D_n\}^*$. We are interested in the effective damping matrix D_{workpart} of the workpart at a point O . (Figure 1 depicts an example with two fingers.)

Determining Effective Workpart Damping

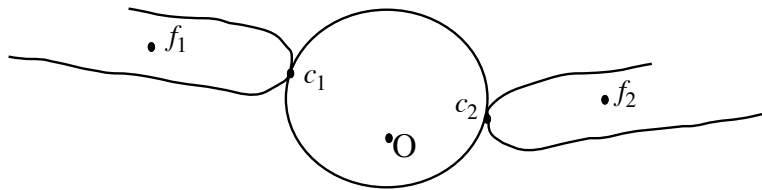


Figure 1. Two fingers grasp a disk-shaped workpart at contact points c_1 and c_2 . The damping matrices of the fingers are known at points f_1 and f_2 . How do we determine the effective damping matrix of the workpart at point O ?

It can be shown that the effective workpart damping matrix at point O is given by

$$D_{\text{workpart}} = \sum_{i=1}^n (H_{c_i}^T J_i^{-1})^T D_i (H_{c_i}^T J_i^{-1}) \quad (5)$$

Where H is a matrix representing the point contact constraints on moments and rotations for hard, point contact [16]. These constraint matrices “reduce” the finger damping matrices, leaving only the purely translational elements nonzero. We note that

$$H_{\text{planar}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad H_{\text{spatial}} = \begin{bmatrix} 0_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \quad (6a, 6b)$$

* Recall that we mean by this that the damping matrices of the fingers *referred* to coordinate systems fixed at points $\{f_1, f_2, \dots, f_n\}$ are $\{D_1, D_2, \dots, D_n\}$.

${}^O_{c_i} \mathbf{J}$ is the Cartesian transformation matrix (following the notation in [2]) which relates the instantaneous velocity from O to c_i (so that $\mathbf{v}_O = {}^O_{c_i} \mathbf{J} \mathbf{v}_{c_i}$). For instance, the transformation matrix from A to B in spatial cases is given by

$${}^B_A \mathbf{J} = \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{R}_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix} \quad \text{where} \quad \mathbf{R}_{3 \times 3} = \begin{bmatrix} 0 & -r_{B/Az} & r_{B/Ay} \\ r_{B/Az} & 0 & -r_{B/Ax} \\ -r_{B/Ay} & r_{B/Ax} & 0 \end{bmatrix} \quad (7)$$

For velocity vectors as defined in section 2.1 and where $r_{B/Ax}$, $r_{B/Ay}$, and $r_{B/Az}$ are the components of vector $\mathbf{r}_{B/A}$, from A to B.

4.2 “Vectorizing” the matrices; the geometry transform matrix B

Ordinarily, matrices map vectors to vectors. Here, we are interested in how matrices map to other matrices, particularly how finger damping matrices map to workpart damping matrices. In order to use familiar tools in our study of matrix to matrix mappings, we may recast our subject matrices as vectors. We then study the larger matrices which relate them.

For notational convenience, we define a “vectorize” function \mathbf{vec} . Suppose we have a $p \times q$ matrix A . Then we say that

$$\mathbf{vec}(A) = [a_{11} \ a_{12} \ \dots \ a_{pq}]^T \quad (8)$$

We will apply this to $\mathbf{D}_{\text{workpart}}$ to create, for example, a 36-vector out of a 6x6 matrix.

We are also interested in vectorizing the damping matrices of the fingers. To vectorize a *set* of matrices, we expand our definition of \mathbf{vec} somewhat. Let $\{\mathbf{D}_i\}, i = 1 \dots n$, be a set of n matrices whose purely translational submatrices are $\{\mathbf{T}_i\}, i = 1 \dots n$. For example, if

$$\mathbf{D}_1 = \begin{bmatrix} d_{\omega\tau} & d_{x\tau} & d_{y\tau} \\ d_{\omega x} & d_{xx} & d_{yx} \\ d_{\omega y} & d_{xy} & d_{yy} \end{bmatrix} \quad \text{then} \quad \mathbf{T}_1 = \begin{bmatrix} d_{xx} & d_{yx} \\ d_{xy} & d_{yy} \end{bmatrix} \quad (9)$$

For such a *set* of matrices, we define the vectorize function to be

$$\mathbf{vec}\{\mathbf{D}_i\} = [\mathbf{vec}(\mathbf{T}_1) \ \mathbf{vec}(\mathbf{T}_2) \ \dots \ \mathbf{vec}(\mathbf{T}_n)]^T \quad (10)$$

So that $\mathbf{vec}\{\mathbf{D}_i\}$ is a vector of all of the purely translational elements in the set of matrices $\{\mathbf{D}_i\}$.

