Impending Motion Direction of Contacting Rigid Bodies

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Contents

List of Figures

Abstract

Three different formulations are presented for expressing the initial motion direction of a system of contacting frictionless rigid bodies under gravity. The bodies are assumed to have no initial velocity. The first formulation expresses the accelerations of the bodies in terms of the contact forces between the bodies. The contact forces are themselves expressed as the solution to a quadratic programming problem. The second formulation expresses the accelerations of the bodies according to Gauss' "principle of least constraint." This principle is well-known to apply to systems with holonomic motion constraints (such as joints or hinges); in this paper, we show that the principle extends to the nonholonomic constraints that arise due to contact between bodies. The third formulation is conceptually the simplest; it says simply that the initial acceleration of the system is in the direction that most quickly decreases the gravitational potential energy of the system without violating the contact constraints between the bodies.

1 Introduction

In this paper we consider the dynamics of a collection of frictionless rigid bodies with contact constraints. All bodies are initially motionless and are acted upon by an external force $m\mathbf{g}$ where m is a body's mass and $\mathbf{g} \in \mathbf{R}^3$ indicates a gravity field. One or more of the bodies are assumed to be fixed in place. Since the bodies are initially motionless, the impending motion for each body is in the direction of the initial acceleration of that body. In this paper, we show the equivalence of three different formulations for expressing the initial direction of acceleration. The first two formulations characterize not only the direction of the initial acceleration, but the magnitude as well.

The first formulation for the acceleration of rigid bodies with contact constraints has appeared several times in the literature[5, 6, 9, 4, 1]. This formulation expresses the acceleration of each body as a function of the net force and torque acting on each body. Since the external gravity force $m\mathbf{g}$ acting on each body is known *a priori*, only the unknown contact forces that arise between bodies at contact points need to be determined. The contact forces can be expressed in terms of the solution to a convex quadratic programming problem. Solving convex quadratic programs is a polynomial-time problem[7].

The second formulation is an application of Gauss' principle of least constraint to the collection of rigid bodies. Gauss' principle expresses the acceleration of systems with holonomic motion constraints as the solution to a minimization problem. We have not encountered any application of Gauss' principle to systems with nonholonomic constraints in the literature. We will show that Gauss' principle applies to the nonholonomic motion constraints that prevent interpenetration between contacting bodies. In the second formulation, the acceleration of the bodies is expressed as the solution to a convex quadratic programming problem.

The third formulation is in some ways the most attractive and elegant of the three. If we think of gravity as the gradient of a potential energy function, we naturally picture the initial acceleration of the system as a motion that carries the system "downhill," with respect to the potential energy function. We will show that the initial acceleration direction of the system is parallel to the steepest descent direction down the potential energy function that does not violate constraints due to contact. This appears to be an obvious statement; what is not so obvious, however, is that this statement is not well-defined until we describe precisely how we measure the steepness of energy descent in a given direction. Such a definition is intimately linked with how we measure distance in the space of motions for our system of rigid bodies.

2 Rigid Body Formulation

2.1 Mass Distribution

Let us describe the mass-distribution of a rigid body in a global frame of reference as a set of mass points, each with location $\mathbf{p}_i$ and mass m_i. The total mass M of a body is

$$M = \sum_i m_i. \tag{1}$$

The vector $\mathbf{c}$ denotes the center of mass of the body; that is, $\mathbf{c}$ satisfies

$$\sum_i m_i(\mathbf{p}_i - \mathbf{c}) = \mathbf{0} \tag{2}$$

for each body. (We will denote row vectors, column vectors, and matrices whose entries are all zero simply by "$\mathbf{0}$" throughout this paper. The dimension of $\mathbf{0}$ should be clear from the context in which it occurs. A scalar value of zero is written simply as "0.")

If we let $\mathbf{r}_i$ denote the world-space displacement of the ith mass point from the center of mass by

$$\mathbf{r}_i = \mathbf{p}_i - \mathbf{c} \tag{3}$$

then the inertia tensor $\mathbf{I}$ of the body is

$$\mathbf{I} = \sum_i m_i \begin{pmatrix} r_{iy}^2 + r_{iz}^2 & -r_{ix}r_{iy} & -r_{ix}r_{iz} \\ -r_{iy}r_{ix} & r_{ix}^2 + r_{iz}^2 & -r_{iy}r_{iz} \\ -r_{iz}r_{ix} & -r_{iz}r_{iy} & r_{ix}^2 + r_{iy}^2 \end{pmatrix}. \tag{4}$$

For a vector $\mathbf{u} \in \mathbf{R}^3$, define $\mathbf{u}^*$ to be the anti-symmetric matrix

$$\mathbf{u}^* = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}. \tag{5}$$

For any vector $\mathbf{v} \in \mathbf{R}^3$, $(\mathbf{u}^*)\mathbf{v} = \mathbf{u} \times \mathbf{v}$. Additionally, the relation $-\mathbf{u}^*\mathbf{u}^* = (\mathbf{u}^T\mathbf{u})\mathbf{1} - \mathbf{u}\mathbf{u}^T$ holds, where $\mathbf{1}$ is the 3×3 identity matrix. Using this relation, it is easy to show that

$$\mathbf{I} = \sum_i m_i((\mathbf{r}_i{}^T\mathbf{r}_i)\mathbf{1} - \mathbf{r}_i\mathbf{r}_i{}^T) = \sum_i -m_i\mathbf{r}_i{}^*\mathbf{r}_i{}^*. \tag{6}$$

In section 4, we will also make use of the relations $\mathbf{u}^*\mathbf{v} = -\mathbf{v}^*\mathbf{u}$ and $\mathbf{u}^{*T} = -\mathbf{u}^*$.