Examining equation (5), we can see that the damping matrix of the workpart $\mathbf{D}_{\text{workpart}}$ is linearly related to the damping matrices of each of the fingers. We can vectorize the workpart damping matrix $\mathbf{D}_{\text{workpart}}$ and the set of n finger damping matrices $\{\mathbf{D}_i\}$, $i = 1 \dots n$. Doing this, we rewrite equation (5) as

$$\mathbf{vec}(\mathbf{D}_{\text{workpart}}) = \mathbf{B} \mathbf{vec}\{\mathbf{D}_i\} \quad (11)$$

We note that, for n fingers grasping a workpart, a planar $\mathbf{vec}(\mathbf{D}_{\text{workpart}})$ is the 9×1 column vector representing the 9 elements of the 3×3 matrix $\mathbf{D}_{\text{workpart}}$ and a spatial $\mathbf{vec}(\mathbf{D}_{\text{workpart}})$ is a 36×1 column vector. Similarly, a planar $\mathbf{vec}\{\mathbf{D}_i\}$ is a $4n \times 1$ column vector of the $4n$ translational elements of $\{\mathbf{D}_i\}$ and a spatial $\mathbf{vec}\{\mathbf{D}_i\}$ is a $9n \times 1$ column vector of the $9n$ translational elements of $\{\mathbf{D}_i\}$. Consequently, a planar \mathbf{B} will be a $9 \times 4n$ geometry transform matrix accounting for the geometry introduced by the grasp in equation (5) while a spatial \mathbf{B} will be a $36 \times 9n$ matrix.

Now, equation (11) is our vectorized version of equation (5) describing the relationship between the damping matrices of the fingers and the resulting effective damping matrix of the workpart. We can now examine this relationship with the well known tools used to analyze vector mappings.

5 Analyzing the geometry transform matrix, \mathbf{B}

We would like to be able to specify the damping characteristics of a grasped workpart by varying the damping characteristics of the fingers which grasp the workpart. By examining properties of \mathbf{B} such as its rank and nullspace we can learn about the limitations of the mapping from $\{\mathbf{D}_i\}$ to $\mathbf{D}_{\text{workpart}}$.

5.1 The rank of \mathbf{B}

If the rank of \mathbf{B} is less than the number of elements in $\mathbf{D}_{\text{workpart}}$ then there are some restrictions on what damping characteristics are attainable by a workpart even if we have full control over the damping characteristics of the fingers.

More formally,

The number of independently specifyable elements in the workpart damping matrix $\mathbf{D}_{\text{workpart}}$ is equal to or less than the rank of the geometry transform matrix \mathbf{B} .

By a simple counting argument, we expect there to be a minimum number of fingers needed to achieve a full rank mapping. For example, consider a planar workpart which has nine elements in its damping

matrix. We would expect that two fingers would be insufficient to allow independent specification of all nine, because the purely translational elements of the two fingers' damping matrices have only eight independent elements themselves. However, we will see that, even with what might seem to be a sufficient number of fingers according to this counting argument, it is possible that the fingers' damping matrices are coupled in some way and do not act independently to allow for a full rank mapping to the workpart's damping matrix.

5.2 Ranks of various B matrices

Here we show the ranks of B for several cases of workparts grasped by different numbers of fingers. We also show the rank needed to achieve a full rank mapping and the difference between the two. The deficiency is the number of restrictions on the part's damping resulting from lack of a full rank mapping. For example, a deficiency of 2 means there are two distinct mathematical restrictions on the elements of D_{workpart} , where each restriction may involve several elements, but not necessarily imply that two of the elements of D_{workpart} are zero.

5.2.1 Planar Cases

Table 1 displays ranks of B matrices for planar geometry. We show the ranks for both symmetric as well as the more general asymmetric finger damping matrices because symmetric impedances are common in applications. Note, however, that the rank shown as needed for a full rank mapping to a symmetric workpart assumes that we are using symmetric fingers. It is possible for there to be symmetric workpart matrices to which we cannot map with two general fingers.

Table 1

| $\#$ fingers n | $\{D_i\}$ | Rank of B | Rank needed for full rank mapping | Rank Deficiency |
|---------------------|-----------|----------------|--------------------------------------|--------------------|
| 2 | general | 7 | 9 | 2 |
| 3 | general | 9 | 9 | none |
| 2 | symmetric | 5 | 6 | 1 |
| 3 | symmetric | 6 | 6 | none |

The general cases are not terribly surprising since they say that three fingers are needed to fully specify the part's damping matrix. We would conclude the same thing from our counting argument.

In the symmetric cases, counting shows that there are six independently choosable elements in the symmetric fingers' matrices and six elements in the workpart's resulting symmetric matrix. Therefore, it might seem possible to specify every element in the workpart's damping matrix when it is grasped with just 2 fingers. However, the rank of \mathbf{B} for this case is five, so there is still one restriction on the elements of the workpart's damping matrix. Three fingers must be used to achieve a full rank mapping, just as in the general case.

5.2.2 Spatial Cases

Table 2 displays ranks of \mathbf{B} matrices for spatial geometry. As for the planar cases, we show the ranks for both symmetric and general finger damping matrices.

Table 2

| # fingers n | $\{\mathbf{D}_i\}$ | Rank of \mathbf{B} | Rank needed for full rank mapping | Rank Deficiency |
|------------------|--------------------|-------------------------|--------------------------------------|--------------------|
| 3 | general | 24 | 36 | 12 |
| 4 | general | 30 | 36 | 6 |
| 5 | general | 35 | 36 | 1 |
| more than 5 | general | 35 | 36 | 1 |
| 3 | symmetric | 15 | 21 | 6 |
| 4 | symmetric | 18 | 21 | 3 |
| 5 | symmetric | 20 | 21 | 1 |
| more than 5 | symmetric | 20 | 21 | 1 |

There is a particularly interesting phenomenon in the spatial cases which does not occur in the planar cases. In the planar cases, there were rank deficiencies in \mathbf{B} when the workpart was held with too few fingers, but, by adding more fingers (until we had at least three total), we could always achieve a full rank mapping.

Not so in the spatial cases. We see in the above table that, though the five-fingered case shows a deficiency of one in both the asymmetric and the symmetric cases, adding additional fingers does *not* eliminate the deficiency. That is, five fingers is the best we can do and additional fingers will not result in a full rank mapping. There is always at least one restriction on the spatial damping matrices that we can attain, regardless of how many fingers we use. We now turn our attention to determining what that restriction is.

5.3 The Left Nullspace of \mathbf{B}

Since we know from examining the rank of \mathbf{B} that there are matrices that we cannot attain using a given number of fingers, the next obvious question is *what are* those matrices? This is what the left nullspace of \mathbf{B} helps us determine. We can determine orthogonal bases for the left nullspace of \mathbf{B} which reveal the restrictions on the attainable workpart damping matrices.

The property of the left nullspace concerning us is that, for any attainable workpart matrix $\mathbf{D}_{\text{workpart}}$, the dot product of the vectorized $\mathbf{D}_{\text{workpart}}$ with any vector in the left nullspace of \mathbf{B} is zero. That is,

Let $D_{\text{attainable}}$ be the space of all vectorized attainable workpart damping matrices (the *column space* or *range* of \mathbf{B}) and let D_{null} be the set of vectors in the left nullspace of \mathbf{B} , then

$$\forall \mathbf{D}_{\text{workpart}} \ni \text{vec}(\mathbf{D}_{\text{workpart}}) \in D_{\text{attainable}}$$

and $\forall \mathbf{D}_n \ni \text{vec}(\mathbf{D}_n) \in D_{\text{null}}$

$$\text{vec}(\mathbf{D}_{\text{workpart}}) \cdot \text{vec}(\mathbf{D}_n) = 0 \quad (12)$$

A consequence of this property is that there is no way for \mathbf{B} to map any finger matrices into the left nullspace of \mathbf{B} , implying that, for any workpart damping matrix in the left nullspace of \mathbf{B} , there is no possibility of choosing a set of finger damping matrices $\{\mathbf{D}_i\}$ to achieve that workpart matrix.