2.2 Motion Constraints

We will represent possible motions of a system of rigid bodies in terms of virtual displacements of each body. Let $\delta\mathbf{p}_i = (\delta\mathbf{c}_i, \delta\boldsymbol{\theta}_i)$ represent a displacement of the ith body in the system, with $\delta\mathbf{c}_i$ and $\delta\boldsymbol{\theta}_i$ vectors in $\mathbf{R}^3$. The vector $\delta\mathbf{c}_i$ denotes a translational displacement

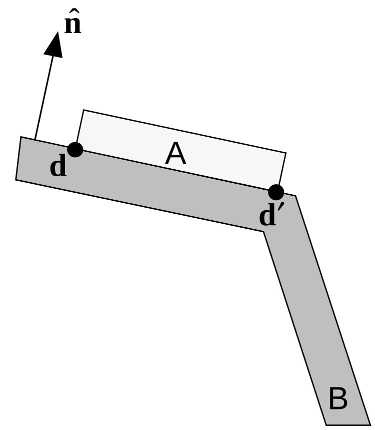

Figure 1: Contact between bodies *A* and *B*. Motion constraints are formulated in terms of the relative motion of the bodies at points $\mathbf{d}$ and $\mathbf{d}'$.

of the ith body, while $\delta\boldsymbol{\theta}_i$ denotes a rotation of magnitude $||\delta\boldsymbol{\theta}_i||$ of the body around its center of mass. The axis of the rotation is along the $\delta\boldsymbol{\theta}_i$ direction.

Contact between bodies generates constraints on the allowable displacements. Consider figure 1 where bodies *A* and *B* contact. If body *A* undergoes a displacement $\delta\mathbf{p}_a = (\delta\mathbf{c}_a, \delta\boldsymbol{\theta}_a)$, then point $\mathbf{d}$, as attached to body *A*, undergoes a particular displacement $\delta\mathbf{d}_a$. Similarly, a displacement $\delta\mathbf{p}_b$ of body *B* causes a displacement $\delta\mathbf{d}_b$ of point $\mathbf{d}$, as attached to body *B*. To prevent interpenetration from occurring, the relative displacement $\delta\mathbf{d}_a - \delta\mathbf{d}_b$ cannot have any component opposite the unit normal direction $\hat{\mathbf{n}}$. We can express this as the constraint

$$\hat{\mathbf{n}} \cdot (\delta\mathbf{d}_a - \delta\mathbf{d}_b) \geq 0. \tag{7}$$

Similarly, we also need to prevent interpenetration from occurring at point $\mathbf{d}'$ by requiring that $\hat{\mathbf{n}} \cdot (\delta\mathbf{d}'_a - \delta\mathbf{d}'_b) \geq 0$. If body *B* was fixed, the motion constraint at $\mathbf{d}$ would simply be

$$\hat{\mathbf{n}} \cdot \delta\mathbf{d}_a \geq 0 \tag{8}$$

and similarly for $\mathbf{d}'$. To simplify bookkeeping, we do not count fixed objects as bodies in our system; rather, we simply note when regular movable objects are in contact with fixed objects, and generate the appropriate motion constraint, such as equation (8).

We will assume that the motion constraints can be expressed by a finite number of constraint inequalities in the form of equation (7) or (8), all of which must be satisfied. (Palmer[11] and Baraff[2] contain further discussion on this issue.) That is, we consider systems whose motion constraints are expressed in terms of m contact points between the bodies. Let the ith contact point of the system be a contact between bodies *A* and *B* at the point $\mathbf{d}_i$ in a global frame of reference. Let $\hat{\mathbf{n}}_i$ denote the unit surface normal, pointing outwards from B towards A at $\mathbf{d}_i$, and let $\mathbf{c}_a$ and $\mathbf{c}_b$ denote the positions of the center of mass of bodies A and B respectively. If A undergoes a displacement $\delta\mathbf{p}_a = (\delta\mathbf{c}_a, \delta\boldsymbol{\theta}_a)$ then $\mathbf{d}_i$, as attached to A, undergoes the displacement

$$\delta\mathbf{c}_a + \delta\boldsymbol{\theta}_a \times (\mathbf{d}_i - \mathbf{c}_a).$$

Similarly, for a displacement $\delta\mathbf{p}_b = (\delta\mathbf{c}_b, \delta\boldsymbol{\theta}_b)$ of body B, $\mathbf{d}_i$'s displacement, as attached to B, is

$$\delta\mathbf{c}_b + \delta\boldsymbol{\theta}_b \times (\mathbf{d}_i - \mathbf{c}_b).$$

The motion constraint at the ith contact point is therefore

$$\hat{\mathbf{n}}_i \cdot (\delta\mathbf{c}_a + \delta\boldsymbol{\theta}_a \times (\mathbf{d}_i - \mathbf{c}_a) - \delta\mathbf{c}_b - \delta\boldsymbol{\theta}_b \times (\mathbf{d}_i - \mathbf{c}_b)) \geq 0. \tag{9}$$

Since each constraint is a linear inequality on the $\delta\mathbf{c}$ and $\delta\boldsymbol{\theta}$ variables, we can express the simultaneous satisfaction of all the constraints as one large linear system. If the vector $\delta\mathbf{p}$ denotes the virtual displacements of the n bodies by writing

$$\delta\mathbf{p} = \begin{pmatrix} \delta\mathbf{c}_1 \\ \delta\boldsymbol{\theta}_1 \\ \vdots \\ \delta\mathbf{c}_n \\ \delta\boldsymbol{\theta}_n \end{pmatrix},$$

then we express all m motion constraints by writing

$$\mathbf{J}\delta\mathbf{p} \geq \mathbf{0} \tag{10}$$

where $\mathbf{J}$ is an $m \times 6n$ matrix. The coefficients of $\mathbf{J}$ are computed according to equation (9). Using this notation, we can say that a legal motion for the system is a displacement $\delta\mathbf{p}$ that satisfies $\mathbf{J}\delta\mathbf{p} \geq \mathbf{0}$. Note that the displacement $\delta\mathbf{p} = \mathbf{0}$ always yields a legal motion (the null-motion).