5.3.1 Grasp-dependent Left Nullspaces

The left nullspace of \mathbf{B} is spanned by basis vectors which are vectorized damping matrices. These nullspace basis vectors are of two types. The first type is grasp-dependent. Such vectors depend on the particular positions of the contact points $\{c_1, c_2, \dots, c_n\}$ at which the fingers grasp the workparts.

Consider a simple example of a grasp-dependent nullspace vector for the case where a planar workpart is grasped by two horizontally opposed symmetric fingers, each located equally distant from the part's center O (so that $r_{2x} = -r_{1x}$ and $r_{1y} = r_{2y} = 0$).

We can determine the grasp-dependent left nullspace vector, $\text{vec}(\mathbf{D}_{\text{null}})$, satisfying equation (12). Recall that vectors here are vectorized matrices; when returned to matrix form, our nullspace vector is

$$\mathbf{D}_{\text{null}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -r_{1x}^2 \end{bmatrix} \quad (13)$$

We can take the dot product of this nullspace vector with the vectorized form of the general 3×3 planar damping matrix presented earlier in equation (1). Doing this, we find

$$d_{\omega\tau} = r_{1x}^2 d_{yy} \quad (14)$$

Which implies that, regardless of the damping matrices of the two symmetric fingers, the torsional resistance to rotation ($d_{\omega\tau}$) will be coupled to the resistance to motion along the y -axis (d_{yy}) by the length of the effective moment arm, which is not surprising.

Unfortunately, the grasp-dependent nullspace vectors are, in general, algebraically complicated and are usually quite challenging to interpret, particularly for the spatial cases.

5.3.2 Grasp-independent nullspaces

The second type of left nullspace vector does not depend upon the particular geometry of the contact points. Nullspace vectors of this type are restrictions on what workpart damping matrices are attainable regardless of where the fingers grasp the workpart.

For example, in the planar case where we specify three symmetric fingers grasping a workpart, the three orthogonal left nullspace vectors (in matrix form) that result for the workpart are given by

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad (15a, 15b, 15c)$$

Dotting these vectors with a general 3×3 matrix, we note that the attainable workpart damping matrices must be such that

$$d_{21} = d_{12} \quad d_{31} = d_{13} \quad \text{and} \quad d_{23} = d_{32} \quad (16a, 16b, 16c)$$

Therefore, such nullspace vectors imply that, regardless of where we place the symmetric fingers, the workpart will have a symmetric damping matrix. Again, a sensible result.

From our examination of ranks, we note that the rank of \mathbf{B} for spatial cases never rises above 35. This implies that, regardless of how many fingers grasp a spatial workpart, we can never achieve a full rank mapping, as we could in the planar case by using three or more fingers to grasp the workpart. An interesting nullspace vector is the one for spatial workparts held by five or more fingers. The matrix form of this vector turns out to be

$$\mathbf{D}_{\text{null}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \square \quad (17)$$

This type of matrix is known as a *circulant* and we will call it \mathbf{C} . It is a nullspace in all of the spatial cases, regardless of how many fingers grasp a workpart or whether or not they are symmetric. From equation (12), its presence as a left nullspace vector implies that

$$\mathbf{vec}(\mathbf{D}_{\text{workpart}}) \cdot \mathbf{vec}(\mathbf{C}) = 0 \quad (18)$$

or, carrying out this dot product with the general spatial damping matrix given in equation (4), we find the restriction that applies to *all* spatial $\mathbf{D}_{\text{workpart}}$

$$d_{v_x \tau_x} + d_{v_y \tau_y} + d_{v_z \tau_z} + d_{\omega_x f_x} + d_{\omega_y f_y} + d_{\omega_z f_z} = 0 \quad (19)$$

$$\text{and} \quad d_{v_x \tau_x} + d_{v_y \tau_y} + d_{v_z \tau_z} = 0 \quad (20)$$

for symmetric matrices. We must note that, although the circulant is its own inverse, the restriction on damping does not imply that the same restriction exists for accommodation.