3 Contact Force Formulation

At each of the m contact points, a contact force may arise between the contacting bodies, to prevent interpenetration. Since we are dealing with frictionless contacts, we know that the contact forces will act normal to the contact surfaces. Thus, at the ith contact point, we consider a contact force $\lambda_i\hat{\mathbf{n}}_i$ that acts on body A of the contact, and a contact force $-\lambda_i\hat{\mathbf{n}}_i$ that acts on body B of the contact, with λ_i the unknown scalar magnitude of the force-pair. Since $\hat{\mathbf{n}}_i$ is directed from B towards A, and since contact forces must be repulsive, we require $\lambda_i \geq 0$ for each contact point.

We define the $6n \times 6n$ block-diagonal generalized mass matrix $\mathbf{M}$ as

$$\mathbf{M} = \begin{pmatrix} M_1\mathbf{1} & \mathbf{0} & \cdots & \\ \mathbf{0} & \mathbf{I}_1 & & \\ \vdots & & \ddots & \vdots \\ & & M_n\mathbf{1} & \mathbf{0} \\ & \cdots & \mathbf{0} & \mathbf{I}_n \end{pmatrix}$$

where $\mathbf{1}$ denotes the 3×3 identity matrix, and M_i and $\mathbf{I}_i$ are the mass and inertia tens
the ith body. A net force and torque of $\mathbf{F}_i$ and $\boldsymbol{\tau}_i$ acting on each body is represented a
generalized force vector $\mathbf{Q}$ of length $6n$, defined by

$$\mathbf{Q} = \begin{pmatrix} \mathbf{F}_1 \\ \boldsymbol{\tau}_1 \\ \vdots \\ \mathbf{F}_n \\ \boldsymbol{\tau}_n \end{pmatrix}.$$

The force exerted on each body by gravity is $M_i\mathbf{g}$ while the torque
gravity field is uniform). Thus, the generalized gravity force $\mathbf{Q}_g$

$$\mathbf{Q}_g = \begin{pmatrix} M_1\mathbf{g} \\ \mathbf{0} \\ \vdots \\ M_n\mathbf{g} \\ \mathbf{0} \end{pmatrix}$$

(with $\mathbf{0} \in \mathbf{R}^3$ for this definition).

If $\mathbf{v}_i$ and $\boldsymbol{\omega}_i$ denote the linear and angular
$\mathbf{a}$ from a generalized force $\mathbf{Q}$ acting on the s

$$\mathbf{a} = \Bigg($$

Let $\boldsymbol{\lambda} \in \mathbf{R}^m$ be the ve
to the contact force pairs

The acceleration $\mathbf{a}$ of the sys

$$\mathbf{a} = \mathbf{M}$$

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Because our system is initially motionless, the constraint $\mathbf{J}\delta\mathbf{p} \geq \mathbf{0}$ on displacements yields the constraint

$$\mathbf{Ja} \geq \mathbf{0} \tag{17}$$

on the accelerations of the bodies. Consider the scalar $(\mathbf{Ja})_i$, which characterizes the relative acceleration in the $\hat{\mathbf{n}}_i$ direction at the ith contact point. If this quantity is positive, contact is being broken. Otherwise, $(\mathbf{Ja})_i = 0$ and contact is not broken. Since frictionless contact forces are workless, the contact force magnitude at the ith contact point must be zero if contact is being broken. We express this constraint by writing

$$\lambda_i(\mathbf{Ja})_i = 0. \tag{18}$$

Since each λ_i is nonnegative, we have $\boldsymbol{\lambda} \geq \mathbf{0}$. Combining this with the constraints $\mathbf{Ja} \geq \mathbf{0}$ and $\lambda_i(\mathbf{Ja})_i = 0$ for all i, we can write

$$\boldsymbol{\lambda} \geq \mathbf{0}, \quad \mathbf{Ja} \geq \mathbf{0} \qquad \text{and} \qquad \boldsymbol{\lambda}^T(\mathbf{Ja}) = 0. \tag{19}$$

Using equation (16), we rewrite this as

$$\boldsymbol{\lambda} \geq \mathbf{0}, \;\; \mathbf{JM}^{-1}\mathbf{J}^T\boldsymbol{\lambda} + \mathbf{JM}^{-1}\mathbf{Q}_g \geq \mathbf{0} \;\text{ and }\; \boldsymbol{\lambda}^T\left(\mathbf{JM}^{-1}\mathbf{J}^T\boldsymbol{\lambda} + \mathbf{JM}^{-1}\mathbf{Q}_g\right) = 0. \tag{20}$$

Equation (20) can be viewed as a quadratic program for the unknown $\boldsymbol{\lambda}$. If the matrix $\mathbf{JM}^{-1}\mathbf{J}^T$ is positive definite then $\boldsymbol{\lambda}$ is unique. Otherwise $\mathbf{JM}^{-1}\mathbf{J}^T$ is positive semidefinite (since $\mathbf{M}$ and thus $\mathbf{M}^{-1}$ is positive definite), and a solution exists for $\boldsymbol{\lambda}$ although it may not be unique. However, Cottle[3] has shown that if $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are solutions to equation (20) then

$$\mathbf{JM}^{-1}\mathbf{J}^T\boldsymbol{\lambda}_1 = \mathbf{JM}^{-1}\mathbf{J}^T\boldsymbol{\lambda}_2 \tag{21}$$

which implies

$$\mathbf{JM}^{-1}\mathbf{J}^T(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) = \mathbf{0}. \tag{22}$$

ultiplying both sides of this equation by $(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)^T$ yields

$$(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)^T\mathbf{JM}^{-1}\mathbf{J}^T(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) = (\mathbf{J}^T(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2))^T\mathbf{M}^{-1}(\mathbf{J}^T(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)) = 0. \tag{23}$$

ce $\mathbf{M}^{-1}$ is positive definite it must be that

$$\mathbf{J}^T(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) = \mathbf{0}, \tag{24}$$

plies $\mathbf{J}^T\boldsymbol{\lambda}_1 = \mathbf{J}^T\boldsymbol{\lambda}_2$. Thus, even if the solution $\boldsymbol{\lambda}$ to equation (20) is not unique, the cceleration of the system

$$\mathbf{M}^{-1}(\mathbf{J}^T\boldsymbol{\lambda}) + \mathbf{M}^{-1}\mathbf{Q}_g$$

(16) and (20) give us our first characterization of the impending motion of Given any $\boldsymbol{\lambda}$ that is a solution to equation (20), equation (16) describes the leration $\mathbf{a}$, and thus the impending motion direction of the system.