One mechanical interpretation of this circulant restriction is that there is no way to force a finger-held workpart to behave as a threaded fastener. For example, suppose we want a workpart to move unimpeded except along and about the x -axis. This implies that we want zero workpart damping elements except for $d_{v_x \tau_x}$ and $d_{\omega_x f_x}$. According to eq. (19), we must then have

$$d_{v_x \tau_x} + d_{\omega_x f_x} = 0 \quad \text{or} \quad d_{v_x \tau_x} = -d_{\omega_x f_x} \quad (21)$$

But this implies that the translational resistance to torques must be the *opposite* of the torsional resistance to translations. For example, suppose we choose $d_{v_x\tau_x}$ positive, so that a positive x -axis rotation maps to a positive x -axis force, as shown in Figure 2A. Then, according to equation (21), a corresponding positive x -axis velocity will necessarily map to a negative x -axis torque (Figure 2B).

Single Axis Interpretation of Circulant Restriction

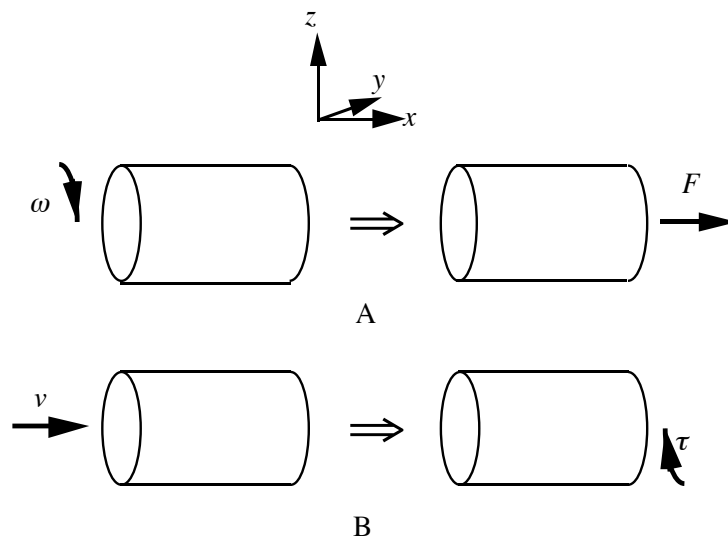


Figure 2. According to the circulant restriction for a system with damping only along and about the x -axis, if a positive rotation results in a positive force, then a positive velocity will result in a negative torque.

6 Conclusion

We have examined how the damping characteristics of point fingers map to damping characteristics of a grasped workpart. By “vectorizing” the workpart and finger damping matrices, we have formulated the mapping in such a way that we can define a geometry transform matrix \mathbf{B} relating the damping matrices of the fingers to the effective damping matrix of the workpart. The rank of \mathbf{B} allows us to determine how many fingers need grasp a workpart to achieve a full rank mapping. (A full rank mapping would mean that we could confer any desired damping matrix upon the workpart by choosing appropriate finger damping matrices.)

We have determined that planar workparts must be held with a minimum of three fingers in order to achieve full rank mappings. We have compared this result to a naive counting argument and found that the counting argument erroneously predicts that only two fingers are needed for planar symmetric full rank mappings.

We have found that, for spatial workparts, a five-fingered grasp allows for the maximum range of attainable matrices. However, there remains one restriction on the attainable workpart damping matrices that persists regardless of the number of fingers that grasp the workpart or where they grasp it.

By examining the left nullspace of \mathbf{B} , we are able to characterize the restrictions on attainable workpart damping matrices when a workpart is grasped with an insufficient number of fingers to achieve a full rank mapping. Examination of \mathbf{B} reveals that the circulant matrix describes the fundamental restriction on all damping matrices attainable by spatial workparts.

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