4 Gauss' Least Constraint Formulation

Gauss' principle of least constraint is a very elegant statement of the acceleration of a system with constraints[8]. Consider a system of particles $\mathbf{p}_i \in \mathbf{R}^3$, each with mass m_i and acted upon by a force $\boldsymbol{f}_i \in \mathbf{R}^3$. Let the scalar quantity Z, called the "constraint" of the system, be defined by

$$Z = \sum_i \frac{1}{2m_i}(\boldsymbol{f}_i - m_i\ddot{\mathbf{p}}_i)^T(\boldsymbol{f}_i - m_i\ddot{\mathbf{p}}_i). \tag{25}$$

Note that Z is nonnegative.

Gauss' principle states very simply that the accelerations $\ddot{\mathbf{p}}_i$ of the particles will minimize Z. If all the particles are completely unconstrained, the principle is obviously true, since the acceleration of every particle satisfies $m_i\ddot{\mathbf{p}}_i = \boldsymbol{f}_i$, yielding $Z = 0$. However, if there are constraints on the particles' accelerations, then the accelerations which minimize Z subject to those constraints are the accelerations that will actually occur.

Gauss' principle is easily shown to apply to systems with holonomic constraints. In such a system, if a motion direction is legal the opposite (or reverse) motion direction is legal as well. This is not necessarily true in our system. We will show however that Gauss' principle can be applied to our problem.

Let us write equation (25) in the form

$$Z = \sum_{j=1}^{n} \sum_i \frac{1}{2m_{ji}}(\boldsymbol{f}_{ji} - m_{ji}\ddot{\mathbf{p}}_{ji})^T(\boldsymbol{f}_{ji} - m_{ji}\ddot{\mathbf{p}}_{ji}) \tag{26}$$

where index j runs over the n bodies, and index i runs over the points of the jth body. The quantities m_{ji}, $\ddot{\mathbf{p}}_{ji}$ and $\boldsymbol{f}_{ji}$ are the mass, acceleration and force acting on the ith point of the jth body.

From equation (3), we can express the location of the ith particle in the jth body in terms of the jth body's center of mass $\mathbf{c}_j$, and the displacement $\mathbf{r}_{ji}$ as

$$\mathbf{p}_{ji} = \mathbf{c}_j + \mathbf{r}_{ji}. \tag{27}$$

The derivative of a vector $\mathbf{r}$ attached to a body with angular velocity $\boldsymbol{\omega}$ is given by $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$, so differentiating equation (27) yields

$$\dot{\mathbf{p}}_{ji} = \mathbf{v}_j + \boldsymbol{\omega}_j \times \mathbf{r}_{ji}. \tag{28}$$

Since $\mathbf{v}_j = \boldsymbol{\omega}_j = \mathbf{0}$, differentiating again yields

$$\ddot{\mathbf{p}}_{ji} = \dot{\mathbf{v}}_j + \dot{\boldsymbol{\omega}}_j \times \mathbf{r}_{ji}. \tag{29}$$

Using the "$*$" notation defined in section 2.1, and the fact that $\mathbf{u}^*\mathbf{v} = -\mathbf{v}^*\mathbf{u}$ we can rewrite equation (26) in the form

$$\begin{aligned} Z &= \sum_{j=1}^{n} \sum_i \frac{1}{2m_{ji}}(\boldsymbol{f}_{ji} - m_{ji}\dot{\mathbf{v}}_j - m_{ji}\dot{\boldsymbol{\omega}}_j \times \mathbf{r}_{ji})^T(\boldsymbol{f}_{ji} - m_{ji}\dot{\mathbf{v}}_j - m_{ji}\dot{\boldsymbol{\omega}}_j \times \mathbf{r}_{ji}) \\ &= \sum_{j=1}^{n} \sum_i \frac{1}{2m_{ji}}(\boldsymbol{f}_{ji} - m_{ji}\dot{\mathbf{v}}_j - m_{ji}\dot{\boldsymbol{\omega}}_j^*\mathbf{r}_{ji})^T(\boldsymbol{f}_{ji} - m_{ji}\dot{\mathbf{v}}_j - m_{ji}\dot{\boldsymbol{\omega}}_j^*\mathbf{r}_{ji}). \end{aligned} \tag{30}$$

Expanding equation (30) using the relations $\dot{\boldsymbol{\omega}}_j^* \mathbf{r}_{ji} = -\mathbf{r}_{ji}^* \dot{\boldsymbol{\omega}}_j$ and

$$\boldsymbol{f}_{ji}^T \dot{\boldsymbol{\omega}}_j^* \mathbf{r}_{ji} = -\boldsymbol{f}_{ji}^T \mathbf{r}_{ji}^* \dot{\boldsymbol{\omega}}_j = (-\boldsymbol{f}_{ji}^T \mathbf{r}_{ji}^* \dot{\boldsymbol{\omega}}_j)^T = -\dot{\boldsymbol{\omega}}_j^T \mathbf{r}_{ji}^{*T} \boldsymbol{f}_{ji} = \dot{\boldsymbol{\omega}}_j^T \mathbf{r}_{ji}^* \boldsymbol{f}_{ji}$$

yields

$$\begin{aligned} Z &= \sum_{j=1}^{n} \sum_i \frac{1}{2m_{ji}} \Big(\boldsymbol{f}_{ji}^T \boldsymbol{f}_{ji} - 2m_{ji} \boldsymbol{f}_{ji}^T \dot{\mathbf{v}}_j - 2m_{ji} \boldsymbol{f}_{ji}^T \dot{\boldsymbol{\omega}}_j^* \mathbf{r}_{ji} \\ &\qquad + 2m_{ji}^2 \dot{\mathbf{v}}_j^T \dot{\boldsymbol{\omega}}_j^* \mathbf{r}_{ji} + m_{ji}^2 \dot{\mathbf{v}}_j^T \dot{\mathbf{v}}_j + m_{ji}^2 (\dot{\boldsymbol{\omega}}_j^* \mathbf{r}_{ji})^T (\dot{\boldsymbol{\omega}}_j^* \mathbf{r}_{ji}) \Big) \\ &= \sum_{j=1}^{n} \sum_i \Big(\frac{\boldsymbol{f}_{ji}^T \boldsymbol{f}_{ji}}{2m_{ji}} - \boldsymbol{f}_{ji}^T \dot{\mathbf{v}}_j - \dot{\boldsymbol{\omega}}_j^T \mathbf{r}_{ji}^* \boldsymbol{f}_{ji} \\ &\qquad - m_{ji} \dot{\mathbf{v}}_j^T \mathbf{r}_{ji}^* \dot{\boldsymbol{\omega}}_j + \tfrac{1}{2} m_{ji} \dot{\mathbf{v}}_j^T \dot{\mathbf{v}}_j + \tfrac{1}{2} m_{ji} (\mathbf{r}_{ji}^* \dot{\boldsymbol{\omega}}_j)^T (\mathbf{r}_{ji}^* \dot{\boldsymbol{\omega}}_j^*) \Big) \quad . \end{aligned}$$

We can break this up into separate sums:

$$\begin{aligned} Z &= \sum_{j=1}^{n} \sum_i \frac{\boldsymbol{f}_{ji}^T \boldsymbol{f}_{ji}}{2m_{ji}} - \sum_{j=1}^{n} \left(\sum_i \boldsymbol{f}_{ji} \right)^T \dot{\mathbf{v}}_j - \sum_{j=1}^{n} \dot{\boldsymbol{\omega}}_j^T \left(\sum_i \mathbf{r}_{ji}^* \boldsymbol{f}_{ji} \right) \\ &\quad - \sum_{j=1}^{n} \dot{\mathbf{v}}_j^T \left(\sum_i m_{ji} \mathbf{r}_{ji}^* \right) \dot{\boldsymbol{\omega}}_j + \sum_{j=1}^{n} \dot{\mathbf{v}}_j^T \left(\sum_i \tfrac{1}{2} m_{ji} \right) \dot{\mathbf{v}}_j \\ &\quad + \sum_{j=1}^{n} \dot{\boldsymbol{\omega}}_j^T \left(\sum_i \tfrac{1}{2} m_{ji} \mathbf{r}_{ji}^{*T} \mathbf{r}_{ji}^* \right) \dot{\boldsymbol{\omega}}_j. \end{aligned} \tag{31}$$

The following identities hold for each body: for any body j, the net force $\mathbf{F}_j$ acting on that body is

$$\mathbf{F}_j = \sum_i \boldsymbol{f}_{ji}. \tag{32}$$

Similarly, the net torque on the jth body is

$$\boldsymbol{\tau}_j = \sum_i \mathbf{r}_{ji} \times \boldsymbol{f}_{ji} = \sum_i \mathbf{r}_{ji}^* \boldsymbol{f}_{ji}. \tag{33}$$

From equations (1) and (2) the relations

$$M_j = \sum m_{ji} \qquad \text{and} \qquad \sum m_{ji} \mathbf{r}_{ji} = \mathbf{0} \tag{34}$$

hold for each body, and using the linearity of the "$*$" operator,

$$\sum m_{ji} \mathbf{r}_{ji}^* = \left(\sum m_{ji} \mathbf{r}_{ji} \right)^* = \mathbf{0}. \tag{35}$$

Last, the inertia tensor $\mathbf{I}_j$ of each body is given by

$$\mathbf{I}_j = \sum -m_{ji} \mathbf{r}_{ji}^* \mathbf{r}_{ji}^*. \tag{36}$$

Using these relations, equation (31) simplifies to

$$Z = \sum_{j=1}^{n} \sum_{i} \frac{\boldsymbol{f}_{ji}^T \boldsymbol{f}_{ji}}{2m_{ji}} - \sum_{j=1}^{n} \dot{\mathbf{v}}_j^T \mathbf{F}_j - \sum_{j=1}^{n} \dot{\boldsymbol{\omega}}_j^T \boldsymbol{\tau}_j + \frac{1}{2} \sum_{j=1}^{n} \dot{\mathbf{v}}_j^T M_j \dot{\mathbf{v}}_j + \frac{1}{2} \sum_{j=1}^{n} \dot{\boldsymbol{\omega}}_j^T \mathbf{I}_j \dot{\boldsymbol{\omega}}_j. \tag{37}$$

If the net force $\mathbf{F}_j$ on each body is $M_j \mathbf{g}$ and the net torque is zero, using the definitions of $\mathbf{Q}_g$, $\mathbf{a}$ and $\mathbf{M}$ from the previous section we can write simply

$$Z = \sum_{j=1}^{n} \sum_{i} \frac{\boldsymbol{f}_{ji}^T \boldsymbol{f}_{ji}}{2m_{ji}} - \mathbf{a}^T \mathbf{Q}_g + \frac{1}{2} \mathbf{a}^T \mathbf{M} \mathbf{a}. \tag{38}$$

We would like to show that the acceleration $\mathbf{a}$ which solves

$$\min Z \quad \text{subject to} \quad \mathbf{Ja} \geq \mathbf{0} \tag{39}$$

is the same as the acceleration $\mathbf{a}$ of the previous section. Since Z is a quadratic function of $\mathbf{a}$, and $\mathbf{M}$ is positive definite, problem (39) is a convex quadratic programming problem with a unique solution $\mathbf{a}$. The first-order optimality conditions, or *KKT* conditions[10], for a constrained optimization problem are both necessary and sufficient when applied to a convex quadratic programming problem. The KKT conditions for $\mathbf{a}$ to be a solution to problem (39) are that there exists $\boldsymbol{\lambda} \in \mathbf{R}^m$ such that

$$\frac{\partial Z}{\partial \mathbf{a}} - \mathbf{J}^T \boldsymbol{\lambda} = \mathbf{0}, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \quad \mathbf{Ja} \geq \mathbf{0}, \quad \text{and} \quad \boldsymbol{\lambda}^T (\mathbf{Ja}) = 0. \tag{40}$$

Differentiating Z with respect to $\mathbf{a}$ yields

$$\frac{\partial Z}{\partial \mathbf{a}} = \mathbf{Ma} - \mathbf{Q}_g. \tag{41}$$

The condition $\partial Z / \partial \mathbf{a} - \mathbf{J}^T \boldsymbol{\lambda} = \mathbf{0}$ is thus

$$\mathbf{Ma} - \mathbf{Q}_g - \mathbf{J}^T \boldsymbol{\lambda} = \mathbf{0} \tag{42}$$

or simply

$$\mathbf{a} = \mathbf{M}^{-1} \mathbf{Q}_g + \mathbf{M}^{-1} \mathbf{J}^T \boldsymbol{\lambda}. \tag{43}$$

This means that the necessary and sufficient conditions for $\mathbf{a}$ to solve problem (39) are simply

$$\mathbf{a} = \mathbf{M}^{-1} \mathbf{Q}_g + \mathbf{M}^{-1} \mathbf{J}^T \boldsymbol{\lambda}, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \quad \mathbf{Ja} \geq \mathbf{0}, \quad \text{and} \quad \boldsymbol{\lambda}^T (\mathbf{Ja}) = 0. \tag{44}$$

owever, this is precisely the same as the formulation for $\mathbf{a}$ given by equations (16) and (20) section 3. As we noted before, the product $\mathbf{J}^T \boldsymbol{\lambda}$ is unique even though $\boldsymbol{\lambda}$ may not be. uation (44) is a direct proof of this, in that a solution $\mathbf{a}$ to equation (44) is unique because s the (unique) minimizer of a positive definite quadratic program, namely problem (39).

5 Steepest Descent Formulation

The formulation of this section is the simplest of the three to state. We claim that a system's initial acceleration is in the direction which most quickly the decreases potential of the system without violating the contact constraints. Although the physical intuition behind this statement is valid, the translation of this statement to a mathematical problem requires care.

Suppose that we describe the geometric state of the n bodies in our system by a vector $\mathbf{x} \in \mathbf{R}^{6n}$. The uniform gravity field acting on our system of objects can be considered the gradient of a potential energy function $U(\mathbf{x})$. We would like to describe the impending motion of our system by saying that the system moves in the direction that most quickly decreases the potential energy, without violating any of the contact constraints; that is, the system moves in the direction of steepest descent (with respect to the function U) that is legal. What precisely do we mean when we say "steepest descent"?

Given two displacement directions $\delta\mathbf{p}_1, \delta\mathbf{p}_2 \in \mathbf{R}^{6n}$ we say that $\delta\mathbf{p}_1$ is a steeper displacement direction if

$$\nabla U(\mathbf{x}) \cdot \delta\mathbf{p}_1 < \nabla U(\mathbf{x}) \cdot \delta\mathbf{p}_2. \tag{45}$$

Clearly, this definition makes no sense unless we are comparing vectors $\delta\mathbf{p}_1$ and $\delta\mathbf{p}_2$ of equal length. If we agree to define a motion direction in our system as a displacement $\delta\mathbf{p}$ of unit length, then we can say that the direction of steepest descent is the unit displacement $\delta\mathbf{p}$ that minimizes

$$\nabla U(\mathbf{x}) \cdot \delta\mathbf{p}.$$

At first glance this appears to adequately define the steepest descent direction. In fact, the definition is still incomplete. Until we specify how we plan to measure distance, that is, what constitutes a unit vector, we still cannot say what the steepest descent direction is. For example, consider figure 2 which shows a potential energy function $U(x, y) = y$. In figure 2a, we have defined length using the two-norm $\|\mathbf{v}\|_2$ of a vector defined by

$$\|\mathbf{v}\|_2 = \sqrt{\mathbf{v}^T\mathbf{v}} = \sqrt{v_x^2 + v_y^2 + v_z^2}. \tag{46}$$

When we use this "standard" distance measure, the set of unit vectors centered at a point (x, y) in the plane traces out a circle in the plane. Using this distance measure, the steepest descent direction at (x, y) is

$$\frac{\nabla U(x, y)}{\|\nabla U(x, y)\|},$$

which is the direction $(0, -1)$ since ∇U points straight upwards (at every point).

In figure 2b however, we are using a different metric for measuring distance. Given a symmetric positive definite matrix $\mathbf{A}$ we define the "$\|\cdot\|_{\mathbf{A}}$-norm" of a vector $\mathbf{v}$ by[1]

$$\|\mathbf{v}\|_{\mathbf{A}} = \sqrt{\mathbf{v}^T\mathbf{A}\mathbf{v}}. \tag{47}$$

[1]The special case when $\mathbf{A}$ is the identity matrix multiplied by some positive scalar α^2 yields $\|\mathbf{v}\|_{\mathbf{A}} = \alpha\|\mathbf{v}\|_2$ for all $\mathbf{v}$. The matrix $\mathbf{A}$ used to define distance in figure 2b is not a scalar multiple of the identity matrix.

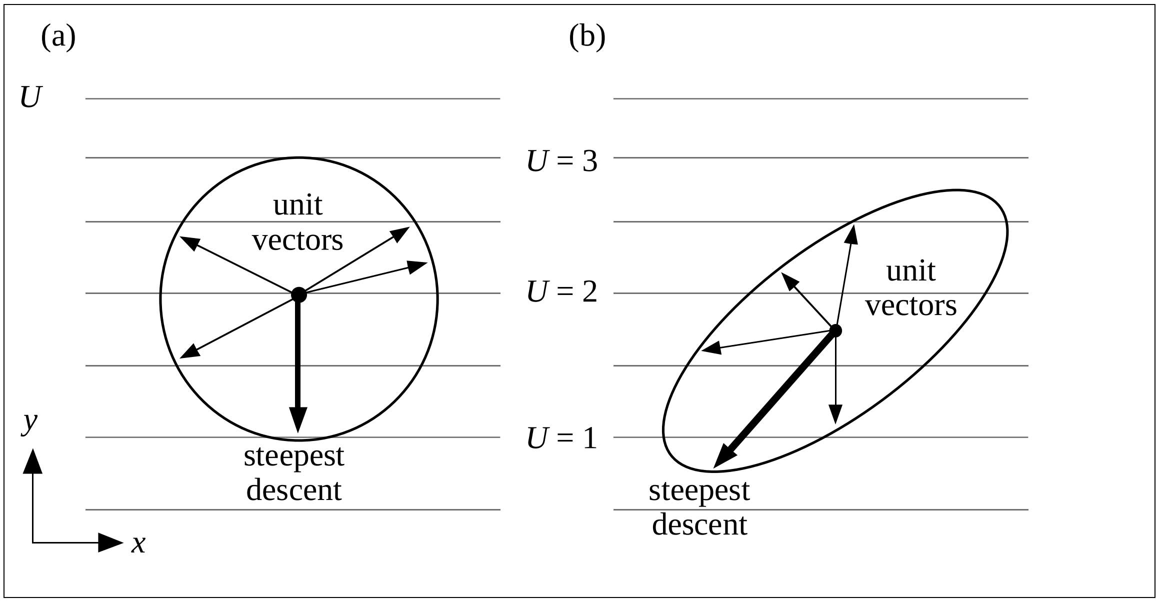

Figure 2: (a) The set of vectors $\mathbf{v}$ defined by $\|\mathbf{v}\|_2 = 1$ forms a circle. The vector in this set for which the largest decrease in U is achieved is indicated by the bold vertical vector. (b) Distance is now defined by the $\|\cdot\|_{\mathbf{A}}$-norm, where $\mathbf{A}$ is a symmetric positive definite matrix. The set of vectors $\mathbf{v}$ satisfying $\|\mathbf{v}\|_{\mathbf{A}} = 1$ is an ellipse, rather than a circle. Under this new distance measure, the unit-length vector for which U decreases the most points down and to the left.

Under this distance metric, the vector $\delta\mathbf{p}$ which satisfies $\|\delta\mathbf{p}\|_{\mathbf{A}} = 1$ and gives the steepest descent (that is, minimizes $\nabla U(x, y) \cdot \delta\mathbf{p}$) is *not* parallel to $\nabla U(x, y)$. Instead, the steepest descent direction points down and to the left.

Why might we prefer to use a norm other than the standard two-norm for measuring distance? Consider a planar rigid body with degrees of freedom x, y, and θ. What constitutes a "unit displacement" of the body? We could consider the set of displacements δx, δy and $\delta\theta$ of the rigid body such that

$$(\delta x)^2 + (\delta y)^2 + (\delta\theta)^2 = 1; \tag{48}$$

that is, all unit two-norm displacements. If we measure x and y in centimeters and θ in radians, then the set of unit displacements is some set V_1. However, if we measure x and y in meters and θ in radians, we get a completely different set of displacements V_2. The sets V_1 and V_2 are fundamentally different, in that neither is a scalar multiple of the other, just as neither the circle nor the ellipse in figure 2 is a scalar multiple of the other. As a result, if we define the steepest descent direction based on a two-norm measure of distance, the steepest descent direction *depends on the units of measurement chosen.* Defining unit displacements of a system of rigid bodies based on the two-norm is therefore a completely arbitrary definition.

We propose that a more natural way to measure the length of a displacement is based on the kinetic energy of a system. If the vector $\mathbf{v}$ is the velocity of our system, that is,

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \boldsymbol{\omega}_1 \\ \vdots \\ \mathbf{v}_n \\ \boldsymbol{\omega}_n \end{pmatrix} \tag{49}$$

then the kinetic energy T of the system is

$$T = \tfrac{1}{2}\mathbf{v}^T\mathbf{M}\mathbf{v} \tag{50}$$

where $\mathbf{M}$ the generalized mass matrix defined in equation (11). We will define distance in terms of the $\|\cdot\|_{\mathbf{M}}$-norm; this allows us a measurement-invariant way of defining t steepest descent direction.[2]

We would like to show that using the $\|\cdot\|_{\mathbf{M}}$-norm we can characterize the i acceleration of our system of objects in a very simple manner. The initial acceler direction is described by simply saying that it is the steepest legal descent direction o the function U, where steepest is with respect to distance measured by the $\|\cdot\|_{\mathbf{M}}$

Mathematically, we will say that if a system with configuration $\mathbf{x}$ has nonzero accel then that acceleration is parallel to the displacement $\delta\mathbf{p}$ which solves

$$\min_{\delta\mathbf{p}} \nabla U(\mathbf{x}) \cdot \delta\mathbf{p} \qquad \text{subject to} \quad \left\{ \begin{array}{l} \mathbf{J}\delta\mathbf{p} \geq \mathbf{0} \text{ and} \\ \|\delta\mathbf{p}\|_{\mathbf{M}}^2 = 1 \end{array} \right\}.$$

In the case of a uniform gravity field, $-\nabla U = \mathbf{Q}_g$ everywhere, so we can say tha acceleration is parallel to the solution $\delta\mathbf{p}$ of

$$\min_{\delta\mathbf{p}} -\delta\mathbf{p}^T\mathbf{Q}_g \qquad \text{subject to} \quad \left\{ \begin{array}{l} \mathbf{J}\delta\mathbf{p} \geq \mathbf{0} \text{ and} \\ \|\delta\mathbf{p}\|_{\mathbf{M}}^2 = 1 \end{array} \right\}.$$

(Note that we do not bother to characterize the acceleration direction of a stable s (that is, a system with acceleration $\mathbf{a} = \mathbf{0}$) in terms of a solution to problem (52). For systems, problem (52) may fail to have a unique solution or any solution at all.)

To prove our claim, we will show that whenever the solution $\mathbf{a}$ to equation (nonzero, there exists a positive scalar α such that

$$\delta\mathbf{p} = \alpha\mathbf{a} \tag{}$$

[2]The *shape* of the set of unit vectors under this norm does *not* depend on the units of measurement u That is, consider two observers with differing measurement systems. Suppose observer A determines th of velocities V_A for which the kinetic energy of the system is some particular value T_A, and suppose obse B determines the set of velocities V_B which yield a kinetic energy of T_B. If the energies T_A and T_B are same (even though A and B may use different units to describe them), then the sets V_A and V_B describe same vectors (even though A's and B's coordinate description of the individual vectors differ, because A B have differing measuring systems). Even if T_A and T_B are not the same, the sets V_A and V_B are the s up to a scalar multiple; that is, every vector in V_A corresponds to a scalar α times a vector in V_B, where a constant depending on T_A and T_B. Thus, two observers can always select vector sets with the same *sh* (but not scale) by having each observer select all vectors that yield unit kinetic energy (or any constant) i measurement units chosen by that observer.

solves problem (52). Let $\mathbf{a} = \mathbf{a}^*$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$ satisfy equation (44), so that $\mathbf{a}^*$ is the actual acceleration of the system, with $\mathbf{a}^*$ nonzero. We claim that

$$\delta\mathbf{p} = \frac{1}{\|\mathbf{a}^*\|_{\mathbf{M}}}\mathbf{a}^* \tag{54}$$

is the solution to problem (52). Since problem (52) is a linear minimization with quadratic constraints, problem (52)'s KKT conditions are both necessary and sufficient for a solution $\delta\mathbf{p}$. The KKT conditions to problem (52) are

$$\mathbf{Q}_g + \mathbf{J}^T\boldsymbol{\lambda} - 2s\mathbf{M}\delta\mathbf{p} = \mathbf{0}, \quad \|\delta\mathbf{p}\|^2_{\mathbf{M}} = 1, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \quad \mathbf{J}\delta\mathbf{p} \geq \mathbf{0}, \quad \text{and} \quad \boldsymbol{\lambda}^T(\mathbf{J}\delta\mathbf{p}) = 0 \tag{55}$$

where s is an unconstrained scalar and $\boldsymbol{\lambda} \in \mathbf{R}^m$. We can rewrite these conditions as

$$2s\delta\mathbf{p} = \mathbf{M}^{-1}\mathbf{Q}_g + \mathbf{M}^{-1}\mathbf{J}^T\boldsymbol{\lambda}, \quad \|\delta\mathbf{p}\|^2_{\mathbf{M}} = 1, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \quad \mathbf{J}\delta\mathbf{p} \geq \mathbf{0}, \quad \text{and} \quad \boldsymbol{\lambda}^T(\mathbf{J}\delta\mathbf{p}) = 0. \tag{56}$$

To see that $\delta\mathbf{p} = \mathbf{a}^*/\|\mathbf{a}^*\|_{\mathbf{M}}$ fulfills the KKT conditions (that is, equation (56)), let $2s = \|\mathbf{a}^*\|_{\mathbf{M}}$. Then since $\mathbf{a}^*$ and $\boldsymbol{\lambda}^*$ satisfy equation (44), choosing $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$ yields

$$\mathbf{M}^{-1}\mathbf{Q}_g + \mathbf{M}^{-1}\mathbf{J}^T\boldsymbol{\lambda} = \mathbf{a}^* = 2s\delta\mathbf{p} \tag{57}$$

s well as

$$\boldsymbol{\lambda} \geq \mathbf{0}. \tag{58}$$

ince $\mathbf{J}\mathbf{a}^* \geq \mathbf{0}$ and $\boldsymbol{\lambda}^{*T}(\mathbf{J}\mathbf{a}^*) = 0$, we have

$$\mathbf{J}\delta\mathbf{p} = \mathbf{J}\frac{\mathbf{a}^*}{\|\mathbf{a}^*\|_{\mathbf{M}}} = \frac{1}{\|\mathbf{a}^*\|_{\mathbf{M}}}\mathbf{J}\mathbf{a}^* \geq \mathbf{0} \tag{59}$$

d

$$\boldsymbol{\lambda}^T(\mathbf{J}\delta\mathbf{p}) = \boldsymbol{\lambda}^T\left(\mathbf{J}\frac{\mathbf{a}^*}{\|\mathbf{a}^*\|_{\mathbf{M}}}\right) = \frac{1}{\|\mathbf{a}^*\|_{\mathbf{M}}}\boldsymbol{\lambda}^{*T}(\mathbf{J}\mathbf{a}^*) = 0. \tag{60}$$

ally,

$$\|\delta\mathbf{p}\|^2_{\mathbf{M}} = \delta\mathbf{p}^T\mathbf{M}\delta\mathbf{p} = \frac{1}{\|\mathbf{a}^*\|^2_{\mathbf{M}}}\mathbf{a}^{*T}\mathbf{M}\mathbf{a}^* = \frac{1}{\|\mathbf{a}^*\|^2_{\mathbf{M}}}\|\mathbf{a}^*\|^2_{\mathbf{M}} = 1. \tag{61}$$

conclude that the direction of acceleration is indeed parallel to the solution $\delta\mathbf{p}$ to blem (52).

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